Inverse scattering at high energies for the multidimensional relativistic Newton equation in a long-range electromagnetic field

Alexandre Jollivet

Laboratoire de Physique Théorique et Modélisation (UMR 8089)
CNRS & Université de Cergy-Pontoise

alexandre.jollivet@u-cergy.fr

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I. Forward problem

• Multidimensional relativistic Newton equation in a static external electromagnetic field [Einstein, 1907]

(1)
$$\dot{p} = -\nabla V(x) + \frac{1}{c} B(x)\dot{x},$$
$$p = \frac{\dot{x}}{\sqrt{1-\frac{|\dot{x}|^2}{c^2}}},\ x(t) \in \mathbb{R}^n,\ n \geq 2.$$

• Smoothness and long range assumptions for the external field

(2)
$$V \in C^2(\mathbb{R}^n, \mathbb{R}),\ B(x) = (B_{i,k}) \in C^1(\mathbb{R}^n, A_n(\mathbb{R})),$$
$$\frac{\partial B_{i,k}}{\partial x_l}(x) + \frac{\partial B_{l,i}}{\partial x_l}(x) + \frac{\partial B_{k,l}}{\partial x_i}(x) = 0,$$
$$V = V^s + V^l,\ B = B^s + B^l,$$

$$|\partial_x^{j_1} V^l(x)| \leq \beta^l_{|j_1|}(1+|x|)^{-\alpha-|j_1|},\ |\partial_x^{j_2} B^l_{i,k}(x)| \leq \beta^l_{|j_2|+1}(1+|x|)^{-\alpha-1-|j_2|},$$

$$|\partial_x^{j_1} V^s(x)| \leq \beta^l_{|j_1|+1}(1+|x|)^{-\alpha-1-|j_1|}, |\partial_x^{j_2} B^s_{i,k}(x)| \leq \beta^s_{|j_2|+2}(1+|x|)^{-\alpha-2-|j_2|},$$

for $|j_1| \leq 2$, $|j_2| \leq 1$, $i, k, l = 1 \ldots n$ and for some $\alpha \in (0, 1]$, where $j = (j^1, \ldots, j^n) \in (\mathbb{N} \cup \{0\})^n$, $|j| = \sum_{i=1}^n j^i$ and where $\beta^s_{|j|+1}$ and $\beta^l_{|j|}$ are positive constants).

- Integral of motion, the energy of the classical relativistic particle

(3) $$E = c^2\sqrt{1+\frac{|p(t)|^2}{c^2}} + V(x(t))$$

- Parametrization of the scattering solutions

For $v \in B_c$, $v \neq 0$, let $z_{\pm}(v,.)$ be a solution of the equation (1) with $F^s \equiv 0$ so that

$$\dot{z}_{\pm}(v,t) \to v, \text{ as } t \to \pm\infty.$$

Then for any $(v_-, x_-) \in B_c \times \mathbb{R}^n$, $v_- \neq 0$, there exists a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ of equation (1) so that

$$x(t) = z_-(v_-, t) + x_- + y_-(t),\ |y_-(t)| + |\dot{y}_-(t)| \to 0, \text{ as } t \to -\infty;$$

and for a.e. $(v_-, x_-) \in B_c \times \mathbb{R}^n$, $v_- \neq 0$,

$$x(t) = z_+(v_+, t) + x_+ + y_+(t),\ |y_+(t)| + |\dot{y}_+(t)| \to 0, \text{ as } t \to +\infty,$$

for some (v_+, x_+), $|v_+| = |v_-|$.

- Scattering map and scattering data for equation (1) :

$$S(v_-, x_-) := (v_+, x_+) =: (v_- + a_{sc}(v_-, x_-), x_- + b_{sc}(v_-, x_-))$$

- Direct problem : $\text{Given } (V, B), \text{ find } S.$

  Inverse problem : $\text{Given } (V^l, B^l, S), \text{ find } (V^s, B^s).$

II. Inverse scattering at high energies

II.1 The "free" solutions

Let $v \in B_c$, $v \neq 0$. When $|v| > \rho(n, c, \beta_1^l, \beta_2^l)$ then there exists a unique solution $z_\pm(v, .)$ of the equation (1) with $F^s \equiv 0$ so that

$$\lim_{t \to \pm\infty} \dot{z}_\pm(v, t) = v, \;\; z_\pm(v, 0) = 0, \qquad \text{and}$$

$$\max\left(\sup_{t \in \mathbb{R}} \frac{|z_\pm(v,t) - tv|}{|t|}, \sup_{t \in \mathbb{R}} |\dot{z}_\pm(v,t) - v| \right) \leq \frac{C n^{\frac{3}{2}} \beta_1^l \sqrt{1 - \frac{|v|^2}{c^2}}}{\alpha |v|}.$$

II.2 Asymptotic of the scattering data

- X-ray transform : $$Pf(\theta, x) = \int_{-\infty}^{+\infty} f(t\theta + x) dt, \;\; (\theta, x) \in T\mathbb{S}^{n-1}.$$

for $f \in C(\mathbb{R}^n, \mathbb{R}^m)$, $f(x) = O(|x|^{-1-\varepsilon})$ when $|x| \to +\infty$, $\varepsilon > 0$,

and where $T\mathbb{S}^{n-1} := \{(\theta', x') \in \mathbb{S}^{n-1} \times \mathbb{R}^n \mid \theta' \cdot x' = 0\}$.

First study and inversion of P in $\mathbb{R}^2$: Radon (1917).

Application to X-ray Tomography : Cormack (1963).

Theorem 1 [J1]. *Let* $(\theta,x)\in T\mathbb{S}^{n-1}$ *and* $0<r\leq 1$, $r<\frac{c}{\sqrt{2}}$. *Under conditions* (2) *we have*

$$\lim_{\substack{\rho\to c\\ \rho<c}}\frac{\rho}{\sqrt{1-\frac{\rho^2}{c^2}}}a_{sc}(\rho\theta,x)=-P(\nabla V)(\theta,x)+\int_{-\infty}^{+\infty}B(x+\tau\theta)\theta d\tau,\qquad and$$

$$\left|\frac{\rho}{\sqrt{1-\frac{\rho^2}{c^2}}}a_{sc}(\rho\theta,x)+P(\nabla V)(\theta,x)-\frac{\rho}{c}\int_{-\infty}^{+\infty}B(x+\tau\theta)\theta d\tau\right|\leq\frac{Cn^4\beta^2(1+\frac{1}{c}+|x|)(\frac{1}{c}+1)\rho}{\alpha^2(\frac{\rho}{2\sqrt{2}}-r)^2(1-r)^{2\alpha+3}}\sqrt{1-\frac{\rho^2}{c^2}}$$

for $\rho_1(c,n,\beta,\alpha,r)<\rho<c$, $(\beta=\max(\beta_1,\beta_2,\beta_3))$ *; In addition*

$$\left|\frac{\rho^2}{\sqrt{1-\frac{\rho^2}{c^2}}}\big(b_{sc}(\rho\theta,x)-W(\rho\theta,x)\big)-\frac{\rho^2}{c^2}PV^s(\theta,x)\theta+\int_{-\infty}^{0}\int_{-\infty}^{\tau}\nabla V^s(\sigma\theta+x)d\sigma d\tau\right.$$

$$\left.-\int_{0}^{+\infty}\int_{\tau}^{+\infty}\nabla V^s(\sigma\theta+x)d\sigma d\tau-\frac{\rho}{c}\int_{-\infty}^{0}\int_{-\infty}^{\tau}B^s(\sigma\theta+x)\theta d\sigma d\tau+\frac{\rho}{c}\int_{0}^{+\infty}\int_{\tau}^{+\infty}B^s(\sigma\theta+x)\theta d\sigma d\tau\right|$$

$$\leq\frac{Cn^4\beta^2\big(\frac{1}{c}+1+|x|\big)(\frac{1}{c}+1)\rho^2\Big(1+\frac{1}{\frac{\rho}{2\sqrt{2}}-r}\Big)^2}{\alpha^2(\frac{\rho}{2\sqrt{2}}-r)^3(1-r)^{2\alpha+2}}\sqrt{1-\frac{\rho^2}{c^2}}$$

for $\rho_2(c,n,\beta,\alpha,r)<\rho<c$.

The vector W is known from F^l and the scattering data:

$$W(v,x) := \int_{-\infty}^{0} \Big(g\big(g^{-1}(v) + \int_{-\infty}^{\sigma} F^l(z_-(v,\tau)+x, \dot{z}_-(v,\tau))d\tau\big) - g\big(g^{-1}(v) + \int_{-\infty}^{\sigma} F^l(z_-(v,\tau), \dot{z}_-(v,\tau))d\tau\big)\Big) d\sigma$$

$$+ \int_{0}^{+\infty} \Big(g\big(g^{-1}(a(v,x)) - \int_{\sigma}^{+\infty} F^l(z_+(a(v,x),\tau), \dot{z}_+(a(v,x),\tau))d\tau\big)$$

$$-g\big(g^{-1}(a(v,x)) - \int_{\sigma}^{+\infty} F^l(z_+(a(v,x),\tau) + x, \dot{z}(a(v,x),\tau))d\tau\big)\Big) d\sigma \qquad \text{for } (v,x) \in \mathcal{D}(S).$$

Proposition 1 [J3]**.** *Under conditions* (2) *we have*

$$P(\nabla V)(\theta,x) = -\frac{1}{2}\left(\omega_1(V,B,\theta,x) + \omega_1(V,B,-\theta,x)\right).$$

for $(\theta,x) \in T\mathbb{S}^{n-1}$ *; in addition*

$$P(B_{i,k})(\theta,x) = \frac{\theta_k}{2}\left(\omega_1(V,B,\theta,x)_i - \omega_1(V,B,-\theta,x)_i\right)$$

$$-\frac{\theta_i}{2}\left(\omega_1(V,B,\theta,x)_k - \omega_1(V,B,-\theta,x)_k\right)$$

for $i,k = 1\ldots n$ *and for every* $(\theta,x) \in T\mathbb{S}^{n-1}$, $\theta = (\theta_1,\ldots,\theta_n)$ *such that* $\theta_j = 0$ *for* $j \neq i$ *and* $j \neq k$.

II.2 Idea of the proof

Theorem 1 was obtained by developing the method of R. Novikov (1999).
Equation (1) is rewritten in an integral equation and we have

$$(y_-, \dot y_-) = A_{v_-,x_-}(y_-, \dot y_-), \qquad \text{where} \quad A_{v_-,x_-} = (A^1_{v_-,x_-}, A^2_{v_-,x_-})$$

$$\begin{cases} A^1_{v_-,x_-}(f,h)(t) = \int_{-\infty}^t A^2_{v_-,x_-}(f,h)(\sigma)d\sigma, \\ A^2_{v_-,x_-}(f,h)(t) = g\left(g^{-1}(v_-) + \int_{-\infty}^t F(z_-(v_-,\sigma) + x_- + f(\sigma), \dot z_-(v_-,\sigma) + h(\sigma))d\sigma\right) \\ -g\left(g^{-1}(v_-) + \int_{-\infty}^t F^l(z_-(v_-,\sigma), \dot z_-(v_-,\sigma))d\sigma\right), \end{cases}$$

$$\textit{and where } F(x,v) = -\nabla V(x) + \tfrac{1}{c}B(x)v \textit{ for } (x,v) \in \mathbb{R}^n \times \mathbb{R}^n.$$

We consider the operator A_{v_-,x_-} on the complete metric space

$$M_{r,v_-} := \{(f,h) \in C(\mathbb{R},\mathbb{R}^n)^2 \mid \sup_{\mathbb{R}} |\dot z_-(v_-,.) + h| \le c,$$

$$\|(f,h)\| := \max\Big(\sup_{(-\infty,0)} |f|, \sup_{t\in(-\infty,0)} (1-r+(\frac{|v_-|}{2\sqrt{2}}-r)|t|)|h(t)|, \sup_{(0,+\infty)} |h|, \sup_{t\in(0,+\infty)} \frac{|f(t)|}{1+t}\Big) \le r\}, \; 0<r<1.$$

Hence we study a small angle scattering regime compared to the dynamics generated by F^l .

- Quantum analogs : Born, Faddeev (1956), Henkin-Novikov (1988), Enss-Weder (1995),
  H. Ito (1995), Jung (1997).

III. A modified scattering map

III.1 Other "free" solutions

Let $(v,x)\in B_c\times\mathbb{R}^n$, $v\neq 0$, $v\cdot x=0$. When $|v|>\tilde\rho(n,c,\beta_1^l,\beta_2^l,|x|)$ then there exists a unique solution $z_\pm(w,x+q,.)$ of the equation (1) with $F^s\equiv 0$ so that

$$\lim_{t\to\pm\infty}\dot z_\pm(w,x+q,t)=w,\ \ z_\pm(w,x+q,0)=x+q,\qquad \text{and}$$

$$\max\Big(\sup_{t\in\mathbb{R}}\frac{|z_\pm(w,x+q,t)-tw-x-q|}{|t|},\sup_{t\in\mathbb{R}}|\dot z_\pm(w,x+q,t)-w|\Big)\leq\frac{Cn^{\frac{3}{2}}\beta_1^l\sqrt{1-\frac{|v|^2}{c^2}}}{\alpha|v|(1+\frac{|x|}{\sqrt{2}})^\alpha}$$

for $t\in\mathbb{R}$ and for $(w,q)\in B_c\times B(0,\frac{1}{\sqrt{2}}+\frac{|x|}{2\sqrt{2}})$, $|g^{-1}(v)-g^{-1}(w)|\leq\frac{|g^{-1}(v)|}{8\sqrt{n}}$.

Consider $x(t)$ the unique solution of the equation (1) that satisfies

$$x(t)=z_-(v_-,x_-,t)+y_-(t),\ \ |y_-(t)|+|\dot y_-(t)|\to 0,\ \ \text{as } t\to-\infty.$$

When $|v|>\tilde\rho_0(n,c,\beta_1^l,\beta_2^l,|x|)$ then

$$x(t)=z_+(\tilde a(v_-,x_-),\tilde b(v_-,x_-),t)+y_+(t),\ \ |y_+(t)|+|\dot y_+(t)|\to 0,\ \ \text{as } t\to+\infty,$$

for some $(\tilde a(v_-,x_-),\tilde b(v_-,x_-))$.

• Modified scattering map : $\tilde{S}(v_-, x_-) = (\tilde{a}(v_-, x_-), \tilde{b}(v_-, x_-))$

• Direct problem : Given (V, B), find $\tilde{S}$.

• Inverse problem : Given $(V^l, B^l, \tilde{S})$ find (V^s, B^s).

III.2 High energies asymptotics of the modified scattering data

Theorem 2 [J1]. *Let* $(\theta, x) \in T\mathbb{S}^{n-1}$ *and* $0 < r \leq 1$, $r < \max\left(\frac{c}{2\sqrt{2}}, \frac{1}{2} + \frac{|x|}{2\sqrt{2}}\right)$. *Under conditions* (2) *we have*

$$\lim_{\substack{\rho \to c \\ \rho < c}} \frac{\rho}{\sqrt{1 - \frac{\rho^2}{c^2}}} (\tilde{a}_{sc}(\rho\theta, x) - \tilde{W}(\rho\theta, x)) = -P(\nabla V^s)(\theta, x) + \int_{-\infty}^{+\infty} B^s(x + \tau\theta)\theta d\tau, \quad and$$

$$\left| \frac{\rho}{\sqrt{1 - \frac{\rho^2}{c^2}}} (\tilde{a}_{sc}(\rho\theta, x) - \tilde{W}(\rho\theta, x)) + P(\nabla V^s)(\theta, x) - \frac{\rho}{c} \int_{-\infty}^{+\infty} B^s(x + \tau\theta)\theta d\tau \right| \leq \frac{Cn^4\beta^2(1 + c^{-1})^2\rho}{\alpha(\frac{\rho}{2\sqrt{2}} - r)^2(1 + \frac{|x|}{\sqrt{2}})^{2\alpha+1}} \sqrt{1 - \frac{\rho^2}{c^2}}$$

for $\tilde{\rho}_1(c, n, \beta, |x|, \alpha, r) < \rho < c$,

$$\lim_{\substack{\rho\to c\\ \rho<c}} \frac{\rho^2}{\sqrt{1-\frac{\rho^2}{c^2}}}\tilde{b}_{sc}(\rho\theta,x)=\omega_2(V^s,B^s,\theta,x)$$

$$\Big|\frac{\rho^2}{\sqrt{1-\frac{\rho^2}{c^2}}}\tilde{b}_{sc}(\rho\theta,x)-\frac{\rho^2}{c^2}PV^s(\theta,x)\theta+\int_{-\infty}^{0}\int_{-\infty}^{\tau}\nabla V^s(\sigma\theta+x)d\sigma d\tau$$

$$-\int_0^{+\infty}\int_\tau^{+\infty}\nabla V^s(\sigma\theta+x)d\sigma d\tau-\frac{\rho}{c}\int_{-\infty}^{0}\int_{-\infty}^{\tau}B^s(\sigma\theta+x)\theta d\sigma d\tau+\frac{\rho}{c}\int_0^{+\infty}\int_\tau^{+\infty}B^s(\sigma\theta+x)\theta d\sigma d\tau\Big|$$

$$\leq\frac{Cn^4\beta^2\big(\frac{1}{c}+1\big)^2\rho^2\big(1+\frac{1}{\frac{\rho}{2\sqrt{2}}-r}\big)^2}{\alpha^2(\frac{\rho}{2\sqrt{2}}-r)^3(1+\frac{|x|}{\sqrt{2}})^{2\alpha}}\sqrt{1-\frac{\rho^2}{c^2}},$$

for $\tilde{\rho}_2(c,n,\beta,|x|,\alpha,r)<\rho<c.$

The vector $\tilde{W}$ is known from F^l and the scattering data:

$$\tilde{W}(v,x)=g\big(g^{-1}(v)+\int_{-\infty}^{0}F^l(z_-(v,x,\tau),\dot{z}_-(v,x,\tau))d\tau$$

$$+\int_0^{+\infty}F^l(z_+(\tilde{a}(v,x),x,\tau),\dot{z}_+(\tilde{a}(v,x),x,\tau))d\tau\big)-v.$$

Other directions

- Inverse scattering at high energies for the non-relativistic Newton equation in a long-range electromagnetic field

- Inverse scattering at high energies for the N-body problem.

- Inverse scattering at fixed energy in a long range electromagnetic field: similar conjectures to those formulated in [Novikov, 1999] for the short range case in classical non-relativistic mechanics.

References

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