Properties and Inversion of a new Radon Transform on parabolas with fixed axis direction in $\mathbb{R}^2$

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Abstract

Radon transforms, which form the core of I. M. Gel'fand integral geometry, play an essential role in solving inverse problems arising in numerous scientific disciplines, such as geophysics, astrophysics, nuclear medicine, non-destructive testing etc. In particular, they have become key working tools for imaging science at large. In this paper, we study a new class of Radon transform defined on parabolas with fixed axis direction in $\mathbb{R}^2$ and work out the inverse transform. Besides their inherent theoretical interest as new integral transforms, this class may be relevant for atmospheric pollution control. As a by-product, we also consider a variant form of this transform occurring in seismic data processing and show that an inverse transform can be obtained in three specific cases.

1 Introduction

Linear integral transforms, of which a prominent example is the Fourier transform, have long played a central role in the development of mathematics and mathematical physics. Generically the domain of their defining kernel is the entire space, e.g. $\mathbb{R}^2$ for the two-dimensional Fourier transform. However in a seminal paper of 1917, J Radon [1] introduced a new type of integral transform which maps functions onto their integrals over sub-manifolds of space, e.g. curves of $\mathbb{R}^2$. The geometrical nature of this integral transform (and its generalizations) turns out to have profound impacts in many fields of mathematics such as harmonic analysis, group representation theory, etc. This new domain of activity was coined by him as *integral geometry*.

In fact, J Radon has worked out his transform only for straight lines of the plane and gave an explicit inversion formula. At first, this topic did not generate a widespread activity. But after several decades, interest in this transform has been vigorously reactivated by the work of A M Cormack [22], who essentially rediscovered the Radon transform in his research on transmission radiation imaging and found an alternative route to its inversion. His work came into limelight when G Hounsfield, about at the same time, invented independently the X-ray scanner. Both scientists were awarded the Medicine Nobel prize in 1979 for their contributions to medical imaging.

From this date on, the theory of Radon transforms has undergone extensive development along three major directions:

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a) in mathematics and theoretical physics: harmonic analysis on homogeneous spaces [2], [3], integral geometry in spaces of constant curvature [4], generalized Radon transforms in particle transport theory [5],

b) in imaging science: standard Radon transform in Transmission Computed Tomography (CT) and in Single Photon Emission Computed Tomography (SPECT), spherical/circular Radon transforms in Synthetic Aperture Radar (SAR) imaging [6] and in SOund Navigation And Ranging imaging (SONAR) [7], hyperbolic Radon transform in Electric Impedance Tomography (EIT) [8], conical Radon transforms in emission imaging by scattered radiation [9, 10, 11, 12], elliptical Radon transforms in ultrasonic imaging [13],

c) in seismic data treatment and analysis, where generalized Radon transforms are used for filtering coherent noise [14], for removal of multiple reflections, for trace interpolation and for restoration of missing data. The standard Radon transform (or slant stack in geophysical sciences) is a widely used technique for analyzing reflection and refraction seismic data [15]. Thorson [16] used hyperbolic Radon transform for the re-sampling of seismic data. However, a special parabolic Radon transform with fixed axis direction turns out to be well-suited for interpolation of missing traces [17], restoration of missing data [18], multiple reflection suppression [19, 14, 20] and for velocity filtering [21]. The considerable popularity of this special parabolic Radon transform in seismic data treatment and analysis is illustrated by the great number of numerical techniques devoted to its use in seismic data processing.

Thus it seems legitimate to devote a special attention to parabolic Radon transforms in $\mathbb{R}^2$. The only known parabolic Radon transform in the literature is that of A M Cormack [22], who considered families of iso-focal parabolas with swinging axis around their common focal point. Remarkably a connection can be established between this Cormack parabolic Radon transform and the standard Radon transform [23]. In this paper the class of Radon transform on parabolas with fixed axis direction is studied in details and their inversion worked out. The paper is organized as follows. The first section will be devoted to the study of the class of Radon transform on parabolas with symmetry axis parallel to a coordinate axis and rigidly translated in space. In the second section we show that inversion can be achieved with the use of the analog of the so-called *central-slice theorem*. Although it is known on general grounds that such transform may be invertible [24], this paper provides the steps to achieve this inversion. Particular emphasis and care are given to the mathematical structure of the argumentation. In the third section, we examine the class of parabolic Radon transforms used in seismic and discuss several cases of invertibility. A conclusion is given in the last section with possible future research directions. Finally four appendices provide the detailed steps of the proofs of several statements in the text.

2 c - Parabolic Radon Transform

2.1 Parabolas

Geometrically parabolas are second order planar curves that are defined as the locus of points at equal distance from a fixed point, the *focal point* (or *focus*), and a straight line called the *directrix*, see figure 2.1. They depend on four quantities: the two coordinates of the focal point and the two parameters of the directrix. The distance between the directrix and the focal point is denoted by c. Its middle point is the vertex of the parabola. c specifies the curvature (or "opening") of the parabola and a change of sign for c means that the point at infinity of the parabola has flipped over the other side of the directrix causing a change of concavity.

Parabolic Radon transforms are introduced in the literature for the first time by A M Cormack [22], who considered parabolas with variable c, but with axis swinging around their focal point, placed at the origin of a polar coordinate system. This Cormack transform depends effectively only on two parameters: c and θ the angle of the parabola axis with some fixed direction.

In this section we consider parabolas in cartesian coordinates given by the following s-parametric representation

$$\begin{aligned} x &= \xi + s \\ y &= \eta + \frac{1}{2c}\, s^2. \end{aligned} \tag{2.1}$$

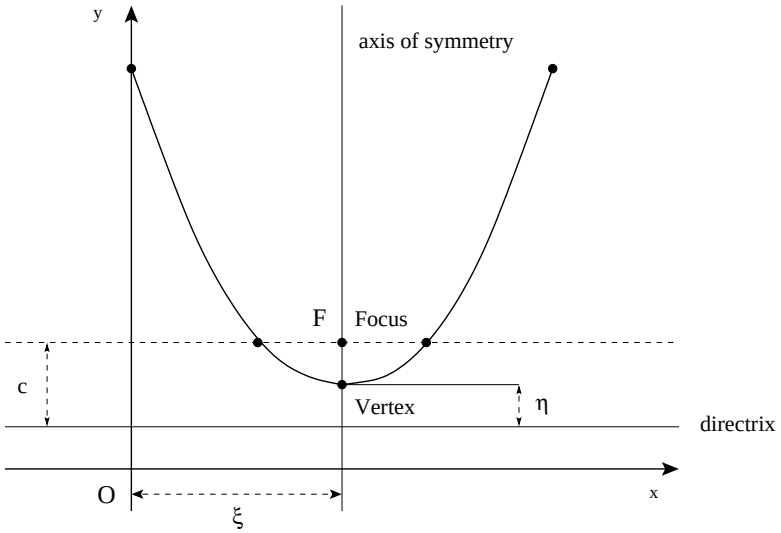

Figure 2.1: Parabola with axis of symmetry parallel to Oy

They are deduced by a translation of vector (ξ, η) from the centered parabola $\mathcal{C}_c := \{(x, x^2/(2c)) \mid x \in \mathbb{R}\}$.

Parabolic Radon transforms defined on parabolas with fixed axis direction and constant focal length c, are from now on called c-parabolic Radon transforms.

2.2 An example from elementary physics

A well-known example of such parabola is the trajectory of a particle under constant earth acceleration $\mathbf{g} = (0, -g)$, where $g > 0$. For an initial velocity $\mathbf{v} = (v\cos\alpha, v\sin\alpha)$, where α is the inclination angle of the velocity $\mathbf{v}$ with respect to the horizontal axis Ox, the projectile parabolic trajectory equations can be derived from elementary mechanics

$$\begin{aligned} x &= x_0 + v\cos\alpha\ t \\ y &= y_0 + v\sin\alpha\ t - \frac{1}{2}\,g\,t^2, \end{aligned} \tag{2.2}$$

where t is the galilean time. The parameters of eq. (2.1) have now the expressions

$$c = -\frac{v^2\cos^2\alpha}{g}, \qquad \xi = x_0 + \frac{v^2}{2g}\sin 2\alpha, \qquad \eta = y_0 + \frac{v^2}{2g}\sin^2\alpha. \tag{2.3}$$

Thus if $v\cos\alpha = v_x$ constant, the set of parabolic trajectories would be of constant focal length c.

As the projectile may be launched from ground and later on recovered also at the same ground level, it is natural to set $y_0 = 0$, (but any other restriction on the three independent quantities (x_0, y_0, v_y) may well be used), so as to have

$$\begin{aligned} \xi &= x_0 + \frac{v_x}{g}\ v_y \\ \eta &= \frac{v_y^2}{2g}, \end{aligned} \tag{2.4}$$

which depend only on the two independent variables (v_y, x_0). This physical choice imposes that $\eta > 0$ and put the parabola vertex in the upper half-plane.

2.3 Definition

As generally known, a Radon transform maps an integrable function $f(x,y)$ on $\mathbb{R}^2$ to its integrals on a set of curves of $\mathbb{R}^2$. The standard Radon transform [1], integrates $f(x,y)$ on straight lines and produce functions of the two variables of a line. These are p, the distance from the coordinate origin to this line and ϕ , the angle made by its unit normal vector with the x-axis. (p,ϕ) may be viewed as the polar coordinates of the projection of the coordinate origin onto the straight line. Analogously circular Radon transforms integrates functions on various two-parameter families of circles in the plane.

If the projectile in section 2.2 is equipped with a device which, in the course of its flight, cumulates (or "*scoops*") the measurements of a physical quantity, (*e.g.* a chemical component density in traversed air), one may obtain, at the recovery of the projectile, the integral of this quantity along a parabolic trajectory. Such type of experiment could be an implementation of the c-parabolic Radon transform.

Thus the c-parabolic Radon transform of $f(x,y)$ on the set of c-parabolas (2.1) are the integrals $\widehat{f}_c(\xi,\eta)$, functions of (ξ,η) coordinates of the c-parabola vertex, given by

$$\widehat{f}_c(\xi,\eta) = \int_{\mathbb{R}} f(\xi+s, \eta+\frac{1}{2c}\, s^2)\, d\sigma, \tag{2.5}$$

where $d\sigma$ is the parabola arc element

$$d\sigma = ds\sqrt{1+(s/c)^2}. \tag{2.6}$$

A more convenient form of this integral is

$$\widehat{f}_c(\xi,\eta) = \int_{-\infty}^{\infty} \sqrt{1+(x-\xi)^2/c^2}\; f(x, \eta+\frac{1}{2c}(x-\xi)^2)\, dx. \tag{2.7}$$

In practice, $f(x,y)$ representing a physical quantity is a non-negative real-valued continuous function with compact support (hence integrable) in $\mathbb{R}^2$.

In this paper we consider the c-parabolic Radon transform either on the set of the bounded real-valued Lebesgue-measurable functions with compact support, which is denoted by $L^\infty_{\text{comp}}(\mathbb{R}^2)$, or on the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ of the infinitely smooth functions that decay faster than any power of x and y at infinity as well as their derivatives (see [29]). We now give some straightforward properties of the c-parabolic Radon transform defined on $L^\infty_{\text{comp}}(\mathbb{R}^2)$ or $\mathcal{S}(\mathbb{R}^2)$ at fixed $c \neq 0$.

1. In both cases (at fixed $c \neq 0$)

$$\big(1+|\xi|\big)^{-1} \widehat{f}_c(\xi,\eta) \in L^\infty(\mathbb{R}_\xi, L^1(\mathbb{R}_\eta)), \tag{2.8}$$

   i.e. $\sup_{\xi\in\mathbb{R}} \big(1+|\xi|\big)^{-1} \int_{-\infty}^{+\infty} |\widehat{f}_c(\xi,\eta)| d\eta < \infty$. This latter range property follows from the estimate $\int_{-\infty}^{+\infty} |\widehat{f}_c(\xi,\eta)| d\eta \leq \int_{\mathbb{R}^2} |f(x,y)| \sqrt{1+\frac{(x-\xi)^2}{c^2}}\, dx\, dy$.

2. The c-parabolic Radon transform $\widehat{f}_c(\xi,\eta)$ is continuous when f is continuous, $f \in L^\infty_{\text{comp}}(\mathbb{R}^2) \cup \mathcal{S}(\mathbb{R}^2)$.

3. The c-parabolic Radon transform $\widehat{f}_c(\xi,\eta)$ is non-negative whenever f is non-negative. We also note that the support of $\widehat{f}_c(\xi,\eta)$ is not a compact subset of $\mathbb{R}^2$ whenever f is non zero and non-negative. This latter statement follows from the equality

$$\int_{-\infty}^{+\infty} \widehat{f}_c(\xi,\eta) d\eta = \int_{\mathbb{R}^2} f(x,y) \sqrt{1+\frac{(x-\xi)^2}{c^2}}\, dx\, dy. \tag{2.9}$$

   Indeed when f is non zero and non-negative then the right-hand side of (2.9) is positive for any $\xi \in \mathbb{R}$, and this shows that the support of $\widehat{f}_c$ is unbounded.

In this paper, the difficult range characterization problem for the c-parabolic Radon transform will not be treated and left for future investigation.

2.4 Kernel

Formula (2.5) can be recast into a Fredholm integral equation of the first kind as

$$\widehat{f_c}(\xi,\eta) = \int_{\mathbb{R}^2} \mathcal{K}_c(\xi,\eta|x,y)\ f(x,y)\,dx\,dy, \tag{2.10}$$

where the kernel $\mathcal{K}_c(\xi,\eta|x,y)$ is singular and given by a delta-function. For the standard Radon transform, we recall that the kernel is the delta function $\delta(p - x\cos\phi - y\sin\phi)$ [26]. In (2.10) we have

$$\mathcal{K}_c(\xi,\eta|x,y) = \delta\left(y - \eta - \frac{(x-\xi)^2}{2c}\right)\ \sqrt{1+(x-\xi)^2/c^2} = \mathcal{K}_c(x-\xi, y-\eta). \tag{2.11}$$

If $f(x,y) = \delta(x-x_S)\delta(y-y_S)$, representing a unit point source at $(x_S,y_S)\in\mathbb{R}^2$, its c-parabolic Radon transform is precisely given by the kernel (2.11), i.e.

$$\widehat{\delta}_c(\xi,\eta) = \mathcal{K}_c(\xi,\eta|x_S,y_S) = \sqrt{1+(\xi-x_S)^2/c^2}\ \delta\left(y_S - \eta - \frac{(\xi-x_S)^2}{2c}\right). \tag{2.12}$$

The image of a point like source at (x_S,y_S) is given by a delta function in Radon space (ξ,η), having support on an inverted parabola. This inverted parabola in Radon space $(\xi,\eta)\in\mathbb{R}^2$ plays the role of a sinogram in the standard Radon transform [26].

2.5 Formal adjoint

The formal adjoint $\hat{f}_c{}^*$ of the c-parabolic Radon transform is given by the formula

$$\hat{f}_c{}^*(x,y) = \int_{-\infty}^{+\infty} f(\xi, y - \frac{1}{2c}(x-\xi)^2)\sqrt{1+(x-\xi)^2/c^2}\,d\xi, \tag{2.13}$$

for $f\in\mathcal{S}(\mathbb{R}^2)\cup L^\infty_{\rm comp}(\mathbb{R}^2)$. Then the c-parabolic Radon transform defined on $L^\infty_{\rm comp}(\mathbb{R}^2)$ can be extended to the set of the compactly supported distributions on $\mathbb{R}^2$ which is denoted by $\mathcal{E}'(\mathbb{R}^2)$. When $f\in\mathcal{E}'(\mathbb{R}^2)$ then $\hat{f}_c$ is a distribution on $\mathbb{R}^2$, and the Fourier transform of f is an analytic function on $\mathbb{C}^2$.

One may prove the injectivity of the c-parabolic Radon transform on $\mathcal{E}'(\mathbb{R}^2)$ using a general injectivity result for translated unbounded curves [24] or using [30].

We recall that the adjoint $\hat{f}^*$ of the standard 2-d Radon transform is related to the back-projection operator and is given by

$$\hat{g}^*(x,y) = \int_0^{2\pi} g(x\cos(\theta) + y\sin(\theta),\theta)\,d\theta$$

for any continuous function g on $\mathbb{R}\times[0,2\pi]$.

2.6 Behavior under scaling transformation

For $s\neq 0$, under the scaling transformation of the coordinates

$$x = \frac{x'}{s}, \quad \text{and} \quad y = \frac{y'}{s}, \tag{2.14}$$

the parabola equation (2.1) is transformed into the equation of a parabola of vertex at (ξ',η') and focal length c', with

$$\eta' = s\,\eta \quad \xi' = s\,\xi \quad \text{and} \quad c' = s\,c. \tag{2.15}$$

Let us denote the transform of a function $f(x,y)$ under the scaling transformation (2.14) by f_s, i.e.

$$f(x,y) = f(\frac{x'}{s},\frac{y'}{s}) = f_s(x',y'). \tag{2.16}$$

Then it can be seen from the definition of the parabolic Radon transform (eq. (2.5)) that the relation between the parabolic Radon transforms on two parabolas of different focal lengths is

$$\widehat{f}_c(\xi,\eta) = \frac{1}{|s|}\,\widehat{h}_{c'}(\xi',\eta'),\ h := f_s. \tag{2.17}$$

In particular, by choosing $s = 1/c$ one has $c' = 1$

$$\widehat{f}_c(\xi,\eta) = |c|\,\widehat{h}_1(\xi',\eta'),\ h := f_{1/c}.$$

This shows that the parabolic Radon transform of a function at unit focal length is related to the parabolic Radon transform of the c-scaled function at c-scaled variables and focal length c.

2.7 Analog of the central slice theorem

A valuable information can be extracted from the relation between the two-dimensional Fourier transform $\mathcal{F}f(k,\omega)$ of $f(x,y)$ and $\widehat{f}_c(\xi,\eta)$, at given c. To stress this point, we review briefly two known cases.

• Let $\widehat{g}(r,\phi)$ be the standard Radon transform of a function $g(x,y)$. The Radon transform $\widehat{g}(r,\phi)$ is the integral of $g(x,y)$ on the straight line of normal unit vector $\mathbf{n} = (\cos\phi, \sin\phi)$ and situated at a distance r from the coordinate origin. Then $\mathcal{F}g(p,q)$ the two-dimensional Fourier transform of $g(x,y)$ is related to the one-dimensional Fourier transform of $\widehat{g}(r,\phi)$ by

$$\mathcal{F}g(\rho\cos\phi, \rho\sin\phi) = \int_{\mathbb{R}} e^{-2i\pi\, r\,\rho}\ \widehat{g}(r,\phi)\, dr. \tag{2.18}$$

This is the so-called central slice theorem, see *e.g.* [26].

Another example of such relation arises from the circular Radon transform on translated circles of fixed radius R. Let $\widehat{g}_R(\xi,\eta)$ be the circular Radon transform of a function $g(x,y)$

$$\widehat{g}_R(\xi,\eta) = \int_0^{2\pi} g(\xi + R\cos t, \eta + R\sin t)\, R\, dt. \tag{2.19}$$

Then $\mathcal{F}g(p,q)$ is related to the two-dimensional Fourier transform of $\widehat{g}_R(\xi,\eta)$ by

$$\mathcal{F}g(p,q)\ 2\pi R\, J_0(2\pi\, R\sqrt{p^2+q^2}) = \int_{\mathbb{R}^2} e^{-2i\pi(p\xi+q\eta)}\ \widehat{g}_R(\xi,\eta)\, d\xi\, d\eta, \tag{2.20}$$

where $J_0(x)$, the Bessel function of the first kind and of order 0, appears as an additional factor as compared to eq. (2.18)

• We now derive the analog of such central slice theorem for the c-parabolic Radon transform. Let us recall the expression of the two-dimensional Fourier transform $\mathcal{F}f(k,\omega)$ when $f(x,y) \in L^\infty_{\text{comp}}(\mathbb{R}^2)$ or when $f(x,y) \in \mathcal{S}(\mathbb{R}^2)$

$$\mathcal{F}f(k,\omega) = \int_{\mathbb{R}^2} e^{-2i\pi(kx+\omega y)}\ f(k,\omega)\, dx\, dy. \tag{2.21}$$

When $f \in L^\infty_{\text{comp}}(\mathbb{R}^2)$, then it follows, in particular, that $\mathcal{F}f$ is infinitely smooth and bounded on $\mathbb{R}^2$ as well as its derivatives. In addition $\mathcal{F}f$ extends to an analytical function on $\mathbb{C}\times\mathbb{C}$ (we remind that the $\mathbb{C}^2$ analytical continuation of $\mathcal{F}f$ is then given by (2.21) where (k,ω) now belongs to $\mathbb{C}^2$).

When $f \in \mathcal{S}(\mathbb{R}^2)$, then $\mathcal{F}f \in \mathcal{S}(\mathbb{R}^2)$.

We denote by $\mathcal{S}'(\mathbb{R}^2)$ the space of tempered distributions, i.e. the space of linear continuous maps from $\mathcal{S}(\mathbb{R}^2)$ to $\mathbb{C}$, where $\mathcal{S}(\mathbb{R}^2)$ is endowed with its standard topology (see, for example, [29] for the basics of the distribution theory). For $g \in \mathcal{S}'(\mathbb{R}^2)$, we recall that the Fourier transform $\mathcal{F}g \in \mathcal{S}'(\mathbb{R}^2)$ of g is defined by

$$< \mathcal{F}g, \phi > := < g, \mathcal{F}\phi >,\ \text{for } \phi \in \mathcal{S}(\mathbb{R}^2), \tag{2.22}$$

where $< g, \mathcal{F}\phi >$ denotes the value of g at $\mathcal{F}\phi$.

Therefore from eq. (2.5) it follows that

$$
\begin{aligned}
&< \mathcal{F}\widehat{f}_c(k,\omega), \phi >= \int_{\mathbb{R}^3} f(x, \eta + \frac{1}{2c}(x-\xi)^2)\sqrt{1+\frac{(x-\xi)^2}{c^2}}\mathcal{F}\phi(\xi,\eta)dxd\xi d\eta \\
&= |c| \int_{\mathbb{R}} \sqrt{1+\nu^2}\Big(\int_{\mathbb{R}^2} f(x,\eta)\mathcal{F}\phi(x-c\nu, \eta - \frac{c}{2}\nu^2)dxd\eta\Big)d\nu \\
&= |c| \int_{\mathbb{R}} \sqrt{1+\nu^2}\Big(\int_{\mathbb{R}^2} e^{2i\pi ck\nu + i\pi c\omega\nu^2}\mathcal{F}f(k,\omega)\phi(k,\omega)dkd\omega\Big)d\nu
\end{aligned}
$$

(we performed the change of variables $\nu = c^{-1}(-\xi + x)$, $|c|d\nu = d\xi$, and then $\eta = \eta + \frac{c\nu^2}{2}$). We obtained

$$\mathcal{F}\widehat{f}_c(k,\omega) = \mathcal{F}f(k,\omega)\,|c|\,\mathcal{P}(ck, c\omega), \tag{2.23}$$

where $\mathcal{P}$ is the tempered distribution defined by

$$< \mathcal{P}, \phi > \;:=\; \int_{\mathbb{R}^2} \sqrt{1+\nu^2}e^{2i\pi k\nu}\Big(\int_{-\infty}^{+\infty} e^{i\pi\omega\nu^2}\phi(k,\omega)d\omega\Big)\,dk\,d\nu \tag{2.24}$$

$$= \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^2}\phi(k,\omega)\big(\int_{-\infty}^{+\infty}\sqrt{1+\nu^2}e^{-\varepsilon\nu^2+2i\pi(k\nu+\frac{\omega}{2}\nu^2)}d\nu\big)\,dk\,d\omega, \tag{2.25}$$

for $\phi \in \mathcal{S}(\mathbb{R}^2)$. (The function $\int_{-\infty}^{+\infty} e^{i\pi\omega\nu^2}\phi(k,\omega)\,d\omega$ belongs to $\mathcal{S}(\mathbb{R}^2_{(k,\nu)})$ for $\phi \in \mathcal{S}(\mathbb{R}^2)$, thus the integral in the (k,ν) variables on the right-hand-side of (2.24) is absolutely convergent.)

Eq. (2.23) is the analog of the central slice theorem for the c-parabolic Radon transform. For eq. (2.23) to be useful, $\mathcal{P}(k,\omega)$ should not be a too bad generalized function, [4], (compare with eqs. (2.18,2.20)). Its nature and properties will be analyzed in section 3.

2.8 Formal expression of $\mathcal{P}(k,\omega)$ as a convolution

From eq. (2.5), one can see $\widehat{f}_c$ as the convolution of the function $f(x,y)$ with the tempered distribution $\delta_{\mathcal{C}_{-c}}(x,y)\sqrt{1+x^2/c^2}$.

Thus from the definition of $\mathcal{P}(k,\omega)$ it follows that $|c|\mathcal{P}(ck,c\omega) = \mathcal{F}(\sqrt{1+\xi^2/c^2}\delta_{\mathcal{C}_{-c}}(\xi,\eta))(k,\omega)$ and

$$\mathcal{P} = \mathcal{F}^{-1}\big(\delta_{\mathcal{C}}(\xi,\eta)\sqrt{1+\xi^2}\big), \tag{2.26}$$

where $\mathcal{C} := \{(\xi, \xi^2/2) \mid \xi \in \mathbb{R}\}$. Then formally $\mathcal{P}$ can be seen as the convolution of $\mathcal{F}^{-1}\delta_{\mathcal{C}}$ and $\mathcal{F}^{-1}(\sqrt{1+\xi^2})$.

We give explicit expressions for $\mathcal{F}^{-1}\delta_{\mathcal{C}}$ and $\mathcal{F}^{-1}(\sqrt{1+\xi^2})$. We recall the following formula (which can be proved by complex analysis)

$$\int_{-\infty}^{\infty}\exp 2i\pi(k\zeta + \frac{\omega}{2c}\zeta^2)\,d\zeta = \sqrt{\Big|\frac{c}{\omega}\Big|}\;e^{i\mathrm{sgn}\left(\frac{c}{\omega}\right)\frac{\pi}{4}}\;e^{-i\pi\frac{k^2}{\omega}c}, \tag{2.27}$$

(where the integral on the left-hand-side of (2.27) is semi convergent and is related to the limit at $\pm\infty$ of the well-known Fresnel integrals, [27]). Then the inverse Fourier transform of $\delta_{\mathcal{C}}(\xi,\eta)$ is given by

$$\mathcal{F}^{-1}\big(\delta_{\mathcal{C}}\big)(k,\omega) = \int_{-\infty}^{\infty}\exp 2i\pi(k\zeta + \frac{\omega}{2}\zeta^2)\,d\zeta = |\omega|^{-\frac{1}{2}}\;e^{i\mathrm{sgn}(\omega)\frac{\pi}{4}}\;e^{-i\pi\frac{k^2}{\omega}}, \tag{2.28}$$

while the inverse Fourier transform of $\sqrt{1+\xi^2}$ is (see [27])

$$\mathcal{F}^{-1}(\sqrt{1+\xi^2})(k,\omega) = -\pi^{-1}\delta(\omega)\;\text{F.P.}\frac{K_1(2\pi|k|)}{|k|}, \tag{2.29}$$

where $K_1(\sigma)$ is a modified Bessel function of order 1 [28], given by its integral representation $K_1(\sigma) = \int_0^{+\infty} \cosh(t) e^{-\sigma\cosh(t)} dt$ for $\sigma > 0$, and the finite part (F.P.) of the function $\frac{K_1(2\pi|k|)}{|k|}$ is the tempered distribution defined by

$$< \text{F.P.} \frac{K_1(2\pi|k|)}{|k|}, \phi >= \int_{-\infty}^{+\infty} \frac{K_1(2\pi|k|)}{|k|} (\phi(k) - \phi(0)) dk - \pi\phi(0), \tag{2.30}$$

for $\phi \in \mathcal{S}(\mathbb{R}_k)$. We remind that $\lim_{\sigma\to 0^+} \sigma K_1(\sigma) = 1$, $\lim_{\sigma\to+\infty} \sigma K_1(\sigma) = 0$ and $K_1 \in C^\infty((0,+\infty))$, and thus the integral on the right-hand-side of (2.30) is absolutely convergent. Moreover we have $\int_0^{+\infty} \left(\frac{K_1(2\pi|\sigma|)}{|\sigma|} - \frac{1}{2\pi\sigma^2}\right) d\sigma = -\pi$.

3 Properties of $\mathcal{P}(k,\omega)$ and the invertibility of the c-parabolic Radon transform on $L^\infty_{\text{comp}}(\mathbb{R}^2)$ or on $\mathcal{S}(\mathbb{R}^2)$

3.1 An alternative expression for $\mathcal{P}(k,\omega)$

We rewrite $\mathcal{P}(k,\omega)$ under the form of a sum of two distributions (denoted by $\mathcal{P}_{reg}$ and $\mathcal{Q}$), and we will prove that one of them ($\mathcal{P}_{reg}$) can be identified to a Lebesgue measurable function while the other one ($\mathcal{Q}$) can not.

First, we use the fact that the integrand in the ν variable in (2.24) is even in ν and rewrite it as

$$< \mathcal{P}, \phi >= 2\int_{-\infty}^{+\infty} \int_0^{\infty} \sqrt{1+\nu^2} \left(\int_{-\infty}^{+\infty} \cos(2\pi k\nu)\ e^{2i\pi(\frac{\omega}{2}\nu^2)} \phi(k,\omega) d\omega \right) d\nu\, dk, \tag{3.1}$$

for $\phi \in \mathcal{S}(\mathbb{R}^2)$.

Next $\mathcal{P}(k,\omega)$ may be written as the sum of

$$\mathcal{P}(k,\omega) = \mathcal{P}_{reg}(k,\omega) + \mathcal{Q}(k,\omega), \tag{3.2}$$

where $\mathcal{P}_{reg}$ and $\mathcal{Q}$ are the tempered distributions defined by

$$< \mathcal{P}_{reg}, \phi > = 2\int_{-\infty}^{+\infty} \int_0^{\infty} (\sqrt{1+\nu^2} - \nu) \left(\int_{-\infty}^{+\infty} \cos(2\pi k\nu)\ e^{2i\pi(\frac{\omega}{2}\nu^2)} \phi(k,\omega) d\omega \right) d\nu\, dk, \tag{3.3}$$

$$< \mathcal{Q}, \phi > = 2\int_{-\infty}^{+\infty} \int_0^{\infty} \nu \left(\int_{-\infty}^{+\infty} \cos(2\pi k\nu)\ e^{2i\pi(\frac{\omega}{2}\nu^2)} \phi(k,\omega) d\omega \right) d\nu\, dk, \tag{3.4}$$

for $\phi \in \mathcal{S}(\mathbb{R}^2)$.

• We first consider $\mathcal{P}_{reg}$. The function $\sqrt{1+\nu^2} - \nu$ has the limit $\lim_{\nu\to+\infty}(\sqrt{1+\nu^2} - \nu) = 0$ and $\frac{d}{d\nu}(\sqrt{1+\nu^2} - \nu) = \frac{\nu}{\sqrt{1+\nu^2}} - 1$. In addition $\frac{\nu}{\sqrt{1+\nu^2}} - 1 = O(\nu^{-2})$ as $\nu \to +\infty$, thus $\frac{\nu}{\sqrt{1+\nu^2}} - 1$ is absolutely integrable on $(0,+\infty)$. Note also that for $\nu \in \mathbb{R}$

$$\int_0^{\nu} \cos(2\pi k\sigma) e^{i\pi\omega\sigma^2} d\sigma = e^{-i\frac{\pi k^2}{\omega}} \sqrt{\frac{1}{8|\omega|}} \Big(E_{\text{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} + k\sqrt{\frac{2}{|\omega|}}\big) + E_{\text{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} - k\sqrt{\frac{2}{|\omega|}}\big), \tag{3.5}$$

where $E_\pm(x) = C(x) \pm i\, S(x)$ are Fresnel exponentials defined as complex combinations of the Fresnel integrals $C(x) := \int_0^x \cos\left(\frac{\pi}{2}\sigma^2\right) d\sigma$ and $S(x) = \int_0^x \sin\left(\frac{\pi}{2}\sigma^2\right) d\sigma$ for $x \in \mathbb{R}$. We remind that the Fresnel integrals are uniformly bounded on $\mathbb{R}$, and we have in particular $\sup_{x\in(0,+\infty)} C(x) = C(1) < 1$ and $\sup_{x\in(0,+\infty)} S(x) = S(\sqrt{2}) < \sqrt{2}$ (both functions C and S are odd in x and $\min(C(x), S(x)) > 0$ for $x > 0$). Thus

$$\left| \int_0^{\nu} \cos(2\pi k\sigma) e^{i\pi\omega\sigma^2} d\sigma \right| \leq \sqrt{\frac{2}{|\omega|}}, \tag{3.6}$$

and we recall that $|\omega|^{-\frac{1}{2}}$ is absolutely integrable on $\mathbb{R}_\omega$.

Finally integrating by parts on the ν-variable on the right-hand-side of (3.3), we obtain

$$< \mathcal{P}_{reg}, \phi >= \int_{\mathbb{R}^2} \phi(k,\omega) e^{-i\frac{\pi k^2}{\omega}} \sqrt{\frac{1}{2|\omega|}} \tag{3.7}$$
$$\times \int_0^\infty (1 - \frac{\nu}{\sqrt{1+\nu^2}}) \Big(E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} + k\sqrt{\frac{2}{|\omega|}}\big) + E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} - k\sqrt{\frac{2}{|\omega|}}\big)\Big)\, d\nu\, dk\, d\omega,$$

for $\phi \in \mathcal{S}(\mathbb{R}^2)$. Therefore $\mathcal{P}_{reg}$ has the expression

$$\mathcal{P}_{reg}(k,\omega) = e^{-i\frac{\pi k^2}{\omega}} \sqrt{\frac{1}{2|\omega|}} \int_0^\infty (1 - \frac{\nu}{\sqrt{1+\nu^2}}) \Big(E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} + k\sqrt{\frac{2}{|\omega|}}\big) + E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} - k\sqrt{\frac{2}{|\omega|}}\big)\Big) d\nu, \tag{3.8}$$

for $(k,\omega) \in \mathbb{R}^2$, $\omega \neq 0$ and the integral in the ν-variable is absolutely convergent.

• We now consider the tempered distribution $\mathcal{Q}$. By setting $\lambda = \nu^2$ in (3.4), we obtain

$$< \mathcal{Q}, \phi >= \int_{-\infty}^{+\infty} \int_0^{+\infty} \nu \, \Big(\int_{-\infty}^{+\infty} \cos\left(2\pi k\sqrt{\lambda}\right) \, e^{i\pi\omega\lambda} \phi(k,\omega) d\omega \Big)\, d\nu\, dk, \tag{3.9}$$

for $\phi \in \mathcal{S}(\mathbb{R}^2)$. We next use the results given on pages 122-123 of [27], on the Fourier transforms of $\cos b\sqrt{|x|}$ and $\mathrm{sgn}x \cos b\sqrt{|x|}$ to write $\mathcal{Q}(k,\omega)$ as

$$< \mathcal{Q}, \phi >= \int_{-\infty}^{+\infty} \frac{1}{2} \int_{-\infty}^{+\infty} \Big[\cos(2\pi k\sqrt{|\lambda|}) + \mathrm{sgn}\lambda \, \cos(2\pi k\sqrt{|\lambda|})\Big] \int_{-\infty}^{+\infty} e^{i\pi\omega\,\lambda} \phi(k,\omega)\, d\omega\, d\lambda\, dk.$$

Thus $\mathcal{Q}(k,\omega)$ can be explicitly evaluated as

$$\mathcal{Q}(k,\omega) = \frac{i}{\pi} \mathrm{P.V.}\frac{1}{\omega} + \mathrm{F.P.}\left[\frac{k\sqrt{2}}{|\omega|^{\frac{3}{2}}} e^{-i\pi\frac{k^2}{\omega}} \, E_{\mathrm{sgn}\omega}\left(k\sqrt{\frac{2}{|\omega|}}\right)\right], \tag{3.10}$$

where the principal value $\mathrm{P.V.}\frac{1}{\omega}$ is the tempered distribution defined by

$$< \mathrm{P.V.}\frac{1}{\omega}, \phi >= \int_{-\infty}^{+\infty} \left(\int_0^{+\infty} \frac{\phi(k,\omega) - \phi(k,-\omega)}{\omega} d\omega \right) dk, \tag{3.11}$$

for $\phi \in \mathcal{S}(\mathbb{R}^2)$, and where the finite part $\mathrm{F.P.}\left[\frac{k\sqrt{2}}{|\omega|^{\frac{3}{2}}} e^{-i\pi\frac{k^2}{\omega}} \, E_{\mathrm{sgn}\omega}\left(k\sqrt{\frac{2}{|\omega|}}\right)\right]$ is the tempered distribution defined by

$$< \mathrm{F.P.}\left[\frac{k\sqrt{2}}{|\omega|^{\frac{3}{2}}} e^{-i\pi\frac{k^2}{\omega}} \, E_{\mathrm{sgn}\omega}\left(k\sqrt{\frac{2}{|\omega|}}\right)\right], \phi >= \int_{-\infty}^{+\infty} \phi(k,0) dk \tag{3.12}$$
$$+ \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \frac{k\sqrt{2}}{|\omega|^{\frac{3}{2}}} e^{-i\pi\frac{k^2}{\omega}} \, E_{\mathrm{sgn}\omega}\left(k\sqrt{\frac{2}{|\omega|}}\right) (\phi(k,\omega) - \phi(k,0)) d\omega \right) dk,$$

for $\phi \in \mathcal{S}(\mathbb{R}^2)$. Both integrals on the right-hand-side of (3.12) are absolutely convergent, and we recall that 1 is the value of the finite part in the sense of Hadamard (see [29]) of the integral $\int_{-\infty}^{+\infty} \frac{k\sqrt{2}}{|\omega|^{\frac{3}{2}}} e^{-i\pi\frac{k^2}{\omega}} \, E_{\mathrm{sgn}\omega}\left(k\sqrt{\frac{2}{|\omega|}}\right) d\omega$,

i.e. $1 = \lim_{\epsilon\to 0^+} \int_{\substack{\omega\in\mathbb{R} \\ |\omega|>\epsilon}} \frac{k\sqrt{2}}{|\omega|^{\frac{3}{2}}} e^{-i\pi\frac{k^2}{\omega}} \left(E_{\mathrm{sgn}\omega}\left(k\sqrt{\frac{2}{|\omega|}}\right) - \frac{\mathrm{sgn}(k) e^{i\mathrm{sgn}(\omega)\frac{\pi}{4}}}{\sqrt{2}} \right) d\omega$. This integral over $\{\omega \in \mathbb{R} \mid |\omega| > \epsilon\}$ is absolutely convergent and $\frac{\mathrm{sgn}(k) e^{i\mathrm{sgn}(\omega)\frac{\pi}{4}}}{\sqrt{2}} = \lim_{\eta\to 0^+} E_{\mathrm{sgn}\omega}\left(k\sqrt{\frac{2}{\eta}}\right)$ for any $k \in \mathbb{R}$, $k \neq 0$.

Note that unlike $\mathcal{P}_{reg}$, the distribution $\mathcal{Q}$ cannot be identified to a Lebesgue measurable function since neither P.V.$\frac{1}{\omega}$ nor F.P. $\left[\frac{k\sqrt{2}}{|\omega|^{\frac{3}{2}}}\, e^{-i\pi\frac{k^2}{\omega}}\ E_{\mathrm{sgn}\omega}\left(k\sqrt{\frac{2}{|\omega|}}\right)\right]$ can be identified to a Lebesgue measurable function.

Finally combining (3.8) and (3.10), we obtain the following final expression for $\mathcal{P}(k,\omega)$

$$\mathcal{P}(k,\omega) = e^{-i\frac{\pi k^2}{\omega}}\sqrt{\frac{1}{2|\omega|}}\int_0^\infty (1-\frac{\nu}{\sqrt{1+\nu^2}})\Big(E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|}+k\sqrt{\frac{2}{|\omega|}}\big)+E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|}-k\sqrt{\frac{2}{|\omega|}}\big)\Big)d\nu$$
$$+\frac{i}{\pi}\mathrm{P.V.}\frac{1}{\omega}+\mathrm{F.P.}\left[\frac{k\sqrt{2}}{|\omega|^{\frac{3}{2}}}\, e^{-i\pi\frac{k^2}{\omega}}\ E_{\mathrm{sgn}\omega}\left(k\sqrt{\frac{2}{|\omega|}}\right)\right]. \tag{3.13}$$

3.2 Regularity of $\mathcal{P}(k,\omega)$

We introduce some new notations. Let Δ be the straight line in $\mathbb{R}^2$ defined by $\Delta := \{(k,0)\mid k\in\mathbb{R}\}$. We denote by $\mathcal{D}(\mathbb{R}^2\backslash\Delta)$ the set of infinitely smooth functions ϕ on $\mathbb{R}^2$ that are compactly supported inside the open set $\mathbb{R}^2\backslash\Delta$, and we denote by $\mathcal{D}'(\mathbb{R}^2\backslash\Delta)$ the set of distributions on $\mathcal{D}(\mathbb{R}^2\backslash\Delta)$, i.e. $\mathcal{D}'(\mathbb{R}^2\backslash\Delta)$ is the set of the linear continuous functionals from $\mathcal{D}(\mathbb{R}^2\backslash\Delta)$ to $\mathbb{C}$ (where $\mathcal{D}(\mathbb{R}^2\backslash\Delta)$ is endowed with its usual topology, see [29]).

We have seen that $\mathcal{Q}$ cannot be identified to a Lebesgue measurable function. Thus from (3.13), it follows that $\mathcal{P}$ cannot be identified to a Lebesgue measurable function also. However roughly speaking $\mathcal{P}$ restricted to $\mathbb{R}^2\backslash\Delta$ is an infinitely smooth function $\tilde{\mathcal{P}}$ with a nonvanishing property (see (3.16) given below). We make now this latter statement more precise.

We denote by $\tilde{\mathcal{P}}$ the distribution of $\mathcal{D}'(\mathbb{R}^2\backslash\Delta)$ induced by $\mathcal{P}$, i.e. $\tilde{\mathcal{P}}$ is defined by

$$<\tilde{\mathcal{P}},\phi>:=<\mathcal{P},\phi>, \tag{3.14}$$

for $\phi\in\mathcal{D}(\mathbb{R}^2\backslash\Delta)$ (we remind that $\mathcal{D}(\mathbb{R}^2\backslash\Delta)\subset\mathcal{S}(\mathbb{R}^2)$). Then from (3.11), (3.12) and (3.13) it follows that $\tilde{\mathcal{P}}$ is the pointwise defined function given by

$$\tilde{\mathcal{P}}(k,\omega) = e^{-i\frac{\pi k^2}{\omega}}\sqrt{\frac{1}{2|\omega|}}\int_0^\infty (1-\frac{\nu}{\sqrt{1+\nu^2}})\Big(E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|}+k\sqrt{\frac{2}{|\omega|}}\big)+E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|}-k\sqrt{\frac{2}{|\omega|}}\big)\Big)d\nu$$
$$+\frac{i}{\pi\omega}+\frac{k\sqrt{2}}{|\omega|^{\frac{3}{2}}}\, e^{-i\pi\frac{k^2}{\omega}}\ E_{\mathrm{sgn}\omega}\left(k\sqrt{\frac{2}{|\omega|}}\right), \tag{3.15}$$

for $(k,\omega)\in\mathbb{R}^2\backslash\Delta$.

The function $\tilde{\mathcal{P}}(k,\omega)$ is even in the k-variable ($\tilde{\mathcal{P}}(k,\omega)=\tilde{\mathcal{P}}(-k,\omega)$) and satisfies $\overline{\tilde{\mathcal{P}}(k,\omega)}=\tilde{\mathcal{P}}(k,-\omega)$ for $(k,\omega)\in\mathbb{R}^2\backslash\Delta$. In addition $\tilde{\mathcal{P}}$ has the following properties

$$\tilde{\mathcal{P}}\in C^\infty(\mathbb{R}^2\backslash\Delta) \text{ and } \Re\Big[e^{i\pi\frac{k^2}{\omega}}\tilde{\mathcal{P}}(k,\omega)\Big]>0 \text{ for any } (k,\omega)\in\mathbb{R}^2\backslash\Delta, \tag{3.16}$$

(where $\Re z$ denotes the real part of a complex number z) and $\tilde{\mathcal{P}}$ satisfies the following rather straightforward asymptotic formulas and global estimate

$$\tilde{\mathcal{P}}(k,\omega)=|k|e^{i\mathrm{sgn}(\omega)\frac{\pi}{4}-i\frac{\pi k^2}{\omega}}|\omega|^{-\frac{3}{2}}+\frac{i}{\pi}(1+k)\omega^{-1}+o(|\omega|^{-\frac{1}{2}}),\ \text{as } \omega\to 0,\ \omega\neq 0, \tag{3.17}$$

for any $k\in\mathbb{R}$,

$$\tilde{\mathcal{P}}(k,\omega)=e^{i\mathrm{sgn}(\omega)\frac{\pi}{4}-i\frac{\pi k^2}{\omega}}|\omega|^{-\frac{1}{2}}+o(|\omega|^{-\frac{1}{2}}),\ \text{as } \omega\to\pm\infty,\ \text{for } k\in\mathbb{R}, \tag{3.18}$$

and

$$|\tilde{\mathcal{P}}(k,\omega)|\leq\frac{2|k|}{|\omega|^{\frac{3}{2}}}+\frac{1}{\pi|\omega|}+\frac{2}{\sqrt{|\omega|}},\quad \text{for } (k,\omega)\in\mathbb{R}^2\backslash\Delta. \tag{3.19}$$

The properties (3.16), (3.17), (3.18) and (3.19) are proved in appendix A.

The asymptotic formula (3.17) shows that $\tilde{\mathcal{P}}(k,\omega)$ blows up when ω gets closer and closer to 0, while the asymptotic formula (3.18) (or (3.19)) shows that $\tilde{\mathcal{P}}(k,\omega)$ goes to 0 as ω goes to $\pm\infty$. The nonvanishing property $\Re\Big[e^{i\pi\frac{k^2}{\omega}}\tilde{\mathcal{P}}(k,\omega)\Big] > 0$ shows that $\tilde{\mathcal{P}}(k,\omega)$ is nowhere zero and thus $\tilde{\mathcal{P}}^{-1}$ is a well-defined and smooth function on $\mathbb{R}^2\backslash\Delta$.

3.3 Partial differential equation for $\mathcal{P}(k,\omega)$

In this subsection, we show that $\mathcal{P}(k,\omega)$ is the unique solution of a Cauchy problem for a 1-dimensional Schrödinger equation. In this equation ((3.21) given below) the ω-variable plays the role of the time variable while k plays the role of the spatial variable. The initial value condition is taken at "time $\omega = 0$" and roughly speaking we have to take the value of $\mathcal{P}$ at $\omega = 0$. As $\mathcal{P}$ is a distribution and not a pointwise defined function, we first need to give a meaning to $\mathcal{P}(k,0)$ and to introduce the right distribution subset on which the initial value problem is considered. This subset is denoted by $\mathcal{W}$ and defined by

$$L \in \mathcal{W} \Leftrightarrow \left\{ \begin{array}{l} L \in \mathcal{S}'(\mathbb{R}^2) \\ \text{and there exists a tempered distribution } u \in \mathcal{S}'(\mathbb{R}) \text{ such that for all } \psi \in C_0^\infty(\mathbb{R}), \\ \displaystyle\int_{\mathbb{R}} \psi(\sigma)d\sigma = 1 \text{ and for all } \phi \in \mathcal{S}(\mathbb{R}) \text{ the limit } \lim_{\varepsilon\to 0^+} < L(k,\omega), \varepsilon^{-1}\psi(\varepsilon^{-1}\omega)\phi(k) > \\ \text{exists in } \mathbb{C} \text{ and is equal to } < u, \phi > \end{array} \right. \tag{3.20}$$

(we denote by $C_0^\infty(\mathbb{R})$ the set of the compactly supported and infinitely smooth functions defined on $\mathbb{R}$). For $L \in \mathcal{W}$, we denote by $L(k,0)$ the distribution $u \in \mathcal{S}'(\mathbb{R})$ related to L by (3.20). When $L \in \mathcal{S}'(\mathbb{R}^2)$ and L is continuous on $\mathbb{R}^2$ then $L(k,0)$ is the usual restriction of the function L to $\mathbb{R}_k \times \{0\}_\omega$. Note that $\mathcal{W} \neq \mathcal{S}'(\mathbb{R}^2)$ (see appendix C for more details).

The distribution $\mathcal{P}(k,\omega)$ belongs to $\mathcal{W}$ and is the unique solution of the following initial value problem in $\mathcal{W}$

$$\frac{\partial}{\partial\omega}\mathcal{P}(k,\omega) = \frac{1}{4i\pi}\frac{\partial^2}{\partial k^2}\mathcal{P}(k,\omega), \tag{3.21}$$

$$\mathcal{P}(k,0) = \int_0^{+\infty} \big(1 - \frac{\nu}{\sqrt{1+\nu^2}}\big)\frac{\sin(2\pi k\nu)}{\pi k}\,d\nu - \frac{1}{2\pi^2}\big(\text{P.V.}\frac{1}{k}\big)', \tag{3.22}$$

where the integral on the right-hand-side of (3.22) is absolutely convergent for $k \neq 0$, and where $\big(\text{P.V.}\frac{1}{k}\big)'$ denotes the derivative of the principal value distribution and is given by

$$< \big(\text{P.V.}\frac{1}{k}\big)', \phi > = - < \text{P.V.}\frac{1}{k}, \phi'(k) > = -\int_0^{+\infty} \frac{\phi'(k) - \phi'(-k)}{k}\,dk,\ \phi \in \mathcal{S}(\mathbb{R}). \tag{3.23}$$

The proof of (3.22) (and of the statement $\mathcal{P}(k,\omega) \in \mathcal{W}$) is given in appendix B while the integral structure of $\mathcal{P}(k,\omega)$ (see (2.26)) shows the validity of (3.21) in $\mathcal{S}'(\mathbb{R}^2)$. Then the smooth function $\tilde{\mathcal{P}}$ also satisfies (3.21) in a classical sense on $\mathbb{R}^2\backslash\Delta$.

In appendix C we shall prove the existence and uniqueness of the Cauchy problem for (3.21) in the space $\mathcal{W}$ with an arbitrary initial condition taken at "time $\omega = 0$". An alternative expression for $\mathcal{P}(k,0)$ is given by

$$\mathcal{P}(k,0) = -\frac{1}{\pi}\,\text{F.P.}\Big[\frac{K_1(2\pi|k|)}{|k|}\Big], \tag{3.24}$$

where the right-hand-side is defined by (2.30).

The distribution $\mathcal{P}$ is also subjected to the following boundary condition

$$\mathcal{P}(0,\omega) = \sqrt{\frac{2}{|\omega|}}\int_0^{+\infty} (1 - \frac{\nu}{\sqrt{1+\nu^2}})E_{\text{sgn}(\omega)}\big(\nu\sqrt{2|\omega|}\big)\,d\nu + \frac{i}{\pi}\text{P.V.}\frac{1}{\omega} + \delta(\omega), \tag{3.25}$$

where $\mathcal{P}(0,\omega) \in \mathcal{S}'(\mathbb{R})$ is defined by

$$< \mathcal{P}(0,\omega), \phi > := \lim_{\varepsilon\to 0^+} < \mathcal{P}(k,\omega), \phi(k)\varepsilon^{-1}\psi(\varepsilon^{-1}\omega) >, \phi \in \mathcal{S}(\mathbb{R}), \tag{3.26}$$

for any $\psi \in C_0^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \psi(\sigma)d\sigma = 1$ (the integral in ν on the right-hand-side of (3.25) is bounded by $\sqrt{2}\int_0^{+\infty}(1-\frac{\nu}{\sqrt{1+\nu^2}})\,d\nu < \infty$). For the proof of (3.25) (with the proof that the limit in (3.26) exists) see appendix B.

3.4 Inversion

From (2.23) it follows that the Fourier transform of $\hat{f}_c$ as a distribution of $\mathcal{D}'(\mathbb{R}^2\backslash\Delta)$ is a smooth function on $\mathbb{R}^2\backslash\Delta$ which is equal to $|c|\tilde{\mathcal{P}}(ck,c\omega)\mathcal{F}f(k,\omega)$. Let us denote by g this function, $g(k,\omega) = |c|\tilde{\mathcal{P}}(ck,c\omega)\mathcal{F}f(k,\omega)$, $(k,\omega)\in\mathbb{R}^2\backslash\Delta$. The function g is known from the data $\hat{f}_c$, and using property (3.16) and the injectivity of the Fourier transform on $L^2(\mathbb{R}^2)$ we obtain that the c-parabolic Radon transform is one-to-one on $\mathcal{S}(\mathbb{R}^2)$ or on $L^\infty_{\text{comp}}(\mathbb{R}^2)$. We remind that the injectivity of the c-parabolic Radon transform on $\mathcal{E}'(\mathbb{R}^2)$ may be found in [24, 30]. Inverting the Fourier transform yields the inversion formula

$$f = \mathcal{F}^{-1}\Big(\frac{g(k,\omega)}{|c|\tilde{\mathcal{P}}(ck,c\omega)}\Big). \tag{3.27}$$

When $f\in\mathcal{S}(\mathbb{R}^2)$, then the inversion (3.27) holds pointwise. When $f\in L^\infty_{\text{comp}}(\mathbb{R}^2)$, then the inversion (3.27) holds in $L^2(\mathbb{R}^2)$.

Formula (2.23) also provides approximate solutions for the problem of the reconstruction of f from $\hat{f}_c$. Indeed let M be a fixed positive constant and let $\chi\in\mathcal{D}(\mathbb{R}^2\backslash\Delta)$, $\sup_{(k,\omega)\in\mathbb{R}^2}|\chi(k,\omega)|\le M$. Then eq. (2.23) becomes

$$|c|\chi(k,\omega)\tilde{P}(ck,c\omega)\mathcal{F}f(k,\omega) = \chi(k,\omega)g(k,\omega) \tag{3.28}$$

As we multiply the data g by the cut-off function χ, we are actually filtering the data by cutting the low and high frequencies of the data. Then inverting the Fourier transform we obtain the following approximate solution

$$\big(\mathcal{F}^{-1}\chi\big) * f = \mathcal{F}^{-1}\Big(\frac{\chi(k,\omega)g(k,\omega)}{|c|\tilde{\mathcal{P}}(ck,c\omega)}\Big) \text{ pointwise}, \tag{3.29}$$

where $*$ denotes the convolution product. Then $\big(\mathcal{F}^{-1}\chi\big)*f$ converges to f in the L^2 sense when χ converges to 1 point-wise on $\mathbb{R}^2\backslash\Delta$.

Remark 1. This inversion of the c-parabolic Radon transform is achieved through Fourier transform via the analog of the central slice theorem. Another way to approach its inversion is to use the standard Radon transform. Let us introduce polar coordinates for the c-parabola vertex by ($\xi = r\cos\phi, \eta = r\sin\phi$) and set $\widehat{f}_c(\xi,\eta) = \widetilde{f}_c(r,\phi)$. From (2.23) it follows that

$$\widehat{f}_c = \mathcal{F}^{-1}\big(\mathcal{F}f(k,\omega)\,|c|\,\mathcal{P}(ck,c\omega)\big), \tag{3.30}$$

and using polar coordinates eq. (3.30) becomes

$$\widehat{f}_c(\xi,\eta) = \widetilde{f}_c(r,\phi) = \int_{\mathbb{R}^2} \mathcal{F}f(k,\omega)\; e^{2i\pi r(k\cos\phi+\omega\sin\phi)}\; |c|\mathcal{P}(ck,c\omega)\,dk\,d\omega. \tag{3.31}$$

Now after performing the one-dimensional Fourier transform on both sides of this equation, we get

$$\int_{\mathbb{R}} dr\; e^{-2i\pi rp}\; \widetilde{f}_c(r,\phi) = \int_{\mathbb{R}^2} \delta(p-k\cos\phi-\omega\sin\phi)\; \mathcal{F}f(k,\omega)\; |c|\mathcal{P}(ck,c\omega)\,dk\,d\omega. \tag{3.32}$$

The left-hand-side of eq. (3.32) is the r-Fourier transform of the c-parabolic Radon data. Its right-hand-side is just the standard Radon transform of the product $\mathcal{F}f(k,\omega)\;|c|\mathcal{P}(ck,c\omega)$. Thus by applying the inverse standard Radon transform on this equation one would be able to retrieve this product and then reobtain $f(x,y)$ as above. This is an amusing connection, which is worth mentioning and akin to the connection found by Denecker in the Cormack parabolic Radon transform [23].

Remark 2. The invertibility of the c-parabolic Radon transform suggests an application in atmospheric pollution control. One may send projectiles from the ground in the atmosphere to collect

data on polluting agent concentration along parabolic trajectories as described in subsection 2.2. The pollution level by specific substance can then be retrieved through the c-parabolic inverse Radon transform.

No other cases of invertibility resulting from other constraints imposed on the set of variables (ξ, η, c) has been found.

4 Seismic parabolic Radon transforms

4.1 Background

In geophysical imaging, the aim is to determine the structure of the earth interior from data obtained on its surface [31]. Reflection seismology uses inverse methods to find structure, composition, and source parameters by processing the received seismograms. However one of the problems in handling seismic data is the elimination of multiple reflection events. To perform this task, a special parabolic Radon transform has been advocated by many workers (see e.g. [32], [33]). It is based on the assumption that the arrival time t of a seismic signal is given by a "parabolic" relation of the form

$$t = p + \frac{(x-\xi)^2}{2c}, \tag{4.1}$$

where $(x-\xi)$ is the so-called "off-set" distance from the source at abscissa ξ [14], p is the travel time of the signal at $x = \xi$, and c is a parameter related to signal propagation, assumed to be constant.

In fact events with one reflection obey a hyperbolic relation time *vs.* space. But after appropriate "corrections", a parabolic relation of the type (4.1) is obtained. Thus $u_c(t,x)$ may represent an energy detected at time t and site x. Thus if one sums up all the recorded signals along x, one obtains the seismic c-parabolic Radon transform $\widehat{u}_c(\xi,p)$ of $u_c(t,x)$. The inverse problem consists of obtaining $u(t,x)$ from the set of values of $\widehat{u}_c(\xi,p)$. This would help to create the missing data, when necessary and also to reject data due to multiple reflections.

Here we adopt the seismic notations in cartesian coordinates, by putting t on the vertical axis. This seismic c-parabolic Radon transform of a function $u(x,t)$ defined in $\mathbb{R}^2$, takes the form

$$\widehat{u}_c(\xi,p) = \int_{-\infty}^{\infty} u(x, p + \frac{1}{2c}(x-\xi)^2)\, dx, \tag{4.2}$$

as dictated by the measurement procedure. As we can see, this transform differs from the previous c-parabolic Radon transform and will be mathematically simpler to handle.

An alternative form of this integral transform may be given in terms of the two-dimensional Fourier transform $\mathcal{F}u(k,\omega)$ of a function $u(x,t)$

$$\mathcal{F}u(k,\omega) = \int_{\mathbb{R}^2} dx\, dt\ e^{-2i\pi(kx+\omega t)}\ u(x,t). \tag{4.3}$$

Eq. (4.2) becomes now

$$\widehat{u}_c(\xi,p) = \int_{\mathbb{R}^2} e^{2i\pi(k\xi+\omega p)}\ \mathcal{F}u(k,\omega)\, dk\, d\omega \int_{-\infty}^{\infty} \exp 2i\pi(k\zeta + \frac{\omega}{2c}\zeta^2)\, d\zeta. \tag{4.4}$$

4.2 Inversion of the seismic parabolic transform

In this section, we discuss three cases of invertibility. There may be other cases, but they are not known at present. We assume that the function u we want to reconstruct belongs to $\mathcal{S}(\mathbb{R}^2)$ (then $\mathcal{F}u(k,\omega) \in \mathcal{S}(\mathbb{R}^2_{k,\omega})$).

4.2.1 c constant

This simple case can be inverted by using a version of the central slice theorem. Using eq. (2.27), this analog of the central slice theorem takes the form

$$\mathcal{F}u(k,\omega)\ \sqrt{\left|\frac{c}{\omega}\right|}\ e^{i\mathrm{sgn}\left(\frac{c}{\omega}\right)\frac{\pi}{4}} e^{-i\pi\frac{k^2}{\omega}c} = \mathcal{F}(\widehat{u}_c)(k,\omega) \tag{4.5}$$

(we have the following range property $\widehat{u}_c(\xi,\eta) \in L^\infty(\mathbb{R}_\xi, L^1(\mathbb{R}_\eta))$). Thus $u(x,t)$ can be recovered by Fourier inverse as

$$u(x,t) = \int_{\mathbb{R}^2} e^{2i\pi(kx+\omega t)} \sqrt{\left|\frac{\omega}{c}\right|}\, e^{-i\mathrm{sgn}\left(\frac{c}{\omega}\right)\frac{\pi}{4}}\; e^{i\pi\frac{k^2}{\omega}c}\, dk\, d\omega \int_{\mathbb{R}^2} e^{-2i\pi(k\xi+\omega p)} \widehat{u}_c(\xi,p)\, d\xi\, dp. \tag{4.6}$$

Using the Fourier transform of an imaginary gaussian function (see (2.27)), we can explicitly evaluate the k-integration in eq. (4.6)

$$\int_{\mathbb{R}^2} e^{2i\pi(k(x-\xi)+\omega(t-p))} \sqrt{\left|\frac{\omega}{c}\right|}\, e^{-i\mathrm{sgn}\left(\frac{c}{\omega}\right)\frac{\pi}{4}}\; e^{i\pi\frac{k^2}{\omega}c}\, dk\, d\omega = |c| \int_{\mathbb{R}} |\lambda|\; e^{i\pi\,\lambda\, X}\, d\lambda, \tag{4.7}$$

where $X = \big(2(t-p)c - (x-\xi)^2\big)$. The final result, after performing the ω-integration and the p-integration, may be cast as a derivative of a Hilbert transform. Hence a further integration on ξ yields the form

$$u(x,t) = -\frac{1}{2\pi^2|c|} \int_{\mathbb{R}_\xi} \frac{\partial}{\partial t} \left(\text{P.V.} \int_{-\infty}^{+\infty} \frac{\widehat{u}_c(\xi,p)}{p - \big(t - \frac{(x-\xi)^2}{2c}\big)}\, dp \right) d\xi. \tag{4.8}$$

In formula (4.8), the integral over $\mathbb{R}_\xi$ is understood as $\lim_{M\to+\infty} \int_{|\xi|\le M}$. A proof of (4.8) using Radon's method [1] is given in Appendix D.

Here the integrand on the right-hand-side of eq. (4.8) is a function on $\mathbb{R}^2$ of the form

$$F(\xi, t - \frac{1}{2c}\,(x-\xi)^2),$$

which takes a constant value on a parabola in (x,t)-space of the form $t = t_0 + \frac{1}{2c}\,(x-\xi)^2$ with t_0 being given. Hence we may say that the seismic parabolic Radon data $\widehat{u}_c(\xi,p)$ has been back-projected on such (x,t)-space parabola and the inversion formula of eq. (4.6) may be viewed also as a summation image of back-projected data. This was not possible for the c-parabolic Radon transform.

Remark

As in subsection 3.4, we can set $(\xi = r\cos\phi, \eta = r\sin\phi)$ and call $\widehat{u}_c(\xi,p) = \widetilde{u}_c(r,\phi)$ in eq. (4.4). Then performing the r-Fourier transform on both sides, we get

$$\int_{\mathbb{R}} e^{-2i\pi\,\rho\, r}\; \widetilde{u}_c(r,\phi)\, dr = \int_{\mathbb{R}^2} \delta(\rho - k\cos\phi - \omega\sin\phi)\; \mathcal{F}u(k,\omega)\; \sqrt{\left|\frac{c}{\omega}\right|}\; e^{i\mathrm{sgn}\left(\frac{c}{\omega}\right)\frac{\pi}{4}}\; e^{-2i\pi\frac{k^2}{2\omega}c}\, dk\, d\omega. \tag{4.9}$$

The right-hand-side is precisely the standard Radon transform of

$$\mathcal{F}u(k,\omega)\; \sqrt{\left|\frac{c}{\omega}\right|}\; e^{i\mathrm{sgn}\left(\frac{c}{\omega}\right)\frac{\pi}{4}}\, e^{-2i\pi\frac{k^2}{2\omega}c},$$

which can be inverted in a conventional way to obtain $\mathcal{F}u(k,\omega)$ and hence $u(x,t)$ via inverse two-dimensional Fourier transform as before. This connection is similar to the one in the Remark of section 3.4.

In the next two cases of inversion, the focal parameter c of the parabolas is no longer fixed.

4.2.2 $\xi = \pm\xi_0$ constant

Let $\xi_0 > 0$ be a positive constant. First note that $\widehat{u}_c(\pm\xi_0, p) \in L^\infty(\mathbb{R}_c, L^1(\mathbb{R}_p))$, and $\widehat{u}_c(\pm\xi_0, p)$ defines a distribution of $\mathcal{S}'(\mathbb{R}^2_{c,p})$. We perform a Fourier transform on equation (4.4) with respect to p after setting $\xi = \xi_0$

$$\int_{-\infty}^{\infty} e^{-2i\pi\omega' p}\; \widehat{u}_c(\xi_0,p)\, dp = \int_{\mathbb{R}} e^{2i\pi k\xi_0}\; \mathcal{F}u(k,\omega')\; \sqrt{\left|\frac{c}{\omega'}\right|}\; e^{i\mathrm{sgn}\left(\frac{c}{\omega'}\right)\frac{\pi}{4}}\, e^{-2i\pi\frac{k^2}{2\omega'}c}\, dk. \tag{4.10}$$

We apply a 1d-Fourier transform in the variable c to equation (4.10), after shifting the factor $\sqrt{|\frac{c}{\omega'}|}\, e^{i\mathrm{sgn}\left(\frac{c}{\omega'}\right)\frac{\pi}{4}}$ to the left-hand-side. This yields a delta function $\delta(\eta - \frac{k^2}{2\omega})$ on the right-hand-side of equation (4.10)

$$\int_{-\infty}^{\infty} e^{2i\pi c\eta}\, \sqrt{\Big|\frac{\omega}{c}\Big|}\, e^{-i\mathrm{sgn}\left(\frac{c}{\omega}\right)\frac{\pi}{4}}\, dc \int_{-\infty}^{\infty} e^{-2i\pi\omega p}\, \widehat{u}_c(\xi_0, p)\, dp = \int_{-\infty}^{\infty} \mathcal{F}u(k,\omega)\; e^{2i\pi k\xi_0}\; \delta(\eta - \frac{k^2}{2\omega})\, dk. \tag{4.11}$$

Then rewriting the delta function as a function of the integration variable k

$$\delta(\eta - \frac{k^2}{2\omega}) = |\frac{\omega}{k}|\, \Big(\delta(k - \sqrt{2\omega\eta}) + \delta(k + \sqrt{2\omega\eta})\Big) \;\text{ for } \omega\eta > 0, \tag{4.12}$$

we get

$$\int_{-\infty}^{\infty} e^{2i\pi c\eta}\, \sqrt{\Big|\frac{\omega}{c}\Big|}\, e^{-i\mathrm{sgn}\left(\frac{c}{\omega}\right)\frac{\pi}{4}}\, dc \int_{-\infty}^{\infty} e^{-2i\pi\omega p}\, \widehat{u}_c(\xi_0, p)\, dp =$$

$$\sqrt{\frac{\omega}{2\eta}}\; \Big(\mathcal{F}u(\sqrt{2\omega\eta}, \omega)\, e^{2i\pi\xi_0\sqrt{2\omega\eta}} + \mathcal{F}u(-\sqrt{2\omega\eta}, \omega)\, e^{-2i\pi\xi_0\sqrt{2\omega\eta}}\Big)\, \chi(\omega\eta), \tag{4.13}$$

where $\chi(s) = 0$ for $s \leq 0$ and $\chi(s) = 1$ otherwise.

By repeating the same reasoning with $-\xi_0$ instead of ξ_0, we obtain a similar equation, from which one can compute explicitly $\mathcal{F}u(\sqrt{2\omega\eta}, \omega)$ and $\mathcal{F}u(-\sqrt{2\omega\eta}, \omega)$ in terms of $\widehat{u}_c(\xi_0, p)$ and $\widehat{u}_c(-\xi_0, p)$. This gives the necessary data to recover $u(x,t)$ by inverse two-dimensional Fourier transform.

4.2.3 $p = p_0$ constant

Let p_0 be a constant. First note that $\widehat{u}_c(\xi, p_0) \in L^\infty(\mathbb{R}^2_{c,\xi})$, and $\widehat{u}_c(\xi, p_0)$ defines a distribution of $\mathcal{S}'(\mathbb{R}^2_{c,\xi})$. We perform a Fourier transform in ξ on equation (4.2) after setting $p = p_0$ and get

$$\int_{-\infty}^{\infty} e^{-2i\pi k\xi}\; \widehat{u}_c(\xi, p_0)\, d\xi = \int_{\mathbb{R}^2} e^{-2i\pi k(x+y)} u\big(x, p_0 + \frac{y^2}{2c}\big)\, dx\, dy. \tag{4.14}$$

This formula holds pointwise for $(k, c) \in \mathbb{R}^2$, $c \neq 0$ (and both integrals in equality (4.14) are absolutely convergent). Thus for $(k, c) \in \mathbb{R}^2$, $k \neq 0$ and $c > 0$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-2i\pi k\xi}\; \widehat{u}_{\pm c}(\xi, p_0)\, d\xi &= \sqrt{2c} \int_{\mathbb{R}^2} e^{-2i\pi(kx + \sqrt{2c}ky)} u\big(x, p_0 \pm y^2\big)\, dx\, dy \\ &= \sqrt{2c} \int_{\mathbb{R}^2} e^{-2i\pi(kx - \sqrt{2c}ky)} u\big(x, p_0 \pm y^2\big)\, dx\, dy. \end{aligned}$$

Thus for $(k, \eta) \in \mathbb{R}^2$, $k \neq 0$, $\eta \neq 0$, $(c = \eta^2/(2k^2))$ we have

$$\Big|\frac{k}{\eta}\Big| \int_{-\infty}^{\infty} e^{-2i\pi k\xi}\; \widehat{u}_{\pm\frac{\eta^2}{2k^2}}(\xi, p_0)\, d\xi = \int_{\mathbb{R}^2} e^{-2i\pi(kx + \eta y)} u\big(x, p_0 \pm y^2\big)\, dx\, dy. \tag{4.15}$$

Then we can recover the function $u\big(x, p_0 \pm y^2\big)$, where $(x, y) \in \mathbb{R}^2$, by inverse two-dimensional Fourier transform. This proves that the function u can be reconstructed from the data $\widehat{u}_c(\xi, p_0)$ for $(c, \xi) \in \mathbb{R}^2$, $c \neq 0$.

5 Conclusion

This work illustrates the versatile and ingenious concept of Radon transform in I. M. Gel'fand integral geometry. The original "linear" Radon transform has evolved into a major tool in mathematics as well as in transmission and emission imaging science by ionizing radiation. Non-trivial extensions to second order curves in $\mathbb{R}^2$ are made first on circles, which are relevant for radar, sonar and electric impedance imaging, and subsequently on parabolas. In a pioneering work, A. M. Cormack has considered Radon transform on swinging parabolas around their focal point but with variable focal

length and has suggested an implementation with stream of particles scattering a dust medium along parabolas [34]. In this work, we have succeeded in inverting the class of c-parabolic Radon transforms (with fixed axis direction) as well as the class of seismic parabolic Radon transforms. These results open the way to a deepened study of their properties, to extending their working to other function spaces in which questions on range are to be tackled as well as to their generalizations in higher dimensional spaces.

A Proof of properties (3.16), (3.17), (3.18) and (3.19) for the function $\tilde{\mathcal{P}}$

Proof of (3.16)**.** From (3.15) it follows that

$$\begin{aligned}\tilde{\mathcal{P}}(k,\omega) &= \frac{e^{-i\frac{\pi k^2}{\omega}}}{2|\omega|}\int_{k\sqrt{\frac{2}{|\omega|}}}^{\infty}\left(1-\frac{\nu-k\sqrt{\frac{2}{|\omega|}}}{\sqrt{2|\omega|+\left(\nu-k\sqrt{\frac{2}{|\omega|}}\right)^2}}\right)E_{\mathrm{sgn}(\omega)}(\nu)\,d\nu \\ &+\frac{e^{-i\frac{\pi k^2}{\omega}}}{2|\omega|}\int_{-k\sqrt{\frac{2}{|\omega|}}}^{\infty}\left(1-\frac{\nu+k\sqrt{\frac{2}{|\omega|}}}{\sqrt{2|\omega|+\left(\nu+k\sqrt{\frac{2}{|\omega|}}\right)^2}}\right)E_{\mathrm{sgn}(\omega)}(\nu)\,d\nu \\ &+\frac{i}{\pi\omega}+\frac{k\sqrt{2}}{|\omega|^{\frac{3}{2}}}\,e^{-i\pi\frac{k^2}{\omega}}\;E_{\mathrm{sgn}_\omega}\left(k\sqrt{\frac{2}{|\omega|}}\right),\end{aligned} \tag{A.1}$$

for $(k,\omega)\in\mathbb{R}^2\backslash\Delta$. Using (A.1) and the boundedness and smoothness of $E_{\pm}(\eta)$ on $\mathbb{R}_\eta$, one can see that $\tilde{\mathcal{P}}\in C^\infty(\mathbb{R}^2\backslash\Delta)$.

We now prove the nonvanishing property of (3.16). Let $\omega\in\mathbb{R}$, $\omega\neq 0$, be fixed. Since $\tilde{\mathcal{P}}$ is even in the k-variable, we just need to consider the case when $k\geq 0$. From (A.1) it follows that

$$\begin{aligned}\Re\Big[|\omega|e^{i\pi\frac{k^2}{\omega}}\tilde{\mathcal{P}}(k,\omega)\Big] &= \frac{1}{2}\int_{\eta}^{\infty}\left(1-\frac{\nu-\eta}{\sqrt{2|\omega|+(\nu-\eta)^2}}\right)C(\nu)\,d\nu \\ &+\frac{1}{2}\int_{-\eta}^{\infty}\left(1-\frac{\nu+\eta}{\sqrt{2|\omega|+(\nu+\eta)^2}}\right)C(\nu)\,d\nu \\ &-\frac{\sin\left(\pi\frac{\eta^2}{2}\right)}{\pi}+\eta C(\eta),\ \text{where } \eta=k\sqrt{\frac{2}{|\omega|}},\end{aligned} \tag{A.2}$$

for $k\in[0,+\infty)$, where $C(x)=\int_0^x\cos(\frac{\pi\sigma^2}{2})\,d\sigma$, $x\in\mathbb{R}$. We remind that $C(x)$ is odd in x, $C(0)=0$ and $C(x)>0$ for $x>0$, $\sup_{x\in(0,+\infty)}C(x)=C(1)\leq 1$, and $C(x)\to 1/\sqrt{2\pi}$ as $x\to+\infty$. Using the property $C(x)>0$ for $x>0$, we have

$$\int_{\eta}^{\infty}\left(1-\frac{\nu-\eta}{\sqrt{2|\omega|+(\nu-\eta)^2}}\right)C(\nu)d\nu+\int_{0}^{\infty}\left(1-\frac{\nu+\eta}{\sqrt{2|\omega|+(\nu+\eta)^2}}\right)C(\nu)d\nu>0 \text{ for } \eta\geq 0. \tag{A.3}$$

Hence setting $k=0$ in (A.3), we have

$$\Re\Big[|\omega|\tilde{\mathcal{P}}(0,\omega)\Big]=\int_{0}^{\infty}\left(1-\frac{\nu}{\sqrt{2|\omega|+\nu^2}}\right)C(\nu)d\nu>0. \tag{A.4}$$

In addition combining (A.2) and (A.3) we obtain

$$\Re\Big[|\omega|e^{i\pi\frac{k^2}{\omega}}\tilde{\mathcal{P}}(k,\omega)\Big]\geq \mathrm{P}\Big(k\sqrt{\frac{2}{|\omega|}}\Big),\ k>0, \tag{A.5}$$

where $\mathrm{P} \in C^\infty(\mathbb{R})$ is defined by

$$\mathrm{P}(\eta) = \frac{1}{2}\int_{-\eta}^{0} \left(1 - \frac{\nu+\eta}{\sqrt{2|\omega| + (\nu+\eta)^2}}\right) C(\nu)d\nu - \frac{\sin\left(\pi\frac{\eta^2}{2}\right)}{\pi} + \eta C(\eta), \tag{A.6}$$

for $\eta \in \mathbb{R}$. Taking the η-derivative of (A.6), we have

$$\begin{aligned}\frac{d\mathrm{P}}{d\eta}(\eta) &= C(\eta) + \frac{C(-\eta)}{2} - |\omega|\int_{-\eta}^{0} \frac{C(\nu)}{(2|\omega| + (\nu+\eta)^2)^{\frac{3}{2}}}d\nu \\ &= \frac{C(\eta)}{2} + |\omega|\int_{0}^{\eta} \frac{C(\nu)}{(2|\omega| + (-\nu+\eta)^2)^{\frac{3}{2}}}d\nu \\ &> 0 \text{ for } \eta > 0. \end{aligned} \tag{A.7}$$

Hence $\mathrm{P}(\eta) > \mathrm{P}(0) = 0$ for $\eta > 0$. Therefore from (A.5) it follows that

$$\Re\Big[|\omega| e^{i\pi\frac{k^2}{\omega}} \tilde{\mathcal{P}}(k,\omega)\Big] > 0,\ k > 0. \tag{A.8}$$

Then the second part of (3.16) follows from (A.4) and (A.8). ■

Proof of (3.17)**.** We remind that $\sup_{x\in\mathbb{R}} |E_\pm(x)| \le \sqrt{2}\max\big(\sup_{x\in\mathbb{R}}|C(x)|, \sup_{x\in\mathbb{R}}|S(x)|\big) < 2$. Thus

$$(1 - \frac{\nu}{\sqrt{1+\nu^2}})\Big|E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} + k\sqrt{\frac{2}{|\omega|}}\big) + E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} - k\sqrt{\frac{2}{|\omega|}}\big)\Big| \le 2\sqrt{2}(1 - \frac{\nu}{\sqrt{1+\nu^2}}), \tag{A.9}$$

for $(k,\omega) \in \mathbb{R}^2\backslash\Delta$ and for $\nu \in (0,+\infty)$. Note that the right-hand-side of (A.9) is an integrable function of the ν-variable on $(0,+\infty)_\nu$, which does not depend on (k,ω).

Let $k \in \mathbb{R}$, $k \neq 0$, be fixed. At fixed $\nu \in (0,+\infty)$ we have

$$\Big(E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} + k\sqrt{\frac{2}{|\omega|}}\big) + E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} - k\sqrt{\frac{2}{|\omega|}}\big)\Big) = o(1),\ \text{as } \omega \to 0, \omega \neq 0 \tag{A.10}$$

for $k \in \mathbb{R}$, where we used that $\lim_{\eta\to+\infty} E_\pm(\varepsilon\eta) = \varepsilon\frac{e^{\pm i\frac{\pi}{4}}}{\sqrt{2}}$ for $\varepsilon \in \{-1,1\}$, and $\lim_{\eta\to+\infty} E_\pm(\eta) + \lim_{\eta\to+\infty} E_\pm(-\eta)) = 0$. Then using (A.9) and Lebesgue dominated convergence theorem, we obtain that

$$\begin{aligned}&\frac{e^{-i\frac{k^2}{\omega}}}{\sqrt{2|\omega|}}\int_0^{+\infty}(1 - \frac{\nu}{\sqrt{1+\nu^2}})\Big(E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} + k\sqrt{\frac{2}{|\omega|}}\big) + E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} - k\sqrt{\frac{2}{|\omega|}}\big)\Big)d\nu \\ =\ & o(|\omega|^{-\frac{1}{2}}),\ \text{as } \omega \to \pm\infty. \end{aligned} \tag{A.11}$$

We use the fact that

$$\Big|E_\pm(\eta) - \frac{e^{\pm i\frac{\pi}{4}}}{\sqrt{2}} - (\pm i)\frac{e^{\pm i\frac{\pi}{2}\eta^2}}{\pi\eta}\Big| \le \frac{2}{\pi^2\eta^3}, \tag{A.12}$$

for $\eta > 0$ (this estimate is obtained by integration by parts). And we obtain

$$\begin{aligned}E_{\mathrm{sgn}(\omega)}\Big(k\sqrt{\frac{2}{|\omega|}}\Big) &= \mathrm{sgn}(k)E_{\mathrm{sgn}(\omega)}\Big(|k|\sqrt{\frac{2}{|\omega|}}\Big) \\ &= \mathrm{sgn}(k)\left[\frac{e^{i\mathrm{sgn}(\omega)\frac{\pi}{4}}}{\sqrt{2}} + (\mathrm{sgn}(\omega)i)|\omega|^{\frac{1}{2}}\frac{e^{i\frac{\pi k^2}{\omega}}}{\sqrt{2}\pi|k|}\right] + O(|\omega|^{\frac{3}{2}}), \end{aligned} \tag{A.13}$$

as $\omega \to 0$, $\omega \neq 0$. Hence

$$\frac{k\sqrt{2}}{|\omega|^{\frac{3}{2}}}e^{-i\pi\frac{k^2}{\omega}}E_{\mathrm{sgn}(\omega)}\Big(k\sqrt{\frac{2}{|\omega|}}\Big) = |k|e^{i\mathrm{sgn}(\omega)\frac{\pi}{4} - i\pi\frac{k^2}{\omega}}|\omega|^{-\frac{3}{2}} + \frac{i}{\pi}\omega^{-1} + O(1), \tag{A.14}$$

as $\omega \to 0$, $\omega \neq 0$. Then (3.17) for $k \neq 0$ follows from (3.15), (A.11) and (A.13).

Consider the case $k = 0$. Then from (3.15) it follows that

$$\tilde{\mathcal{P}}(0,\omega) = \sqrt{\frac{2}{|\omega|}} \int_0^\infty (1 - \frac{\nu}{\sqrt{1+\nu^2}}) E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|}\big) d\nu + \frac{i}{\pi\omega}, \tag{A.15}$$

for $\omega \neq 0$. Then note that at fixed $\nu \in (0, +\infty)$

$$E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|}\big) = E_{\mathrm{sgn}(\omega)}(0) + o(1) = o(1), \text{ as } \omega \to 0, \omega \neq 0. \tag{A.16}$$

Thus using (A.9) and Lebesgue dominated convergence theorem, we have

$$\int_0^\infty (1 - \frac{\nu}{\sqrt{1+\nu^2}}) E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|}\big) d\nu = o(1), \text{ as } \omega \to 0, \omega \neq 0. \tag{A.17}$$

Therefore combining (A.15) and (A.17) we obtain

$$\tilde{\mathcal{P}}(0,\omega) = \frac{i}{\pi\omega} + o(1), \text{ as } \omega \to 0, \omega \neq 0. \tag{A.18}$$

This implies (3.17) for $k = 0$. ■

Proof of (3.18)**.** At fixed $\nu \in (0, +\infty)$ we have

$$\Big(E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} + k\sqrt{\frac{2}{|\omega|}}\big) + E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} - k\sqrt{\frac{2}{|\omega|}}\big)\Big) = \sqrt{2} e^{i\mathrm{sgn}(\omega)\frac{\pi}{4}} + o(1), \text{ as } \omega \to \pm\infty, \tag{A.19}$$

for $k \in \mathbb{R}$, where we used that $\lim_{\eta\to+\infty} E_\pm(\eta) = \frac{e^{\pm i\frac{\pi}{4}}}{\sqrt{2}}$. Then using (A.9) and the Lebesgue dominated convergence theorem, we obtain that at fixed $k \in \mathbb{R}$

$$\begin{aligned} &\frac{e^{-i\frac{k^2}{\omega}}}{\sqrt{2|\omega|}} \int_0^{+\infty} (1 - \frac{\nu}{\sqrt{1+\nu^2}}) \Big(E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} + k\sqrt{\frac{2}{|\omega|}}\big) + E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} - k\sqrt{\frac{2}{|\omega|}}\big)\Big) d\nu \\ = \quad & e^{i\mathrm{sgn}(\omega)\frac{\pi}{4} - i\frac{k^2}{\omega}} \sqrt{\frac{1}{|\omega|}} \int_0^{+\infty} (1 - \frac{\nu}{\sqrt{1+\nu^2}}) d\nu + o(|\omega|^{-\frac{1}{2}}), \text{ as } \omega \to \pm\infty. \end{aligned} \tag{A.20}$$

Note also that

$$\Big| \frac{i}{\pi\omega} + \frac{k\sqrt{2}}{|\omega|^{\frac{3}{2}}} e^{-i\pi\frac{k^2}{\omega}} \, E_{\mathrm{sgn}_\omega}\left(k\sqrt{\frac{2}{|\omega|}}\right) \Big| \leq \frac{1}{\pi|\omega|} + \frac{2|k|}{|\omega|^{\frac{3}{2}}}, \tag{A.21}$$

for $(k,\omega) \in \mathbb{R}^2 \backslash \Delta$. Then (3.18) follows from (3.15), (A.20) and (A.21) and the integral value $\int_{-\infty}^{\infty} \Big(1 - \frac{\nu}{\sqrt{1+\nu^2}}\Big) d\nu = 1$. ■

Proof of (3.19)**.** From (A.9) it follows that

$$\frac{1}{\sqrt{2|\omega|}} \int_0^\infty (1 - \frac{\nu}{\sqrt{1+\nu^2}}) \Big(E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} + k\sqrt{\frac{2}{|\omega|}}\big) + E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} - k\sqrt{\frac{2}{|\omega|}}\big)\Big) d\nu \leq \frac{2}{\sqrt{|\omega|}}, \tag{A.22}$$

for $(k,\omega) \in \mathbb{R}^2 \backslash \Delta$, and where we use the value of the integral $\int_{-\infty}^{\infty} \Big(1 - \frac{\nu}{\sqrt{1+\nu^2}}\Big) d\nu = 1$. Combining (3.15), (A.22) and (A.21) we obtain (3.19). ■

B The initial and boundary values $\mathcal{P}(k,0)$ and $\mathcal{P}(0,\omega)$

Let $\psi \in C_0^\infty(\mathbb{R})$ be such that $\int_{\mathbb{R}} \psi(x)dx = 1$, where $C_0^\infty(\mathbb{R})$ denotes the set of infinitely smooth and compactly supported functions on $\mathbb{R}$. For $\varepsilon \in (0,+\infty)$ we denote by ψ_ε the function of $C_0^\infty(\mathbb{R})$ defined by $\psi_\varepsilon(\sigma) = \varepsilon^{-1}\psi(\varepsilon^{-1}\sigma)$, $\sigma \in \mathbb{R}$.

Proof of (3.22)**.** Let $\phi \in \mathcal{S}(\mathbb{R})$. From (2.24) it follows that

$$< \mathcal{P}, \phi(k)\psi_\varepsilon(\omega) >= \int_{\mathbb{R}} \sqrt{1+\nu^2}\big(\mathcal{F}_1^{-1}\phi\big)(\nu)\Big(\int_{-\infty}^{+\infty} e^{i\pi\omega\nu^2}\psi_\varepsilon(\omega)d\omega\Big)d\nu, \tag{B.1}$$

where $\mathcal{F}_1^{-1}\phi$ denotes the one-dimensional inverse Fourier transform of the function ϕ, i.e. $\mathcal{F}_1^{-1}\phi(\nu) = \int_{-\infty}^{+\infty} e^{2i\pi k\nu}\phi(k)dk$, $\nu \in \mathbb{R}$. We remind that $\mathcal{F}_1^{-1}\phi \in \mathcal{S}(\mathbb{R})$ and that $\int_{-\infty}^{+\infty}\psi_\varepsilon(\omega)f(\omega)d\omega \to f(0)$ as $\varepsilon \to 0^+$ for any locally integrable function f on $\mathbb{R}$ such that f is continuous at 0. Thus we have

$$\lim_{\varepsilon\to 0^+}\int_{-\infty}^{+\infty} e^{i\pi\omega\nu^2}\psi_\varepsilon(\omega)d\omega = 1, \tag{B.2}$$

and by Lebesgue dominated convergence theorem

$$\lim_{\varepsilon\to 0^+} < \mathcal{P}, \phi(k)\psi_\varepsilon(\omega) >= \int_{\mathbb{R}} \sqrt{1+\nu^2}\Big(\mathcal{F}_1^{-1}\phi\Big)(\nu)d\nu =< u_1+u_2, \phi >, \tag{B.3}$$

where

$$< u_1, \phi >= \int_{\mathbb{R}} \big(\sqrt{1+\nu^2}-|\nu|\big)\Big(\mathcal{F}_1^{-1}\phi\Big)(\nu)d\nu, \quad < u_2, \phi >= \int_{\mathbb{R}} |\nu|\Big(\mathcal{F}_1^{-1}\phi\Big)(\nu)d\nu. \tag{B.4}$$

The study of u_1 is similar to the study of $\mathcal{P}_{reg}$. First we note that

$$< u_1, \phi >= 2\int_0^{+\infty} \big(\sqrt{1+\nu^2}-\nu\big)\left(\int_{-\infty}^{+\infty}\cos(2\pi k\nu)\phi(k)dk\right)d\nu. \tag{B.5}$$

Then integrating by parts on the ν-variable we have

$$< u_1, \phi >= \int_0^{+\infty} \big(1-\frac{\nu}{\sqrt{1+\nu^2}}\big)\left(\int_{-\infty}^{+\infty}\frac{\sin(2\pi k\nu)}{\pi k}\phi(k)dk\right)d\nu. \tag{B.6}$$

Note that

$$\Big|\frac{\sin(2\pi k\nu)}{\pi k}\Big| \le \sqrt{\Big|\frac{\sin(2\pi k\nu)}{2\pi k\nu}\Big|}\times\sqrt{2}\times\sqrt{|\sin(2\pi k\nu)|}\times(\pi|k\nu|)^{-\frac{1}{2}} \le \sqrt{2}(\pi|k\nu|)^{-\frac{1}{2}}, \text{ for } (k,\nu)\in(\mathbb{R}\backslash\{0\})^2,$$

and

$$\Big|\big(1-\frac{\nu}{\sqrt{1+\nu^2}}\big)\frac{\sin(2\pi k\nu)}{\pi k}\Big| \le \sqrt{\frac{2}{\pi}}|k|^{-\frac{1}{2}}\big(1-\frac{\nu}{\sqrt{1+\nu^2}}\big)|\nu|^{-\frac{1}{2}}, \text{ for } (k,\nu)\in(\mathbb{R}\backslash\{0\})^2 \tag{B.7}$$

Hence the function $\big(1-\frac{\nu}{\sqrt{1+\nu^2}}\big)\frac{\sin(2\pi k\nu)}{\pi k}$ is absolutely integrable on $\mathbb{R}^2$ and from (B.6) it follows that u_1 is the Lebesgue measurable function given by

$$u_1(k) = \int_0^{+\infty}\big(1-\frac{\nu}{\sqrt{1+\nu^2}}\big)\frac{\sin(2\pi k\nu)}{\pi k}d\nu, \text{ for } k\in\mathbb{R}\backslash\{0\}. \tag{B.8}$$

We now study u_2. From (B.4) and the equality $\nu\Big(\mathcal{F}_1^{-1}\phi\Big)(\nu) = \frac{1}{2i\pi}\Big(\mathcal{F}_1^{-1}\phi'\Big)(\nu)$ for $\nu\in\mathbb{R}$ where $\phi'(k) = \frac{d}{dk}\phi(k)$, $k\in\mathbb{R}$, we obtain

$$\begin{aligned} < u_2, \phi > &= \int_0^{+\infty}\nu\Big(\Big(\mathcal{F}_1^{-1}\phi\Big)(\nu)+\Big(\mathcal{F}_1^{-1}\phi\Big)(-\nu)\Big)d\nu = \frac{1}{2i\pi}\int_0^{+\infty}\Big(\Big(\mathcal{F}_1^{-1}\phi'\Big)(\nu)-\Big(\mathcal{F}_1^{-1}\phi'\Big)(-\nu)\Big)d\nu \\ &= \frac{1}{\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty}\sin(2\pi k\nu)\phi'(k)dkd\nu. \end{aligned} \tag{B.9}$$

Then we have

$$\begin{aligned} < u_2, \phi > &= \frac{1}{\pi} \lim_{M\to+\infty} \int_{-\infty}^{+\infty} \int_0^M \sin(2\pi k\nu)\phi'(k)dk = \frac{1}{\pi} \lim_{M\to+\infty} \int_{-\infty}^{+\infty} \frac{1-\cos(2\pi kM)}{2\pi k}\phi'(k)dk \\ &= \frac{1}{2\pi^2} \lim_{M\to+\infty} \int_0^{+\infty} (1-\cos(2\pi kM))\frac{\phi'(k)-\phi'(-k)}{k}dk. \end{aligned} \tag{B.10}$$

As $\phi \in \mathcal{S}(\mathbb{R})$ we have $\frac{\phi'(k)-\phi'(-k)}{k} \in \mathcal{S}(\mathbb{R})$ and by the Lebesgue-Riemann Lemma we have

$$\lim_{M\to+\infty} \int_0^{+\infty} \cos(2\pi kM)\frac{\phi'(k)-\phi'(-k)}{k}dk = 0.$$

Therefore

$$< u_2, \phi >= \frac{1}{2\pi^2} \int_0^{+\infty} \frac{\phi'(k)-\phi'(-k)}{k}dk. \tag{B.11}$$

Combining (B.3), (B.8) and (B.11) we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0^+} < \mathcal{P}, \phi(k)\psi_\varepsilon(\omega) > &= \int_{-\infty}^{+\infty} \phi(k) \left(\int_0^{+\infty} \big(1 - \frac{\nu}{\sqrt{1+\nu^2}}\big) \frac{\sin(2\pi k\nu)}{\pi k} d\nu \right) dk \\ &\quad + \frac{1}{2\pi^2} \int_0^{+\infty} \frac{\phi'(k)-\phi'(-k)}{k}dk. \end{aligned} \tag{B.12}$$

The limit (3.22) is proven. ■

Proof of (3.25)**.** Let $\phi \in \mathcal{S}(\mathbb{R})$. From (3.8), (3.10), and with $\int_{-\infty}^{+\infty} \psi_\varepsilon(\sigma)d\sigma = 1$, it follows that

$$< \mathcal{P}_{reg}(k,\omega), \psi_\varepsilon(k)\phi(\omega) >= \int_{(\nu,\omega)\in(0,+\infty)\times\mathbb{R}} \phi(\omega)(1 - \frac{\nu}{\sqrt{1+\nu^2}})\Big(\int_{-\infty}^{+\infty} \psi_\varepsilon(k)\Phi_1(\nu,k,\omega)dk\Big)d\nu d\omega, \tag{B.13}$$

and

$$\begin{aligned} < \mathcal{Q}, \psi_\varepsilon(k)\phi(\omega) > &= \frac{i}{\pi} \int_0^{+\infty} \frac{\phi(\omega)-\phi(-\omega)}{\omega}d\omega + \phi(0) \\ &\quad + \int_{-\infty}^{+\infty} \frac{\phi(\omega)-\phi(0)}{|\omega|^{\frac{3}{2}}} \left(\int_{-\infty}^{+\infty} \psi_\varepsilon(k)\Phi_2(k,\omega)dk \right) d\omega, \end{aligned} \tag{B.14}$$

where

$$\Phi_1(\nu,k,\omega) := e^{-i\frac{\pi k^2}{\omega}} \sqrt{\frac{1}{2|\omega|}} \Big(E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} + k\sqrt{\frac{2}{|\omega|}}\big) + E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|} - k\sqrt{\frac{2}{|\omega|}}\big)\Big), \tag{B.15}$$

for $(\nu,k,\omega) \in (0,+\infty)\times\mathbb{R}^2$, $\omega \neq 0$, and

$$\Phi_2(k,\omega) := k\sqrt{2}e^{-i\pi\frac{k^2}{\omega}} E_{\mathrm{sgn}\omega}\left(k\sqrt{\frac{2}{|\omega|}}\right), \tag{B.16}$$

for $(k,\omega) \in \mathbb{R}^2$, $\omega \neq 0$. Then we remind that $\int_{-\infty}^{+\infty} \psi_\varepsilon(k)f(k)dk \to f(0)$ as $\varepsilon \to 0^+$ for any locally integrable function f on $\mathbb{R}$ such that f is continuous at 0. Here at fixed $(\nu,\omega) \in (0,+\infty)\times\mathbb{R}$, $\omega \neq 0$, the functions $\Phi_1(\nu,.,\omega)$ and $\Phi_2(.,\omega)$ are continuous on $\mathbb{R}$, thus we have

$$\lim_{\varepsilon\to 0^+} \int_{-\infty}^{+\infty} \psi_\varepsilon(k)\Phi_1(\nu,k,\omega)dk = \Phi_1(\nu,0,\omega) = \sqrt{\frac{2}{|\omega|}} E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|}\big), \tag{B.17}$$

$$\lim_{\varepsilon\to 0^+} \int_{-\infty}^{+\infty} \psi_\varepsilon(k)\Phi_2(k,\omega)dk = \Phi_2(0,\omega) = 0. \tag{B.18}$$

Hence using Lebesgue dominated convergence Theorem we obtain that

$$\lim_{\varepsilon\to 0^+} < \mathcal{P}_{reg}, \psi_\varepsilon(k)\phi(\omega) > = \int_{-\infty}^{+\infty} \sqrt{\frac{2}{|\omega|}}\phi(\omega) \int_0^{+\infty} (1 - \frac{\nu}{\sqrt{1+\nu^2}}) E_{\mathrm{sgn}(\omega)}\big(\nu\sqrt{2|\omega|}\big) d\nu d\omega, \quad \text{(B.19)}$$

$$\lim_{\varepsilon\to 0^+} < \mathcal{Q}, \psi_\varepsilon(k)\phi(\omega) > = \frac{i}{\pi}\int_0^{+\infty} \frac{\phi(\omega)-\phi(-\omega)}{\omega} d\omega + \phi(0). \quad \text{(B.20)}$$

Combining (3.2) , (B.19) and (B.20) we obtain (3.25). ■

C The Cauchy initial value problem for equation (3.21)

We first make some comments on $\mathcal{W}$. First $\mathcal{W} \neq \mathcal{S}'(\mathbb{R}^2)$. Indeed we give the following two examples of tempered distributions which do not belong to $\mathcal{W}$.

Example 1: The distribution $\delta(k,\omega)$ does not belong to $\mathcal{W}$. Indeed for $\psi \in C_0^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \psi(x)dx = 1$, and for $\phi \in \mathcal{S}(\mathbb{R})$ we have

$$< \delta(k,\omega), \varepsilon^{-1}\psi(\varepsilon^{-1}\omega)\phi(k) >= \varepsilon^{-1}\phi(0)\psi(0) \to \begin{cases} \pm\infty, \text{ when } \pm\,\phi(0)\psi(0) > 0, \\ 0, \text{ otherwise,} \end{cases} \quad \text{as } \varepsilon \to 0^+.$$

Example 2: The distribution $\mathrm{sgn}(k)\mathrm{sgn}(\omega)$ does not belong to $\mathcal{W}$. Indeed for $\psi \in C_0^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \psi(x)dx = 1$, and for $\phi \in \mathcal{S}(\mathbb{R})$ we have

$$< \mathrm{sgn}(k)\mathrm{sgn}(\omega), \varepsilon^{-1}\psi(\varepsilon^{-1}\omega)\phi(k) >= \int_{\mathbb{R}} \mathrm{sgn}(k)\phi(k)dk \int_0^{+\infty} (\psi(\omega) - \psi(-\omega))d\omega.$$

Therefore we have $< \mathrm{sgn}(k)\mathrm{sgn}(\omega), \varepsilon^{-1}\psi(\varepsilon^{-1}\omega)\cdot >$ converges in $\mathcal{S}'(\mathbb{R})$ to the distribution $\mathrm{sgn}(k)\int_0^{+\infty}(\psi(\omega)-\psi(-\omega))d\omega$. As the value of $\int_0^{+\infty}(\psi(\omega)-\psi(-\omega))d\omega$ varies with ψ, we conclude that $\mathrm{sgn}(k)\mathrm{sgn}(\omega) \notin \mathcal{W}$.

Let $\mathcal{F}_1$ denote the one-dimensional Fourier transform with respect to the k-variable:

$$< \mathcal{F}_1 L, \phi >:=< L, \mathcal{F}_1\phi >,$$

for $(L,\phi) \in \mathcal{S}(\mathbb{R}^2)' \times \mathcal{S}(\mathbb{R}^2)$ where $\mathcal{F}_1\phi(k,\omega) = \int_{-\infty}^{+\infty} e^{-i2\pi kx}\phi(x,\omega)dx$, for $(k,\omega) \in \mathbb{R}^2$.

It is straightforward from the definition of $\mathcal{W}$ that $\mathcal{F}_1$ preserves $\mathcal{W}$, i.e. $\mathcal{F}_1\mathcal{W} \subset \mathcal{W}$. Since $\mathcal{F}_1^{-1}\phi(k,\omega) = \mathcal{F}_1\phi(-k,\omega)$ for $\phi \in \mathcal{S}(\mathbb{R}^2)$ and $(k,\omega) \in \mathbb{R}^2$ we have

$$\mathcal{F}_1\mathcal{W} = \mathcal{W}.$$

We now prove the uniqueness and existence of a solution for (C.1) with an arbitrary initial condition $u \in \mathcal{S}'(\mathbb{R})$

$$\frac{\partial}{\partial\omega}L(k,\omega) = \frac{1}{4i\pi}\frac{\partial^2}{\partial k^2}L(k,\omega), \text{ in } \mathcal{W}, \quad \text{(C.1)}$$

$$L(k,0) = u. \quad \text{(C.2)}$$

Let $u \in \mathcal{S}'(\mathbb{R})$.

Uniqueness. Assume that $L \in \mathcal{W}$ solves (C.1)–(C.2). Then $\mathcal{F}_1 L$ satisfies

$$\frac{\partial}{\partial\omega}\Big(e^{-i\pi k^2\omega}\mathcal{F}_1 L(k,\omega)\Big) = 0, \text{ in } \mathcal{W}, \quad \text{(C.3)}$$

$$\mathcal{F}_1 L(k,0) = \mathcal{F}_1 u, \quad \text{(C.4)}$$

where we also denote by $\mathcal{F}_1$ the fourier transform on $\mathcal{S}'(\mathbb{R})$. The solutions of the equation $\frac{\partial}{\partial\omega}\tilde{L} = 0$, $\tilde{L} \in \mathcal{S}'(\mathbb{R}^2)$ are well known. They are the distributions $\tilde{L}$ such that there exists $l \in \mathcal{S}(\mathbb{R})'$,

$< \tilde{L}, \phi(k,\omega) >=< l, \int_{-\infty}^{+\infty} \phi(k,\omega) d\omega >$ for $\phi \in \mathcal{S}(\mathbb{R}^2)$. Hence from (C.3) and (C.4) it follows that $\mathcal{F}_1 \mathcal{L} = e^{i\pi k^2 \omega} \mathcal{F}_1 u$, and then

$$\begin{aligned} < L, \phi > &= < L, \mathcal{F}_1 \mathcal{F}_1^{-1} \phi >=< e^{i\pi k^2 \omega} \mathcal{F}_1 u, \mathcal{F}_1^{-1} \phi > \\ &= < u, \int_{-\infty}^{+\infty} \mathcal{F}_1 \Big[e^{i\pi \cdot^2 \omega} \big(\mathcal{F}_1^{-1} \phi \big)(\cdot, \omega) \Big](k) d\omega >, \text{ for } \phi \in \mathcal{S}(\mathbb{R}^2). \end{aligned} \tag{C.5}$$

Thus we have proven that there exists a unique solution for (C.3)–(C.4). ■

Existence. The distribution $L \in \mathcal{S}(\mathbb{R}^2)'$ defined by (C.5) is the solution of the initial value problem (C.3)–(C.4). ■

D Proof of formula (4.8)

Let $c \neq 0$ and let $u \in \mathcal{S}(\mathbb{R}^2)$. Let $(x,t) \in \mathbb{R}^2$.

Let $M > 0$ and $\varepsilon > 0$, $\varepsilon < M$, $M > |x|$. Then consider the real number $I(\varepsilon, M)$ defined by

$$I(\varepsilon, M) := \int_{-M}^{M} \int_{|p|>\varepsilon} \frac{\frac{\partial \hat{u_c}}{\partial t}(\xi, t - \frac{(x-\xi)^2}{2c} + p)}{p} dp\, d\xi. \tag{D.1}$$

From (4.2) it follows that

$$I(\varepsilon, M) = \int_{-\infty}^{+\infty} \int_{-M}^{M} \int_{|p|>\varepsilon} \frac{\frac{\partial u}{\partial t}\big(x + s + \xi, t + p - \frac{(x-\xi)^2}{2c} + \frac{(x+s)^2}{2c}\big)}{p}\, dp\, d\xi\, ds. \tag{D.2}$$

Note that the integral over (ξ, p, s) is absolutely convergent since we assume that $u \in \mathcal{S}(\mathbb{R}^2)$. Performing the change of variables $(y_1, y_2) = (s + \xi, p - \frac{(x-\xi)^2}{2c} + \frac{(x+s)^2}{2c})$ $(d\xi\, dp = dy_1\, dy_2)$ we obtain

$$I(\varepsilon, M) = \int_{\mathbb{R}^2} \frac{\partial u}{\partial t}(x + y_1, t + y_2)\ J(\varepsilon, M, y_1, y_2)\, dy_1\, dy_2, \tag{D.3}$$

where

$$J(\varepsilon, M, y_1, y_2) := \int_{s \in E(\varepsilon, M, y_1, y_2)} \frac{ds}{y_2 - \frac{(x+s)^2}{2c} + \frac{(x-y_1+s)^2}{2c}}, \tag{D.4}$$

for $(y_1, y_2) \in \mathbb{R}^2$ where $E(\varepsilon, M, y_1, y_2) := \Big\{ s \in \mathbb{R} \mid |s - y_1| \leq M \text{ and } |y_2 - \frac{(x+s)^2}{2c} + \frac{(x-y_1+s)^2}{2c}| > \varepsilon \Big\}$. We split the right-hand-side of (D.3) into a sum of the following two terms

$$I(\varepsilon, M) = (I_1 + I_2)(\varepsilon, M), \tag{D.5}$$

where

$$I_1(\varepsilon, M) = \int_{\substack{(y_1,y_2)\in\mathbb{R}^2 \\ c y_1 \geq 0}} \frac{\partial u}{\partial t}(x + y_1, t + y_2)\ J(\varepsilon, M, y_1, y_2)\, dy_1\, dy_2, \tag{D.6}$$

$$I_2(\varepsilon, M) = \int_{\substack{(y_1,y_2)\in\mathbb{R}^2 \\ c y_1 \leq 0}} \frac{\partial u}{\partial t}(x + y_1, t + y_2)\ J(\varepsilon, M, y_1, y_2)\, dy_1\, dy_2. \tag{D.7}$$

Consider first $I_1(\varepsilon, M)$, $I_2(\varepsilon, M)$ can be studied in a similar way.

Let $(y_1, y_2) \in \mathbb{R}^2$, $c\, y_1 > 0$. Then performing the change of variables "s"$= s + x$ on the right-hand-side of (D.4) we obtain

$$J(\varepsilon, M, y_1, y_2) = \int_{s \in \tilde{E}(\varepsilon, M, y_1, y_2)} \frac{ds}{\frac{y_1^2}{2c} + y_2 - \frac{y_1}{c} s}, \tag{D.8}$$

where $\tilde{E}(\varepsilon, M, y_1, y_2) := \Big\{ s \in \mathbb{R} \mid y_1 - M + x < s < y_1 + M + x \text{ and } \big|s - \frac{y_1}{2} - \frac{cy_2}{y_1}\big| \geq \frac{c\varepsilon}{y_1} \Big\}$. Thus

$$\begin{aligned} J(\varepsilon, M, y_1, y_2) &= -\frac{c}{y_1} \int_{s \in \tilde{E}(\varepsilon, M, y_1, y_2)} \frac{d}{ds} \ln\Big(\big|\frac{y_1^2}{2c} + y_2 - \frac{y_1}{c} s\big|\Big) ds, \\ &= -\frac{c}{y_1} \left[\ln\left(\left| \frac{\min\Big(-\frac{y_1^2}{2c} + y_2 + \frac{y_1}{c}(M-x), \max\big(\varepsilon, -\frac{y_1^2}{2c} + y_2 - \frac{y_1}{c}(M+x)\big)\Big)}{-\frac{y_1^2}{2c} + y_2 + \frac{y_1}{c}(M-x)} \right| \right) \right. \\ &\quad \left. + \ln\left(\left| \frac{\frac{y_1^2}{2c} - y_2 + \frac{y_1}{c}(M+x)}{\min\Big(\frac{y_1^2}{2c} - y_2 + \frac{y_1}{c}(M+x), \max\big(\varepsilon, \frac{y_1^2}{2c} - y_2 - \frac{y_1}{c}(M-x)\big)\Big)} \right| \right) \right] \end{aligned} \tag{D.9}$$

Therefore from (D.6) it follows that

$$I_1(\varepsilon, M) = I_{1,1}(M) + I_{1,2}(\varepsilon, M), \tag{D.10}$$

where

$$I_{1,1}(M) := \int_{\substack{(y_1,y_2)\in\mathbb{R}^2 \\ cy_1 \geq 0}} \frac{c}{y_1} \frac{\partial u}{\partial t}(x + y_1, t + y_2) \ \ln\left(\left| \frac{\frac{y_1^2}{2c} - y_2 + \frac{y_1}{c}(-M+x)}{\frac{y_1^2}{2c} - y_2 + \frac{y_1}{c}(M+x)} \right| \right) dy_1\, dy_2, \tag{D.11}$$

$$\begin{aligned} I_{1,2}(\varepsilon, M) &:= -\int_{\substack{(y_1,y_2)\in\mathbb{R}^2 \\ cy_1 \geq 0}} \frac{c}{y_1} \frac{\partial u}{\partial t}(x + y_1, t + y_2) \\ &\quad \times \ln\left(\left| \frac{\min\Big(-\frac{y_1^2}{2c} + y_2 + \frac{y_1}{c}(M-x), \max\big(\varepsilon, -\frac{y_1^2}{2c} + y_2 - \frac{y_1}{c}(M+x)\big)\Big)}{\min\Big(\frac{y_1^2}{2c} - y_2 + \frac{y_1}{c}(M+x), \max\big(\varepsilon, \frac{y_1^2}{2c} - y_2 - \frac{y_1}{c}(M-x)\big)\Big)} \right| \right) dy_1\, dy_2. \end{aligned} \tag{D.12}$$

The aim is to prove that

$$\lim_{M\to+\infty} I_{1,1}(M) = -|c|\pi^2 u(x,t), \tag{D.13}$$

$$\lim_{\varepsilon\to 0^+} I_{1,2}(\varepsilon, M) = 0. \tag{D.14}$$

Then combining (D.13), (D.14) and (D.10) we obtain $\lim_{M\to+\infty} \lim_{\varepsilon\to 0^+} I_1(\varepsilon, M) = -|c|\pi^2 u(x,t)$. Similarly we have $\lim_{M\to+\infty} \lim_{\varepsilon\to 0^+} I_2(\varepsilon, M) = -|c|\pi^2 u(x,t)$, and using (D.5) and (D.1) we finally obtain

$$\lim_{M\to+\infty} \lim_{\varepsilon\to 0^+} \int_{-M}^{M} \int_{|p|>\varepsilon} \frac{\frac{\partial \hat{u_c}}{\partial t}(\xi, t - \frac{(x-\xi)^2}{2c} + p)}{p} dp\, d\xi = -2|c|\pi^2 u(x,t), \tag{D.15}$$

which proves formula (4.8) for $u \in \mathcal{S}(\mathbb{R}^2)$.

We now prove first (D.13) and then (D.14). Performing the change of variables "y_2"$= \frac{1}{M}\big(\frac{cy_2}{y_1} - \frac{y_1}{2} - x\big)$ on the right-hand-side of (D.11) we obtain

$$I_{1,1}(M) = M \int_{\substack{(y_1,y_2)\in\mathbb{R}^2 \\ cy_1 \geq 0}} \frac{\partial u}{\partial t}\big(x + y_1, t + \frac{y_1 x}{c} + \frac{y_1^2}{2c} + \frac{M y_1 y_2}{c}\big) \ \ln\left(\left| \frac{1+y_2}{1-y_2} \right| \right) dy_1\, dy_2. \tag{D.16}$$

Then performing the change of variables "y_1"$= My_1y_2$ we obtain

$$I_{1,1}(M) = \int_{\substack{(y_1,y_2)\in\mathbb{R}^2 \\ cy_1y_2 \geq 0}} \frac{\partial u}{\partial t}\big(x + \frac{y_1}{My_2}, t + \frac{y_1 x}{cMy_2} + \frac{y_1^2}{2cM^2y_2^2} + \frac{y_1}{c}\big) \ \frac{1}{|y_2|} \ln\left(\left| \frac{1+y_2}{1-y_2} \right| \right) dy_1\, dy_2. \tag{D.17}$$

Note that $\int_{\mathbb{R}} |y_2|^{-1} \left| \ln\left(\left| \frac{1+y_2}{1-y_2} \right| \right) \right| dy_2 < \infty$. Thus from the Lebesgue dominated convergence theorem

we have

$$\begin{aligned}
&\lim_{M\to+\infty} I_{1,1}(M) = \int_{\substack{(y_1,y_2)\in\mathbb{R}^2\\ cy_1y_2\geq 0}} \frac{\partial u}{\partial t}\big(x,t+\frac{y_1}{c}\big)\ \frac{1}{|y_2|}\ln\left(\left|\frac{1+y_2}{1-y_2}\right|\right)\,dy_1\,dy_2\\
=\ &\int_{\substack{y_1\in\mathbb{R}\\ cy_1\geq 0}} \frac{\partial u}{\partial t}\big(x,t+\frac{y_1}{c}\big)\,dy_1\ \int_0^{+\infty}\frac{1}{y_2}\ln\left(\left|\frac{1+y_2}{1-y_2}\right|\right)\,dy_2\\
&+\int_{\substack{y_1\in\mathbb{R}\\ cy_1\leq 0}} \frac{\partial u}{\partial t}\big(x,t+\frac{y_1}{c}\big)\,dy_1\int_{-\infty}^{0}\frac{1}{|y_2|}\ln\left(\left|\frac{1+y_2}{1-y_2}\right|\right)\,dy_2.\\
=\ &\Big(\int_{\substack{y_1\in\mathbb{R}\\ cy_1\geq 0}} \frac{\partial u}{\partial t}\big(x,t+\frac{y_1}{c}\big)\,dy_1\ -\int_{\substack{y_1\in\mathbb{R}\\ cy_1\leq 0}} \frac{\partial u}{\partial t}\big(x,t+\frac{y_1}{c}\big)\,dy_1\Big)\int_0^{+\infty}\frac{1}{y_2}\ln\left(\left|\frac{1+y_2}{1-y_2}\right|\right)\,dy_2. \quad \text{(D.18)}
\end{aligned}$$

Then (D.13) follows from (D.18) and the value of the integral $\int_0^{+\infty} y_2^{-1}\ln\left(\left|\frac{1+y_2}{1-y_2}\right|\right)\,dy_2 = 16\int_0^{+\infty}\frac{\sigma}{e^{2\sigma}-e^{-2\sigma}}\,d\sigma = \frac{\pi^2}{2}$.

We now prove (D.14). From (D.12) it follows that

$$I_{1,2}(\varepsilon,M) = I_{1,2,1}(\varepsilon,M) + I_{1,2,2}(\varepsilon,M), \tag{D.19}$$

where

$$\begin{aligned}
I_{1,2,1}(\varepsilon,M) \ :=\ &-\int_{\substack{(y_1,y_2)\in\mathbb{R}^2\\ \frac{y_1}{c}M\geq\varepsilon}} \frac{c}{y_1}\frac{\partial u}{\partial t}(x+y_1,t+y_2) \qquad \text{(D.20)}\\
&\times\ln\left(\left|\frac{\min\Big(-\frac{y_1^2}{2c}+y_2+\frac{y_1}{c}(M-x),\max\big(\varepsilon,-\frac{y_1^2}{2c}+y_2-\frac{y_1}{c}(M+x)\big)\Big)}{\min\Big(\frac{y_1^2}{2c}-y_2+\frac{y_1}{c}(M+x),\max\big(\varepsilon,\frac{y_1^2}{2c}-y_2-\frac{y_1}{c}(M-x)\big)\Big)}\right|\right)\,dy_1\,dy_2,\\
I_{1,2,2}(\varepsilon,M) \ :=\ &-\int_{\substack{(y_1,y_2)\in\mathbb{R}^2\\ 0<\frac{y_1}{c}M\leq\varepsilon}} \frac{c}{y_1}\frac{\partial u}{\partial t}(x+y_1,t+y_2) \qquad \text{(D.21)}\\
&\times\ln\left(\left|\frac{\min\Big(-\frac{y_1^2}{2c}+y_2+\frac{y_1}{c}(M-x),\max\big(\varepsilon,-\frac{y_1^2}{2c}+y_2-\frac{y_1}{c}(M+x)\big)\Big)}{\min\Big(\frac{y_1^2}{2c}-y_2+\frac{y_1}{c}(M+x),\max\big(\varepsilon,\frac{y_1^2}{2c}-y_2-\frac{y_1}{c}(M-x)\big)\Big)}\right|\right)\,dy_1\,dy_2.
\end{aligned}$$

We first study $I_{1,2,1}(\varepsilon,M)$. We perform the change of variables "y_2"$= y_2-\frac{y_1^2}{2c}$ and we obtain

$$\begin{aligned}
I_{1,2,1}(\varepsilon,M) = &-\int_{\substack{y_1\in\mathbb{R}\\ \frac{y_1}{c}M\geq\varepsilon}}\frac{c}{y_1}\left(\int_{\frac{y_1}{c}(M+x)-\varepsilon}^{\frac{y_1}{c}(M+x)+\varepsilon}\frac{\partial u}{\partial t}(x+y_1,t+y_2+\frac{y_1^2}{2c}) \qquad \text{(D.22)}\right.\\
&\left.\times\ln\left(\frac{\varepsilon}{|-y_2+\frac{y_1}{c}(M+x)|}\right)\,dy_2\right)\,dy_1\\
&-\int_{\substack{y_1\in\mathbb{R}\\ \frac{y_1}{c}M\geq\varepsilon}}\frac{c}{y_1}\left(\int_{-\frac{y_1}{c}(M-x)-\varepsilon}^{-\frac{y_1}{c}(M-x)+\varepsilon}\frac{\partial u}{\partial t}(x+y_1,t+y_2+\frac{y_1^2}{2c})\ \ln\left(\frac{|y_2+\frac{y_1}{c}(M-x)|}{\varepsilon}\right)\,dy_2\right)\,dy_1,
\end{aligned}$$

We perform the changes of variables "y_2"$= y_2-\frac{y_1}{c}(M+x)$ (resp. "y_2"$= y_2+\frac{y_1}{c}(M-x)$) on the first (resp. second) integral on the right hand side of (D.22), and we obtain

$$\begin{aligned}
I_{1,2,1}(\varepsilon,M) = &-\int_{\substack{y_1\in\mathbb{R}\\ \frac{y_1}{c}M\geq\varepsilon}}\frac{c}{y_1}\ \int_{-\varepsilon}^{\varepsilon}\ln\left(\frac{\varepsilon}{|y_2|}\right) \qquad \text{(D.23)}\\
&\times\left(\frac{\partial u}{\partial t}\big(x+y_1,t+y_2+\frac{y_1^2}{2c}+\frac{y_1}{c}(M+x)\big)-\frac{\partial u}{\partial t}\big(x+y_1,t+y_2+\frac{y_1^2}{2c}-\frac{y_1}{c}(M-x)\big)\right)\,dy_2\,dy_1,
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{\partial u}{\partial t}\big(x+y_1,t+y_2+\frac{y_1^2}{2c}+\frac{y_1}{c}(M+x)\big)-\frac{\partial u}{\partial t}\big(x+y_1,t+y_2+\frac{y_1^2}{2c}-\frac{y_1}{c}(M-x)\big)\\
=\ &\frac{y_1M}{c}\int_0^1\frac{\partial^2 u}{\partial t^2}\big(x+y_1,t+y_2+\frac{y_1^2}{2c}+\frac{xy_1}{c}+\eta\,\frac{y_1M}{c}\big)\,d\eta,
\end{aligned}$$

for $(y_1, y_2) \in \mathbb{R}^2$, $y_1 \neq 0$. Since $u \in \mathcal{S}(\mathbb{R}^2)$, there exists a positive constant C such that

$$\begin{aligned} & \left| \frac{\partial u}{\partial t}\big(x + y_1, t + y_2 + \frac{y_1^2}{2c} + \frac{y_1}{c}(M + x)\big) - \frac{\partial u}{\partial t}\big(x + y_1, t + y_2 + \frac{y_1^2}{2c} - \frac{y_1}{c}(M - x)\big) \right| \\ \leq \quad & C(1 + y_1^2)^{-1} \frac{y_1 M}{c}, \end{aligned} \tag{D.24}$$

for $(y_1, y_2) \in \mathbb{R}^2$, $y_1 \neq 0$. Therefore using (D.23) and (D.24) we have

$$|I_{1,2,1}(\varepsilon, M)| \leq MC \int_{\mathbb{R}_{y_1}} (1 + y_1^2)^{-1} \, dy_1 \int_{-\varepsilon}^{\varepsilon} \ln\left(\frac{\varepsilon}{|y_2|}\right) dy_2 \to 0^+, \text{ as } \varepsilon \to 0^+. \tag{D.25}$$

We consider $I_{1,2,2}(\varepsilon, M)$. We perform the change of variables "y_2"$= y_2 - \frac{xy_1}{c} - \frac{y_1^2}{2c}$ on the right-hand-side of (D.21), and we obtain

$$\begin{aligned} & I_{1,2,2}(\varepsilon, M) = - \int_{\substack{(y_1,y_2)\in\mathbb{R}^2 \\ 0 < \frac{y_1}{c} M \leq \varepsilon}} \frac{c}{y_1} \frac{\partial u}{\partial t}(x + y_1, t + y_2 + \frac{y_1^2}{2c} + \frac{xy_1}{c}) \\ & \times \ln\left(\left| \frac{\min\left(y_2 + \frac{y_1}{c}M, \max\left(\varepsilon, y_2 - \frac{y_1}{c}M\right)\right)}{\min\left(-y_2 + \frac{y_1}{c}M, \max\left(\varepsilon, -y_2 - \frac{y_1}{c}M\right)\right)} \right| \right) dy_1 \, dy_2 \\ = \quad & - \int_{\substack{y_1\in\mathbb{R} \\ 0 < \frac{y_1}{c} M \leq \varepsilon}} \frac{c}{y_1} \int_{-\frac{y_1 M}{c} + \varepsilon}^{\frac{y_1 M}{c} + \varepsilon} \frac{\partial u}{\partial t}(x + y_1, t + y_2 + \frac{y_1^2}{2c} + \frac{xy_1}{c}) \ \ln\left(\frac{\varepsilon}{|y_2 - \frac{y_1}{c}M|}\right) dy_2 \, dy_1 \\ & - \int_{\substack{y_1\in\mathbb{R} \\ 0 < \frac{y_1}{c} M \leq \varepsilon}} \frac{c}{y_1} \int_{\frac{y_1 M}{c} - \varepsilon}^{-\frac{y_1 M}{c} + \varepsilon} \frac{\partial u}{\partial t}(x + y_1, t + y_2 + \frac{y_1^2}{2c} + \frac{xy_1}{c}) \ln\left(\frac{|y_2 + \frac{My_1}{c}|}{|y_2 - \frac{My_1}{c}|}\right) dy_2 \, dy_1 \\ & - \int_{\substack{y_1\in\mathbb{R} \\ 0 < \frac{y_1}{c} M \leq \varepsilon}} \frac{c}{y_1} \int_{-\frac{y_1 M}{c} - \varepsilon}^{\frac{y_1 M}{c} - \varepsilon} \frac{\partial u}{\partial t}(x + y_1, t + y_2 + \frac{y_1^2}{2c} + \frac{xy_1}{c}) \ \ln\left(\frac{|y_2 + \frac{y_1}{c}M|}{\varepsilon}\right) dy_2 \, dy_1 \end{aligned}$$

Let $C_1 := \sup_{y\in\mathbb{R}^2} |\frac{\partial u}{\partial t}(y)| < \infty$. Then we have

$$\begin{aligned} |I_{1,2,2}(\varepsilon, M)| \quad \leq \quad & 2C \int_{\substack{y_1\in\mathbb{R} \\ 0 < \frac{y_1}{c} M \leq \varepsilon}} \frac{c}{y_1} \int_{-\frac{y_1 M}{c} + \varepsilon}^{\frac{y_1 M}{c} + \varepsilon} \left| \ln\left(\frac{\varepsilon}{|y_2 - \frac{y_1}{c}M|}\right) \right| dy_2 \, dy_1 \\ & + C \int_{\substack{y_1\in\mathbb{R} \\ 0 < \frac{y_1}{c} M \leq \varepsilon}} \frac{c}{y_1} \int_{\frac{y_1 M}{c} - \varepsilon}^{-\frac{y_1 M}{c} + \varepsilon} \left| \ln\left(\frac{|y_2 + \frac{My_1}{c}|}{|y_2 - \frac{My_1}{c}|}\right) \right| dy_2 \, dy_1 \\ \leq \quad & 2C \int_{\substack{y_1\in\mathbb{R} \\ 0 < \frac{y_1}{c} M \leq \varepsilon}} \frac{c}{y_1} \int_{-\frac{y_1 M}{c} + \varepsilon}^{\frac{y_1 M}{c} + \varepsilon} \left| \ln\left(\frac{\varepsilon}{|y_2 - \frac{y_1}{c}M|}\right) \right| dy_2 \, dy_1 \\ & + C \int_{\substack{y_1\in\mathbb{R} \\ 0 < \frac{y_1}{c} M \leq \varepsilon}} \frac{c}{y_1} \int_{-\varepsilon}^{\varepsilon} \left| \ln\left(\frac{|y_2 + \frac{My_1}{c}|}{|y_2 - \frac{My_1}{c}|}\right) \right| dy_2 \, dy_1 \end{aligned} \tag{D.26}$$

With the change of variables "y_1"$= \varepsilon^{-1} y_1$ and "y_2"$= \varepsilon^{-1} y_2$ in the first integral on the right-hand-side of (D.26), we obtain

$$\begin{aligned} \int_{\substack{y_1\in\mathbb{R} \\ 0 < \frac{y_1}{c} M \leq \varepsilon}} \frac{c}{y_1} \int_{-\frac{y_1 M}{c} + \varepsilon}^{\frac{y_1 M}{c} + \varepsilon} \left| \ln\left(\frac{\varepsilon}{|y_2 - \frac{y_1}{c}M|}\right) \right| dy_2 \, dy_1 \quad &= \quad \varepsilon \int_{\substack{y_1\in\mathbb{R} \\ 0 < \frac{y_1}{c} M \leq 1}} \frac{c}{y_1} \int_{-\frac{y_1 M}{c} + 1}^{\frac{y_1 M}{c} + 1} \left| \ln\left(\frac{1}{|y_2 - \frac{y_1}{c}M|}\right) \right| dy_2 \, dy_1 \\ & \to \quad 0^+, \text{ as } \varepsilon \to 0^+. \end{aligned} \tag{D.27}$$

We also perform a last change of variables "y_1"$= \frac{y_1}{y_2}$ in the second integral on the right-hand-side of

(D.26) to obtain

$$\int_{-\varepsilon}^{\varepsilon}\int_{y_1\in\mathbb{R}}|y_1|^{-1}\left|\ln\left(\frac{|y_2+\frac{My_1}{c}|}{|y_2-\frac{My_1}{c}|}\right)\right|dy_1dy_2=2\varepsilon\int_{y_1\in\mathbb{R}}|y_1|^{-1}\left|\ln\left(\frac{|1+\frac{My_1}{c}|}{|1-\frac{My_1}{c}|}\right)\right|dy_1\to 0^+,\ \text{as}\ \varepsilon\to 0^+. \tag{D.28}$$

Combining (D.26)–(D.28) we obtain that $\lim_{\varepsilon\to 0^+} I_{1,2,2}(\varepsilon, M) = 0$. Using also (D.19) and (D.25) we obtain (D.14). □

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