Pre-homogeneous spaces and stratified Kähler spaces

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Abstract

A representation of a reductive Lie group is a *pre-homogeneous space* provided it has a Zariski-open orbit. A *stratified Kähler space* is a stratified symplectic space together with a complex analytic structure which is compatible with the stratified symplectic structure; in particular each stratum is a Kähler manifold in an obvious fashion. The notion of stratified Kähler space establishes an intimate relationship between nilpotent orbits, singular reduction, invariant theory, reductive dual pairs, Jordan triple systems, symmetric domains, and pre-homogeneous spaces. The purpose of the talk is to illustrate the geometry of stratified Kähler spaces.

Examples of stratified Kähler spaces abound. The closure of a holomorphic nilpotent orbit carries a normal Kähler structure. Symplectic reduction carries a Kähler manifold to a normal stratified Kähler space in such a way that the sheaf of germs of polarized functions coincides with the ordinary sheaf of germs of holomorphic functions. Projectivization of holomorphic nilpotent orbits yields exotic stratified Kähler structures on complex projective spaces and on certain complex projective varieties including complex projective quadrics, Scorza varieties and their secant varieties. Physical examples are reduced spaces arising from angular momentum.

The appearance of singularities in classical phase spaces is the rule rather than the exception. While within canonical quantization the implementation of singularities is far from being clear the possible impact of classical phase space singularities on quantum problems can be explored via quantization on stratified Kähler spaces.

Contents

1 Origins and motivation

Origin of the notion of *stratified Kähler space*: Quantization in the presence of singularities

Message today:

Stratified Kähler spaces establish an intimate relationship between pre-homogeneous spaces, nilpotent orbits, singular reduction, invariant theory, reductive dual pairs, JORDAN triple systems, symmetric domains.

2 Vinberg's theorem

Given a graded Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

$\mathfrak{g}_1$ *acquires a* G_0*-pre-homogeneous structure*

Attractive mathematics behind

PURPOSE OF MY TALK: Unveil some of this attractive mathematics

GUIDING EXAMPLE,

illustration of VINBERG's theorem:

$$\mathfrak{sp}(\ell,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}^+,\ \mathfrak{k} = \mathfrak{u}(\ell),\ \mathfrak{p}^+ \cong \mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$$

$$\mathfrak{g} = \mathfrak{sp}(\ell,\mathbb{C}) \cong \mathfrak{p}^- \oplus \mathfrak{gl}(\ell,\mathbb{C}) \oplus \mathfrak{p}^+$$

$$\mathfrak{p}^- \cong \mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$$

$\mathfrak{p}^+ \cong \mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$ a pre-homogeneous space, the $\mathrm{GL}(\ell,\mathbb{C})$-action on $\mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$ being given by

$$x \cdot S = xSx^t, \quad x \in \mathrm{GL}(\ell,\mathbb{C}),\ S \in \mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^\ell]$$

second symmetric power of defining representation of $\mathrm{GL}(\ell,\mathbb{C})$

Lurking behind grading: JORDAN triple system

3 Singularities, stratified Kähler spaces

Issue of singularities of true classical physical phase spaces not academic:
Singularities rule rather than the exception.
Simple mecanical systems and solution spaces of field theories come with singularities.
Examples:
— ℓ particles in $\mathbb{R}^s$ with total angular momentum zero;
reduced classical phase space: space of complex symmetric $(\ell\times\ell)$-matrices of rank at most equal to $\min(s,\ell)$.
— Special case $s = 3$ our solar system.
— ℓ harmonic oscillators in $\mathbb{R}^s$ with total angular momentum zero and constant energy: exotic projective variety
DEFINITION. A *stratified Kähler space* consists of a complex analytic space N, with
(i) a stratification, finer than complex analytic
(ii) a stratified symplectic structure

$$(C^\infty N, \{\cdot,\cdot\})$$

compatible with complex analytic structure. That is: $(C^\infty N, \{\cdot, \cdot\})$ is a POISSON algebra of continuous functions on N; on each stratum, an ordinary symplectic Poisson algebra which combines with the complex analytic structure to a KÄHLER structure.

$C^\infty N$ an algebra of *continuous* functions, not necessarily ordinary smooth functions, referred to as a *smooth* structure on N.

4 Lie algebras of hermitian type

(Semisimple) Lie algebra of *hermitian type*: pair $(\mathfrak{g}, z)$ real semisimple Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and central element z of $\mathfrak{k}$, referred to as an *H-element*, such that:

$J_z = \mathrm{ad}(z)\big|_{\mathfrak{p}}$ a complex structure on $\mathfrak{p}$

Reductive Lie algebra of hermitian type: reductive Lie algebra $\mathfrak{g}$ together with an element $z \in \mathfrak{g}$ whose constituent z' in the semisimple part $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ an H-element for $[\mathfrak{g}, \mathfrak{g}]$

Notation:

G Lie group having $\mathfrak{g} = \mathrm{Lie}(G)$

$K \subset G$ compact connected subgroup having $\mathfrak{k} = \mathrm{Lie}(K)$

$(\mathfrak{g}, z)$ of hermitian type equivalent to G/K a (non-compact) hermitian symmetric space with complex structure induced by z.

complex structure J_z on $\mathfrak{p}$ necessarily K-invariant

E. Cartan's infinitesimal classification of ir-

reducible hermitian symmetric spaces:
complete list of simple hermitian Lie algebras

$$
\begin{aligned}
\mathfrak{su}(p,q) &\quad (A_n, n \geq 1,\ n+1 = p+q)\\
\mathfrak{so}(2,2n-1) &\quad (B_n, n \geq 2)\\
\mathfrak{sp}(n,\mathbb{R}) &\quad (C_n, n \geq 2)\\
\mathfrak{so}(2,2n-2) &\quad (D_{n,1}, n > 2)\\
\mathfrak{so}^*(2n) &\quad (D_{n,2}, n > 2)\\
\mathfrak{e}_{6(-14)} &\\
\mathfrak{e}_{7(-25)} &
\end{aligned}
$$

5 The closure of a holomorphic nilpotent orbit

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ hermitian
Adjoint orbit $\mathcal{O} \subseteq \mathfrak{g}$ *pseudoholomorphic*: projection from $\mathfrak{g}$ to $\mathfrak{p}$, restricted to $\mathcal{O}$, a diffeomorphism onto its image
A pseudoholomorphic orbit $\mathcal{O}$ inherits
— a complex structure from the embedding into $\mathfrak{p}$
— viewed as a coadjoint orbit by means of (a positive multiple of) the KILLING form, $\mathcal{O}$ inherits a symplectic structure, the KOSTANT-KIRILLOV-SOURIAU-form
the two combine to a (not necessarily positive) KÄHLER structure
Choose positive multiple of the KILLING form
Define: pseudoholomorphic orbit $\mathcal{O}$ to be *holomorphic*:
KÄHLER structure on $\mathcal{O}$ positive
Given a holomorphic *nilpotent* orbit $\mathcal{O}$,
$C^\infty(\overline{\mathcal{O}})$ algebra of *Whitney smooth functions* on topological closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ (not Zariski clo-

sure) resulting from embedding of $\overline{\mathcal{O}}$ into $\mathfrak{g}^*$
Lie bracket on $\mathfrak{g}$ passes to *Poisson* bracket $\{\cdot,\cdot\}$ on $C^\infty(\overline{\mathcal{O}})$
Poisson bracket turns $\overline{\mathcal{O}}$ into *stratified symplectic space*

Theorem. *Given a holomorphic nilpotent orbit $\mathcal{O}$, the closure $\overline{\mathcal{O}}$ is a union of finitely many holomorphic nilpotent orbits. Moreover, the diffeomorphism from $\mathcal{O}$ onto its image in $\mathfrak{p}$ extends to a homeomorphism from the closure $\overline{\mathcal{O}}$ onto its image in $\mathfrak{p}$, this homeomorphism turns $\overline{\mathcal{O}}$ into a complex affine variety, and the complex analytic structure, in turn, combines with the Poisson structure $(C^\infty(\overline{\mathcal{O}}),\{\cdot,\cdot\})$ to a normal stratified Kähler structure.*

Let r be the real rank of $\mathfrak{g}$. There are $r+1$ holomorphic nilpotent orbits $\mathcal{O}_0,\dots,\mathcal{O}_r$, and these are linearly ordered in such a way that

$$\{0\} = \mathcal{O}_0 \subseteq \overline{\mathcal{O}_1} \subseteq \ldots \subseteq \overline{\mathcal{O}_r} = \mathfrak{p}$$

Cartan decomposition induces decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^- \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^+$$

of complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$, $\mathfrak{p}^+$ and $\mathfrak{p}^-$ being the holomorphic and antiholomorphic constituents, respectively, of $\mathfrak{p}^{\mathbb{C}}$

Thus we are back in situation of Vinberg's theorem.

Theorem. *The projection from $\overline{\mathcal{O}_r}$ to $\mathfrak{p}$ is a homeomorphism onto $\mathfrak{p}$, and the G-orbit stratification of $\overline{\mathcal{O}_r}$ passes to the $K^{\mathbb{C}}$-orbit stratification of $\mathfrak{p} \cong \mathfrak{p}^+$. Thus, for $1 \leq s \leq r$, restricted to $\mathcal{O}_s$, this homeomorphism is a K-equivariant diffeomorphism from $\mathcal{O}_s$ onto its image in $\mathfrak{p}^+$, and this image is a $K^{\mathbb{C}}$-orbit in $\mathfrak{p}^+$.*

Guiding example

$$\mathfrak{sp}(\ell, \mathbb{R}) = \mathfrak{u}(\ell) \oplus \mathrm{S}^2_{\mathbb{C}}[\mathbb{C}^{\ell}],\ r = \ell,$$

$\mathcal{O}_s$ space of complex symmetric $(\ell \times \ell)$-matrices of rank $s \leq \ell$;

$\overline{\mathcal{O}_s}$ space of complex symmetric $(\ell \times \ell)$-matrices of rank at most $s \leq \ell$

SUMMARY: $\mathcal{O}$ holomorphic nilpotent orbit
— $\overline{\mathcal{O}} \subseteq \mathfrak{g}^*$: stratified symplectic structure
— $\overline{\mathcal{O}} \subseteq \mathfrak{p}$: complex analytic structure
combine to a stratified KÄHLER structure
PROJECTIVIZATION: Given the hermitian Lie algebra $\mathfrak{g}$, with CARTAN decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the ascending sequence

$$\{0\} = \mathcal{O}_0 \subseteq \overline{\mathcal{O}_1} \subseteq \ldots \subseteq \overline{\mathcal{O}_r} = \mathfrak{p}$$

of closures of holomorphic nilpotent orbits descends to an ascending sequence

$$Q_1 \subseteq \ldots \subseteq Q_r = \mathbb{P}(\mathfrak{p})$$

of projective varieties where $\mathbb{P}(\mathfrak{p})$ is the projective space on $\mathfrak{p}$ and where each Q_s $(1 \leq s \leq r)$ arises from $\overline{\mathcal{O}_s}$ by projectivization. The stratified KÄHLER structures on the $\overline{\mathcal{O}_s}$ descend to stratified KÄHLER structures on the Q_s, and all stratified KÄHLER structures in sight are normal.

We refer to a stratified KÄHLER structure of this kind on a projective variety as being *exotic* since this kind of structure can *not* be induced from a Fubini-Study structure on a projective space.

Example of ℓ harmonic oscillators in $\mathbb{R}^s$ with total angular momentum zero and constant energy:

exotic projective variety arising from space of complex symmetric $(\ell \times \ell)$-matrices of rank at most equal to $\min(s, \ell)$ by projectivization

The above ascending sequence of the projectivizations of the closures of holomorphic nilpotent orbits admits an attractive interpretation in terms of Severi and Scorza varieties.

Theorem. *Let $k \geq 2$, let X be a k-Scorza variety, and let $\mathbb{P}^m\mathbb{C}$ be the ambient complex projective space. This projective space carries an exotic normal stratified Kähler structure with the following properties:*

(1) The closures of the strata constitute an ascending sequence

$$Q_1 \subseteq Q_2 \subseteq \ldots \subseteq Q_{k+1} = \mathbb{P}^m\mathbb{C}$$

of normal stratified Kähler spaces where the closed stratum Q_1 coincides with the given Scorza variety and where, complex algebraically, for $1 \leq \rho \leq k$, Q_ρ is a projective determinantal variety in $Q_{k+1} = \mathbb{P}^m\mathbb{C}$; in particular, in the regular case, Q_k is a projective degree $k+1$ hypersurface.

(2) For $1 \leq \rho \leq k$, $Q_{\rho+1}$ is the ρ'th secant

variety $S^\rho(Q_1)$ *of* Q_1 *in* $Q_{k+1} = \mathbb{P}^m\mathbb{C}$.
(3) For $2 \le \rho \le k+1$, $Q_{\rho-1}$ *is the singular locus of* Q_ρ, *in the sense of stratified Kähler spaces.*
(4) The exotic stratified Kähler structure on $\mathbb{P}^m\mathbb{C}$ *restricts to an ordinary Kähler structure on* Q_1 *inducing, perhaps up to rescaling, the standard hermitian symmetric space structure.*

The regular rank 3 case is somewhat special. This case corresponds to the *critical* SEVERI varieties and includes the exceptional case of the SEVERI variety arising from the *octonions*.

Addendum. *For* $m = 5, 8, 14, 26$, *the complex projective space* $\mathbb{P}^m\mathbb{C}$ *carries an exotic normal stratified Kähler structure with the following properties:*
(1) The closures of the strata constitute an ascending sequence

$$Q_1 \subseteq Q_2 \subseteq Q_3 = \mathbb{P}^m\mathbb{C}$$

of normal stratified Kähler spaces where, complex algebraically, Q_1 is a (critical) Severi variety and Q_2 a projective cubic hypersurface, the chordal variety of Q_1.
(2) The singular locus of Q_3, in the sense of stratified Kähler spaces, is the hypersurface Q_2, and that of Q_2 (still in the sense of stratified Kähler spaces) is the non-singular variety Q_1; furthermore, Q_1 is as well the complex algebraic singular locus of Q_2.
(3) The exotic stratified Kähler structure on $\mathbb{P}^m\mathbb{C}$ restricts to an ordinary Kähler structure on Q_1 inducing, perhaps up to rescaling, the standard hermitian symmetric space structure.

In the case involving the octonions, the term "determinantal variety" refers to FREUDENTHAL's determinant.

7 Quantization in the presence of singularities

"Singular points in a quantum problem are a set of measure zero so cannot possibly be important."

EMMRICH-RÖMER: *Wave functions may congregate near singular points.*

Ignoring singularities: inconsistencies

(1) Can we unveil a structure on the quantum level which has the classical phase space singularities as its shadow?

(2) Do the strata carry physical information?

Concerning (1):

SCHRÖDINGER quantization fails

Developing KÄHLER quantization on a stratified KÄHLER space I isolated the concept of a **costratified Hilbert space**

Exploiting this notion:

Answer to (2): for $K = \mathrm{SU}(2)$ *tunneling probabilities among strata* [HRS]

Beginning of a huge research program

Stratified KÄHLER spaces arising from pre-homogeneous spaces yield interesting special cases of quantization in the presence of singularities.

8 Description of Severi and Scorza varieties

Let $m \geq 2$. A SEVERI VARIETY is a nonsingular variety X in complex projective m-space $\mathbb{P}^m\mathbb{C}$ having the property that, for some point $O \notin X$, the projection from X to $\mathbb{P}^{m-1}\mathbb{C}$ is a closed immersion. Let X be a SEVERI variety and n the dimension of X. The critical cases are when $m = \frac{3}{2}n + 2$. ZAK proved that only the following *four* critical cases occur:
$X = \mathbb{P}^2\mathbb{C} \subseteq \mathbb{P}^5\mathbb{C}$ (VERONESE)
$X = \mathbb{P}^2\mathbb{C} \times \mathbb{P}^2\mathbb{C} \subseteq \mathbb{P}^8\mathbb{C}$ (SEGRE)
$X = \mathrm{G}_2(\mathbb{C}^6) = \mathrm{U}(6)/(\mathrm{U}(2) \times \mathrm{U}(4)) \subseteq \mathbb{P}^{14}\mathbb{C}$ (PLÜCKER)
$X = \mathrm{Ad}(\mathfrak{e}_{6(-78)})/(\mathrm{SO}(10,\mathbb{R}) \cdot \mathrm{SO}(2,\mathbb{R})) \subseteq \mathbb{P}^{26}\mathbb{C}$.

These varieties arise from the projective planes over the *four* real normed division algebras (reals, complex numbers, quaternions, octonions) by complexification.

SCORZA varieties generalize the critical SEVERI varieties (ZAK).

Given a non-singular variety X in $\mathbb{P}^m\mathbb{C}$, for $0 \leq k \leq m$, the k'th *secant variety* $S^k(X)$ is the projective variety in $\mathbb{P}^m\mathbb{C}$ which arises as the closure of the union of all k-dimensional projective spaces in $\mathbb{P}^m\mathbb{C}$ that contain $k+1$ independent points of X. Then $S^0(X) = X$ and $S^1(X)$ is the ordinary secant variety, referred to as well as *chordal variety*. For $k \geq 2$, a k-SCORZA VARIETY is defined to be a non-singular complex projective variety of maximal dimension among those varieties whose $(k-1)$-secant variety is not the entire ambient projective space.

The last SEVERI variety is a very special 2-SCORZA variety. Let $k \geq 2$. We now list the other k-SCORZA varieties.

$X = \mathbb{P}^k\mathbb{C} \subseteq \mathbb{P}^{\frac{k(k+3)}{2}}\mathbb{C}$ (VERONESE)

$X = \mathbb{P}^k\mathbb{C} \times \mathbb{P}^k\mathbb{C} \subseteq \mathbb{P}^{k(k+2)}\mathbb{C}$ (SEGRE)

$X = \mathbb{P}^k\mathbb{C} \times \mathbb{P}^{k+1}\mathbb{C} \subseteq \mathbb{P}^{k^2+3k+1}\mathbb{C}$ (SEGRE)

$X = \mathrm{G}_2(\mathbb{C}^{2(k+1)}) \subseteq \mathbb{P}^{k(2k+3)}\mathbb{C}$ (PLÜCKER)

$X = \mathrm{G}_2(\mathbb{C}^{2k+3}) \subseteq \mathbb{P}^{2k^2+5k+2}\mathbb{C}$ (PLÜCKER)

The critical SEVERI varieties are exactly the 2-SCORZA varieties that are regular in a sense not made precise here.

When the chordal variety of a non-singular projective variety Q in $\mathbb{P}^5\mathbb{C}$ is a hypersurface (and not the entire ambient space) the projection from a generic point gives an embedding in $\mathbb{P}^4\mathbb{C}$. A classical result of Severi says that the Veronese surface is the only surface (not contained in a hyperplane) in $\mathbb{P}^5\mathbb{C}$ with this property. This is the origin of the terminology "Severi variety". The terminology "Scorza variety" has been introduced by ZAK to honor SCORZA's pioneering work on linear normalization of varieties of small codimension.

The above two results exhibit an interesting geometric feature of SCORZA varieties, in particular of the critical SEVERI varieties, in the world of singular POISSON-KÄHLER geometry.

There are exactly *four* simple regular rank 3 hermitian Lie algebras over the reals; these result from the euclidean JORDAN algebras of hermitian (3×3)-matrices over the *four* real normed division algebras by the superstructure construction; and *each critical Severi variety arises from the minimal holomorphic nilpotent orbit in such a Lie algebra.*

The resulting geometric insight into the four critical SEVERI varieties has been made explicit in the Addendum above. For higher rank, that is, when the rank r (say) is at least equal to 4, there are exactly *three* simple regular rank r hermitian Lie algebras over the reals; these result from the euclidean JORDAN algebras of hermitian $(r \times r)$-matrices over the *three associative* real normed division algebras by the superstructure construction.

Thus we see in particular that the classification of (regular) SCORZA varieties parallels that of regular pre-homogeneous spaces. The singular POISSON-KÄHLER point of view exhibits interesting additional geometric features.