ON CYCLIC COVERS OF THE RIEMANN SPHERE AND A RELATED CLASS OF CURVES

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ABSTRACT. This note consists of two short parts. The first part is semi-expository and summarizes some known and less well-known classification results about moduli and automorphisms of prime cyclic covers of the Riemann sphere. In the second part, we restrict attention to those curves affording fixed-point free induced actions on their vector space of holomorphic differentials. These curves correspond to those with all cyclic actions ramifying over the sphere. We describe them completely in terms of their affine equations.

1. AN ILLUSTRATIVE EXAMPLE

A Riemann surface in the sense of Riemann is the collection of all branches of a multi-valued algebraic function $w = f(z)$ obtained by solving an irreducible polynomial equation

$$P(z, w) = a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0,$$

where the $a_i(z)$ are polynomials in z. In the sense of Poincaré however, a Riemann surface (of genus $g > 1$) is the quotient of the upper half plane $\mathcal{H}$ by a discrete torsion free subgroup Γ of $\mathrm{Aut}(\mathcal{H})$ (i.e. a Fuchsian group). A standard difficulty in the theory of Riemann surfaces is going back and forth between Riemann's approach and Poincaré's (for recent work in that direction see [2]). Below is a leisurely example of how this correspondence sometimes work.

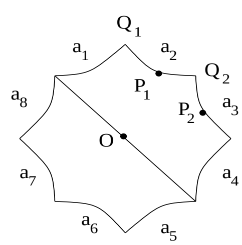

FIGURE 1. Fundamental polygon $P(2)$ for a genus 2 (compact) curve

Let C be a closed Riemann surface (or curve for short) of genus $g = 2$ uniformized by a Fuchsian group Γ with fundamental domain in the hyperbolic plane given by a regular polygon as shown in figure 1, with 8 sides labeled $a_1, \ldots, a_8$ and ordered clockwise. All angles are equal to $2\pi/8$ and all sides have equal length. The group Γ is generated by elements $\gamma_1, \dots \gamma_4$, where γ_i is defined by the conditions $\gamma_i(P) \cap int(P) = \emptyset$, $\gamma_i(a_i) = a_{i+4}$ if $i = 1, 3$, and $\gamma_i(a_{i+4}) = a_i$ if $i = 2, 4$. The polygon $P(2)$ is *canonical* in the sense of Schaller [24]. That Γ has fundamental domain $P(2)$ is a consequence of a classical theorem of Poincaré (cf. [24], section 3). Note that all the vertices Q_i are identified under the action of Γ and they map to a unique Q on the surface $C = \mathcal{H}/\Gamma$.

Rotation about the center of the polygon O gives an action of the cyclic group $G = \mathbb{Z}_8$ on $P(2)$ respecting identifications, and hence an action on C. By closing down sides, the quotient surface of C under the action of G is obviously $\mathbb{P}^1$. Consider then the sequence of quotient maps

$$P(2) \xrightarrow{q} C \xrightarrow{\pi} \mathbb{P}^1$$

Naturally the origin O (or its image $q(O)$ in C) is a fixed point of the action. It is hence a *ramification point* of π and its image in $\mathbb{P}^1$ a *branching* point. Since Q_1 and all of its translates $Q_i = T^i(Q_1)$ get identified under q, their image in C is also a fixed point of G. Similarly P_i and $P_{4+i} = T^4 P_i$ get identified under q so that $q(P_i)$ is a fixed point for the subgroup of order 2 in G. These are the only fixed points of the action and we get in total six ramification points on the surface.

On the other hand, there are only three branched points in $\mathbb{P}^1$ given by $\pi(q(O))$, $\pi(q(Q_1))$ and $\pi(q(P_1))$. The ramification about the points $q(O), q(Q_1)$ and $q(P_1)$ is such that there are 8 sheets coming together at O, the same number at Q_1 and only 2 sheets at P_1. That is the "ramification numbers" of the action are $2, 8, 8$. This is naturally all consistent with the Riemann Hurwitz formula relating the genus g of C to the genus h of C/G and the ramification numbers[1] n_y of each point y in the branch locus B; i.e.

$$2 - 2g = |G| \left(2 - 2h - \sum_{y \in B} \left(1 - \frac{1}{n_y} \right) \right)$$

Next we make use of Galois theory for coverings to describe a form for the affine equation $f(x, y) = 0, (x, y) \in \mathbb{C}^2$ that the curve satisfies (this form is not unique). Given a curve C, we consider its field of meromorphic functions $\mathcal{M}(C) = \{f : C \longrightarrow \mathbb{C} \text{ meromorphic}\}$. Note that $\mathcal{M}(\mathbb{P}^1)$ is $\mathbb{C}(x)$ the field of rational functions in one variable. Since $\pi : C \longrightarrow \mathbb{P}^1$ is a covering, we get by precomposition a map $\mathbb{C}(x) \longrightarrow \mathcal{M}(C)$ which is an inclusion of fields. This exhibits $\mathcal{M}(C)$ as a field extension of $\mathbb{C}(x)$ which in fact is Galois if the original cover is Galois. In the case at hand, $\mathbb{P}^1$ is the quotient of C by the action of $G = \mathbb{Z}_8$ and the cover is by definition Galois.

The fact that our curve is dimension one (complex), $\mathcal{M}(C)$ is necessarily an algebraic function field in one variable (Siegel), or more precisely it is a finite field extension of $\mathbb{C}(x)$. We can write $\mathcal{M}(C) = \mathbb{C}(x, y)$ with x, y satisfying an equation $F(x, y) = a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$ with $a_i(x) \in \mathbb{C}(x)$, and n is the degree of the covering. Now theorem 6.2 in [18] (chap. VIII,§6) states that if K is a degree

[1] These numbers correspond to the order of the stabilizer subgroups at the fixed points, and these subgroups are necessarily cyclic.

n cyclic field extension of k, n prime to char(k) and k containing a primitive n-th root, then there is $\alpha \in K$ such that $K = k(\alpha)$ and α satisfies an equation $Y^n - a = 0$ for $a \in k$. In our case, $k = \mathbb{C}(x)$, $n = 8$ and $a = f(x)$ so that $\mathcal{M}(C) = \mathbb{C}(x, y)$ with $y^8 = f(x)$. The curve C is the locus of this polynomial and the cyclic quotient $C \to \mathbb{P}^1$ is the restriction to C of the projection $\mathbb{C}^2 \to \mathbb{C}$, $(x, y) \mapsto x$. Note that $F(x, y) = y^8 - f(x)$ is necessarily irreducible since we started with a connected Riemann surface.

We can set our three branched points[2] to be $0, 1$ and -1. These points correspond to the zeros of $f(x)$, so that our equation becomes $y^8 = x^a(x + 1)^b(x - 1)^c$. The numbers a, b, c relate to the ramification indexes in an interesting way. Notice that in a small neighborhood of either O, Q_1 or P_1, the action of G is rotation by a multiple of $2\pi/8$. These multiples can be chosen to be a, b, c (respectively) and are such that

$$a + b + c \equiv 0 \bmod 8 \ , 0 < a, b, c < 8 \ , \quad [a, 8] = [b, 8] = 1 \ , [c, 8] = 4$$

The first congruence is necessary to avoid having ramification over the point at infinity in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ (there are only 3 branched points as we pointed out). The triple (a, b, c) is well defined up to a multiple k prime to 8 (since of course the numbering of the sheets of $C \longrightarrow \mathbb{P}^1$ is only well-defined up to permutation). If we choose $a = 1$, then necessarily $b = 3$ and $c = 4$. Our surface has affine equation

$$y^8 = x(x - 1)^3(x + 1)^4$$

This curve was studied by Kulkarni for instance (cf. [14], [15]) who determined its full automorphism group $Aut(C) = GL_2(\mathbb{F}_3)$ (the general linear group over the finite field $\mathbb{F}_3$). Exceptionally, this curve is completely determined by the fact that it is of genus two and that it admits a $\mathbb{Z}_8$-action.

Notice that since our curve is hyperelliptic (with involution rotation by π about the axis through Q_4 and Q_8 in figure 1), we could have searched for an equation of the form $y^2 = g(x)$. But the reduced group of automorphisms of this curve; i.e. the quotient by the central involution, is $GL_2(\mathbb{F}_3)/\mathbb{Z}_2 = S_4$ (see Coxeter-Moser, p:96), and by the classication result of Bolza (§3.1), another equation for C is $y^2 = x(x^4 - 1)$.

2. Synopsis

We say C is p-elliptic if it admits an action of the cyclic group $\mathbb{Z}_p$ with quotient $\mathbb{P}^1$. When p is a prime, these curves admit affine equations of the form

$$w^p = (z - e_1)^{a_1} \times (z - e_2)^{a_2} \times \cdots \times (z - e_r)^{a_r} \tag{1}$$

where $e_1, \dots, e_r$ are distinct complex numbers, and r is related to p by the formula $2g = (r - 2)(p - 1)$. We can assume (without any loss of generality) that $a_1, \dots, a_r$ are integers satisfying $1 \leq a_i \leq p - 1$ and $\sum_{i=1}^{r} a_i \equiv 0 \pmod p$. The projection $\pi : (w, z) \mapsto z$ is of course *branched* over the e_i's. The last condition ensures that there is no branching over ∞ in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.

[2]Recall that the action of $Aut(\mathbb{P}^1)$ on the Riemann sphere is 3-transitive and hence any choice for the 3 branched points yield isomorphic coverings.

In the first part of this note we collect and slightly expand on several disseminated facts about the automorphism groups and moduli of p-elliptic curves. Interestingly for example, and combining results in [11] and [14], we obtain

Proposition 1. *Suppose $g > (p-1)^2$, and p prime. Then a p-elliptic curve C_g admits a unique normal subgroup $\mathbb{Z}_p$ which acts with quotient the Riemann sphere.*

This is the analog for $p > 2$ of the known fact that hyperelliptic curves ($p = 2$) admit a *unique* hyperelliptic involution.

For a cyclic n-fold cover of the line (n not necessarily prime), it is believed that the branch data fully determines the curve (up to isomorphism). Nakajo [21] and Gabino-Diez [12] verified this directly for p prime. A calculation of Lloyd [19] on the other hand gives a count for the distinct isomorphism classes of p-elliptic curves of a given genus. The following, of which we give a slightly novel proof, summarizes the situation

Proposition 2. *: Assume $2g = (r-2)(p-1)$. Then the moduli space $\mathbf{M}(g,p)$ of genus g prime Galois covers of the sphere splits into N_r disjoint copies of $C_{r-2}(\mathbb{C})$, the configuration space of unordered $r-2$ tuples of distinct complex numbers, where N_r is obtained from the generating function*

$$\sum_r N_r x^r = \frac{1}{p-1}\left\{\frac{1}{p}\left[\frac{1}{(1-x)^{p-1}} + (p-1)\frac{(1-x)}{(1-x^p)}\right] + \sum_{\substack{ll'=p-1\\ l\neq 1}} \phi(l)(1-x^l)^{-l'}\right\}$$

Remarks.

(a) In the case $p = 2$ ($r = 2g+2$), the series reduces to $1 + x^2 + x^4 + \cdots$ and hence $N_r = 1$ (for any given $g > 1$). The moduli space is connected and coincides with the set of conformal classes of hyperelliptic curves (classical).

(b) When $r = 3$, $g = (p-1)/2$ and the curve is isomorphic to one with equation $y^p = x^a(x-1)$ for some a, $1 \leq a \leq \frac{p-1}{2}$ (see [14]). Such curves are usually called *Lefschetz*. The following count of conformal classes of distinct Lefschetz curves can be deduced from Proposition 2 and appears for instance in [22]:

$$N_3 = \begin{cases} (p+1)/6, \text{if } p \equiv 2 \ (mod\ 3) \\ (p+5)/6, \text{if } p \equiv 1 \ (mod\ 3) \end{cases}$$

Recall that any genus g curve C admits g-independent holomorphic one forms (Riemann). Let V be this vector space, and suppose $G \times C \to C$ is an action of the finite group G on the curve C. We say that the induced action on V is fixed point free if $+1$ is not an eigenvalue of any non-identity element $g \in G$. Such an element does act fixed point freely on $V^* = V\backslash 0$.

It has been observed in [13] that the associated linear representation $\rho : G \to GL(V)$ is fixed point free if and only if the genus of each orbit surface $C/\mathbb{Z}_p$ is zero for every subgroup $\mathbb{Z}_p \subseteq G$, where p is a prime dividing the order of G. A group with this property is said to have a "genus-zero action" on C. Such a property imposes strong Sylow conditions on G, which in fact have allowed the authors in [13] to completely classify these groups.

Theorem 3. [13] *The groups admitting genus-zero actions on surfaces of genus $g > 1$ are: the cyclic groups $\mathbb{Z}_{p^e}$ for primes $p \geq 2$ and exponents $e \geq 1$; the cyclic groups $\mathbb{Z}_{pq}$ for distinct primes p, q; the generalized quaternion groups $Q(2^n)$ for $n \geq 3$; and the ZM groups $G_{p,4}(-1)$ for odd primes p.*

For the definition of ZM (Zassenhaus metacyclic) groups see §5 or [27]. Our goal here is to give equations and complete listing for the surfaces admitting fixed point free actions on their vector space of differentials. We can summarize our calculations in one main theorem which we split into the following three propositions (cf. sections 4 and 4):

Proposition 4. *Let C be a Riemann surface of genus $g > 1$ admitting a genus zero action by $G = \mathbb{Z}_{pq}$. Then*

(1) *Either $g = \frac{1}{2}(p-1)(q-1)$, and C is isomorphic to the Fermat curve $w^p = z^q - 1$,*
(2) *or $g = (p-1)(q-1)$, and C is isomorphic to the surface with the equation $w^p = \dfrac{z^q - 1}{z^q - \lambda}$ (with $\lambda \neq 0$ and $\lambda^q \neq 1$).*

Proposition 5. *Let C be a Riemann surface of genus g admitting a genus zero action by $G = G_{p,4}(-1) = \langle x, y | x^p = 1,\ y^4 = 1,\ yxy^{-1} = x^{-1} \rangle$. Then $g = p - 1$ and C is isomorphic to the surface $w^2 = z^{2p} - 1$.*

Similar results are obtained for the cyclic groups $\mathbb{Z}_{p^e}$, $e > 1$, and for the quaternionic groups as summarized towards the end of the paper. Here's a sample:

Proposition 6. $Q_8 = \langle A, B \mid A^4 = 1, A^2 = B^2, BAB^{-1} = A^{-1} \rangle$ *admits a genus zero action on the genus 4 surface*

$$w^2 = z(z^4 - 1)(z^4 + 1) = z(z^8 - 1)$$

as follows: $A(x, y) = (-x, iy)$ and $B(x, y) = \big(-1/x, y/x^5\big)$. The induced action on the space of holomorphic differentials is fixed-point free.

3. Part I : On Cyclic Covers of the Sphere, their Automorphisms and Moduli

Automorphisms. One is tempted to classify all finite groups that can arise as automorphisms of p-elliptic curves $C_p : y^p = f(x)$ in terms of p and the polynomial f. We assume the genus g of C to be bigger than 2 so that the group $Aut(C)$ of all automorphisms of C is finite. It is clear that for generic curves, $Aut(C)$ is reduced to just $\mathbb{Z}_p$ (cf. [8] for instance). The answer is also known for both small and "sufficiently" large genera.

For curves of genus $g = 2$ and $g = 3$ the classification of the automorphism groups is completely known (whether the curve is p-elliptic or not). For $g = 3$ see [4], [17], [20]. Genus two curves are necessarily hyperelliptic (i.e. 2-elliptic) and have affine equations of the form $y^2 = f(x)$, $(x, y) \in \mathbb{C}^2$. The involution $(x, y) \mapsto (x, -y)$ is always central; see [10]. O. Bolza (1888) seems to have been first to determine the groups acting on genus two curves and write equations for them (see [1], chapter 1 for example). One of Bolza's curves has full automorphism group $GL_2(\mathbb{F}_3)$ (reduced group $GL_2(\mathbb{F}_3)/\mathbb{Z}_2 = S_4$) and has affine equation $w^2 = z(z^4-1)$. There is on the other hand a single isomorphism class of curves with automorphisms $\mathbb{Z}_{10}$; it has equation $w^2 = z^5 - 1$. The other possible full automorphism groups that can occur are $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, D_8 and D_{12} (see [23] for example).

Proposition 1 addresses the problem of determining $\mathrm{Aut}(C)$ when the number of branchpoints of f in the defining equation $C : y^p = f(x)$ is large enough (so is the genus).

PROOF (of Proposition 1) Let $C : y^p = f(x)$ be a p-elliptic curve, and denote by r the number of distinct roots of f (i.e. the branchpoints). This is related to g and p by the formula $2g = (r-2)(p-1)$. According to theorem 4 of [14], the p-cyclic subgroup G acting on a p-elliptic curve $C : y^p = f(x)$ becomes normal in $\mathrm{Aut}(C)$ as soon as r exceeds $2p$. On the other hand, a theorem of Gabino-Diez [11] asserts that any other cyclic subgroup G' of order p acting on C with quotient the Riemann sphere must be conjugate to G. Since G is normal, G' must coincide with G and hence the uniqueness statement. ∎

REMARKS AND COROLLARY:
(a) The theorem above implies that for p-elliptic curves with large enough genus $g > (p-1)^2$, $\mathrm{Aut}(C)$ is an extension of $\mathbb{Z}_p$ by a polyhedral group (i.e. a finite subgroup of SO(3)). The polyhedral groups are: the finite cyclic groups, the dihedral groups D_{2n} of order $2n$, the tetrahedral group A_4, the octahedral group S_4 and the icosahedral group A_5.
(b) The theorem is true for "general curves" of genus $g > p-1$, or equivalently for the number of branched points r exceeding 4. (A curve $y^p = f(x)$ is general for an open dense choice of branched points in S^2). This is essentially the statement of the main lemma in [8].
(c) Finally the conclusion of the theorem is not anymore true for smaller genus since the Fermat curve $x^p + y^p = 1$ has genus $(p-1)(p-2)/2$ but affords two distinct cyclic p-actions ramifying over the sphere (the obvious ones) and which are conjugate under the involution $(x,y) \mapsto (y,x)$.

Proposition 7. *The following cyclic p-covers admit actions by the corresponding polyhedral groups:*
(1) Dihedral D_{2n}: $y^p = x^n - 1$, $p|n$.
(2) Octohedral (and Tetrahedral): $y^{2p} = x^{2p} - 1$, p odd.

PROOF: The action of D_{2n} is easy enough (cf. [14]). The involution in this case acts on $y^p = x^n - 1$ by $(x,y) \mapsto (x/y^m, 1/y)$, with $m = n/p$. For the tetrahedral group A_4 we use the $(2,3,3)$ triangle group presentation

$$\left\langle R, S, T \;\middle|\; R^2 = S^3 = T^3 = RST = 1 \right\rangle .$$

Then an action of A_4 on $y^{2p} = x^{2p} - 1$, whenever p is odd, is given by

$$R(x,y) = (x, -y),\ S(x,y) = (\imath/y, \imath x/y),\ T(x,y) = (-y/x, \imath/x).$$

This action extends to an action of

$$S_4 = \langle R, S, T \mid R^2 = S^3 = T^4 = RST = 1 \rangle$$

as follows: $R(x,y) = (x/y, 1/y)$, $S(x,y) = (y/x, \imath/x)$, $T(x,y) = (\imath y, \imath x)$. ∎

It turns out that it is possible to completely classify $\mathrm{Aut}(C)$ when the number of branched points is less than 4; in which case we are dealing with surfaces of the form

$$C_n(a,b,c) : y^n = x^a(x-1)^b(x+1)^c, \quad a+b+c \equiv 0 \ (mod\ n) \tag{2}$$

In [14], a classification of all groups $Aut(C_n(a,b,c))$ in terms of n, a, b and c is given. Here one assumes $n \geq 4$ (so that $g \geq 2$) and $1 \leq a, b, c < n$. When n is not prime, one also assumes $GCD(n,a,b,c) = 1$ to ensure connectedness. The following definition is needed for the rest of the paper.

Definition. The mod-n Nielsen class of a tuple $(a_1, \ldots, a_k)$ of integers consists of all $(a'_1, \ldots, a'_k)$ such that $(a'_1, \ldots, a'_k) = (ka_{\tau(1)}, \ldots, ka_{\tau(k)})$ for some permutation $\tau \in S_k$ and k prime to n. We write

$$(a'_1, \ldots, a'_k) \sim_\tau (a_1, \ldots, a_k) \ \mathrm{mod}(n)$$

(and $\sim$ instead of $\sim_\tau$ when there is no need for explicit mention of τ). Note that triples in the same class yield isomorphic curves in (2) (a classic observation of Nielsen). We will see shortly and similarly that Nielsen classes together with the cross ratio of the branched points is all that determines the isomorphism type of Galois p-covers.

Theorem 8. [14] *Suppose n is odd and denote by $A : B$ a non-split extension of A by B. Then $Aut(C_n(a,b,c))$ is determined by the Nielsen class of (a,b,c) as follows:*

1. *$Aut(C)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_n$ if $(a,b,c) \sim (1,1,n-2)$.*
2. *$Aut(C)$ is the metacyclic group $\mathbb{Z}_n : \mathbb{Z}_2$ if $(a,b,c) \sim (1,b,n-1-b)$, where $GCD(b,n) = 1, b \neq 1, 8 \not| n, b^2 \equiv 1 \ (mod\ n)$.*
3. *$Aut(C)$ is the metacyclic group $\mathbb{Z}_n : \mathbb{Z}_3$ if $(a,b,c) \sim (1,b,b^2)$, where $b \neq 1$ and $GCD(b,n) = 1$.*
4. *$Aut(C)$ is $PSL_2(7)$ if $n = 7$ and $(a,b,c) \sim (1,2,4)$. C is the Klein curve.*
5. *$Aut(C) = \mathbb{Z}_n$ in all other cases. The situation for n even is equally well understood* [14].

The Moduli Space. Unlike the general moduli space of all closed curves, the moduli space of cyclic coverings of the line has a simple description.

Theorem 9. (Equivalence of cyclic p-Coverings)

1. *([8]) Fix some branch set $B = \{q_1, \ldots, q_r\}$ in a curve X, and consider two $\mathbb{Z}_p$-coverings C_1 and C_2 over X, branched over the q_i's with multiplicities $(k_1, \ldots, k_r)$ and $(l_1, \ldots, l_r)$ respectively. Then C_1 is isomorphic to C_2 (as branched covers) if and only if $(k_1, \ldots, k_r) \sim (l_1, \ldots, l_r)$ mod-p.*

(2) *([21]) Now Suppose $X = \mathbb{P}^1$; C_1 is branched over $B_1 \subset X$, with branch points q_i of multiplicities k_i, $1 \leq i \leq r$; and C_2 is branched over $B_2 \subset X$ with branch points p_i of multiplicities l_i, $1 \leq i \leq r$. Then C_1 is isomorphic to C_2 if and only if there is $\sigma \in PGL_2(\mathbb{C})$ and $\tau \in \Sigma_r$ such that $p_{\tau(i)} = \sigma q_i$ and $(k_1, \ldots, k_r) \sim_\tau (l_1, \ldots, l_r)$.*

The sufficiency part of these assertions is a direct consequence of Riemann's extension theorem. Riemann's theorem asserts that a ramified cover $\pi : X \longrightarrow Y$ branched over $B \subset Y$ is determined by the étale cover $\pi' : X - \pi^{-1}(B) \longrightarrow Y - B$ and by the way the sheets of π "come together" at the ramification points. This is specified by the monodromy around the branch points. More precisely

(Riemann) Suppose X is a curve, $B \subset X$ is a finite subset and $x_0 \notin B$. Then there exists a correspondence

Degree d (algebraic) branched covers $C \longrightarrow X$, branched over B, modulo covering transformations	$\longleftrightarrow$	*Transitive representations $\pi_1(X - B, x_0) \longrightarrow \Sigma_d$, modulo equivalence of representations*

By elementary covering space theory, the étale cover is uniquely determined (up to equivalence of covers) by the kernel of the monodromy $\rho : \pi_1(X - B, x_0) \longrightarrow \Sigma_d$. If the cover is connected, then the image of ρ (the monodromy sugbroup) acts transitively on the cover. The "existence" part stipulates then that an étale cover of curves $\pi' : C - B_0 \longrightarrow X - B$, B a finite set covered by B_0, always extends to an analytic (and hence algebraic) ramified cover $C \longrightarrow X$. Part(1) and Part(2) (only if part) of theorem 9 is an immediate consequence of the RET after observing that for $\mathbb{Z}_p$-Galois coverings over $\mathbb{P}^1$, we can replace Σ_p by $\mathbb{Z}_p$, and that automorphisms of p-cyclic groups are given by raising elements to a power k prime to p. The permutation τ entering in the Nielsen class comes from the fact again that the sheets of the covering can only be numbered up to a permutation.

Part (2) of the above theorem as observed by Nakajo (see also [11]) asserts that any abstract isomorphism of prime cyclic covers of the line is in fact an equivalence of branched coverings. We can see this statement through the eyes of uniformization theory as follows. Let Π denote the fundamental group of a p-elliptic curve C, and P a fundamental polygon for C in the upper half-plane $\mathcal{H}$. The polygon P then affords a $\mathbb{Z}_p$ symmetry (see §1), and in fact there is another Fuchsian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ and a short exact sequence (called a skep)

$$1 \longrightarrow \Pi \longrightarrow \Gamma \xrightarrow{\theta} \mathbb{Z}_p \longrightarrow 1$$

uniformizing the action. The group Γ has *signature* $(0|\overbrace{p, \ldots, p}^{r})$. This means that as an abstract group it has the presentation

$$\left\langle x_1, x_2, \ldots, x_r \;\middle|\; x_1^p = x_2^p = \cdots = x_r^p = x_1 x_2 \cdots x_r = 1 \right\rangle. \tag{4.2}$$

The x_i are called the elliptic generators. Describing the $\mathbb{Z}_p$ action on C is done in a standard way and comes down to choosing a realization of the abstract group Γ as a Fuchsian group (i.e embedding it in $\mathrm{PSL}_2(\mathbb{R})$ up to conjugation) and then specifying the epimorphism $\theta : \Gamma \to \mathbb{Z}_p$ with torsion free kernel Π. Pick a generator $T \in \mathbb{Z}_p$. Any epimorphism $\theta : \Gamma \to \mathbb{Z}_p$ is described on the elliptic elements by:

$$\theta(x_i) = T^{a_i}, \text{ where } 1 \leq a_i \leq p-1, \ 1 \leq i \leq r, \text{ and } \sum_{i=1}^{r} a_i \equiv 0 \ (mod \ p).$$

The conditions guarantee that θ is a well defined epimorphism with torsion free kernel. The number of elliptic generators r corresponds to the number of fixed points of $T : C \to C$.

An isomorphism of Fuchsian groups is an abstract isomorphism of groups that is induced from an element of $\mathrm{PSL}_2(\mathbb{R})$ (that is both groups can be embedded in $\mathrm{PSL}_2(\mathbb{R})$ as conjugates). We say that two skeps θ_1 and θ_2 are equivalent if the corresponding extensions are equivalent; that is if there are vertical isomorphisms of Fuchsian groups making the diagram commute

$$\begin{tikzcd}
\Pi_1 \arrow[r] \arrow[d, "\lambda"] & \Gamma_1 \arrow[r, "\theta_1"] \arrow[d, "\alpha"] & \mathbb{Z}_p \arrow[d, "\beta"] \\
\Pi_2 \arrow[r] & \Gamma_2 \arrow[r, "\theta_2"] & \mathbb{Z}_p
\end{tikzcd}$$

Lemma 10. *: There is a 1-1 correspondence between equivalence classes of skeps $\Gamma \to \mathbb{Z}_p$ and isomorphism classes of p-elliptic curves.*

Proof: Consider an isomorphism class of skeps where α is induced from $\tau : \mathcal{H} \to \mathcal{H}$. Here $\ker \theta_i = \Pi$ as abstract groups and λ is an automorphism of Π. Both copies of Π (embedded in $\mathrm{PSL}_2(\mathbb{R})$) are conjugate by $\tau \in \mathrm{PSL}_2(\mathbb{R})$ and hence $\mathcal{H}/\Pi$ is a well defined Riemann surface (up to isomorphism). The correspondence so indicated is well-defined. To see that it is bijective, start with two isomorphic p-elliptic curves $f : C_1 \xrightarrow{\cong} C_2$ and $\mathbb{Z}_p$ actions $S_i : \mathbb{Z}_p \times C_i \longrightarrow C_i$. Consider the diagram of branched coverings

$$\begin{tikzcd}
\mathcal{H} \arrow[r, dashed] \arrow[d] & \mathcal{H} \arrow[d] \\
C_1 \arrow[r, "f"] \arrow[d, "\pi_1"] & C_2 \arrow[d, "\pi_2"] \\
\Sigma \arrow[r, dashed, "A"] & \Sigma
\end{tikzcd}$$

where π_i are the projections $C_i \to C_i/\mathbb{Z}_p = \Sigma$. If the diagram extends at the bottom, i.e. if there is a linear fractional transformation $A : \Sigma \to \Sigma$ making the bottom diagram commute, then the diagram extends at the top as well, and the commutative diagram so obtained readily proves the lemma. It is not possible in general to compress f to A for this means that necessarily $\pi_2(f(g_1(x)) = \pi_2(f(x))$ (that is that $f(g_1(x)) = g_2^k f(x)$ for some k prime to p; here we write $g_i^k(x) = S_i(g^k, x)$ where g is the generator of $\mathbb{Z}_p$). However one can replace f by an isomorphism f' with such a property (and hence a commuting diagram as above exists with f replaced by f'). To this end, we pull back the action of $\mathbb{Z}_p$ on C_2 via f to an action on C_1; $S : \mathbb{Z}_p \times C_1 \to C_1$ determined by $S(g, x) := g_s(x) = f^{-1} g_2(f(x))$. But the two actions S and S_1 are necessarily conjugate in $\mathrm{Aut}(C_1)$ according to a beautiful result [11] (see remark below), and hence there is $h \in \mathrm{Aut}C_1$ such that $h^{-1}Sh = S_1$; i.e. such that $h^{-1}g_s h(x) = g_1(x)$. Define $f' = f \circ h$ from C_1 to C_2. Then we check

$$\begin{aligned} \pi_2(f'(g_1(x)) &= \pi_2[fhg_1(x)] = \pi_2[fg_sh(x)] = \pi_2[g_2f(h(x))] \\ &= \pi_2[fh(x)] = \pi_2[f'(x)] \end{aligned}$$

which precisely states that f' descends to a holomorphic 1-1 map (necessarily a fractional linear transformation) A defined by $A(x) = \pi_2(f'\pi^{-1}(x))$. The claim follows. ∎

Proof (of Proposition 2) According to Lemma 10, we see that an isomorphism class of p-elliptic curves corresponds to an isomorphism class of skeps $\Gamma(0, \underbrace{p, \dots, p}_{r}) \to \mathbb{Z}_p$ for some r. Such a class is determined by the orbit of r-points in Σ under the action of $\mathrm{PGL}_2(\mathbb{C})$ and by the automorphism $\beta : \mathbb{Z}_p \to \mathbb{Z}_p$. Since automorphisms of $\mathbb{Z}_p$ are always of the form $\tau \longrightarrow \tau^k$ for some $1 \le k < p$, β is uniquely determined by some integer k prime to p, as pointed out earlier.

On the other hand, an automorphism of Γ as in 3.2 must be given on generators by $\theta(x_i) = \lambda_i x_j \lambda_i^{-1}$ (see [19]). A diagram of skeps as in 2.5 then determines a permutation on the generators x_i of Γ and an integer k. It follows that for a given projective class of r points in S^2, there are N_r distinct isomorphism classes of skeps $\Gamma \to \mathbb{Z}_p$ where N_r is the number of equivalence classes of tuples of distint integers $(k_1, \dots, k_r)$, $1 \le k_i \le r$ as in Theorem 9 above. Using generating series, Lloyd [19] was now able to compute the numbers N_r for all r and his result is summarized in Proposition 2. ∎

Remark 11. A pivotal result in the proof of Lemma 10 is the result of Gabino-Diez that if two p-cyclic groups (p prime) act on C with quotient $\mathbb{P}^1$, then necessarily the two groups are conjugate in $\mathrm{Aut}(C)$. It is interesting to compare this to a result of Nielsen which states that two orientation preserving periodic maps on a topological surface C are conjugate (in $Homeo^+(C)$) if and only if they have the same period and the same fixed point data.

4. Part II: Fixed point free representations

This part occupies the rest of the paper. If a finite group G acts on a Riemann surface C, it is often very useful to consider the induced action on the complex vector space $V = H^0(C, w_c)$ of holomorphic differentials. The complex dimension of V is the genus g. Thus we have an associated linear representation $\rho : G \to GL_g(\mathbb{C})$ which is faithful whenever the quotient $C/G \cong \mathbb{P}^1$ and $g \ge 2$. Thus the action of G on C induces an embedding $\mathrm{Aut}(C) \hookrightarrow GL_g(\mathbb{C})$.

The holomorphic differentials are usually computed as follows. Let C be the (smooth) curve with affine equation $p(x,y) = 0$, $x, y \in \mathbb{C}$. Then a basis of differentials is given by

$$w_{r,s} = \frac{x^r y^s}{p'_y} dx \ \ | \ r, s \ge 0, r + s \le n - 3$$

where $n = \deg p_y$ and p'_y is the partial with respect to y. A count of differentials gives $g = (1/2)(n-1)(n-2)$ as is well-known for smooth curves in $\mathbb{P}^2$. For the Fermat curve for instance ($x^n + y^n = 1$), we get the forms $w_{r,s} = \frac{x^{r-1}y^{s-1}}{y^{n-1}}$, $1 \le r, s \le n$.

Example 12. The special case of curves $C : y^n = f(x)$ is treated in [7] for example. When $f(x) = \prod(x - e_i)$ is a polynomial with distinct roots and $\deg f = kn$, then the $w_{ij} = x^i y^{j-n+1} dx$, $i + kj \le k(n-1) - 2$ form a basis of $V = H^0(C, w_c)$. If $n = p$ is a prime then the action of $\mathbb{Z}_p$ at a fixed point is given by rotation by $2\pi/p$. This immediately gives the action on $H^0(C, w_c)$: for $w = f(z)dz$, $\sigma^*(w) = f(\zeta z)\zeta dz$, where σ is a generator for $\mathbb{Z}_p$ and ζ a primitive p-th root of unity.

A finite group G acts on C with the genus zero property (GZP) if for any non-trivial subgroup $H \subset G$, $C/H = \mathbb{P}^1$. This is equivalent to the induced action $G \times H^0(C, w_c) \longrightarrow H^0(C, w_c)$ being fixed-point free (see §2). It turns out that groups of this type are rare. When $g > 1$ for example, the only abelian groups having this property are $\mathbb{Z}_{p^e}$ and $\mathbb{Z}_{pq}$ where p and q are distinct primes.

The case $g = 1$ is easy. There are four finite groups which can act on a torus C with the GZP, namely $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$, and $\mathbb{Z}_6$. If C admits a genus zero action by $\mathbb{Z}_3$ then it also admits one by $\mathbb{Z}_6$. In fact all genus zero actions on C contain the hyperelliptic involution (which in terms of the equation $w^2 = 4z^3 - g_2 z - g_3$ corresponds to $(z, w) \to (z, -w)$). Let $S : C \to C$ denote an automorphism of order 4 or 6.

Lemma 13. *For the groups $G \cong \mathbb{Z}_4$ or $G \cong \mathbb{Z}_6$ there is a unique torus with a genus zero action by G. The tori and actions are given by*
(1) For $G \cong \mathbb{Z}_4$: $w^2 = 4z^3 - z$, $S(z, w) = (-z, \imath w)$.
(2) For $G \cong \mathbb{Z}_6$: $w^2 = 4z^3 - 1$, $S(z, w) = (\zeta z, -w)$ where $\zeta = e^{2\pi\imath/3}$.

Proof: If C has an automorphism of order 4 then the lattice Λ must admit a rotation of order 4, that is $\imath\Lambda = \Lambda$. From this it follows that $g_3 = 0$, so the torus must have the equation $w^2 = 4z^3 - g_2 z$ for some $g_2 \in \mathbb{C}$, $g_2 \neq 0$. Moreover, multiplication by $\imath : \Lambda \to \Lambda$ then corresponds to $(z, w) \to (-z, \imath w)$. The torus is unique because the elliptic modular invariant $J(\Lambda) = 1$ for any lattice Λ satisfying $\imath\Lambda = \Lambda$, and therefore there is no loss of generality in assuming $g_2 = 1$. Similarly if C has an automorphism of order 3, $\zeta\Lambda = \Lambda$. Here $g_2 = 0$, $J(\Lambda) = 0$, and we can take $g_3 = 1$. ∎

More generally now, and as was observed in [13], if G has the GZP than necessarily its Sylow p-subgroups (for $p > 2$) are cyclic. Such groups include the cyclic, dihedral and generalized quaternionic groups. As it turns out, only the cyclic groups $\mathbb{Z}_{p^e}$ and $\mathbb{Z}_{pq}$, for p and q distinct primes, admit genus zero actions on Riemann surfaces of genus $g > 1$. The only genus zero action the dihedral group has is on the Riemann sphere.

Finite groups having all Sylow subgroups cyclic are called the Zassenhauss metacyclic groups $G_{m,n}(r)$ (or ZM for short) and are described as follows: $G_{m,n}(r)$ *is* *the group with generators A, B and relations*

$$A^m = 1,\ B^n = 1,\ BAB^{-1} = A^r$$

satisfying $GCD((r-1)n, m) = 1$ and $r^n \equiv 1$ $(mod\ m)$. These are the groups described by extensions $1 \to \mathbb{Z}_m\{A\} \to G_{m,n}(r) \to \mathbb{Z}_n\{B\} \to 1$, where the cyclic group generated by A, $\mathbb{Z}\{A\}$, is the commutator subgroup of G. Only $G_{p,4}(-1)$ (p is an odd prime) turns out to act with genus zero (Theorem 3).

In the next few paragraphs we give representative equations for Riemann surfaces of genus $g \geq 1$ admitting genus zero actions by the cyclic groups, the metacyclic group $G_{p,4}(-1)$ and the quaternion groups.

The Cyclic Case. In this section we analyze actions $G \times C \to C$, where G is cyclic and $g > 1$. According to Theorem 3 either $G \cong \mathbb{Z}_{p^e}$ or $G \cong \mathbb{Z}_{pq}$, where p and q are distinct primes. Let $S \in G$ denote a generator. The following was proved in [13]:

Proposition 14. *Suppose p, q are distinct primes. Then the genus-zero actions of $\mathbb{Z}_{pq}$ have signature and corresponding genus given by:*
(1) sig$(\Gamma) = (0|pq, pq)$, in which case $g = 0$.
(2) sig$(\Gamma) = (0|p, q, pq)$, in which case $g = \dfrac{1}{2}(p-1)(q-1)$.
(3) sig$(\Gamma) = (0|p, p, q, q)$, in which case $g = (p-1)(q-1)$.

According to the theorem above there are 2 possibilities we need consider in this section; either $g = \dfrac{1}{2}(p-1)(q-1)$ or $g = (p-1)(q-1)$.

Proposition 15. *Let C denote a surface of genus $g = \dfrac{1}{2}(p-1)(q-1)$ admitting a genus-zero action by $G = \mathbb{Z}_{pq}$. Then C is isomorphic to the Fermat curve $w^p = z^q - 1$, and the action is $S(z, w) = (\eta z, \zeta w)$, where η, ζ are respectively a primitive q^{th} root of unity and a primitive p^{th} root of unity.*

Proof: The automorphism $T = S^q : C \to C$ has order p and the quotient of the action is $\mathbb{P}^1$. As pointed out earlier, this implies that C is the curve associated to an equation $w^p = \prod_{i=1}^{r}(z - e_i)^{a_i}$, where $e_i \neq e_j$ if $i \neq j$, $1 \leq a_i \leq p-1$ and $\sum_{i=1}^{r} a_i = kp$. We write $T(z, w) = (z, \kappa w)$ for some primitive p^{th} root of unity κ. Applying Lemma 20 in the appendix we have $S(z, w) = (Z, W)$ where

$$Z = A(z) = \frac{az + b}{cz + d},\ W = \frac{\mu}{(cz + d)^k} w$$

and μ satisfies $\mu^p = \prod_{i=1}^{r}(a - ce_i)^{a_i}$. Here A has order q since $S^q(z, w) = (z, \kappa w) = (A^q(z), \kappa w)$, and $A \neq I$. Moreover, A must permute the e_i and, if $A(e_i) = e_j$ then $a_i = a_j$. The Riemann-Hurwitz formula gives $r = q + 1$, and therefore the only possibility is that one of the e_i is fixed by A and the other q form a complete cycle under the action of A. For argument's sake let's assume

$$A(e_1) = e_2, A(e_2) = e_3, \cdots, A(e_{q-1}) = e_q, A(e_q) = e_1 \text{ and } A(e_{q+1}) = e_{q+1}.$$

It follows that $a_1 = a_2 = \cdots = a_q = a$ and $qa + a_{q+1} = kp$.

Up to conjugation we can assume A has the form $A(z) = \lambda z$, where λ is a primitive q^{th} root of unity. That is $A = \begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix}$, where $\xi^2 = \lambda$. Therefore $e_{q+1} = 0$ and $\{e_i\}_{1 \le i \le q} = \{\lambda^i e\}_{1 \le i \le q}$ for some $e \ne 0$. It follows that the equation for C is

$$w^p = z^{a_{q+1}}(z-e)^a(z-\lambda e)^a \cdots (z-\lambda^{q-1}e)^a = z^{a_{q+1}}(z^q - e^q)^a,$$

and $S : C \to C$ is given by $S(z,w) = (Z,W)$ where $Z = \lambda z$ and $W = \mu \xi^k w$ for some μ satisfying $\mu^p = \xi^{kp}$. Therefore, $\mu = \rho\xi^k$ for some p^{th} root of unity ρ. In fact ρ must be a primitive p^{th} root of unity, for otherwise S would not have order pq. Therefore $S(z,w) = (\lambda z, \rho \xi^{2k} w) = (\lambda z, \rho \lambda^k w)$.

The projection $\psi : C \to \Sigma$, $\psi : (z,w) \to z$, has no branching over ∞ since $qa + a_{q+1} = kp$. If we make the change of variables $x = z^{-1}$, $y = z^{-k}w$ then the equation becomes $y^p = (1 - e^q x^q)^a$ and there is now branching over ∞. The formula for the action by G is $S(x,y) = (\lambda^{-1}x, \rho y)$. We can make another change of variables so that C is $v^p = (u^q - 1)^a$ and $S(u,v) = (\lambda^{-1}u, \rho v)$. Finally we make the change of variables

$$w = \frac{v^l}{(u^q-1)^m}, z = u \text{ where } l, m \text{ are chosen so that } la - mp = 1.$$

Then C has the equation $w^p = z^q - 1$ and $S(z,w) = (\lambda^{-1}z, \rho^k w) = (\eta z, \zeta w)$. ∎

In much the same manner we can prove the following result:

Proposition 16. *Let C denote a surface of genus $g = (p-1)(q-1)$ admitting a genus zero action by $G = \mathbb{Z}_{pq}$, where p and q are distinct primes. Then C is isomorphic to the curve with the equation $w^p = \dfrac{z^q - 1}{z^q - \lambda}$, where $\lambda \ne 0$ and $\lambda^q \ne 1$. Moreover, under this isomorphism, $S(z,w) = (\eta z, \zeta w)$, where η, ζ are respectively a primitive q^{th} root of unity and a primitive p^{th} root of unity.*

Now we consider genus-zero actions of $G = \mathbb{Z}_{p^e}$, $e \ge 2$. Let S be a generator of G, let $T = S^{p^{e-1}}$ and set H equal to the subgroup of order p generated by T. Suppose $G \times C \to C$ is a genus-zero action on a surface of genus g. There is a short exact sequence $1 \to \Pi \to \Gamma \xrightarrow{\theta} G \to 1$, where the signatures of Π and Γ are $(g|-)$ and $(0|\overbrace{p, \cdots, p}^{r}, p^e, p^e)$ respectively, and the genus is $g = \frac{1}{2}r(p^e - p^{e-1})$.

Proposition 17. *C and the action are given by*

$$w^p = z^a \prod_{i=1}^{r} \left(z^{p^{e-1}} - f_i\right)^{a_i}, \; S(z,w) = (\eta z, \lambda w),$$

where (i) $1 \le a \le p-1$ and $1 \le a_i \le p-1$ for $1 \le i \le r$,
(ii) $a + a_1 + \cdots + a_r \not\equiv 0$ (mod p),
(iii) η is a primitive root of unity of order p^{e-1} and λ is any complex number so that $\lambda^p = \eta^a$,
(iv) the f_i are distinct non-zero complex numbers.

PROOF: As before C is given by

$$w^p = \prod_{i=1}^{n} (z-e_i)^{a_i}, \text{ where } 1 \le a_i \le p-1 \text{ and } \sum_{i=1}^{n} a_i = kp.$$

The projection map $\psi : C \to \Sigma$, $\psi : (z,w) \to z$, is branched over the n points $\{e_i\}_{1\le i\le n}$, with all branching of order $p-1$. It is not branched over ∞. By Riemann-Hurwitz we have $g = 1-p+\frac{1}{2}n(p-1)$. On the other hand $g = \frac{1}{2}r(p^e - p^{e-1})$ and therefore $n = 2 + rp^{e-1}$. Thus C and the action of T can be described by the equations:

$$w^p = \prod_{i=1}^{2+rp^{e-1}} (z-e_i)^{a_i}, \; T(z,w) = (z, \zeta w),$$

where ζ is a primitive p^{th} root of unity. From Lemma 20 we have $S(z,w) = (A(z), W)$, where $A(z) = \frac{az+b}{cz+d}$, $W = \frac{\mu}{(cz+d)^k} w$ and μ is a complex number so that $\mu^p = \prod_{i=1}^{2+rp^{e-1}} (a-ce_i)^{a_i}$. Moreover A will have order p^{e-1}.

Now A permutes the $2+rp^{e-1}$ points e_i and therefore must fix two of them, say $e_{1+rp^{e-1}}$ and $e_{2+rp^{e-1}}$. This is because the order of A is p^{e-1}. The remaining e_i must fall into r orbits, each of length p^{e-1}. We may assume that $e_1, \cdots, e_r$ are representatives of the orbits. Moreover, if $A(e_i) = e_j$, then $a_i = a_j$, and therefore

$$w^p = (z-e_{1+rp^{e-1}})^{a_{1+rp^{e-1}}} (z-e_{2+rp^{e-1}})^{a_{2+rp^{e-1}}} \prod_{i=1}^{r} \prod_{j=1}^{p^{e-1}} (z - A^j(e_i))^{a_i},$$

$$\mu^p = (a-ce_{1+rp^{e-1}})^{a_{1+rp^{e-1}}} (a-ce_{2+rp^{e-1}})^{a_{2+rp^{e-1}}} \prod_{i=1}^{r} \prod_{j=1}^{p^{e-1}} (a - cA^j(e_i))^{a_i}$$

Now make the change of variables

$$\begin{aligned} x &= L(z) = \frac{z-e_{1+rp^{e-1}}}{z-e_{2+rp^{e-1}}}, \; y = \frac{(x-1)^k w}{\nu}, \text{ where} \\ \nu^p &= (e_{2+rp^{e-1}} - e_{1+rp^{e-1}})^{a_{1+rp^{e-1}}+a_{2+rp^{e-1}}} \prod_{i=1}^{r} \prod_{j=1}^{p^{e-1}} \left(e_{2+rp^{e-1}} - A^j(e_i)\right)^{a_i} \end{aligned}$$

Then LAL^{-1} has fixed points $x = 0, \infty$ and therefore $LAL^{-1}(x) = \eta x$, where η is a primitive p^{e-1} root of unity. Now it is routine to check that in these variables the surface C has the equation

$$\begin{aligned} y^p &= x^{a_{1+rp^{e-1}}} \prod_{i=1}^{r} \prod_{j=1}^{p^{e-1}} \left(x - LA^j(e_i)\right)^{a_i} = x^a \prod_{i=1}^{r} \prod_{j=1}^{p^{e-1}} \left(x - LA^jL^{-1}(e_i')\right)^{a_i} \\ &= x^a \prod_{i=1}^{r} \prod_{j=1}^{p^{e-1}} \left(x - \eta^j e_i'\right)^{a_i} = x^a \prod_{i=1}^{r} \left(x^{p^{e-1}} - f_i\right)^{a_i}, \text{ where} \\ a &= a_{1+rp^{e-1}}, \; e_i' = L(e_i) \text{ and } f_i = L(e_i)^{p^{e-1}}, \; 1 \le i \le r. \end{aligned}$$

One can also check that $S(x,y) = (\eta x, \lambda y)$. ∎

The Metacyclic Case. In this section we give equations for genus-zero actions by the metacyclic group $G = G_{p,4}(-1)$ presented by

$$G = \left\langle X, Y | X^p = 1,\ Y^4 = 1,\ YXY^{-1} = X^{-1} \right\rangle.$$

It was shown in [13] that any surface C admitting a genus-zero action by G is associated to a short exact sequence $1 \to \Pi \to \Gamma \xrightarrow{\theta} G \to 1$, where the genus of C is $g = p - 1$ and the signature of Γ is $(0|4, 4, p)$.

Proposition 18. *The surface C is equivalent to the surface $w^2 = z^{2p} - 1$ and the action is given by $X(z, w) = (\zeta z, \epsilon w)$, $Y(z, w) = \left(\frac{1}{z}, \frac{\epsilon w}{(\imath z)^p}\right)$, where ζ is a primitive p^{th} root of unity and $\epsilon = \pm 1$.*

Proof: C is hyperelliptic, with hyperelliptic involution Y^2, and therefore we can present the surface and action of Y^2 by

$$w^2 = \prod_{i=1}^{2g+2} (z - e_i) = \prod_{i=1}^{2p} (z - e_i),\ Y^2(z, w) = (z, -w),$$

where the e_i are distinct complex numbers. The automorphism X then has the form

$$\begin{aligned} X(z, w) &= (A(z), W),\ \text{where } A(z) = \frac{az + b}{cz + d} \text{ has order } p, \text{ and} \\ W &= \frac{\mu}{(cz + d)^p} w \text{ for some } \mu \text{ satisfying } \mu^2 = \prod_{i=1}^{2p} (a - ce_i). \end{aligned}$$

Since A has order p and permutes the $2p$ numbers e_i, $1 \le i \le 2p$, we see that the e_i must fall into 2 orbits with respect to the action of A. Suppose e_1 and e_2 are in different orbits. Then we have

$$\begin{aligned} \mu^2 &= \prod_{i=1}^{2p} (a - ce_i) = \prod_{j=1}^{p} \left(a - cA^{-j}(e_1)\right) \prod_{j=1}^{p} \left(a - cA^{-j}(e_2)\right) \\ &= \epsilon^2 = 1 \text{ (by Lemma 22 applied to } A^{-1}). \end{aligned}$$

Moreover, $X^p(z, w) = \left(z, \frac{\mu^p}{\epsilon^p} w\right) = (z, w)$, and therefore $\mu = \epsilon$. The same considerations apply to the automorphism Y, that is $Y(z, w) = (B(z), V)$, where $B(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$ has order 2 and $V = \frac{\lambda}{(\gamma z + \delta)^p} w$ for some λ satisfying $\lambda^2 = \prod_{i=1}^{2p} (\alpha - \gamma e_i)$. We must have $\delta = -\alpha$ and so $Y(z, w) = \left(B(z), \frac{\lambda}{(\gamma z - \alpha)^p} w\right)$.

The automorphism B permutes the e_i, $1 \le i \le 2p$, and in fact fixes 2 of them and pairs off the remaining $2p - 2$. To see this note that

$$(\gamma B(z) - \alpha)(\gamma z - \alpha) = -1, \text{ and therefore } Y^2(z, w) = (z, -\lambda^2 w)$$

Thus $\lambda^2 = 1$ since Y^2 is the hyperelliptic involution. If $B(e_i) \ne e_i$ then the contribution of $\{e_i, B(e_i)\}$ to λ^2 would be $(\alpha - \gamma e_i)(\alpha - \gamma B(e_i)) = -1$. Thus if B did not fix any of the e_i we would have $\lambda^2 = (-1)^p = -1$, a contradiction. Therefore,

B must fix some of the e_i, and in fact 2 of them. Finally, the fixed points of B are $\dfrac{\alpha \pm \imath}{\gamma}$ and their contribution to λ^2 is $\left(\alpha - \gamma\dfrac{\alpha+\imath}{\gamma}\right)\left(\alpha - \gamma\dfrac{\alpha-\imath}{\gamma}\right) = 1$.

We may alter A within its conjugacy class and therefore there is no loss of generality in assuming $A = \begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix}$, where $\xi^2 = \zeta$ is a primitive p^{th} root of unity. The relation $XYX = Y$ holds in G and therefore, by calculation we see that $\alpha = 0$. This relation implies that $ABA = B$, and from this we see that the fixed points of B are in different orbits with respect to A.

For argument's sake suppose the fixed points of B are $e_1 = \dfrac{\imath}{\gamma}$, $e_2 = -\dfrac{\imath}{\gamma}$. By a change of variables we may assume $e_1 = 1$ and $e_2 = -1$, that is we may assume $\gamma = \imath$. Therefore

$$\{e_i\}_{1 \le i \le 2p} = \{\zeta^j | 1 \le j \le p\} \cup \{-\zeta^j | 1 \le j \le p\}$$

We then see that the equation for C is

$$w^2 = \prod_{i=1}^{2p}(z - e_i) = \prod_{j=1}^{p}(z - \zeta^j)\prod_{j=1}^{p}(z + \zeta^j) = \prod_{j=1}^{p}(z^2 - \zeta^{2j}) = z^{2p} - 1.$$

With these choices the generators of G are acting as stated in the theorem. ∎

The Quaternionic Case. In this section we give equations for genus-zero actions by the generalized quaternion groups $Q = Q(2^n)$, $n \ge 3$. A presentation of Q is

$$Q = \left\langle A, B \mid A^{2^{n-1}} = 1, A^{2^{n-2}} = B^2, BAB^{-1} = A^{-1} \right\rangle.$$

Genus-zero actions on Q have been fully described in [13]. In particular actions by $Q(2^n)$ have signature $(0|\overbrace{2,\ldots,2}^{r}, 4, 4, 2^{n-1})$, where r is odd. The genus is $g = 2^{n-2}(r+1)$.

Proposition 19. *Let C be a Riemann surface with a genus zero action by Q. Then C is conformally equivalent to the surface with the equation*

$$w^2 = z(z^{2^{n-1}} - \beta^{2^{n-1}})\prod_{i=1}^{r}\left(z^{2^{n-1}} - \left(e_i + \frac{1}{e_i}\beta^{2^{n-1}}\right)z^{2^{n-2}} + \beta^{2^{n-1}}\right),$$

where $\beta \ne 0$ and the e_i are distinct non-zero complex numbers $\ne \pm\beta^{2^{n-2}}$. The action of Q on C is given by:

$$\begin{aligned} A(z,w) &= (\eta z, \lambda w), \text{ where } \eta^{2^{n-2}} = 1 \text{ and } \lambda^2 = \eta, \\ B(z,w) &= \left(-\frac{\beta^2}{z}, \frac{\mu}{z^m}w\right), \text{ where } m = 2^{n-2}(r+1) + 1 \text{ and } \mu = \pm\beta^m. \end{aligned}$$

Proof: The subgroup $< A > \cong \mathbb{Z}_{2^{n-1}}$ has a genus-zero action on C and so according to Proposition 17, we have

$$w^2 = z\prod_{i=1}^{2(r+1)}\left(z^{2^{n-2}} - f_i\right), \quad A(z,w) = (\eta z, \lambda w) \tag{5.1}$$

where η is a primitive 2^{n-2} root of 1, $\lambda^2 = \eta$, and the f_i are distinct non-zero complex numbers. The curve C is hyperelliptic with corresponding involution $A^{2^{n-2}}(z,w) = (z,-w)$. The orbit surface $C/ < A^{2^{n-2}} >\cong \mathbb{P}^1$ and the quotient map $\psi : C \to \mathbb{P}^1$ can be identified with $\psi(z,w) = z$. Now the automorphism $B : C \to C$ commutes with the hyperelliptic involution and therefore there is a linear fractional transformation $N(z) = \dfrac{\alpha z+\beta}{\gamma z+\delta}$ of order 2 so that $\psi \circ B = N \circ \psi$.

Thus $B(z,w) = (N(z), W(z,w))$ for some $W(z,w)$. The automorphism B normalizes the subgroup $< A >$, the orbit surface $C/ < A >\cong \mathbb{P}^1$ and the quotient map $\phi : C \to \mathbb{P}^1$ can be identified with the map $\phi : C \to \mathbb{P}^1$, $\phi(z,w) = z^{2^{n-2}}$. Therefore there is a linear fractional transformation $N'(z) = \dfrac{\alpha' z+\beta'}{\gamma' z+\delta'}$ so that $\rho \circ N = N' \circ \rho$, where $\rho(z) = z^{2^{n-2}}$. Thus we must have

$$\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)^{2^{n-2}} = \frac{\alpha' z^{2^{n-2}}+\beta'}{\gamma' z^{2^{n-2}}+\delta'}.$$

This last equation implies either $\alpha' = \delta' = 0$ and $\alpha = \delta = 0$, or $\beta' = \gamma' = 0$ and $\beta = \gamma = 0$. If $\beta' = \gamma' = 0$ and $\beta = \gamma = 0$ then $N(z) = \dfrac{\alpha z}{\delta} = -z$ (since $\alpha + \delta = 0$), and so $B(z,w) = (-z, W)$. Substituting into the equation (5.1) for C we see that $W = \pm \imath w$. Checking this against the relation $ABA = B$ gives a contradiction and therefore $\alpha' = \delta' = 0$ and $\alpha = \delta = 0$. Thus $N(z) = \dfrac{\beta}{\gamma z} = -\dfrac{\beta^2}{z}$ since $\beta\gamma = -1$.

Now substitute $B(z,w) = \left(-\dfrac{\beta^2}{z}, W\right)$ into the equation for C:

$$\begin{aligned}
W^2 &= -\frac{\beta^2}{z}\prod_{i=1}^{2(r+1)}\left(\left(-\frac{\beta^2}{z}\right)^{2^{n-2}} - f_i\right)\\
&= -\frac{\beta^2}{z^{2m-1}}\prod_{i=1}^{2(r+1)}\left(\beta^{2^{n-1}} - f_i z^{2^{n-2}}\right)\\
&= -\left(\beta^2\prod_{i=1}^{2(r+1)} f_i\right) z^{-2m} z \prod_{i=1}^{2(r+1)}\left(z^{2^{n-2}} - \frac{1}{f_i}\beta^{2^{n-1}}\right)\\
&= \mu^2 z^{-2m} w^2, \text{ where } \mu^2 = -\beta^2\prod_{i=1}^{2(r+1)} f_i.
\end{aligned}$$

To see the last line note that as sets we have

$$\{f_i \mid 1 \le i \le 2(r+1)\} = \left\{\frac{1}{f_i}\beta^{2^{n-1}} \;\middle|\; 1 \le i \le 2(r+1)\right\}$$

because both represent the Weirstrass points of C. Therefore $B(z,w) = \left(-\dfrac{\beta^2}{z}, \dfrac{\mu}{z^m} w\right)$.

Next we check to see what conditions are imposed by the relations in Q. For example $B^2(z,w) = (z,-w)$ implies $\mu^2 = \beta^{2m}$, so $\mu = \pm\beta^m$. The relation $A^{2^{n-1}} = 1$ is clearly satisfied, and one can check that the relation $ABA = B$ is also satisfied.

Finally we consider the conditions that follow from equations (3), (3) above and from and $\mu = \pm\beta^m$. First note that (3) and $\mu = \pm\beta^m$ imply $\prod_{i=1}^{2(r+1)} f_i = -\beta^{2m-2} = -\beta^{2^{n-1}(r+1)}$. Now suppose $f_i \neq \frac{1}{f_i}\beta^{2^{n-1}}$ for $1 \leq i \leq 2(r+1)$. Then equation (3) gives $\prod_{i=1}^{2(r+1)} f_i = \beta^{2^{n-1}(r+1)}$, a contradiction. Thus there is at least one i, and in fact 2, so that $f_i = \frac{1}{f_i}\beta^{2^{n-1}}$. For these values of i we have $f_i = \pm\beta^{2^{n-2}}$. Let the other f_i be denoted $e_i, 1 \leq i \leq r$. Then substituting into (5.1) we get the equation in the statement of the theorem. ∎

5. Appendix

In this appendix we collect some technicalities and proofs. Consider genus-zero actions $G \times C \to C$, where G has a normal subgroup H of prime order p. Let $T \in H$ denote a generator and suppose $S \in G$ is an element of order n, not in H. Then C and the action of T are given by:

$$w^p = \prod_{1\leq i\leq r} (z - e_1)^{a_1},\ T(z,w) = (z, \zeta w),\ 1 \leq a_i \leq p-1 \text{ and } \sum_{i=1}^{r} a_i = kp.$$

Lemma 20. *There exists an element $A(z) = \dfrac{az+b}{cz+d}$ in $PSL_2(\mathbb{C})$ so that*

$$\begin{aligned} S(z,w) &= (Z,W),\ \textit{where } Z = A(z) \textit{ and} \\ W &= \frac{\mu}{(cz+d)^k} w \textit{ for some } \mu \textit{ satisfying } \mu^p = \prod_{i=1}^{r}(a - ce_i)^{a_i}. \end{aligned}$$

Moreover $A^n = 1$, A must permute the e_i, $1 \leq i \leq r$, and if $A(e_i) = e_j$ then $a_i = a_j$.

Proof: Let $\psi : C \to \mathbb{P}^1$ be the quotient map $C \to C/H \cong \mathbb{P}^1$, that is $\psi(z,w) = z$. Since H is normal in G there is an element $A \in PSL_2(\mathbb{C})$ so that $A \circ \psi = \psi \circ S$. Now S has order n so that $A^n = 1$. The branching of $\psi : C \to \mathbb{P}^1$ is preserved by A, and therefore A must permute the e_i. This implies that $a - ce_i \neq 0$ for $1 \leq i \leq r$ (to see this note that $A(\infty) = a/c$). The orders of the branch points are also preserved and so $a_i = a_j$ if $A(e_i) = e_j$. Since $(Z, W) \in C$ we have

$$\begin{aligned} W^p &= \prod_{i=1}^{r} (Z - e_i)^{a_i} = \prod_{i=1}^{r} \left(\frac{az+b}{cz+d} - e_i \right)^{a_i} \\ &= \frac{1}{(cz+d)^{kp}} \prod_{i=1}^{r} ((a - ce_i)z - (de_i - b))^{a_i} \\ &= \frac{\lambda}{(cz+d)^{kp}} \prod_{i=1}^{r} \left(z - A^{-1}(e_i)\right)^{a_i}, \text{ where } \lambda = \prod_{i=1}^{n} (a - ce_i)^{a_i}. \end{aligned}$$

Therefore $W^p = \dfrac{\lambda}{(cz+d)^{kp}} \displaystyle\prod_{i=1}^{r} (z - e_i)^{a_i} = \dfrac{\lambda}{(cz+d)^{kp}} w^p$. It follows that $W = \dfrac{\mu}{(cz+d)^k} w$, where μ is a complex number satisfying $\mu^p = \lambda = \prod_{i=1}^{r} (a - ce_i)^{a_i}$. ∎

The next lemma follows easily by induction.

Lemma 21. *In the notation above $S^l(z, w)$ equals*

$$\left(A^l(z), \frac{\mu}{\left(cA^{l-1}(z) + d\right)^k} \times \frac{\mu}{\left(cA^{l-2}(z) + d\right)^k} \times \cdots \times \frac{\mu}{(cz+d)^k} \times w \right).$$

For the next lemma let $\tilde{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{C})$ be a matrix representative of $A \in PSL_2(\mathbb{C})$ and assume the order of A is n. Then $\tilde{A}^n = \epsilon I$, where ϵ is ± 1.

Lemma 22. $\left(cA^{n-1}(z) + d\right) \times \left(cA^{n-2}(z) + d\right) \times \cdots \times (cz + d) = \epsilon$ *for all* $z \in \mathbb{C}$ *for which the left hand side has no zero terms.*

Proof: We have a telescoping product

$$\begin{aligned} &(cz+d) \times (cA(z)+d) \times \cdots \times \left(cA^{n-1}(z) + d\right) = \\ &(cz+d) \times \left(c\frac{az+b}{cz+d} + d \right) \times \cdots \times \left(cA^{n-1}(z) + d\right) = \\ &(cz+d) \times \left(\frac{c(az+b) + d(cz+d)}{cz+d} \right) \times \cdots \times \left(cA^{n-1}(z) + d\right) \end{aligned}$$

in which every numerator cancels the next denominator, leaving only the last numerator. Since $\tilde{A}^{n-1} = \epsilon \tilde{A}^{-1} = \begin{bmatrix} \epsilon d & -\epsilon b \\ -\epsilon c & \epsilon a \end{bmatrix}$ as matrices in $SL_2(\mathbb{C})$ we get

$$\begin{aligned} cA^{n-1}(z) + d &= c \times \frac{\epsilon dz - \epsilon b}{-\epsilon cz + \epsilon a} + d \\ &= \frac{c(\epsilon dz - \epsilon b) + d(-\epsilon cz + \epsilon a)}{-\epsilon cz + \epsilon a} \\ &= \frac{\epsilon}{-\epsilon cz + \epsilon a}. \end{aligned}$$

Thus the last numerator is ϵ. ∎

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