COMBINATORIAL INVARIANTS OF STRATIFIABLE SPACES

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Abstract. We extend the well-known construction of the Grothendieck ring of varieties to categories whose objects can be partitioned into predefined strata (i.e. stratifiable spaces). To this end, we introduce "Lego categories" which are subcategories of the category of spaces stratifiable by locally compact strata in Euclidean space. Restricting to strata that are cohomologically of finite type, as well as are their closures, we construct a well-defined motivic morphism on the associated Grothendieck ring which coincides with the Euler characteristic with compact supports on locally compact spaces. This "categorification" allows streamlined combinatorial derivations of Euler characteristics, be they topological or with compact supports. Main applications pertain to spaces stratified by configuration spaces, with new results including the computation of the Grothendieck class of graph configuration spaces, of orbit configuration spaces and of finite subset spaces.

1. Introduction

Computing the topological Euler characteristic of various functorial constructions appearing in geometry, topology and arithmetics is both an important and useful endeavor (a representative recent sample is in [17, 6, 23, 29, 34, 43, 73]). Often as it turns out, it is very convenient, in fact indispensable, to compute a "combinatorial Euler characteristic" χ_c first in order to get to the topological Euler characteristic χ. The combinatorial part lies in the fact that χ_c is additive on suitable stratifications of locally compact spaces (or LC-spaces) by locally closed subspaces, making it much more amenable to computations than χ. The typical combinatorial characteristic used in the literature is the compactly supported Euler characteristic which is defined for locally contractible LC-spaces that have finitely generated cohomology and are of finite cohomological dimension (i.e. of cohomological finite type) [75], and which is obtained from either cohomology with compact supports or Borel-Moore homology.

In this paper, we work at the level of strata directly and lift the computation of χ_c to a computation in an appropriate Grothendieck ring for a category $\mathcal{M}$ whose objects are subspaces of a Hausdorff LC-space that can be stratified by "LC locally contractible subspaces of cohomological finite type" in Euclidean space (i.e. LCFT-stratifiable spaces §4), with extra mild restrictions on closures of strata. The associated Grothendieck ring is denoted by $K_0(\mathcal{M})$ and is the natural niche for a well-defined "motivic morphism" $\langle - \rangle$ which is a ring morphism from $K_0(\mathcal{M})$ into $\mathbb{Z}$ (Theorem 5.2). This morphism coincides with the Euler characteristic with compact supports on classes of LC-spaces, and it recovers the topological Euler characteristic under favorable circumstances like compactness or manifold stratifications. It is fitting to observe that the compactly supported Euler characteristic, when defined for non-LC spaces, fails to be additive anymore (i.e. is no longer "combinatorial"). This motivates and also underscores the usefulness of the K_0-formalism developed in this paper.

There is at least a triple advantage in this categorification and in working with $K_0(\mathcal{M})$. First of all, we are working with strata directly rather than with their invariants. Breaking a space into fundamental "lego" pieces makes the combinatorics and the formulas in $K_0(\mathcal{M})$ much more meaningful. Furthermore, the combinatorial aspect is not restricted to LC spaces exclusively, as is the case of the combinatorial χ_c, but extends to the larger collection of spaces that are stratifiable by those. Finally, we collect various tools that make computations easier than what is done conventionally. This streamlines several old computations of Euler characteristics, and produces new formulas. Note that a related but less general

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combinatorial class which doesn't necessarily require local compactness is the one based on O-minimal sets (a survey in [23]. See also discussion in Example 3.1).

The larger body of this work is about applications and consists in computing the Grothendieck class of various constructions used in geometry and topology. Under favorable conditions, the topological Euler characteristic can be obtained as a direct consequence. Many of the constructions we consider are stratified by the classical configuration spaces $\mathrm{Conf}(X, n)$ consisting of ordered pairwise distinct points on X, or by their unordered analogs, or by finite coverings of those. If $F(X) \in \mathcal{M}$ is such a construction associated to $X \in \mathcal{M}$, then its class $[F(X)] \in K_0(\mathcal{M})$ can be expressed as a polynomial of the form

$$[F(X)] = \sum a_N \prod_{i=0}^{N}([X] - i) \tag{1}$$

or a product of those. These constructions include the finite subset spaces $\mathrm{Sub}_n(X)$, with X of the homotopy type of a finite CW-complex (Proposition 8.1), the orbit configuration spaces $\mathrm{Conf}_G(X, n)$ for a finite group G acting on X, and their relation to the orbit stratification for X (Theorem 11.3), or the bounded multiplicity configuration spaces (Proposition 12.1). This latter computation recovers a special case of a more general but much more tedious computation in [29]. The obtained formulas for the Euler characteristics of the orbit configuration spaces and the finite subset spaces are new.

The chromatic polynomial of graphs has the form (1) and naturally appears as the combinatorial class of the graph configuration spaces (Theorem 10.2), extending results in [27] and recovering an old result of Rota after applying a general Möbius inversion formula for the class of subspace complements (Theorem 9.1). The flexibility of our "lego" formalism allows fast derivations of a number of well-known Grothendieck class formulas and Euler characteristics, many of which are generally obtained through homology computations. These formulas are disseminated throughout the text.

This work is divided up in two parts: part I (§2-§6) is foundational, while part II (§7-§12) deals with applications. In §3-4 we define Lego categories and their Grothendieck rings. They are viewed as subcategories of the category $\mathcal{U}$ of all LC-stratifiable subspaces in Euclidean space. These are identified with the constructible subspaces (Corollary 3.3), and interestingly as it turns out, they have trivial K_0. A non-trivial lego category $\mathcal{F}$ of LCFT-stratifiable spaces is then constructed and refined to give the desired $\mathcal{M}$ §5. The organigram in Figure 1 helps read through the text.

$$\begin{tikzcd}[arrows=dash]
 & \mathcal{U} \arrow[d] & \\
 & \mathcal{F} \arrow[dl] \arrow[dr] & \\
\mathcal{M} \arrow[dr] & & \mathcal{M}_{cell} \arrow[dl] \\
 & \mathcal{O} &
\end{tikzcd}$$

FIGURE 1. All Lego categories and subcategories considered in this paper, ordered bottom to top by inclusion: $\mathcal{U}$ is the Lego category of LC-stratifiable spaces, $\mathcal{F}$ is the subcategory of LCFT-stratifiable spaces, $\mathcal{M}_{cell}$ is the subcategory of cell-stratifiable spaces, $\mathcal{O}$ is any subcategory of O-minimal sets, and finally $\mathcal{M}$ is the subcategory of LCFT-stratifiable spaces with finiteness conditions on closures of strata. The motivic morphism $\langle - \rangle$ is well-defined on $K_0(\mathcal{M})$ and on $K_0(\mathcal{O})$.

In §6 we formulate some general properties of group actions and stratifications. Starting in §7 and §8, we give streamlined derivations of some classical invariants. In §9 we import techniques from poset topology in order to compute the class of the complement of an arrangement in an LCFT-space. Special cases of these complements are the "generalized configuration spaces" which appear in various guises in the literature, and which we formalize in §9. These turn out to be stratifiable in terms of the classical configuration spaces of distinct points, so we have some ease in computing their Grothendieck classes as pointed out earlier. This is done in the remaining sections §10-§12.

Lego categories will be further investigated in a sequel, with additional applications. The present paper is written in semi-expository style and many of its applications center around the fruitful interplay between stratifications, arrangements and posets.

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CONTENTS

2. PRELIMINARIES

Unless stated explicitly otherwise, all spaces considered in this work are Hausdorff and locally contractible, and all stratifications are, by definition, *finite* (i.e. have a finite number of strata).

Our aim in the first part of this work (§2-§6) is to construct a ring, whose generators are isomorphism classes of objects in a predefined category, so as to realize the Euler characteristic with compact supports, defined for locally compact (i.e. LC) spaces of finite cohomological type as a ring morphism into $\mathbb{Z}$. We will accomplish this, and more, in steps.

Let X be a topological space. Then cohomology groups with compact supports $H_c^*(X)$, and constant coefficients, can be defined as follows: start by forming a direct system for the compact subsets of X under inclusion. For $K_1 \subset K_2$, the inclusion induced homomorphism $H^i(X, X \setminus K_1) \to H^i(X, X \setminus K_2)$ yields a directed system of abelian groups. One then defines

$$H_c^i(X) = \lim_{K \subset X} H^i(X, X \setminus K) \tag{2}$$

A space X is "cohomologically of finite type" (or cft) if its *singular* cohomology groups with compact supports $H_c^*(X, \mathbb{Z})$ have finite cohomological dimension, and if they are finitely generated [75]. A space in this paper is said to be *LCFT* if it is locally contractible, locally compact and cft.

For LCFT spaces, the Euler characteristic with compact supports can be defined as the alternating sum

$$\chi_c(X) = \sum_{k\geq 0}(-1)^k \mathrm{rank}_{\mathbb{Z}}\, H^k_c(X,\mathbb{Z}) \ \in \mathbb{Z} \tag{3}$$

A remarkable property of χ_c is that it is additive on partitions of X by LCFT spaces, provided that X is LC and locally contractible to begin with. More explicitly, let's use the notation $\bigsqcup$ for "disjoint union as sets" or partition. It will also be convenient to write a stratification $\mathcal{S}$ of X as its set of strata, so $\mathcal{S} = \{X_i\}$ where, as a set, $X = \bigsqcup X_i$. If $\mathcal{S} = \{X_i\}$ is any such finite partition of X into LCFT strata (we say in this case that $\mathcal{S}$ is an *LCFT-stratification* of X), and X is LC and locally contractible, then X is automatically LCFT and

$$\chi_c(X) = \sum \chi_c(X_i) \tag{4}$$

This is the combinatorial aspect of χ_c on this class of spaces. For A closed in X, with both spaces locally contractible, the additivity property is an immediate consequence of the long exact sequence in cohomology with compact supports ([66], §2)[1]

$$\cdots \to H^i_c(X\setminus A) \to H^i_c(X) \to H^i_c(A) \to H^{i+1}_c(X\setminus A) \to \cdots \tag{5}$$

To prove additivity in general, one can use a spectral sequence argument as in [56] (equation (3)).

It is crucial to point out that if X fails to be LC, or to be locally contractible, then the long sequence (5) may not be exact anymore, and the Euler characteristic with compact supports χ_c may no longer be additive. This point is generally neglected in the literature, and Example 2.1 gives a simple illustration of this odd behavior.

For non-LC spaces that are LCFT-stratifiable, we can still define

$$\langle X\rangle := \sum \chi_c(X_i) \tag{6}$$

if $\{X_i\}$ is any LCFT-stratification of X. Of course $\langle X\rangle = \chi_c(X)$ if X is LC. It is not clear a priori what $\langle -\rangle \in \mathbb{Z}$ is an invariant of, and whether it depends or not on the stratification chosen. An interesting relevant discussion is in [7]. We show in Theorem 5.2 that, under mild conditions, the computation of $\langle X\rangle$ does not depend on the stratification as long as X is in an ambiant LC-space.

FIGURE 2. An example of a non-LC space (left) decomposable into LC-strata (right): X is the closed triangle and $Y \cong (0,1)$ is the diagonal segment without its endpoints, both in $\mathbb{R}^2$. Then $X\setminus Y$ (left figure) is not LC but it is LC-stratified as $X\setminus Y = S^0 \sqcup (X\setminus \overline{Y})$ (right figure), where $\overline{Y}$ is the closure of Y. By construction $\langle X\setminus Y\rangle = \langle S^0\rangle + \langle X\rangle - \langle \overline{Y}\rangle = 2+1-1 = 2$. We can also write directly $\langle X\setminus Y\rangle = \langle X\rangle - \langle Y\rangle = 1-(-1) = 2$.

Example 2.1. Consider the spaces X and Y as depicted in Figure 2, with $\langle X\setminus Y\rangle = 2$. It can be checked that $H^*_c(X\setminus Y) = 0$ using (2) and the observation that any compact K in $X\setminus Y$ is contained in a compact K' with contractible complement. This shows that $\langle X\setminus Y\rangle \neq \chi_c(X\setminus Y)$, and so the "measure" $\langle -\rangle$ is not always a characteristic but has its own interpretation (see §5).

[1]For this long exact sequence to exist, we need spaces to be HLC "homologically locally connected" ([15],§1). Local contractibility implies HLC. See §3.

2.1. **Topological Properties.** We collect needed point-set topological properties. As is standard, a space is LC (locally compact) if every $x \in X$ has an open neighborhood U whose closure in X is compact.

Properties I:

(i) Closed and open subsets of LC spaces are LC.
(ii) The intersection of two LC subspaces in a Hausdorff space is again LC.
(iii) A *locally closed* subspace A of a space X is the intersection of a closed and open subspace of X. Equivalently, the set A is locally closed in X if and only if $\overline{A} \cap A^c$ is closed. Here $\overline{A}$ is the closure of A in X. In other words, the part of the boundary of A in X that is in the complement must be closed. We write $\underline{A} = \overline{A} \setminus A = \overline{A} \cap A^c$. Note that $\underline{A}$ is written $\breve{A}$ in [1].
(iv) Any locally closed subspace in an LC Hausdorff space X is again LC. This gives an alternative definition of local compactness in this case.

The collection of LC spaces has the "defect" of not being closed under pushout (union) nor under complementation, i.e. the complement of an LC-subspace in an LC space is not necessarily LC as already illustrated in the Example of Figure 2. All these defects are remedied when working with LC-stratifiable spaces or spaces that can be partitioned into LC strata.

3. LC-Stratifiable Spaces and Grothendieck Rings

We will be working exclusively with subspaces of Euclidean space. We define $\mathcal{U}_n$ to be the collection of all LC-stratifiable subspaces of $\mathbb{R}^n$, and we let $\mathcal{U} := \bigcup_{n\geq 1} \mathcal{U}_n$ be the direct limit obtained from fixed coordinate plane inclusions $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ for each n.

Remark 3.1. There is a criterion (due to Allouche [1], see also [42]) for when a subspace A of an LC-space X is LC-stratifiable. For $A \subset X$, define $\underline{A}_n$ to be an iteration of the construction $\underline{A}$ in Properties I (iii), so that $\underline{\underline{A}} = \underline{A}_2$, etc. If A is LC, then $\underline{A}_2 = \emptyset$. With this notation, A is LC-stratifiable if and only if $\underline{A}_n = \emptyset$ for some n. In particular, if $\underline{A}_{2n} = \emptyset$, then A has the LC-stratification (Theorem, page 5 of [1])

$$A = \left(A \setminus \underline{A}\right) \sqcup \left(\underline{A} \setminus \underline{A}_2\right) \sqcup \cdots \sqcup \left(\underline{A}_{2n-2} \setminus \underline{A}_{2n-1}\right)$$

We can relate our collection $\mathcal{U}$ to the more familiar notion of *constructible sets*. We define a constructible set in a topological space to be a finite union (as sets) of locally closed subsets.

Proposition 3.2. *Let X be locally compact and A constructible in X. Then both A and $X \setminus A$ are LC-stratifiable (canonically so, once we pick a representation $A = \bigcup U_i$, U_i locally closed).*

Proof. We fix the notation: for $A, B \subset X$, we write $A \setminus B$ to mean $A \setminus A \cap B$ and set as before $\underline{U} = \overline{U} \setminus U = \overline{U} \cap U^c$. Since A is constructible, it takes the form $A = U_1 \cup \cdots \cup U_n$ for locally closed subspaces U_i in X, $1 \leq i \leq n$. As indicated in Properties I (iv), we can use LC here to mean both locally closed or locally compact. We start by stratifying $X \setminus A$. When $n = 1$, A is itself LC and the decomposition $X \setminus A = (X \setminus \overline{A}) \sqcup \underline{A}$ is an open-closed union, so is an LC-stratification. For $n > 1$, we can write $X \setminus \bigcup U_i = \bigcap X \setminus U_i$. Since each $X \setminus U_i$ is LC-stratified, and since the intersections of LC subspaces is again LC, we get automatically an LC-stratification of $\bigcap X \setminus U_i$. It is possible to write what this canonical stratification is

$$\begin{aligned} X \setminus A &= \left(X \setminus \bigcup_{i=1}^n \overline{U_i}\right) \sqcup \bigsqcup_{I \subset \{1,\ldots,n\}} \left(\bigcap_{i_r \in I} \underline{U}_{i_r} \setminus \bigcup_{j_s \in I^c} \overline{U}_{j_s}\right) \\ &= \left(X \setminus \bigcup_{i=1}^n \overline{U_i}\right) \sqcup \bigsqcup_{i=1}^n \left(\underline{U}_i \setminus \bigcup_{j \neq i} \overline{U_j}\right) \sqcup \cdots \sqcup \left(\underline{U}_1 \cap \cdots \cap \underline{U}_n\right) \qquad (7) \end{aligned}$$

Each term of this decomposition is LC because closed and open subsets of LC spaces are LC (Properties I (i)). If A is not a subspace of X, then the above decomposition remains valid after replacing $\bigcap_{i_r\in I}\underline{U}_{i_r}$ by $\bigcap_{i_r\in I}\underline{U}_{i_r}\cap X$ throughout. For example $X\setminus A=(X\setminus\overline{A})\sqcup(\underline{A}\cap X)$.

We can now decompose A as follows

$$\begin{aligned} A=\bigcup_{i=1}^n U_i &= \bigsqcup_i\left(U_i\setminus\bigcup_{i\neq j}U_j\right)\sqcup\bigsqcup_{i,j}\left(U_i\cap U_j\setminus\bigcup_{k\notin\{i,j\}}U_k\right)\sqcup\cdots\sqcup(U_1\cap U_2\cap\cdots\cap U_n)\\ &= \bigsqcup_{I\subset\Omega}\left(\bigcap_{i\in I}U_i\setminus\bigcup_{j\in I^c}U_j\right)\qquad I\text{ a subset of }\Omega=\{1,\dots,n\},\ I^c=\Omega\setminus I \end{aligned}\tag{8}$$

To get the desired LC-stratification of A, we further decompose each factor, which is the complement of a constructible set in an LC-space. The resulting stratification is the canonical LC-decomposition for A. For example, for $n=2$, $A=U_1\cup U_2=(U_1\setminus\overline{U_2})\sqcup(\underline{U_2}\cap U_1)\sqcup(U_1\cap U_2)\sqcup(U_2\setminus\overline{U_1})\sqcup(\underline{U}_1\cap U_2)$. $\square$

Let $\mathcal{A}_n$ be the collection of constructible subsets in $\mathbb{R}^n$. If a set is constructible in $\mathbb{R}^n$, then it is constructible as a subspace of $\mathbb{R}^{n+1}$ under a hyperplane inclusion $\mathbb{R}^n\hookrightarrow\mathbb{R}^{n+1}$. Let $\mathcal{A}=\bigcup_{n\geq 1}A_n$. Each $\mathcal{A}_n$ is a Boolean algebra (i.e. closed under finite intersections, finite unions and complements).

Corollary 3.3. *$\mathcal{U}$ is a Boolean algebra, and $\mathcal{U}=\mathcal{A}$.*

We can turn $\mathcal{U}$ into a subcategory of Top by choosing the morphisms to be the continuous maps. This category is small and will be the "niche" category where all other constructions will live.

Definition 3.4. An *Euler category* is any subcategory $\mathcal{C}$ of $\mathcal{U}$ which is not necessarily full and whose object set is both (i) a Boolean algebra, and (ii) cartesian closed.

Naturally $\mathcal{U}$ itself is Euler.

To an Euler category $\mathcal{C}$, we associate a "Grothendieck" ring $K_0(\mathcal{C})$ constructed in the standard way. More precisely, since the category $\mathcal{C}$ is small, its isomorphism classes form a set. Take the free abelian group generated by the isomorphism classes $[X]$, for $X\in Ob(\mathcal{C})$, subject to the relation

$$[X\setminus Y]=[X]-[Y]\ \ ,\text{ if } Y\subset X\tag{9}$$

and to the relation $[X\times Y]=[X][Y]$. This is a well-defined ring whose zero element is $[\emptyset]$ (also written 0) and whose identity for multiplication is the class of a point $[pt]$. In this ring, if $X=\bigsqcup X_i$ is a finite partition, then $[X]=\sum[X_i]$.

Remark 3.5. In the literature, "stratified categories" exist whereby objects are stratified spaces and morphisms respect stratifications in some form or another. In this paper, we do not necessarily have a specific stratification once we pick an object X in the category.

We next describe the most important family of Euler categories used in the literature.

3.1. **Cell-stratifiable spaces and O-minimal structures.** A space is "cell-stratifiable" if it can be stratified by open cells. A cell is any space homeomorphic to $\mathbb{R}^n$ for some n. The product of cells is again a cell. Simplicial complexes or CW complexes are special examples but there are of course many more.

We can consider the subcategory $\mathcal{M}_{cell}$ of all cell-stratifiable subspaces in $\mathcal{U}$. This is not an Euler category because it is not Boolean: let X be the square $]0,1[\times[-1,1]$ and let Y be the graph of the topologist's sine curve $x\to\sin\left(\frac{1}{x}\right)$ in X. This graph is homeomorphic to $]0,1[$, so it is a cell, but $X\setminus Y$ is a collection of an infinite number of cells.

Another interesting category is described in [72] where cells come equipped with controlled attaching maps.

We will write $\mathcal{O}$ for an Euler category of cell-stratifiable spaces (no particular one in mind). There is of course the trivial category reduced to a point.

Remarkably, there is a very interesting family of non-trivial $\mathcal{O}$ categories called the O-minimal structures [22, 74, 23, 37]. An O-minimal structure on $\mathbb{R}$ consists of subcollections S_n of subsets of $\mathbb{R}^n$ for each $n \in \mathbb{N}$, each being Boolean, and satisfying certain compatibility conditions (see in particular [37],§2). Their union forms a collection $\mathcal{S}$ that is Boolean and cartesian closed. Elements of $\mathcal{S}$ are called *definable sets* [22, 37] or *tame sets* [23]. In the literature, morphisms are chosen to be definable morphisms (i.e functions whose graphs are definable). Such maps may fail to be continuous. In order to have an Euler category of definable sets, we will restrict to morphisms that are either the continuous maps, or the continuous definable maps. Such a category is written **Sp** in [37], and it is small.

The key representative of an $\mathcal{O}$-structure are the semialgebraic sets. A semialgebraic set of $\mathbb{R}^n$ is a set belonging to the Boolean algebra generated by subsets defined by polynomial equalities and inequalities.

The main theorem of the theory is the *Cell Decomposition Theorem* ([74], page 4, or [23], Theorem 3.1) which states that any definable set can be partitioned into (cylindrical) cells. In other words, an O-minimal structure produces an Euler category of cell-stratifiable spaces.

For all those categories, the following holds

Lemma 3.6. $K_0(\mathcal{O}) \cong \mathbb{Z}[pt]$.

Proof. An $\mathcal{O}$-structure starts with subintervals and points of $\mathbb{R}$ and then builds definable sets in all dimensions. The partition of $\mathbb{R}$ as $\mathbb{R}^{<0} \sqcup \{0\} \sqcup \mathbb{R}^{>0}$ gives immediately that $[\mathbb{R}] = -[pt]$, and thus $[\mathbb{R}^n] = [\mathbb{R}]^n = (-1)^n$. Stratify now an object X of $\mathcal{O}$ by cells. For σ a cell, $\sigma \cong \mathbb{R}^n$, $n = \dim \sigma$, so that $[\sigma] = (-1)^{\dim \sigma}[pt]$. It then follows that $[X] = \sum[\sigma_i] = \sum(-1)^{\dim \sigma_i}[pt]$, the sum being taken over the open cells making up the partition of X. This shows that $K_0(\mathcal{O})$ is freely generated by the class of the point. $\square$

Surprisingly perhaps, the Grothendieck ring of $\mathcal{U}$ is trivial (i.e. there are "too many" spaces for this ring to be of interest).

Lemma 3.7. $K_0(\mathcal{U}) = 0$.

Proof. In the category $\mathcal{U}$ there are LC-stratifiable spaces (objects) that are homeomorphic to a punctured version of themselves. For instance let $X = \mathbb{R}^2 - Z$, where Z is the integers viewed as a discrete countable subset. Then X is LC, being open, and $X \cong X - \{p\}$, which gives that $[X] = [X] - 1$ or $1 = 0$. There is only one ring with this property, it is the trivial ring. $\square$

4. Lego Categories

Since $\mathcal{U}$ has trivial K_0, we have to seek a more interesting subcategory of LC-stratifiable spaces on which to do the combinatorics, and the expected idea is to consider LCFT-stratifiable subspaces in Euclidean space (§2). There are however subtleties when working with this category.

Demanding that strata be cft means in particular having finite cohomological dimension. This can be ensured by requiring that the space X be *topologically of finite type* [75], which is the case for instance when X is locally homeomorphic to an open subspace of a geometric simplicial complex. Interestingly, for compact X, tft guarantees cft by a Theorem of Wilder ([75], §1 or [14], §1.3). There are surprising examples of compact non-tft spaces (in fact non-locally contractible spaces) that have infinitely many singular non-trivial cohomology groups, like the Barratt-Milnor example [5]. Also, the condition of being tft is not always well behaved with respect to taking closures; eg. the closure of the topologist sine curve on $(0, 1]$ in the plane.

Let $\mathcal{F}$ be the collection of all LCFT-stratifiable subspaces. In particular all strata are locally contractible. Examples exist nonetheless that prevent this collection from being Boolean. For example, we can take in $\mathbb{R}^2$ the two closed cones of a suspension, C^+X and C^-X, on the non-cft non-compact $X = \{(n, 0), n \in \mathbb{Z}\}$. Both cones are cft (as can be checked in the same way as for Remark 2.1), but their intersection X is non-cft. This means that $\mathcal{F}$ is not closed under intersection.

The above examples and pathologies indicate that finite type conditions (tft, cft) are not always preserved under set operations (closure, complement and intersection). The notion of a "Lego category" is precisely the notion that allows us to still consider all these stratifiable collections, eventhough they are not boolean, and yet be able to define a perfectly good K_0.

The product of two stratifications $\mathcal{S}_1 = \{X_i\}$ of X and $\mathcal{S}_2 = \{Y_j\}$ of Y is the stratification $\mathcal{S}_1 \times \mathcal{S}_2 = \{X_i \times Y_j\}_{i,j}$ of $X \times Y$. Removing a stratum (say X_i) from a stratification $\mathcal{S}$ means considering the stratification $\mathcal{S} \setminus \{X_i\}$ of $X \setminus X_i$.

Definition 4.1. Consider the collection

$$\{(X, \mathcal{S}) \mid X \in \mathcal{U} \text{ and } \mathcal{S} \text{ is an LC-stratification of } X\}$$

- A *Lego collection* $\mathcal{L}$ is any subcollection of the above that is closed under cartesian product and removing strata.
- We say that X has an *$\mathcal{L}$-stratification* $\mathcal{S}$ if $(X, \mathcal{S}) \in \mathcal{L}$. Equivalently, X is *$\mathcal{L}$-stratifiable* if there is a stratification $\mathcal{S}$ such that $(X, \mathcal{S}) \in \mathcal{L}$.
- Given a lego collection $\mathcal{L}$, we define a lego category by taking the object set to be all $\mathcal{L}$-stratifiable sets X, and the morphisms to be the continuous maps. This is a small subcategory of Top, and we give it the same name $\mathcal{L}$. Note that if $\mathcal{S} = \{X_i\}$ is an $\mathcal{L}$-stratification of X, then all strata X_i are objects in $\mathcal{L}$.

Example 4.2. A most natural way of obtaining Lego collections and categories is as follows. Let U_n be the collection of all LC-subspaces of $\mathbb{R}^n$ and let $U = \bigcup_{n\geq 0} U_n \subset \mathbb{R}^\infty$. Let L be any subcollection of U that is closed under cartesian product. Then

$$\mathcal{L} = \{(X, \{X_i\})) \mid X_i \in L\}$$

is the *Lego collection of all L-stratifiable subspaces*. Main examples are when L is the subcollection of LCFT subspaces, so $\mathcal{L} = \mathcal{F}$ consists of the LCFT-stratifiable subspaces. Another example is when L is the subcollection of spaces homeomorphic to open cells, and $\mathcal{L} = \mathcal{M}_{cell}$ is the Lego collection of all cell-stratifiable subspaces. Not all Lego categories can be obtained this way.

Definition 4.3. For a Lego category $\mathcal{L}$, $K_0(\mathcal{L})$ is the quotient of the free abelian group on isomorphism classes $[X]$ of objects $X \in \mathcal{L}$ by the subgroup generated by $[X] - [Y] - [X \setminus Y]$, whenever Y is a stratum of an $\mathcal{L}$-stratification of X, and by $[X][Y] - [X \times Y]$.

Remark 4.4. The important point in Definition 4.3 is that it does not require $[X \setminus Y] = [X] - [Y]$ for all pairs (X, Y) in $Obj(\mathcal{L})$, only for pairs where Y is a stratum of an adapted stratification. Clearly when $\mathcal{L}$ is Boolean, this definition of K_0 coincides with that in (9). A lego category is an example of a *distributive category* [53] for which K_0 can be constructed. This is also sometimes called the "Burnside ring" of the category [65].

We conclude this section with a useful property used in applications.

Definition 4.5. An open cover of X by U_α's is of "finite type" if the U_α's are all open and cft. We then say that $X \to Y$ is a "finite type d-covering" if it is a trivial covering over the open sets of a finite type open cover $\{U_\alpha\}$ of Y.

Lemma 4.6. *If $\pi : X \to Y$ is a finite type degree d covering, then $X, Y \in \mathcal{F}$ and $[X] = d[Y]$ in $K_0(\mathcal{F})$.*

Proof. Let $\{U_\alpha\}$ be a finite type trivializing open cover of Y. We claim that the associated stratification of $\bigcup U_\alpha$ in (8) is an LCFT-stratification. Indeed, each term of that stratification is of the form $Y_I = \bigcap_{i\in I} U_i \setminus \bigcup_{j\in I^c} U_j$, for I a subset of $\Omega = \{1, \ldots, n\}$, and $I^c = \Omega \setminus I$, and thus is the complement of an open cft subspace in another cft subspace, so must be cft. Now, since π is trivial over the U_α's, it must be trivial over the strata Y_I of this stratification. Therefore $\pi^{-1}(Y_I)$ is d copies homeomorphic to Y_I, for all I, and $[\pi^{-1}(Y_I)] = d[Y_I]$. These preimages are cft necessarily and they stratify X. Summing over all strata gives the desired formula. □

Example 4.7. $[S^1] = 0 \in K_0(\mathcal{F})$. This is due to the existence of a covering $S^1 \to S^1$ of degree 2, so that $[S^1] = 2[S^1]$ and thus $[S^1] = 0$.

5. The Category $\mathcal{M}$ and its Motivic Measure

If $\mathcal{L}$ is a lego category, and R is a commutative ring with unit, then an R-valued *motivic morphism* is a ring morphism $\langle\ \rangle : K_0(\mathcal{L}) \longrightarrow R$ mapping unit to unit, $\langle [pt] \rangle = 1$. It is not clear in general that an interesting motivic morphism exists, meaning that it is well-defined and non-trivial. The point is that any such morphism on $X \in \mathcal{L}$ must be independent of the way we stratify X. In this section we construct a Lego category $\mathcal{M}$ which realizes the formula (6) as a motivic morphism out of $K_0(\mathcal{M})$.

Definition 5.1. Define

$$\begin{aligned}\mathcal{M} \ \ := \ \ &\{(X,\mathcal{S}) \in \mathcal{F} \text{ with } X \subset \mathbb{R}^n \text{ for some } n \text{ and } \mathcal{S} = \{X_i\} \text{ such that all possible unions}\\ &\text{of closures } \textstyle\bigcup \overline{X}_i \text{ in } \mathbb{R}^n \text{ are cft and locally contractible}\}\end{aligned}$$

As indicated earlier, if $(X,\mathcal{S}) \in \mathcal{M}$, then we say that $\mathcal{S}$ is an $\mathcal{M}$-stratification of X. This is an LCFT-stratification with additional conditions on the closures of strata. So for example, $(X, \{X_1, X_2, X_3\})$ is in $\mathcal{M}$ means each X_i is LCFT and all closures in the ambiant Euclidean space; $\overline{X}_i$, all of $\overline{X}_i \cup \overline{X}_j$ for $i,j \in \{1,2,3\}$, and $\overline{X}_1 \cup \overline{X}_2 \cup \overline{X}_3$ are cft and locally contractible. Note that if closures are cft and their union is cft, then so is their intersection by iterated use of (5). We also require local contractibility throughout so that (5) is indeed valid.

The collection $\mathcal{M}$ is Lego.

If $(X, \{X_i\})$ is in $\mathcal{F}$, then each $\chi_c(X_i)$ is well defined and we set as before

$$\langle X \rangle := \sum \chi_c(X_i) \in \mathbb{Z} \tag{10}$$

If X is LC to begin with, then $\langle - \rangle = \chi_c$ and (10) is independent of the stratification. We claim that independence holds for general $X \in \mathcal{M}$ and this is the key result of this section.

Theorem 5.2. *$\langle - \rangle$ gives a well-defined ring morphism between unitary commutative rings $K_0(\mathcal{M}) \to \mathbb{Z}$.*

Proof. We must show that (10) neither depends on the choice of an $\mathcal{M}$-stratification of X (this is claim 1), nor does it depend on the homeomorphism type of X, so can be defined on the class $[X]$ by setting $\langle [X] \rangle := \langle X \rangle$ (claim 2). We set $\langle pt \rangle = 1$.

We can use the following two properties ([75], §1):

- Let X be LC and Y open (resp. closed) in X. If two among X, Y and $X \setminus Y$ are cft, then so is the third.
- If all finite intersections of cft open (resp. closed) subspaces of X are cft, then their entire union is cft.

Let $(X, \{X_i\}_{1 \le i \le n}) \in \mathcal{M}$ and assume $X \subset \mathbb{R}^n$. As in (7), consider the decomposition

$$\mathbb{R}^n \setminus X = \left(\mathbb{R}^n \setminus \bigcup_{i=1}^n \overline{X_i}\right) \sqcup \bigsqcup_{I \subset \{1,\ldots,n\}} \left(\bigcap_{i_r \in I} \underline{X}_{i_r} \setminus \bigcup_{j_s \in I^c} \overline{X}_{j_s}\right) \tag{11}$$

We claim that this is an LCFT-stratification of the complement of X. The first term in the decomposition is cft since the complement in $\mathbb{R}^n$ of a closed cft is cft. To show that the other terms are cft, it is enough to show that an arbitrary intersection $\bigcap_j \underline{X}_j$ is cft. Each $\underline{X}_j = \overline{X}_j \setminus X_j$ is cft as the complement of an open cft subset in a cft space. On the other hand, because the X_i's are disjoint,

$$\bigcap \underline{X}_i = \begin{cases} \bigcap \overline{X}_i & \text{if all of the } X_i\text{'s are not closed} \\ \emptyset & \text{if one of the } X_i\text{'s is closed.} \end{cases}$$

Since intersections of closures of strata are cft, all terms $\bigcap \underline{X}_i$ are cft, and thus the decomposition (11) is of finite type as desired.

To proceed with the proof of claim 1, let's write the decomposition (11) as $\mathbb{R}^n \setminus X = \bigsqcup T_i$. Given now *any other* $\mathcal{M}$-stratification $X = \bigsqcup Y_i$, thus LCFT, then

$$\mathbb{R}^n = \bigsqcup Y_i \sqcup \bigsqcup T_j$$

is an LCFT-stratification of $\mathbb{R}^n$, so that $\langle X\rangle = \sum\langle Y_j\rangle = \langle\mathbb{R}^n\rangle - \sum_j\langle T_j\rangle = (-1)^n - \sum_j\langle T_j\rangle$ is obviously independent of the LCFT-stratification of X. This proves our first claim.

To prove claim 2, let $Y\in[X]$, then by definition there is a homeomorphism $f: Y\to X$. Let $\{Y_i\}$ be any $\mathcal{M}$-stratification of Y. Then $\{f(Y_i)\}$ is an $\mathcal{M}$-stratification of X, and since χ_c is a homeomorphism invariant, we can write $\langle Y\rangle = \sum\chi_c(Y_i) = \sum\chi_c(f(Y_i)) = \langle X\rangle$, the last equality being now a consequence of claim 1 earlier. This completes the proof of the claim, and thus of the Theorem. $\square$

Since $\langle -\rangle$ is well-defined and clearly surjective, the following is an immediate non-trivial consequence.

Corollary 5.3. $K_0(\mathcal{M})\neq 0$.

Remark 5.4. (Measure) If $\{X\setminus Y, Y\setminus X, X\cap Y\}$ forms an LCFT-stratification of $X\cup Y$, then

$$[X\cup Y] = [X] + [Y] - [X\cap Y]$$

When this stratification is an $\mathcal{M}$-stratification, then the same formula holds replacing $[-]$ with $\langle -\rangle$.

The next proposition shows that the O-minimal sets (Remark 3.1) form a subcategory of $\mathcal{M}$, as depicted in the organigram of Figure 1.

Proposition 5.5. *Let $\mathcal{O}$ be a category of O-minimal sets. Then $\mathcal{O}\subset\mathcal{M}$, $\langle -\rangle$ is a measure and $\langle -\rangle : K_0(\mathcal{O})\to\mathbb{Z}$ is an isomorphism.*

Proof. An element of $\mathcal{O}$ is called "definable". By the cell decomposition theorem, any definable set is partitioned into cells which are also definable. It is known that the closure of definable sets are definable ([22], Proposition 1.12). Moreover, definable sets form a Boolean algebra, so the union of closures of cells in a cell decomposition must also be in $\mathcal{O}$, and thus in $\mathcal{M}$ as well. Finally $\langle -\rangle : K_0(\mathcal{O})\to\mathbb{Z}$ sends $[X]\mapsto\sum_{\sigma\subset X}(-1)^{\dim\sigma}$, the sum being over a cell decomposition, and this is an isomorphism by Lemma 3.6. $\square$

5.1. **The Euler Characteristic.** From the the motivic morphism $\langle -\rangle$, we can recover the topological Euler characteristic under favorable circumstances.

- If $X\in\mathcal{M}$ is compact, then $\langle X\rangle = \chi_c(X) = \chi(X)$.
- If X is a manifold (compact or not), Poincaré duality with compact supports holds $H^i_c(M,\mathbb{Z}_2)\cong H_{n-i}(M,\mathbb{Z}_2)$, so that $\langle M\rangle = \chi_c(M) = (-1)^{\dim M}\chi(M)$.

More generally we have

Corollary 5.6. *If a compact and locally contractible $X\subset\mathbb{R}^n$ is stratified by manifolds M_i and compact spaces B_j, all of finite cohomological type, then the topological Euler characteristic is given by*

$$\chi(X) = \sum\chi(B_i) + \sum(-1)^{\dim M_j}\chi(M_j)$$

Example 5.7. Any complex algebraic variety X can be stratified by even dimensional manifolds, so that $\chi_c(X) = \chi(X)$ (see [34], Proposition 3). This is a well-known and widely used result.

The existence of $\langle -\rangle$ gives remarkably short proofs of known classical results in the literature. We illustrate with two "baby" examples: Corollary 5.8 below whose standard derivation requires Lefshetz duality, and Corollary 5.9 which requires Smith's theory of fixed points of periodic maps.

Corollary 5.8. *Let X be a compact manifold (with or without boundary),* $\dim X = n$ *and A a closed subset of X that doesn't meet the boundary of X. Then*

$$\chi(X-A) = \chi(X) - (-1)^n\chi(A)$$

Proof. This is simply a special case of Corollary 5.6 applied to the LC-stratification

$$X = (X\setminus\partial X\cup A)\sqcup A\sqcup\partial X \tag{12}$$

The stratum $X\setminus\partial X\cup A$ is a manifold, while both A and ∂X are compact. Moreover $X\setminus\partial X\cup A$ is homotopic to $X\setminus A$, so that both spaces have the same χ. We have

$$\chi(X) = \chi(X\setminus\partial X) = (-1)^n\chi_c(X\setminus\partial X) = (-1)^n(\chi_c(X) - \chi_c(\partial X)) = (-1)^n(\chi(X) - \chi(\partial X))$$

so that $\chi(\partial X) = (1-(-1)^n)\chi(X)$. Applying now χ_c to (12) and using duality and compactness, we obtain

$$\chi(X) = (-1)^n\chi(X \setminus \partial X \cup A) + \chi(A) + \chi(\partial X) = (-1)^n\chi(X \setminus A) + (1-(-1)^n)\chi(X) + \chi(A)$$

This is the desired formula after recombination. ☐

Corollary 5.9. *(P.A. Smith) Let p be prime, G a finite p-group and let X be a finite-dimensional G-CW complex. Then $\chi(X^G) \equiv \chi(X)$ modulo p.*

Proof. Here $\langle X\rangle = \chi(X)$ since X is compact. The proof is immediate if $G = \mathbb{Z}_p$ is the cyclic group of prime order p. In this case the action must be semifree in the sense that G acts on $X \setminus X^G$ freely. We have a bundle projection $X \setminus X^G \to (X \setminus X^G)/\mathbb{Z}_p$ and

$$\left\langle \frac{X^p \setminus X}{\mathbb{Z}_p} \right\rangle = \frac{1}{p}\left\langle X^p - X\right\rangle = \frac{1}{p}(\langle X\rangle^p - \langle X\rangle) = \frac{1}{p}(\chi^p - \chi) \tag{13}$$

which must be an integer. For a general p-group G, $|G| \geq p$, we can proceed by induction as in [58]. Choose a proper normal subgroup H of G (this is always possible in a nilpotent group). Then G/H acts on X^H, and so inductively twice

$$\langle X\rangle \cong \langle X^H\rangle \cong \langle (X^H)^{G/H}\rangle = \langle X^G\rangle$$

with both congruences being modulo p. ☐

Example 5.10. As an amusing final example, we can include Fermat's little theorem: for any integer $\chi \in \mathbb{N}$, there is $X \in \mathcal{M}$ such that $\langle X\rangle = \chi$. Let $G = \mathbb{Z}_p$ act on X^p by cyclic permutation of coordinates (see §7). The fixed point set is $(X^p)^{\mathbb{Z}_p} = X$ so that, by (5.9), $\chi^p \equiv \chi$ modulo p.

6. Orbit Type Stratifications

Quotient spaces by finite group actions are ubiquitous in this work. We here discuss properties of $X \in \mathcal{M}$ and its G-action so that the quotient X/G stays in $\mathcal{M}$.

Let G be a group acting on a space X, and denote by $G_x \subset G$ the stabilizer group of any $x \in X$. Consider the subspace

$$X_{(H)} = \{x \in X \mid G_x \text{ is conjugate to } H\}$$

This subspace[2] is a union of orbits since if $x \in X_{(H)}$, $G_{g.x} = gG_xg^{-1}$, and $gx \in X_{(H)}$. In particular, $X_{(H)}$ is a G-space. We have that $X_{(e)}$ is the free part of the action (where G acts freely) and $X_{(G)} = X^G$ is the fixed part of the action.

We show in Proposition 6.1 that under suitable conditions, the spaces $X_{(H)}$ produce an LC-stratification of X, as (H) varies across conjugacy classes of subgroups $H \subset G$, while the quotient strata $X_{(H)}/G$ do also give an LC-stratification of X/G. Note that in the case G acts differentiably and properly on a smooth manifold X, then the $X_{(H)}$ are submanifolds of X, a greatly useful fact in the theory of transformation groups ([25], Proposition 5.13).

We will need the following properties:

Properties II: Throughout X is Hausdorff.

(i) If G is finite (or compact hausdorff topological group), then X is LC if and only if X/G is LC ([25], Proposition 3.6). If the action is proper, then X/G is Hausdorff ([49], Prop. 12.24).
(ii) Suppose X is normal, $G \times X \to X$ a proper action such that X/G is LC. Then X is LC ([46], Lemma 17).
(iii) Let X be LCFT, and G a finite group acting on X. Then X/G is LCFT ([14], Theorem 5.3, Chap. IV).

Proposition 6.1. *Let G be a finite group acting on a metrizable LC-space X. Then:*

[2] $X_{(H)}$ is referred to as the "(H)-orbit bundle" of X, [25], page 3.

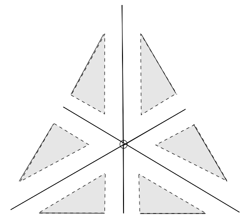


FIGURE 3. The orbit stratification of an equilateral triangle X under the $\mathfrak{S}_3$- action by linear symmetries. There are three non-empty strata: the centerpoint (B) corresponds to $X_{(\mathfrak{S}_3)}$ (the only fixed point), the three axes without B correspond to $X_{((12))}$, where $((12))$ is the conjugacy class containing the transpositions, and $X_{(e)}$ is the rest (the shaded triangles). Here $X_{((123))} = \emptyset$, where $((123)) = \{(123), (132)\}$.

(1) Both $X_{(H)}$ and $X_{(H)}/G$ are LC for $H \subset G$.
(2) If $\{X_{(H)}\}_{(H)}$ is an LCFT-stratification of X (resp. an $\mathcal{M}$-stratification), then $\{X_{(H)}/G\}_{(H)}$ is an LCFT-stratification (resp. an $\mathcal{M}$-stratification) of X/G.

Proof. We first verify that each X_H is LC. Indeed, the fixed point set $X^H = \{x \in X \mid hx = x, \forall h \in H\}$ is a closed subspace of X, and $X_H = \{x \in X \mid G_x = H\}$ is a subset of X^H whose complement $X^H \setminus X_H$ is $\bigcup_K X^K$ for all $K \supsetneq H$. This means that X_H is open in X^H which is closed, therefore it must be LC.

Next consider $X_{(H)} = \bigcup X_{H'}$, H' is conjugate to H. We must show that $\underline{X}_{(H)} := \overline{X}_{(H)} \cap X^c_{(H)}$ is closed (see Properties I, (iii)). Let (x_n) be a sequence in $\underline{X}_{(H)}$ converging to x. Since this sequence is in $\overline{X}_{(H)}$ which is closed, $x \in \overline{X}_{(H)}$. We must show it is in the complement $X^c_{(H)}$ as well. Since $\underline{X}_{(H)} \subset \bigcup \underline{X}_{H'}$, and this is a finite union, there must be a subsequence (y_n) of (x_n) in $\underline{X}_{H'}$ for some $H' \sim H$. This sequence converges in $\underline{X}_{H'}$ as well, being closed, so $x \in \underline{X}_{H'}$. This shows in particular that $G_x \supsetneq H'$ since $x \in X^c_{H'}$. Consequently G_x contains H' as a proper subgroup, it cannot be conjugate to H (not of the same cardinality), so $x \in X^c_{(H)}$, which is what we wanted to prove. In conclusion, $X_{(H)}$ is LC. Since G acts on $X_{(H)}$, then by Properties II (i), $X_{(H)}/G$ is also LC. The orbit type stratifications of X and X/G produce therefore LC-stratifications.

To address the finite type condition, and since G acts on $X_{(H)}$, we can apply Properties II (iii) to deduce that $X_{(H)}/G$ is cft. Notice that the quotient map $X_{(H)} \to X_{(H)}/G$ is a covering of degree $d = |G : H|$.

Finally assume the orbit stratification of G acting on X is an $\mathcal{M}$-stratification. It is easy to see that this stratification has the frontier condition [45], so that the closure of strata is a union of strata as well. This shows that the union of closures is a union of G-invariant strata, so is G invariant. The quotient image is then a union of the closures of the orbit strata of $X_{(H)}/G$, and these must be of finite type by Properties II (iii) again. □

Remark 6.2. An action of a finite group G on X has finite type (or is a *finite type action*) if its orbit stratification $\{X_{(H)}\}_{(H)}$ is an LCFT-stratification. In this paper, all actions will be of finite type. The following example shows that non-finite type action by groups as small as $\mathbb{Z}_2$ exists. This is Bing's action on S^3 [10]. Here, the orbit strata consist of $X_{(e)}$ (the free part) and $X_{\mathbb{Z}_2}$ (the fixed points), with orbit type stratification $S^3 = X_{(e)} \sqcup X_{\mathbb{Z}_2}$. The action Bing constructs has as fixed pointset the Alexander Horned sphere (this is cft being homeomorphic to a sphere S^2) but its complement $X_{(e)}$ has non finitely generated fundamental group hence its abelianization H_1 is not finitely generated either. The orbit type stratification associated to this action is not an LCFT-stratification.

Remark 6.3. To a finite group G, we can associate the lattice of subgroups given by ordering all subgroups of G by inclusion. On the other hand, if X is a G space, then to a subgroup H of G corresponds a fixed pointset X^H such that $H_1 \subset H_2$, implies that $X^{H_2} \subset X^{H_1}$. The poset of subgroups maps to the poset of fixedpoints of the action of G on X, ordered by *reversed inclusion*. They are sometimes isomorphic, and this fact is used in computing the class of the cyclic products in the next section.

The next few sections will apply the Lego formalism to give numerous computations of Grothendieck classes and Euler characteristics. The following notation will be handy.

Notation 6.4. Suppose X is stratified by strata some of which are homeomorphic to other strata. We will write $X \doteqdot \bigsqcup_{i\in I} k_iX_i$ if in the stratification of X there are k_i strata homeomorphic to X_i, for $i \in I$. For example S^n decomposes as $S^n \doteqdot S^{n-1} \sqcup 2D^n$, where D^n is an open disk. This will be a handy notation when we start decomposing spaces in the second part of this paper.

7. Basic Computations I: Classical Constructions

The second half of the paper is about applications. We will work with three combinatorial "invariants": $[-]$ the Grothendieck class, $\langle -\rangle$ its associated measure and χ the topological Euler characteristic. All computations will derive from knowing $[-]$ on constructions defined in $\mathcal{M}$ (or in $\mathcal{F}$ if we don't need to use $\langle -\rangle$).

This section and the next give lightning derivations of the Grothendieck classes of fundamental constructions we use in this paper: the classical configuration spaces of distinct points, some permutation products and the finite cardinality subspaces. Extensive generalizations given in terms of χ_c and Poincaré polynomials are found in [2]. In later sections we will conceptualize these computations.

For a given space $X \in \mathcal{M}$ and integer $n \geq 1$, the space of *ordered* n pairwise distinct points of X is

$$\mathrm{Conf}(X,n) = \{(x_1,\ldots,x_n) \in X^n \mid x_i \neq x_j, i \neq j\}$$

The corresponding space of *unordered* points is written $B(X,n)$ and is the quotient of $\mathrm{Conf}(X,n)$ by the permutation action of symmetric group on n-letters $\mathfrak{S}_n$ ([3]). We have the inclusion $B(X,n) \subset \mathrm{SP}^n(X)$ where $\mathrm{SP}^n(X)$ is the n-th symmetric product defined as the quotient of X^n by the permutation action of $\mathfrak{S}_n$. We adopt the notation $[x_1,\ldots,x_n]$ for an element of $\mathrm{SP}^n(X)$ (i.e an "unordered tuple").

Define as in [2] the *cardinality* subspaces (our notation is different):

$$\mathrm{Card}_{\leq d}(X,n) := \{[x_1,\ldots,x_n] \in \mathrm{SP}^n(X) \mid \mathrm{card}\{x_1,\ldots,x_n\} \leq d\}$$

We can write $\mathrm{Card}_d(X,n) = \mathrm{Card}_{\leq d}(X,n) \setminus \mathrm{Card}_{\leq d-1}(X,n)$. Observe that $\mathrm{SP}^n(X) = \mathrm{Card}_{\leq n}(X,n)$ while $B(X,n) = \mathrm{Card}_n(X,n)$. For $n = 0$, we set all spaces to be the emptyset.

Proposition 7.1. *For $X \in \mathcal{M}$, $n \geq 1$, all above constructions are in $\mathcal{M}$, and*

$$[\mathit{Conf}(X,n)] = \prod_{i=0}^{n-1}([X]-i) \quad , \quad [B(X,n)] = \binom{[X]}{n}$$

$$[\mathit{Card}_k(X,n)] = k^{n-k}\binom{[X]}{k} \quad , \quad [\mathit{Card}_{\leq k}(X,n)] = \sum_i^k i^{n-i}\binom{[X]}{i}$$

In series form, we can write $1 + \sum_{n\geq 1}[B(X,n)]t^n = (1+t)^{[X]} = \sum_{n\geq 0}\binom{[X]}{n}t^n$.

Proof. Suppose first that X is LCFT. Notice that $\mathrm{Conf}(X,n)$ is the complement in X^n of the union of diagonal subspaces $\Delta_{ij} = \{(x_1,\ldots,x_n) \mid x_i = x_j\} \cong X^{n-1}$. These diagonal subspaces are homeomorphic to X^{n-1}, thus are also LCFT, and being closed, the complement of their union is LCFT. So $\mathrm{Conf}(X,n)$ is LCFT. Now suppose X is LCFT-stratifiable, then $\mathrm{Conf}(X,n)$ is LCFT-stratifiable by the configuration spaces of strata of X (see (21)), and thus is in $\mathcal{F}$. If a stratum $X_i \subset X \subset \mathbb{R}^n$ is cft and its closure is cft, then so is $\mathrm{Conf}(X_i,k)$ and its closure $\overline{\mathrm{Conf}}(X_i,k) = \overline{X}_i^k \subset (\mathbb{R}^n)^k$. Similarly for the

[3] In other notation, $\mathrm{Conf}(X,n)$ is $PConf(X,n)$ and $B(X,n)$ is $UConf(X,n)$.

union of strata, so $\mathrm{Conf}(X,n)\in\mathcal{M}$. We leave checking the other cases as an exercise (using Properties II (iii)).

There are various derivations of the expression for $[\mathrm{Conf}(X,n)]$ that can be found in the literature. The shortest argument which we extract and present next is from ([2], §0.1). For another derivation, see Example 9.6. Write

$$\Delta(X)=\{(x_0,\dots,x_n)\in X\times\mathrm{Conf}(X,n)\ |x_0=x_i \text{ for some } i\}$$

This is a closed set in $X\times\mathrm{Conf}(X,n)$ and we have the stratification $X\times\mathrm{Conf}(X,n)=\mathrm{Conf}(X,n+1)\sqcup\Delta(X)$. The projection $\Delta(X)\to\mathrm{Conf}(X,n)$ is a covering of degree n. This gives that

$$[X][\mathrm{Conf}(X,n)]=[\mathrm{Conf}(X,n+1)]+[\Delta(X)]=[\mathrm{Conf}(X,n+1)]+n[\mathrm{Conf}(X,n)]$$

The formula follows now by induction. This gives the proof in the ordered case. The claim for the unordered case follows from the fact that $\mathrm{Conf}(X,n)\to B(X,n)$ is a covering of degree $n!$, so that $[B(x,n)]=\frac{1}{n!}[\mathrm{Conf}(X,n)]=\binom{[X]}{n}$ by Lemma 4.6.

We turn to the cardinality subspaces. Clearly $\mathrm{Card}_1(X,n)=X$, while $\mathrm{Card}_n(X,n)=B(X,n)$. Consider the projection $\mathrm{Card}_k(X,n)\longrightarrow B(X,k)$ which sends a tuple to the set of its entries that are pairwise distinct. We claim that this is a covering of degree k^{n-k}. Indeed, given an element $[x_1,\dots,x_k]\in B(X,k)$, $x_i\neq x_j$, its preimage in Card_k consists of all tuples of the form $[x_1,\dots,x_k,y_1,\dots,y_{n-k}]$, where $y_i\in\{x_1,\dots,x_k\}$. The number of these preimages corresponds to the number of ways we can assign k elements to $n-k$ spots, and this is k^{n-k}. It follows that $\mathrm{Card}_k(X,n)$ is in $\mathcal{M}$ and that $[\mathrm{Card}_k(X,n)]=k^{n-k}[B(X,k)]$. We can now write

$$\mathrm{Card}_{\leq k}(X,n)=\mathrm{Card}_1(X,n)\sqcup\mathrm{Card}_2(X,n)\sqcup\cdots\sqcup\mathrm{Card}_k(X,n)\tag{14}$$

so that $[\mathrm{Card}_{\leq k}(X,n)]$ is the sum of the individual classes of the strata. □

Recovering the topological Euler characteristic from $[\mathrm{Conf}(X,n)]$ can now be quickly done.

Corollary 7.2. *Let X be a manifold (with or without boundary), with $\dim X=m$ and $\chi(X)=\chi$. Then*

$$\chi\left(\mathit{Conf}(X,k)\right)=\prod_{i=0}^{k-1}(\chi-(-1)^m i)$$

When X is closed, $\chi B(X,k)=\begin{pmatrix}\chi\\ k\end{pmatrix}$ if X is even dimensional, and is zero otherwise.

Proof. Applying the measure $\langle -\rangle$ to the formula in Proposition 7.1 we get that

$$\langle\mathrm{Conf}(X,k)\rangle=\prod_{i=0}^{k-1}(\langle X\rangle-i)\quad=\quad\prod_{i=0}^{k-1}((-1)^m\chi(X)-i)=(-1)^{km}\prod_{i=0}^{k-1}(\chi(X)-(-1)^m i)\tag{15}$$

If X is a manifold of dimension m, then $\mathrm{Conf}(X,k)$ is a manifold of dimension mk, and $\langle\mathrm{Conf}(X,k)\rangle=\chi_c(\mathrm{Conf}(X,k))=(-1)^{mk}\chi\mathrm{Conf}(X,k)$. Replacing this formula in (15) gives us the first statement. For manifolds with boundary, by a theorem of Brown [16], every topological manifold M with boundary is collarable (that is the boundary admits a product neighborhood) so that $\mathrm{int}(M)=\mathring{M}$ is homotopy-equivalent to M and in fact $\mathrm{Conf}(\mathring{M},k)\simeq\mathrm{Conf}(M,k)$, so we are back to the case of a manifold without boundary since χ, unlike χ_c, is a homotopy invariant. Finally $\chi B(X,k)=\frac{1}{k!}\chi(\mathrm{Conf}(X,k))$, so that when X is closed of odd dimension, they are all zero, and when it is closed of even dimension, we get $\chi(B(X,k))=\binom{\chi}{k}$. □

Example 7.3. Let $X=\mathbb{R}$, then $\mathrm{Conf}(\mathbb{R},k)$ splits into $k!$-connected components, each being contractible. The homotopy type of this space is a discrete set having $k!$ points, so that $\chi(\mathrm{Conf}(\mathbb{R},k))=k!$, in agreement with the formula.

Remark 7.4. The standard derivation of $\chi(\mathrm{Conf}(X,n))$ given in the literature (eg. [19]) uses the Fadell-Neuwirth projection map $\mathrm{Conf}(X,n) \to \mathrm{Conf}(X,n-1)$, which is a bundle map when X a manifold without boundary, with fiber $X \setminus Q_{n-1}$ where Q_{n-1} is a set of $n-1$ points in X. Using the multiplicativity of χ for such a bundle, one obtains the same formula for $\chi(\mathrm{Conf}(X,n))$.

7.1. **Permutation Products.** The symmetric product is a special example of a *permutation product* obtained as the quotient of X^n by a subgroup of $\Gamma \subset \mathfrak{S}_n$ acting by permutations [51, 45]. We denote such a quotient by $\Gamma P^n(X)$. If $X \in \mathcal{M}$, then $\Gamma P^n(X) \in \mathcal{M}$. We adopt the notation $\Gamma P^n(X) = SP^n(X)$, as before, if $\Gamma = \mathfrak{S}_n$ is the symmetric group, $\Gamma P^n(X) = CP^n(X)$ if $\Gamma = \mathbb{Z}_n$ is the cyclic group, and $\Gamma P^n(X) = AP^n(X)$ if $\Gamma = A_n$ is the alternating group. When $n=0$, $\mathrm{SP}^0(X) = \mathrm{CP}^0(X) = pt$.

Proposition 7.1 gives immediately the well-known formula for $[\mathrm{SP}^n(X)] \in K_0(\mathcal{M})$

$$[\mathrm{SP}^n(X)] = \sum [\mathrm{Card}_i(X,n)] = \sum_{k=1}^{n} k^{n-k}[B(X,k)] = \sum_{k=1}^{n} k^{n-k}\binom{[X]}{k} = \binom{n+[X]-1}{n} \tag{16}$$

with the notation $\binom{n+[X]-1}{n}$ refers to the polynomial $\frac{1}{n!}([X]-1+n)([X]-2+n)\cdots([X]-1)$, where for simplicity a constant k in $K_0(\mathcal{M})$ means $k[pt]$. The above formula translates into the series $\sum_{n\geq 0}[\mathrm{SP}^n X]t^n = (1-t)^{-[X]}$.

Obtaining the series for the alternating products is also immediate. For $n=0,1$, $\mathrm{SP}^n X$ and $\mathrm{AP}^n X$ coincide and when $n=2$, $AP^2(X) = X^2$. For $n \geq 2$, it suffices to consider the projection $\pi : \mathrm{AP}^n(X) \to \mathrm{SP}^n(X)$ which is a ramified covering of degree two. The branched locus is a copy of $B_2(X,n) = \mathrm{SP}^n X - B(X,n)$ which is the fixed pointset of the natural action of $\mathbb{Z}_2$ on $AP^n(X)$ induced from the action of (any) transposition on X^n. Over $B(X,n)$, π is a regular covering of degree 2. This gives that for $n \geq 2$, $[\mathrm{AP}^n(X)] = [B_2(X,n)] + 2[B(X,n)] = [\mathrm{SP}^n X - B(X,n)] + 2[B(X,n)]$ so that $[\mathrm{AP}^n(X)] = [\mathrm{SP}^n(X)] + [B(X,n)]$ and

$$[\mathrm{AP}^n(X)] = \binom{n+[X]-1}{n} + \binom{[X]}{n} \quad n \geq 2$$

This formula, after replacing $[-]$ by $\chi(-)$, can be found in [78] who derived it by computing invariants in the rational cohomology ring of X^n.

The cyclic products are slightly more intricate. When $n = p$ is a prime, it is easy to see that

$$[\mathrm{CP}^p(X)] = \frac{p-1}{p}[X] + \frac{1}{p}[X]^p \tag{17}$$

To see this indeed, let $Y \in \mathcal{M}$ be any space with $\mathbb{Z}_p$-action. Then one stratifies Y by the fixed points of the action; i.e. $Y = Y^{\mathbb{Z}_p} \sqcup (Y \setminus Y^{\mathbb{Z}_p})$. This stratification is $\mathbb{Z}_p$-equivariant, so we get a stratification at the level of quotient spaces. Since the action of $\mathbb{Z}_p$ is trivial on the fixed point set, and free on the complement (p being a prime) we get that $\left[\frac{Y}{\mathbb{Z}_p}\right] = [Y^{\mathbb{Z}_p}] + \frac{1}{p}([Y] - [Y^{\mathbb{Z}_p}])$, yielding (17) in the case $Y = X^p$ and the action is by cyclic permutation.

There is a cute generalization of (17). Recall that for integers d, $\phi(d)$ is Euler's totient function that counts the positive integers from 1 to d relatively prime to d.

Proposition 7.5. *For d, m divisors of n, $d|m$, define $\mu_n(d,m) = (-1)^k$ if $\frac{m}{d}$ is the product of k distinct primes, and is 0 otherwise. Then*

$$[CP^n(X)] = \sum_{d|n}\frac{d}{n}\sum_{\substack{m\\ d|m}}\mu_n(d,m)[X]^{\frac{n}{m}} = \frac{1}{n}\sum_{d|n}\phi(d)[X]^{\frac{n}{d}}$$

We can derive this formula using the Burnside formula (Proposition 7 of [34]). We choose however to give in §9 a more compelling derivation using the combinatorics of posets. The original computation of both the Poincaré polynomial and the Euler characteristic of $\mathrm{CP}^n(X)$ uses homology and is carried out in [51] (see Remark 9.9, with generalizations in [2]).

8. Basic Computations II: Finite Subset Spaces

This is one of our most basic illustrative examples. Given a space X, one defines $\mathrm{Sub}_n(X)$ to be the space of all finite subsets of X of cardinality at most n. This space is topologized as a quotient of $\bigsqcup_{i=1}^n X^i$ (see [71] and references therein). There are subtle but important topological differences with the symmetric products since for example, $[x,x,y]$ and $[x,y,y]$ are distinct in $\mathrm{SP}^3(X)$ if $x\neq y$, whereas $\{x,x,y\}$ and $\{x,y,y\}$ are both equal to $\{x,y\}$ in $\mathrm{Sub}_3(X)$. Notice that there is a quotient map $q:\mathrm{SP}^n(X)\rightarrow \mathrm{Sub}_n(X)$ such that $q^{-1}(\mathrm{Sub}_d(X)) = \mathrm{Card}_{\leq d}(X,n)$.

The space $\mathrm{Sub}_n(X)$ has many attractive geometrical properties. For example Tuffley proves that $\mathrm{Sub}_n(S^1)\simeq S^{2n-1}$ [61] and Mostovoy shows that the embedding $\mathrm{Sub}_1(S^1) = S^1\hookrightarrow \mathrm{Sub}_3(S^1)\cong S^3$ is a trefoil knot [54]. We set $\mathrm{Sub}_0(X)=\emptyset$. As far as we know, the following identity is new.

Proposition 8.1. *For $X\in\mathcal{M}$, we have*

$$\sum_{n\geq 1}[Sub_nX]t^n = \frac{(1+t)^{[X]}-1}{1-t}$$

Proof. The space $\mathrm{Sub}_n(X)$ is stratified by the configuration spaces iteratively as follows

$$\mathrm{Sub}_nX = \mathrm{Sub}_{n-1}(X)\sqcup B(X,n)\quad ,\quad n\geq 1$$

with $B(X,n)$ open in Sub_nX. In $K_0(\mathcal{M})$ and inductively, one finds that $[\mathrm{Sub}_nX]=\sum_{k\leq n}[B(X,k)]$. In terms of generating series, this means

$$\sum_{n\geq 1}[\mathrm{Sub}_nX]t^n = (1+t+t^2+\cdots)\sum_{n\geq 1}[B(X,n)]t^n$$

We now replace the righthand term by (7.1) and substitute $1+t+t^2+\cdots = \frac{1}{1-t}$. ∎

Corollary 8.2. *Suppose X is of the homotopy type of a finite complex, then $\chi(Sub_nX) = \sum_{k=1}^n\binom{\chi}{k}$, where $\chi=\chi(X)$. In particular, if $\chi(X)=0$ then $\chi(Sub_n(X))=0$.*

Proof. The functor Sub_n preserves homotopy type, so we can suppose that X is a finite complex, thus $\mathrm{Sub}_n(X)$ is compact for all $n\geq 1$, and

$$\chi(\mathrm{Sub}_n(X)) = \langle \mathrm{Sub}_nX\rangle = \sum_{k\leq n}\langle B(X,k)\rangle = \sum_{k\leq n}\binom{\langle X\rangle}{k}$$

Now replace $\langle X\rangle = \chi(X)$ since X compact. ∎

Remark 8.3. For $\chi>0$, the sum $\sum_{k=1}^n\binom{\chi}{k}$ coincides with the number of binary words of length χ with at least one and no more than n 1's. This is also the same as some "egg-drop" number [13]. There is no known closed formula for this sum for general n (see [13]).

Example 8.4. It is clear that $\mathrm{Sub}_2X\cong \mathrm{SP}^2X$. This checks obviously with the formula in Proposition 16 since $[\mathrm{Sub}_2X]=[X]+\binom{[X]}{2} = \binom{[X]+1}{2} = [\mathrm{SP}^2X]$. When $X=S^1$ on the other hand, $\chi(\mathrm{Sub}_n(S^1))=0$ in accordance to Tuffley's equivalence $\mathrm{Sub}_n(S^1)\simeq S^{2n-1}$.

9. Subspace Arrangements

Stratifications and arrangements are tied to combinatorics through their associated posets and Möbius functions. In this section, we clarify these constructions in the context of this paper, and relate them to computations in the Grothendieck ring $K_0(\mathcal{M})$. Much of our terminology is borrowed from [52, 68].

For a locally compact X, define an arrangement of X to be a *finite* collection $\mathcal{A}=\{A_i\}_{i\in\Omega}$ in X, $\Omega=\{1,\ldots,n\}$, of locally compact subspaces of X. We assume that no A_i is contained in some A_j.

Examples of arrangements are the subspace arrangements in affine space, the *simplicial* arrangements where the A_i's are subcomplexes of an ambient simplicial complex X, or the *manifold* arrangements where each A_i is a submanifold of an ambiant manifold X [18, 26]. Obviously $\bigcup_{i\in\Omega} A_i$ is a constructible subset, and we sometimes refer to this union as being the arrangement as well.

The combinatorial object associated to an arrangement $\mathcal{A}$ is its poset of intersections $L_{\mathcal{A}} := \left\{\bigcap_{i\in J} A_i \mid J \subseteq \Omega\right\}$, ordered by *reversed inclusion.* This is a semi-lattice with bottom element $\hat{0} = X$, which is the term corresponding to $J = \emptyset$. It is a lattice if the top element $\hat{1} = \bigcap_{i\in\Omega} A_i$ is non-empty. Not every intersection is non-empty, and no two intersections need to be different. We write I for the indexing set of the poset (indexing distinct elements), with $\alpha \in I$, A_α the corresponding subspace of $L_{\mathcal{A}}$ and $\alpha \leq \beta$ if $A_\beta \supset A_\alpha$. We will write μ the Möbius function of this lattice.

By definition, an LCFT-arrangement $\{A_i\}$ of X is an arrangement such that all A_i are closed, locally contractible and cft. This will be the case of all the arrangements in this paper. The complement of this arrangement $X - \bigcup A_i$ is LCFT-stratifiable (i.e. in $\mathcal{F}$) and $[X - \bigcup A_i] = [X] - [\bigcup A_i] \in K_0(\mathcal{F})$.

Theorem 9.1. *(Inclusion-exclusion for posets) Let $\mathcal{A} = \{A_i\}_{i\in\Omega}$ be an LCFT-arrangement in an LCFT space X. Then $X \setminus \bigcup_i A_i$ is in $\mathcal{F}$ and we have the general formula*

$$\left[X \setminus \bigcup A_i\right] = \sum_{\hat{0}\leq\alpha\leq\hat{1}} \mu(\hat{0},\alpha)[A_\alpha]$$

Proof. That $X \setminus \bigcup_i A_i$ is in $\mathcal{F}$ is a consequence of (7) and follows by the exact same argument in the proof of Proposition 7.1. Write

$$A_\alpha^+ = A_\alpha - \bigcup_{A_\beta \subsetneq A_\alpha} A_\beta = A_\alpha - \bigcup_{\beta\gneq\alpha} A_\beta \tag{18}$$

with $\alpha \in I$ the indexing set of the intersection poset of the arrangement. Since the A_β^+ for $\beta \geq \alpha$ form a stratification of A_α, $[A_\alpha] = \sum_{\beta\geq\alpha}[A_\beta^+]$. Möbius inversion ([63], Theorem 2.4 (2)) gives that $[A_\alpha^+] = \sum_{\beta\geq\alpha} \mu(\alpha,\beta)[A_\beta]$. Since $A_\emptyset^+ = X \setminus \bigcup_i A_i$, we obtain the result. □

Theorem 9.1 holds true if we replace $[-]$ by $\langle - \rangle$ (the motivic measure) when working in $\mathcal{M}$. This formalism gives instantaneous derivations of useful formulas in combinatorics. Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$ and $M_{\mathcal{A}}$ its complement in $\mathbb{R}^n$. The following is Corollary 3.8 of [12] given for affine subspace arrangements and which took a few paragraphs to derive (see also [11], Thm. 7.3.1).

Corollary 9.2. [12] *For any affine subspace arrangement $\mathcal{A}$ in $\mathbb{R}^n$*

$$\chi(M_{\mathcal{A}}) = \sum_{x\in L_{\mathcal{A}}} (-1)^{n-\dim x}\mu(\hat{0},x)$$

Proof. If $\mathcal{A}$ is a subspace arrangement in affine space $X = \mathbb{R}^n$, then

$$\left\langle \mathbb{R}^n \setminus \bigcup A_i \right\rangle = \sum_{\hat{0}\leq\alpha\leq\hat{1}} \mu(\hat{0},\alpha)\langle A_\alpha\rangle = \sum_{\hat{0}\leq\alpha\leq\hat{1}} \mu(\hat{0},\alpha)(-1)^{\dim A_\alpha} = \chi(\mathcal{A},-1)$$

where $\chi(\mathcal{A},q) := \sum_{x\in L_{\mathcal{A}}} \mu(\hat{0},x)q^{\dim x}$ is known as the "characteristic polynomial" of the arrangement [3, 55, 59, 68]. Passing to Euler characteristics, and since $\chi\left(\mathbb{R}^n \setminus \bigcup A_i\right) = (-1)^n \left\langle \mathbb{R}^n \setminus \bigcup A_i\right\rangle$, one obtains $\chi\left(\mathbb{R}^n \setminus \bigcup_{A_i\in\mathcal{A}} A_i\right) = (-1)^n\chi(\mathcal{A},-1)$. Since $(-1)^{n+\dim A_\alpha} = (-1)^{n-\dim A_\alpha}$, the claim follows. □

We are interested in applying Theorem 9.1 to configuration spaces at large. Configuration spaces of points are ubiquitous in geometry and topology, as well as in more applied fields. The classical construction $\mathrm{Conf}(X,n)$ and its unordered analog $B(X,n)$, have both been extended in many directions and various authors call their variants "generalized configuration spaces" [57] (also "exotic" or "polychromatic" configuration spaces [6]), leading to a flurry of overlapping terminologies. The following definition is fairly general.

Definition 9.3. For every set partition P of $\Omega = \{1, \ldots, n\}$, define the diagonal subspace

$$\Delta_P := \{(x_1, \ldots, x_n) \in X^n,\ x_i = x_j \text{ if and only if } x_i \text{ and } x_j \text{ are in same block of } P\}$$

The subspace Δ_P is LC if X is LC. For a finite collection $\mathcal{S} = \{P_i\}$ of partitions of Ω, which we require to be closed under coarsening, define $\mathcal{A} := \{\Delta_{P_i}\}_{P_i \in \mathcal{S}}$ to be the *diagonal arrangement* in X^n indexed by $\mathcal{S}$. This is always a central arrangement since the intersection of all subspaces in the arrangement is always non-empty and contains $\Delta_\Omega = \{(x, \ldots, x) \in X^n\}$ (the thin diagonal). An $\mathcal{S}$-*configuration space* of points in X is now the complement of this diagonal arrangement

$$\mathrm{Conf}(X, \mathcal{S}) := X^n \setminus \bigcup_{P_i \in \mathcal{S}} \Delta_{P_i} \tag{19}$$

In contrast to the Euler characteristic of configuration spaces which fails in general to be a function of $\chi(X)$ if X is not a manifold and $n \geq 2$, the class $\langle \mathrm{Conf}(X, \mathcal{S}) \rangle$ only depends on $\langle X \rangle$.

Proposition 9.4. *If $X \in \mathcal{M}$, then $Conf(X, \mathcal{S}) \in \mathcal{M}$, and $[Conf(X, \mathcal{S})] \in K_0(\mathcal{M})$ is a monic polynomial in $[X]$. The motivic class $\langle Conf(X, \mathcal{S}) \rangle$ only depends on $\langle X \rangle$. If X is a manifold, then $\chi(Conf(X, \mathcal{S}))$ only depends on $\chi(X)$.*

Proof. By Theorem 9.1, $[\mathrm{Conf}(X, \mathcal{S})] = \sum_{\hat{0} \leq \alpha \leq \hat{1}} \mu(\hat{0}, \alpha)[\Delta_{P_\alpha}]$, where the sum is over the Möbius function of the poset of intersections of the diagonal arrangement indexed over $\mathcal{S}$ (here $\mathcal{S}$ is the full indexing set since $\mathcal{S}$ is closed under coarsening). This is a lattice since the arrangement is central with $\hat{1} = \bigcap \Delta_{P_\alpha} \neq \emptyset$. Each term Δ_{P_α} is homeomorphic to X^s for some s. This shows that $[\mathrm{Conf}(X, \mathcal{S})]$ is a polynomial in $[X]$, and it is monic starting at $[X^n] = [X]^n$ since $\mu(\hat{0}, \hat{0}) = 1$. We write $[\mathrm{Conf}(X, \mathcal{S})] = \chi(\mathcal{S}, [X])$ this polynomial. Taking motivic measure, which is a ring morphism into $\mathbb{Z}$, we see that $\langle \mathrm{Conf}(X, \mathcal{S}) \rangle = \chi(\mathcal{S}, \langle X \rangle)$, so it only depends on $\langle X \rangle$. When X is a manifold of dimension $m \geq 1$, $\chi(X) = (-1)^m \langle X \rangle$, with $m = \dim X$, and $\chi(\mathrm{Conf}(X, \mathcal{S})) = (-1)^{mn} \langle \mathrm{Conf}(X, \mathcal{S}) \rangle$, so the second claim follows. □

Remark 9.5. Note that for special spaces X, this independence is already true at the level of cohomology groups. More precisely, one says that X is i-acyclic over a ring k if the map $H^*_c(X, k) \to H^*(X, k)$ is trivial [2]. For i-acyclic paracompact locally compact Hausdorff spaces X, and k a field, [57, 2] prove that $H^*_c(\mathrm{Conf}(X, \mathcal{S}), k)$ only depends on the graded vector space $H^*_c(X, k)$.

Examples of configuration spaces are complements of diagonal arrangements indexed over collections defined by graphs or simplicial complexes.

Example 9.6. Consider the arrangement $\mathcal{A} = \{\Delta_{i,j}\}_{1 \leq i < j \leq n}$, where $\Delta_{i,j} := \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j\}$. The intersection poset for this arrangement ordered by reverse inclusion is isomorphic to the poset Π_n of all set partitions $Par(\Omega)$ ordered by refinements (see Fig. 4). For two partitions σ, τ, $\sigma \leq \tau$ means every block of σ is contained in a block of τ, and every block of τ is a disjoint union of blocks of σ. The isomorphism sends a partition P to Δ_P the subspace of all tuples $(x_1, \ldots, x_n)$ such that $x_i = x_j$ if $\{i, j\}$ in the same block of P. This subspace is homeomorphic to X^k if the length of the partition is k.

For $\sigma \in Par(\Omega)$, a set partition of Ω, it is well-known that $\mu(\hat{0}, \sigma) = (-1)^{n-\ell(\sigma)} \prod_i (|\sigma_i| - 1)!$, where $\ell(\sigma)$ is the length of σ. For instance $\mu(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$. Theorem 9.1 implies that

$$[\mathrm{Conf}(X, n)] = \sum_{\sigma \in Par(\Omega)} (-1)^{n-\ell(\sigma)} \prod_i (|\sigma_i| - 1)! [X]^{\ell(\sigma)} = [X]([X] - 1) \cdots ([X] - n + 1)$$

This recovers the formula for $[\mathrm{Conf}(X, n)]$ in Proposition 7.1, although it is certainly not the easiest derivation as we pointed out.

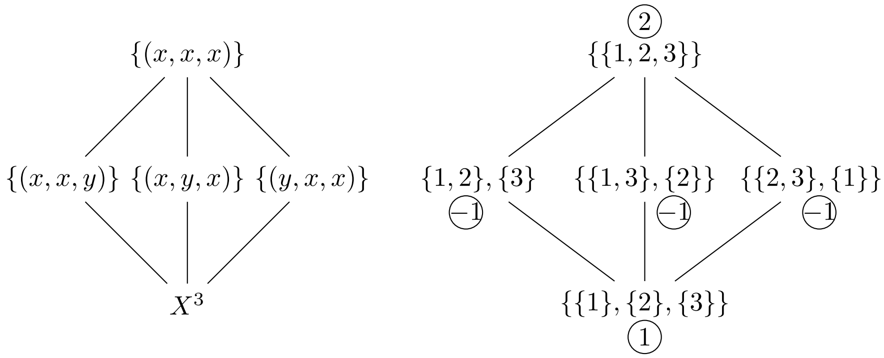
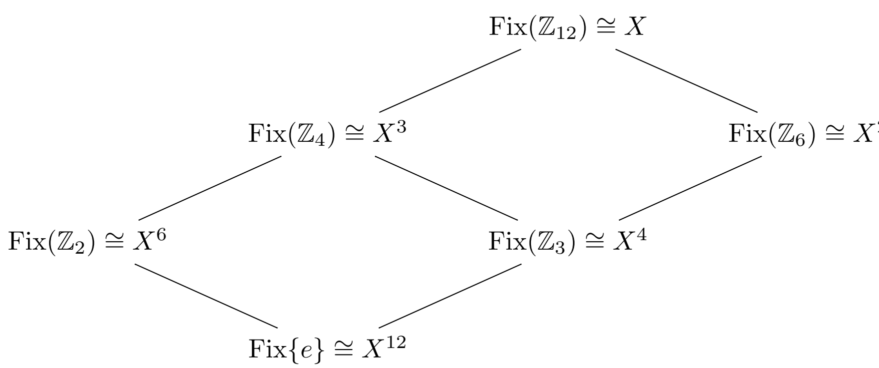


FIGURE 4. For $n=3$, the lattice Π_3 is depicted on the right and the intersection lattice for $\{\Delta_{i,j}(X,3)\}$ on the left. The values of the Möbius function are those within circles so that $[\mathrm{Conf}(X,3)] = [X^3 \setminus \bigcup \Delta_{i,j}(X,3)] = [X]^3 - 3[X]^2 + 2[X] = [X]([X]-1)([X]-2)$.

Example 9.7. (The Cyclic products). Using Theorem 9.1, we give a proof of Proposition 7.5. We recall that the n-th cyclic product of X, $\mathrm{CP}^n(X)$, is the quotient of X^n by the permutation action of the cyclic group $\mathbb{Z}_n$ (§7.1). Break $\mathbb{Z}_n$ into primary cyclic groups $\mathbb{Z}_n = \prod \mathbb{Z}_{p^r}$, where $p|n$, and the product is over distinct primes p. Let $\mathrm{Fix}(\mathbb{Z}_d)$ be the fixed point set of $\mathbb{Z}_d \subset \mathbb{Z}_n$ for $d|n$ acting on X^n. Suppose for example $n=12$. Then $\mathrm{Fix}(\mathbb{Z}_3) = \{(x,y,z,t,x,y,z,t,x,y,z,t)\} \subset X^{12}$. This subspace is homeomorphic to X^4 via the map $\{(x,y,z,t,x,y,z,t,x,y,z,t)\} \mapsto (x,y,z,t)$. The cyclic action of $\mathbb{Z}_{12}$ translates to a cyclic action of $\mathbb{Z}_4$ on the homeomorphic image, so $\mathrm{Fix}(\mathbb{Z}_3)/\mathbb{Z}_{12} \cong X^4/\mathbb{Z}_4 = \mathrm{CP}^4 X$. This is a key observation which is true generally.

Lemma 9.8. *Let $\mathbb{Z}_d \subset \mathbb{Z}_n$ be a subgroup. Then $Fix(\mathbb{Z}_m) \subset Fix(\mathbb{Z}_d)$ if and only if $d|m$. Moreover $Fix(\mathbb{Z}_d) \cong X^{\frac{n}{d}}$, $Fix(\mathbb{Z}_{d_1}) \cap Fix(\mathbb{Z}_{d_2}) = Fix(\mathbb{Z}_{lcm(d_1,d_2)})$ and $Fix(\mathbb{Z}_d)/\mathbb{Z}_n \cong CP^{\frac{n}{d}}(X)$.*

If X is LC, then $\mathcal{A} = \{\mathrm{Fix}(\mathbb{Z}_d)\}_{d|n}$ is an arrangement of closed subspaces of X^n. By Lemma 9.8, the intersection lattice is isomorphic to the divisor lattice for n and is dual to the lattice of subgroups of $\mathbb{Z}_n$ (see Remark 6.3). For example when $n=12$ we obtain



The divisor lattice has $\hat{0}=1$, $\hat{1}=n$ and for a divisor d of n, $\mu_n(m,d) = \mu(\frac{m}{d})$, where $\mu(-)$ is the Möbius arithmetic function defined by $\mu(1)=1$, $\mu(n)=0$ if n is divisible by the square of a prime number, otherwise $\mu(n)=(-1)^k$, where k is the number of prime factors of n. This is all we need to compute $[\mathrm{CP}^n(X)]$.

Proof. (of Proposition 7.5) As in the notation of (18), write $\mathrm{Fix}^+(\mathbb{Z}_d) := \mathrm{Fix}(\mathbb{Z}_d) \setminus \bigcup_{j>d} \mathrm{Fix}(\mathbb{Z}_j)$. The lattice of fixed point sets has Möbius function the divisor lattice (Lemma 9.8). On the other hand, the cyclic action of $\mathbb{Z}_n$ on X^n restricts to both $\mathrm{Fix}(\mathbb{Z}_d)$ and to the stratum $\mathrm{Fix}^+(\mathbb{Z}_d)$. Importantly, the action of $\mathbb{Z}_{\frac{n}{d}}$ on $\mathrm{Fix}(\mathbb{Z}_d)^+$ (viewed as a subspace of $X^{\frac{n}{d}}$) is free. When $d=1$, $\mathrm{Fix}(\mathbb{Z}_1)$ means $\mathrm{Fix}\{e\} = X^n$. We can put all of this together now

$$\begin{aligned}
[CP^n(X)] = \sum_{d|n}\left[\frac{\mathrm{Fix}^+(\mathbb{Z}_d)}{\mathbb{Z}_{\frac{n}{d}}}\right] &= \sum_{d|n}\frac{d}{n}\left[\mathrm{Fix}^+(\mathbb{Z}_d)\right] \\
&= \sum_{d|n}\frac{d}{n}\sum_{\substack{m \\ d|m}}\mu\left(\frac{m}{d}\right)[\mathrm{Fix}(\mathbb{Z}_m)] \qquad \text{(by Theorem (9.1))} \\
&= \sum_{d|n}\frac{d}{n}\sum_{\substack{m \\ d|m}}\mu\left(\frac{m}{d}\right)[X]^{\frac{n}{m}} \qquad \text{(by Lemma 9.8)}
\end{aligned}$$

This proves the claim, and the second equality (with $\phi(d)$) is left as an exercise. □

Remark 9.9. The computation $\chi(\mathrm{CP}^n(X)) = \dfrac{1}{n}\sum_{d|n}\phi(d)[X]^{\frac{n}{d}}$ is due to MacDonald and is derived from the Poincaré polynomial ([51], 8.4) after setting $x=-1$

$$P(\mathrm{CP}^n(X),x) = \frac{1}{n}\sum_{d|n}\phi(d)P(X,(-1)^{d+1}x^d)^{\frac{n}{d}}$$

We refer to Arabia [2] for vast generalizations of these formulas.

The final three sections compute the Grothendieck class of various configuration spaces existing in the literature. An important point we must highlight is that for all these cases, it is easier to construct appropriate stratifications directly in order to compute $[X\setminus\bigcup A_i]$, and consequently deduce motivic measures and Euler characteristics, rather than resort to computing Möbius functions for the corresponding posets (Theorem 9.1).

10. Graph Configuration Spaces

Let X be LCFT, and let Γ be a non-oriented abstract graph with vertex set $V(\Gamma)=\{v_1,\cdots,v_n\}$, $n:=|V|$ and with $E(\Gamma)$ a set of edges. Two vertices sharing an edge are called *adjacent*. The graph is assumed to be simple (no loops and no multiple edges) and connected. Define

$$\mathrm{Conf}(X,\Gamma) = \{(x_1,\cdots,x_n)\in X^{|V|}\mid x_i\neq x_j \text{ if } \{i,j\}\in E(\Gamma)\}$$

This is the configuration space associated to the collection of partitions of Ω having a block of length two for every two adjacent vertices (Definition 9.3). When $\Gamma = K_n$, the complete graph on n-vertices, we recover $\mathrm{Conf}(X,\Gamma)=\mathrm{Conf}(X,n)$ the configuration spaces of distinct points. Two isomorphic graphs yield homeomorphic graph configuration spaces.

Graph configuration spaces have been studied by various authors [4, 8, 27]. In [8] they were studied in connection with Riemann surfaces and were dubbed "partial configuration spaces", and in [4] they were used successfully to prove a conjecture of Bendersky and Gitler on the cohomology of configuration spaces of manifolds. In this section, we relate the class of $\mathrm{Conf}(X,\Gamma)$ to the chromatic polynomial of Γ (a result already noted in [27]) and explain why this result is in fact an incarnation of a much earlier result of Rota [59, 60].

Example 10.1. Consider the $\mathbf{Y}$-graph on four vertices v_1,v_2,v_3 and v_4 (see Fig. 5). The configuration space $\mathrm{Conf}(X,\mathbf{Y})$ consists of all tuples (x_1,x_2,x_3,x_4) with $x_i\neq x_1$, for $i=2,3,4$. Stratify this space as follows

$$\{(y,x,x,x)\}\ ,\ \{(y,x,x,z)\}\ \ ,\ \ \{(y,x,z,x)\}\ \ ,\ \ \{(y,x,z,z)\}\ \ ,\ \ \{(y,x,z,t)\}$$

where different letters mean *distinct* entries, and all letters are free to vary in X. Every stratum in X^4 is homeomorphic to a standard configuration space. The first subspace is homeomorphic to $\mathrm{Conf}(X,2)$, the last subspace is homeomorphic to $\mathrm{Conf}(X, 4)$, while the intermediate subspaces are homeomorphic to $\mathrm{Conf}(X, 3)$. Using Notation 6.4, we have the stratification

$$\mathrm{Conf}(X, \mathbf{Y}) \doteqdot \mathrm{Conf}(X, 2) \sqcup 3\mathrm{Conf}(X, 3) \sqcup \mathrm{Conf}(X, 4) \tag{20}$$

The multiplicities 1, 3 and 1 of the strata have a major significance: the multiplicity of $\mathrm{Conf}(X, 2)$ is equal to the number of ways we can color the graph with exactly two colors (not taking into account permuting colors), while the multiplicity of $\mathrm{Conf}(X, 3)$ is the number of ways we can color the graph with exactly three colors, and so on. Adding up Grothendieck classes of the configuration spaces in (20), we obtain immediately that

$$[\mathrm{Conf}(X, \mathbf{Y})] = [X]([X] - 1)^3$$

This is of course not unexpected since, for closed manifolds, projection onto the first coordinate exhibits $\mathrm{Conf}(X, \mathbf{Y})$ as a bundle over X with fiber $(X - pt)^3$.

As illustrated in the example above, $\mathrm{Conf}(X, \Gamma)$ comes equipped with a natural stratification by configuration spaces indexed by the various colorings. The main result of this section (Theorem 10.2) shows that $[\mathrm{Conf}(X, \Gamma)]$ is none but the chromatic polynomial of the graph. We recover this way a result of [27] (Euler characteristic) and [59] (characteristic polynomial).

10.1. **Graph Coloring.** We recall a few notions rapidly. Given a finite graph Γ, a (proper) coloring is a labeling of the graph's vertices with colors such that no two vertices sharing the same edge have the same color. A coloring using at most k colors is called a *k-coloring*. Consider for example the Y-graph in Fig. 5 and give the central v_1 the color red, and all other vertices the color blue. This is a coloring with $k = 2$ colors. If we permute colors so that v_1 is now blue, and all others red, this is *another* coloring. We wish not to distinguish between these colorings and so we call the equivalence class of these a *coloring configuration*. There is exactly one coloring configuration with two colors for the Y-graph but two different colorings. We will write $a_k(\Gamma)$ the number of coloring configurations with k-colors. So again to illustrate for the Y-graph, $a_3(\Gamma) = 3$ and all three cases are listed below with the different colorings being labeled by different symbols $\square$, $\bigcirc$ and $\bigstar$. The actual number of colorings with exactly three colors is however $a_3(\Gamma) \cdot 3! = 12$, taking into account permuting colors.

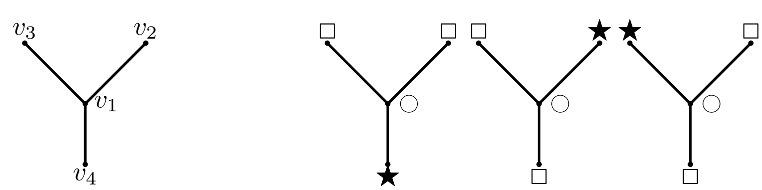


FIGURE 5. The **Y**-graph on the left. The three different coloring configurations of the **Y**-graph on the right.

In general, the number of colorings of a graph Γ by k-colors is $a_k(\Gamma)k!$. If we define the polynomial

$$ch(\Gamma, t) = \sum_{r=1}^{|V|} a_r(\Gamma) \prod_{j=0}^{r-1} (t - j)$$

then we see that $ch(\Gamma, k)$ is the number of colorings of the graph with k colors or less. This is called the *chromatic polynomial* associated to Γ.

Theorem 10.2. *Let Γ be a finite graph on a set of labeled vertices V, $|V| \geq 2$, and let $X \in \mathcal{M}$. Then*

$$[\mathit{Conf}(X, \Gamma)] = ch(\Gamma, [X])$$

Proof. Each coloring configuration gives rise to a stratum in a stratification of $\mathrm{Conf}(X,\Gamma)$. This is done as follows: label the vertices $v_1,\dots,v_n$, $n=|V|$. In X^n, consider the subspace of tuples $(x_1,\dots,x_n)$ where $x_i=x_j$ if v_i,v_j have the same color. If the number of colors is r, then this stratum is a copy of $\mathrm{Conf}(X,r)$. There are $a_r(\Gamma)$ distinct such strata. We then have the stratification $\mathrm{Conf}(X,\Gamma) \doteqdot \bigsqcup_{r=1}^n a_r(\Gamma)\mathrm{Conf}(X,r)$ (see Notation 6.4), from which we deduce immediately that

$$[\mathrm{Conf}(X,\Gamma)] = \sum_{r=1}^n a_r(\Gamma)[\mathrm{Conf}(X,r)] = \sum_{r=1}^n a_r(\Gamma)\prod_{j=0}^{r-1}([X]-j)$$

and this is precisely $ch(\Gamma,[X])$. □

The chromatic polynomial is known for a wide range of graphs and several closed formulas are given using contraction and deletion algorithm.

Example 10.3. Consider the *path graph* P_4 on 4 vertices.

The chromatic polynomial is $ch(P_4,t) = t(t-1)^3 = t^4-3t^3+3t^2-t$. Solving for $a_r(P_4)$ in this expression gives $a_1(P_4)=0$, $a_2(P_4)=1$, $a_3(P_4)=3$ and $a_4(P_4)=1$. The corresponding strata in $\mathrm{Conf}(X,\Gamma)$, as subsets of X^4, are the emptyset for $a_1(P_4)=0$, the stratum $\{(x,y,x,y)\}$ corresponding to $a_2(P_4)=1$, where again different letters stand for necessarily different values, and the three strata $\{(x,y,x,z)\}$, $\{(x,y,z,x)\}$ and $\{(x,y,z,y)\}$ for $a_3(P_4)=3$. There is finally the stratum $\mathrm{Conf}(X,4)=\{((x,y,z,t)\}$ corresponding to $a_4(P_4)=1$. We have

$$[\mathrm{Conf}(X,P_4)] = [X]^4-3[X]^3+3[X]^2-[X]$$

As a corollary to Theorem 10.2, we obtain a quick derivation of the Euler characteristic.

Corollary 10.4. *Let M be an n-dimensional topological manifold (with or without boundary), and let Γ be a simple finite connected graph on the vertex set V. Then*

$$\chi(\mathit{Conf}(M,\Gamma)) = (-1)^{n|V|}ch(\Gamma,(-1)^n\chi(M))$$

Proof. Assume M is a boundaryless manifold first. As before, apply the motivic measure to the equality in Theorem 10.2 to get $\langle\mathrm{Conf}(M,\Gamma)\rangle = ch(\Gamma,\langle M\rangle) = ch(\Gamma,(-1)^n\chi(M))$. But $\mathrm{Conf}(M,\Gamma)$ is an open submanifold of $M^{|V|}$, of dimension $n|V|$, so that $\langle\mathrm{Conf}(M,\Gamma)\rangle = (-1)^{n|V|}\chi(\mathrm{Conf}(M,\Gamma))$. If M has a collared boundary, and $\mathring{M}$ is its interior, then $\mathrm{Conf}(\mathring{M},\Gamma)\simeq\mathrm{Conf}(M,\Gamma)$, and we're back to the earlier case. □

Remark 10.5. The above corollary recovers Corollary 7.2 when Γ is the complete graph, and recovers a computation of [27] in case M is a closed orientable manifold.

We now discuss the *graph arrangement* associated to this configuration space and determine its intersection lattice. Consider $\mathcal{A}=\{\Delta_{i,j}\}_{\{i,j\}\in E(\Gamma)}$, the indexing being over all adjacent vertices, where once more $\Delta_{i,j}=\{(x_1,\dots,x_n)\mid x_i=x_j\}$, $n=|V|$ being the number of vertices. A *bond* of Γ is a partition of its vertices such that all vertices in the same block are connected within the graph (see Fig. 6). It is easy to see that the intersection poset (or the configuration lattice) of the graph arrangement is a lattice isomorphic to the so-called *bond lattice* of the graph which is the sublattice Π_Γ of the lattice of partitions Π_n obtained by taking all bonds of the graph. To see this, simply observe that every bond σ of Γ corresponds to the subspace of X^n of all tuples where $x_i=x_j$ if i,j are in the same block of σ, or in other words, if the corresponding vertices v_i,v_j are connected by a path. This is precisely a subspace of the intersection lattice.

Applying Theorem 9.1 to the graph arrangement, and using the computation in Theorem 10.2, yield immediately the following classical result.

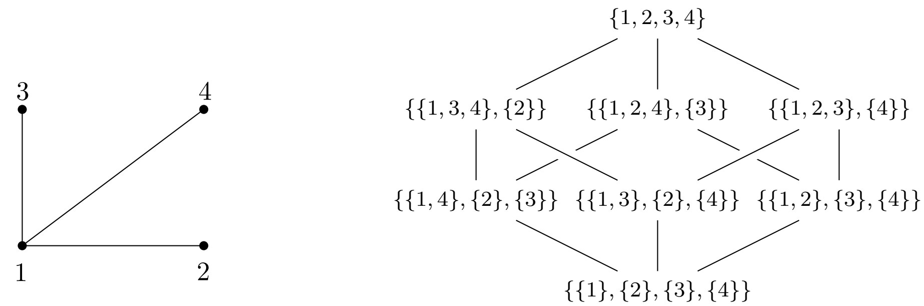


FIGURE 6. A graph with 4 vertices and 3 edges and its corresponding bond lattice

Corollary 10.6. *(Rota) Given a simple graph* Γ*,* Π_Γ *its bond lattice, and* μ *its associated Möbius function, then* $ch(\Gamma, [X]) = \sum_{\sigma \in \Pi_\Gamma} \mu(\hat{0}, \sigma)[X]^{|\sigma|}$.

Finally, and as an application of Theorem 10.2, we consider the *cyclic configuration spaces* studied in [30] in relation to billiard trajectories. They are defined to be

$$Cyc(X, n) = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_{i+1} \ , \ i = 1, \ldots, n \ , \ \text{with } x_{n+1} := x_1\}$$

Let C_n be the *cycle graph* whose vertices are $v_k = e^{i\frac{2k\pi}{n}}$, $1 \leq k \leq n$, on the circle, with edges $[v_k, v_{k+1}]$. Clearly $\mathrm{Conf}(X, C_n) = Cyc(X, n)$. The chromatic polynomial for the cycle graph is known for general n and we can use it to derive the corollary next.

Corollary 10.7. *Let* $Cyc(X, n)$ *be the cyclic configuration space,* $n \geq 2$*. Then*

$$[Cyc(X, n)] = ([X] - 1)^n + (-1)^n([X] - 1)$$

Remark 10.8. Our methods can be reversed to give a proof that the chromatic polynomial of cyclic graphs has the above expression, adding another one to [41]. The idea is to start with a stratification of $Cyc(X, n)$ in terms of configuration spaces and then compute directly $[Cyc(X, n)]$. This is left as an exercise.

11. Orbit Configuration Spaces

This is a family of configuration spaces that does not fit the general framework (Definition 9.3) but is nonetheless a fairly important class of spaces which has been well studied, eg. [9, 19, 20]. Let G be a group, which we assume finite, acting properly on X. We define

$$\mathrm{Conf}_G(X, n) = \{(x_1, \ldots, x_n) \in X^n \mid Gx_i \cap Gx_j = \emptyset \ , \ \text{if } i \neq j\}$$

where Gx is the orbit of x. If the action of G is trivial (i.e. fixes every point), we clearly recover $\mathrm{Conf}(X, n)$. For this space not to be empty, n must be less than the number of orbits of the action.

We recall that the action of G is of finite type if the orbit type stratification of X is an LCFT-stratification (Remark 6.2).

Lemma 11.1. *If* X *is LCFT with a finite type* G*-action, then* $\mathrm{Conf}_G(X, n)$ *is LCFT-stratifiable.*

Proof. Let $q : X \to X/G$ be the quotient map by the action. By definition, there is strict pullback diagram

$$\begin{tikzcd}
\mathrm{Conf}_G(X, n) \arrow[r] \arrow[d, "\pi"] & X^n \arrow[d, "q^n"] \\
\mathrm{Conf}(X/G, n) \arrow[r] & (X/G)^n
\end{tikzcd}$$

The left vertical map is a covering map if the action is free, and more generally, it is a covering map over strata. More precisely, write $X/G = \bigsqcup_{(H)} X_{(H)}/G$, the union running over conjugacy classes (H) of subgroups $H \subset G$ (see §6). This is an LCFT-stratification of X/G since the action is of finite, and it induces in turn an LCFT-stratification of the configuration space $\mathrm{Conf}(X/G, n)$ by strata homeomorphic to

$$\mathrm{Conf}(X_{(H_1)}/G, n_1) \times \cdots \times \mathrm{Conf}(X_{(H_k)}/G, n_k) \tag{21}$$

for some partition $n_1 + \cdots + n_k = n$ and for pairwise distinct conjugacy classes $(H_1), \cdots, (H_k)$. Since $X_{(H)} \to X_{(H)}/G$ is a covering of degree $|G : H| = \frac{|G|}{|H|}$, the map π restricted to any such stratum is a covering of degree

$$\left(\frac{|G|}{|H_1|}\right)^{n_1} \cdots \left(\frac{|G|}{|H_k|}\right)^{n_k} = \frac{|G|^n}{|H_1|^{n_1} \cdots |H_k|^{n_k}}$$

On the other hand, all strata are products of configuration spaces (21), and since the map π is a covering map over these strata, the total space is in $\mathcal{F}$ (i.e. is LCFT-stratifiable) if X is LCFT. □

Our aim is to compute $[\mathrm{Conf}_G(X, n)]$ for a finite G acting on X. When the action of G is free, there is only one stratum $X_{(e)} = X$ (here $X_{(G)} = \emptyset$) so the computation is immediate. The projection $X \to X/G$ is a principal G-bundle, and thus $\mathrm{Conf}_G(X, n) \to \mathrm{Conf}(X/G, n)$ is a G^n-bundle, implying that

$$\begin{aligned}
[\mathrm{Conf}_G(X, n)] &= |G|^n\, [\mathrm{Conf}(X/G, n)] = |G|^n \prod_{i=0}^{n-1} ([X/G] - i) \qquad \text{by Proposition 7.1} \\
&= |G|^n \prod_{i=0}^{n-1} \left(\frac{[X]}{|G|} - i\right) = \prod_{i=0}^{n-1} ([X] - i|G|) \qquad (22)
\end{aligned}$$

This is equivalent to its series form given in [9]

$$\sum_{n=0}^{\infty} [\mathrm{Conf}_G(X, n)]\, \frac{t^n}{n!} = (1 + |G|t)^{\frac{[X]}{G}}$$

The derivation above took a few lines, in striking contrast with the more intricate argument of [9]. We can also extend this derivation to semifree actions, using the same formalism. We recall that an action is semifree if it is free away from the fixed point set X^G (see Corollary 5.9). Below we assume that X is not finite to avoid listing (easy) cases when this is the case.

Proposition 11.2. *Let G be a finite group acting semifreely on LCFT X. Suppose $|X^G| \geq n$ (or infinite), then*

$$[Conf_G(X, n)] = \sum_{r=0}^{n} \binom{n}{r} \prod_{i=0}^{r-1} ([X^G] - i) \prod_{j=0}^{n-r-1} ([X] - [X^G] - j|G|)$$

If $|X^G| < n$, then

$$[Conf_G(X, n)] = \sum_{r=0}^{|X^G|} \binom{n}{r} \prod_{i=0}^{r-1} ([X^G] - i) \prod_{j=0}^{n-r-1} ([X] - [X^G] - j|G|)$$

By convention when $r = 0$, $\binom{n}{0} = 1$ and the term $\prod_{i=0}^{-1}([X^G] - i) = 1$.

When the action is free, $X^G = \emptyset$, $|X^G| = 0$ and $[X^G] = 0$, so we recover (22).

Proof. When G acts semi-freely on X, there are only two conjugacy classes $X = X^G \sqcup X_{(e)}$, where $X^G = X_{(G)}$ is the fixedpoint set and $X_{(e)}$ is the part on which G acts freely.

$$\mathrm{Conf}_G(X, n) \;\doteqdot\; \bigsqcup_{r+s=n} \binom{n}{r} \mathrm{Conf}_G(X^G, r) \times \mathrm{Conf}_G(X_{(e)}, s)$$

We now take the Grothendieck class on both sides, using the facts that $\mathrm{Conf}_G(X^G,r)=\mathrm{Conf}(X^G,r)$, that $[X_{(e)}]=[X]-[X^G]$ and that $[X_{(e)}/G]=\frac{1}{|G|}[X_{(e)}]$. $\square$

To state the general case, we need some notation. Given G, we will denote by $\mathcal{E}=\{(H_1),\dots,(H_{|\mathcal{E}|})\}$ the set of its conjugacy classes of subgroups. We will write $\Omega=\{1,\dots,n\}$ and denote by $Par(\Omega)$ the set of all partitions of Ω, whereby a partition is any element $\{\Omega_1,\dots,\Omega_r\}\in\mathcal{P}(\Omega)$ with $\Omega_i\neq\emptyset$ and $\Omega_1\sqcup\cdots\sqcup\Omega_r=\Omega$. Obviously $r\leq n$. Finally we will write $\alpha:\Omega_1\hookrightarrow\Omega_2$ an injection between finite sets.

Theorem 11.3. *Let G be a finite group acting on $X\in\mathcal{M}$. Then $[Conf_G(X,n)]$ is a polynomial of degree n given by*

$$\frac{[Conf_G(X,n)]}{|G|^n}=\sum_{\substack{1\leq r\leq|\mathcal{E}|\\ \{\Omega_1,\dots,\Omega_r\}\in Par(\Omega)}}\ \sum_{\substack{\alpha:\{1,\dots,r\}\\ \hookrightarrow\{1,\dots,|\mathcal{E}|\}}}\frac{\left[Conf\left(\frac{X_{\left(H_{\alpha(1)}\right)}}{G},|\Omega_1|\right)\right]\cdots\left[Conf\left(\frac{X_{\left(H_{\alpha(r)}\right)}}{G},|\Omega_r|\right)\right]}{|H_{\alpha(1)}|^{|\Omega_1|}\cdots|H_{\alpha(r)}|^{|\Omega_r|}}$$

We must keep in mind that when a term $Conf(Y,k)$ occurs in the above formula with Y a finite set of cardinality $|Y|<k$, then one sets $Conf(Y,k)=\emptyset$ and $[Conf(Y,k)]=0$.

Proof. We appeal to the orbit type stratification discussed in §6 and use it to stratify $\mathrm{Conf}_G(X,n)$ as in the proof of Lemma 11.1. Let us illustrate the argument through one example rather than go through the heavy notation. Pick $n=5$ for example and assume $\mathcal{E}=\{(H_1),\dots,(H_4)\}$ consists of only four conjugacy classes, $|\mathcal{E}|=4$. The following data is precisely what is needed to construct a specific stratum of $\mathrm{Conf}(X/G,5)$:

- $\{\{1,3\},\{2,4\},\{5\}\}$ a partition of $\{1,\dots,5\}$, $\Omega_1=\{1,3\}$, $\Omega_2=\{2,4\}$ and $\Omega_3=\{5\}$,
- A map $\alpha:\{\Omega_1,\Omega_2,\Omega_3\}\hookrightarrow\{H_1,H_2,H_3,H_4\}$ sending $\alpha(\Omega_1)=H_1$, $\alpha(\Omega_2)=H_4$ and $\alpha(\Omega_3)=H_3$.

To this partition and map corresponds the subspace of all $(x_1,x_2,x_3,x_4,x_5)\in\mathrm{Conf}(X/G,5)$ with $(x_1,x_3)\in\mathrm{Conf}(X_{(H_1)}/G,2)$, $(x_2,x_4)\in\mathrm{Conf}(X_{(H_4)}/G,2)$ and $x_5\in X_{(H_3)/G}$. This is an LC stratum of $\mathrm{Conf}(X/G,5)$ homeomorphic to (using notation as in (21))

$$\mathrm{Conf}\left(\frac{X_{(H_1)}}{G},2\right)\times X_{(H_3)}\times\mathrm{Conf}\left(\frac{X_{(H_4)}}{G},2\right)\tag{23}$$

and the corresponding stratum in $\mathrm{Conf}_G(X,5)$ is a covering of (23) of degree $\left(\frac{|G|}{|H_1|}\right)^2\left(\frac{|G|}{|H_4|}\right)^2\left(\frac{|G|}{|H_3|}\right)=\frac{|G|^5}{|H_1|^2|H_4|^2|H_3|}$. For that same partition, we get other strata by changing the choice of α. We can then vary partitions to get all strata. The general proof proceeds exactly in the same way by listing all strata of $\mathrm{Conf}_G(X,n)$ as regular covers over corresponding strata in $\mathrm{Conf}(X,n)$. $\square$

In the case $n=2$ and when X is not the singleton, we can simplify this formula.

Corollary 11.4. *Let X be LC of finite type not reduced to a point, and let G be a finite group acting on X. Then*

$$[Conf_G(X,2)]=[X]^2-\sum_{(H)\in\mathcal{E}}\frac{|G|}{|H|}\left[X_{(H)}\right]$$

Proof. The configuration space of X/G decomposes in this case as follows

$$\mathrm{Conf}(X/G,2)=\bigsqcup_{\substack{((H_1),(H_2))\\ (H_1)\neq(H_2)}}\left(\frac{X_{(H_1)}}{G}\right)\times\left(\frac{X_{(H_2)}}{G}\right)\sqcup\bigsqcup_{(H)}\mathrm{Conf}(X_{(H)}/G,2)$$

The space $\mathrm{Conf}_G(X,2)$ is made out of covers of these strata and one has

$$\begin{aligned}
[\mathrm{Conf}_G(X,2)] &= \sum_{\substack{((H_1),(H_2))\\ (H_1)\neq(H_2)}} \frac{|G|}{|H_1|}\frac{|G|}{|H_2|}\left[\frac{X_{(H_1)}}{G}\right]\left[\frac{X_{(H_2)}}{G}\right] + \sum_{(H)}\frac{|G|^2}{|H|^2}\left[\mathrm{Conf}(X_{(H)}/G,2\right]\\
&= \sum_{\substack{((H_1),(H_2))\\ (H_1)\neq(H_2)}} \frac{|G|}{|H_1|}\frac{|G|}{|H_2|}\left[\frac{X_{(H_1)}}{G}\right]\left[\frac{X_{(H_2)}}{G}\right] + \sum_{(H)}\frac{|G|^2}{|H|^2}\left[\frac{X_{(H)}}{G}\right]\left(\left[\frac{X_{(H)}}{G}\right]-1\right)\\
&= \sum_{((H_1),(H_2))} \frac{|G|}{|H_1|}\frac{|G|}{|H_2|}\left[\frac{X_{(H_1)}}{G}\right]\left[\frac{X_{(H_2)}}{G}\right] - \sum_{(H)}\frac{|G|^2}{|H|^2}\left[\frac{X_{(H)}}{G}\right]
\end{aligned}$$

The first sum of the righthand term can be identified with $[X]^2 = \left[\sum X_{(H)}\right]^2 = \left(\sum \frac{|G|}{|H|}\left[\frac{X_{(H)}}{G}\right]\right)^2$, so that $[\mathrm{Conf}_G(X,2)] = [X]^2 - \sum_{(H)}\frac{|G|^2}{|H|^2}\left[\frac{X_{(H)}}{G}\right] = [X]^2 - \sum_{(H)}\frac{|G|}{|H|}\left[X_{(H)}\right]$ as claimed. □

Example 11.5. When G acts freely, there is only one non-trivial conjugacy class $[X_{(e)}]=[X]$, and the formula above gives $\mathrm{Conf}_G(X,2) = [X]^2 - |G|[X] = [X]([X]-|G|)$ as given in (22).

12. Bounded Multiplicity Configurations

In this last section, we compute the Grothendieck classes of spaces intermediate between the symmetric products $\mathrm{SP}^n(X)$ and the braid spaces $B(X,n)$. These spaces have been considered in a number of references, eg. [45, 47, 73], and in fact the content of this section is an unpublished computation of ours[4]. An interesting aspect is how it relates to recent work of [29].

Let $\mathrm{SP}^n_d(X)$ be the *d-th fat diagonal subspace* in $\mathrm{SP}^n(X)$ consisting of all (unordered) tuples of points $[x_1,\dots,x_n]$ such that at least one x_i has multiplicity at least d. We have a decreasing filtration

$$\mathrm{SP}^n_1(X) = \mathrm{SP}^n(X)\supset \mathrm{SP}^n_2(X)\supset\cdots\supset \mathrm{SP}^n_n(X) = X$$

interpolating between the symmetric product and the thin diagonal. Define

$$B^d(X,n) := \mathrm{SP}^n(X) - \mathrm{SP}^n_{d+1}(X)$$

the complement subspace of all configurations $[x_1,\dots,x_n]$ such that no x_i has multiplicity $d+1$ or more (the so called space of no-$d+1$ equal unordered configurations). In particular $B^1(X,n) = B(X,n)$. We will set $B^d(X,0)=\emptyset$. The following is our main calculation.

Proposition 12.1. *For $X\in\mathcal{M}$, $1\le d\le n$, we have the series*

- $\sum_{n\ge1}[SP^n_d(X)]t^n = (1-t)^{-[X]} - (1-t^d)^{[X]}(1-t)^{-[X]}$
- $1+\sum_{n\ge1}[B^d(X,n)]t^n = (1-t^{d+1})^{[X]}(1-t)^{-[X]}$.

Proposition 12.1 is covered by Theorem 1.9 of [29] who consider a larger family of "0-cycles". It can also be deduced from the results of [17] which were carried out for Hilbert schemes of algebraic surfaces. Our techniques are general since we do not need to restrict to varieties [17] nor do we need to compute homology [29]. Notice that in the Proposition and when $d=1$, we recover the series for the configuration spaces

$$1+\sum_{n\ge1}[B^1(X,n)]t^n = 1+\sum_{n\ge1}[B(X,n)]t^n = (1-t^2)^{[X]}(1-t)^{-[X]} = (1+t)^{[X]}$$

[4] arXiv:1010.1507 (v1, 2014).

Proof. (of Theorem 12.1) As in [17], we stratify $\mathrm{SP}^n_d(X)$ according to the multiplicity of points. A *decomposition* $(\alpha_1, \dots, \alpha_n)$ of n is a string of integers such that $n = \alpha_1 + 2\alpha_2 + \dots + n\alpha_n$, $\alpha_i \geq 0$. To this decomposition, we associate the subspace of $\mathrm{SP}^n(X)$ of all tuples having α_i distinct entries, each having multiplicity i. Of course $\sum i\alpha_i = n$. Let's write this stratum $\mathrm{SP}^n_{(\alpha_1,\dots,\alpha_n)}(X)$. For example, the element

$$[x, y, z, t, s, t, s, s, x]$$

with different letters indicating different points, belongs to $\mathrm{SP}^9_{(2,2,1,0,0,0,0,0,0)}(X)$ since there are $\alpha_1 = 2$ points having multiplicity 1 (i.e. y, z), two points having multiplicity 2 (i.e. x, t) and a single point with multiplicity 3. Similarly

$$[x, y, x, y, z, y, z, y, y] \in \mathrm{SP}^9_{(0,2,0,0,1,0,0,0,0)}(X)$$

It is straightforward to see that

$$\mathrm{SP}^n_{(\alpha_1,\dots,\alpha_n)}(X) \cong \frac{\mathrm{Conf}(X, \sum \alpha_i)}{\mathfrak{S}_{\alpha_{n_1}} \times \dots \times \mathfrak{S}_{\alpha_{n_r}}} \quad , \quad \text{where the } \alpha_{n_i}\text{'s are the non-zero entries} \tag{24}$$

and where the Young subgroup $\mathfrak{S}_{\alpha_{n_1}} \times \dots \times \mathfrak{S}_{\alpha_{n_r}}$ acts freely as a block subgroup of $\mathfrak{S}_{\sum \alpha_i}$. The subspaces (24) are pairwise disjoint if the decompositions are distinct, and they stratify $\mathrm{SP}^n(X)$. Notice that if, for some $i \geq d$, $\alpha_i \neq 0$, then the corresponding stratum is a subspace of $\mathrm{SP}^n_d(X)$. In other words

$$\mathrm{SP}^n_d(X) = \coprod_{\substack{(\alpha_1,\dots,\alpha_n) \\ \exists i \geq d, \alpha_i \neq 0}} \mathrm{SP}^n_{(\alpha_1,\dots,\alpha_n)}(X)$$

We can write in $K_0(\mathcal{M})$

$$\begin{aligned}
[\mathrm{SP}^n_d(X)] = \sum_{\substack{(\alpha_1,\dots,\alpha_n) \\ \alpha_i \neq 0 \text{ for some } i \geq d}} \left[\mathrm{SP}^n_{(\alpha_1,\dots,\alpha_n)}(X)\right] &= \sum_{\substack{(\alpha_1,\dots,\alpha_n) \\ \alpha_i \neq 0 \text{ for some } i \geq d}} \left[\frac{\mathrm{Conf}(X, \Sigma\alpha_{k_i})}{\mathfrak{S}_{\alpha_{k_1}} \times \dots \times \mathfrak{S}_{\alpha_{k_n}}}\right] \\
&= \sum_{\substack{\sum i\alpha_i = n \\ \alpha_i \neq 0 \text{ for some } i \geq d}} \frac{1}{\alpha_1! \cdots \alpha_n!} [\mathrm{Conf}(X, \Sigma\alpha_i)] \\
&= \sum_{\substack{\sum i\alpha_i = n \\ \alpha_i \neq 0 \text{ for some } i \geq d}} \frac{(\Sigma\alpha_i)!}{\alpha_1! \cdots \alpha_n!} \binom{[X]}{\Sigma\alpha_i} \qquad (25)
\end{aligned}$$

To get the claimed series $(1-t)^{-[X]} - (1-t^d)^{[X]}(1-t)^{-[X]}$, we identify the coefficient of t^n in this series with the formula we just computed. To see they are the same it is enough to write

$$(1-t)^{-[X]} - (1-t^d)^{[X]}(1-t)^{-[X]} = (1 + t + t^2 + \cdots)^{[X]} - (1 + t + \dots + t^{d-1})^{[X]}$$

and observe that $\displaystyle\sum_{\sum i\alpha_i = n} \frac{(\sum \alpha_i)!}{\alpha_1! \cdots \alpha_n!} \binom{[X]}{\sum \alpha_i}$ is the coefficient of x^n in the expansion of $(\sum_{i=0}^n x^i)^{[X]}$, where α_i is the number of times the monomial x^i appears in the factorization of $x^n = \prod (x^i)^{\alpha_i}$ ([33], 1.77). This completes the proof of Proposition 12.1. $\square$

Example 12.2. When $d = n$, then $\mathrm{SP}^n_n(X) = \mathrm{SP}^n_{(0,\dots,0,n)}(X) = X$. This fact we can recover from the formula (25) since in this case $\sum \alpha_i = \alpha_n = 1$. When $d = 1$, $\mathrm{SP}^n_1(X) = \mathrm{SP}^n(X)$ and we also recover Proposition 16 since we have the identity

$$[\mathrm{SP}^n(X)] = \sum_{\sum i\alpha_i = n} \frac{(\sum \alpha_i)!}{\alpha_1! \cdots \alpha_n!} \binom{[X]}{\sum \alpha_i} = \binom{n + [X] - 1}{n} \tag{26}$$

the last equality being a consequence of rewriting in two different ways the coefficient of x^n in $(\sum_{i=0}^n x^i)^{[X]}$.

As we explained earlier, computations in the Grothendieck ring yield computations for χ in presence of compactness and homotopy invariance. Suppose X is of the homotopy type of a finite CW complex. Since $\mathrm{SP}^n_d(X)$ is a homotopy functor, we can assume X is a finite CW complex, thus compact. For compact X, $\mathrm{SP}^n_d(X)$ is also compact. We can replace in (25) $[X]$ by $\chi(X)$ and $[\mathrm{SP}^n_d(X)]$ by $\chi(\mathrm{SP}^n_d(X))$. The following corollary generalizes [17] from complex varieties to any CW complex.

Corollary 12.3. *If X is of the homotopy type of a finite CW complex with Euler characteristic χ, then*

$$\chi(SP^n_d(X)) = \sum_{\substack{\sum i\alpha_i = n \\ \alpha_i \neq 0 \text{ for some } i \geq d}} \frac{(\Sigma\alpha_i)!}{\alpha_1! \cdots \alpha_n!} \binom{\chi}{\Sigma\ \alpha_i}$$

13. Open Questions

We conclude with some open questions:

Question 1: Are there any other motivic morphisms out of $K_0(\mathcal{M})$?

Question 2: Can $K_0(\mathcal{M})$ be π_0 of a spectrum? Recent work in [38] seems relevant.

Question 3: Any structure result about $K_0(\mathcal{M})$? Any set of generators?

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