Singular Poisson-Kähler geometry of stratified Kähler spaces and quantization

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Abstract

In the presence of classical phase space singularities the standard methods are insufficient to attack the problem of quantization. In certain situations these difficulties can be overcome by means of *stratified Kähler spaces*. Such a space is a stratified symplectic space together with a complex analytic structure which is compatible with the stratified symplectic structure; in particular each stratum is a Kähler manifold in an obvious fashion.

Examples abound: Symplectic reduction, applied to Kähler manifolds, yields a particular class of examples; this includes adjoint and generalized adjoint quotients of complex semisimple Lie groups which, in turn, underly certain lattice gauge theories. Other examples come from certain moduli spaces of holomorphic vector bundles on a Riemann surface and variants thereof; in physics language, these are spaces of conformal blocks. Still other examples arise from the closure of a holomorphic nilpotent orbit. Symplectic reduction carries a Kähler manifold to a stratified Kähler space in such a way that the sheaf of germs of polarized functions coincides with the ordinary sheaf of germs of holomorphic functions. Projectivization of the closures of holomorphic nilpotent orbits yields exotic stratified Kähler structures on complex projective spaces and on certain complex projective varieties including complex projective quadrics. Other physical examples are reduced spaces arising from angular momentum, including our solar system whose correct reduced phase space acquires the structure of an affine stratified Kähler space.

In the presence of singularities, the naive restriction of the quantization problem to a smooth open dense part, the "top stratum", may lead to a loss of information and in fact to inconsistent results. Within the framework of holomorphic quantization, a suitable quantization procedure on stratified Kähler spaces unveils a certain *quantum structure having the classical singularities as its shadow*. The new structure which thus emerges is that of a *costratified Hilbert space*, that is, a Hilbert space together with a system which consists of the subspaces associated with the strata of the reduced phase space and of the corresponding orthoprojectors. The costratified Hilbert space structure reflects the stratification of the reduced phase space. Given a Kähler manifold, reduction after quantization then coincides with quantization after reduction in the sense that not only the reduced and unreduced quantum phase spaces correspond but the invariant unreduced and reduced quantum observables as well.

We will illustrate the approach with a concrete model: We will present a quantum (lattice) gauge theory which incorporates certain classical singularities. The reduced phase space is a stratified Kähler space, and we make explicit the requisite singular holomorphic quantization procedure and spell out the resulting costratified Hilbert space. In particular, certain tunneling probabilities between the strata emerge, the energy eigenstates can be determined, and corresponding expectation values of the orthoprojectors onto the subspaces associated with the strata in the strong and weak coupling approximations can be explored.

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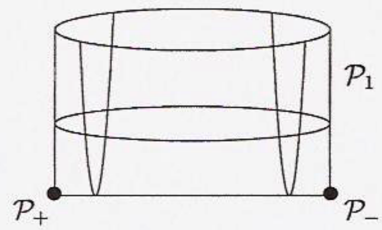
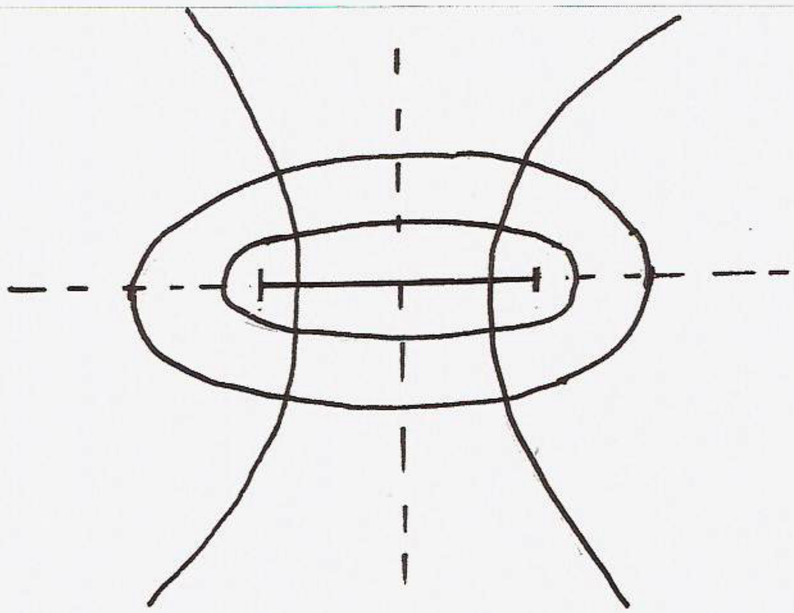


Figure 1: The reduced phase space $\mathcal{P}$ for $K = \mathrm{SU}(2)$.



1 Leitmotiv

The Leitmotiv lurking behind everything I will try to explain in these lectures:
Question:
What is the *quantum structure* having *classical phase space singularities* as its shadow?
Goal of the lectures:
Convince you of a possible answer:
Costratified Hilbert space structure
Pushing through the formalism in a particular model:
Huebschmann, Rudolph, Schmid:
Tunneling effect between quantum objects corresponding to different strata
NOTA BENE: In the presence of singularities, restricting quantization to a smooth open dense stratum, sometimes referred to as “top stratum”, can result in a loss of information and may in fact lead to *inconsistent* results.

2 Singularities

Issue of singularities not academic:
Singularities rule rather than the exception.
Simple mecanical systems and solution spaces of field theories come with singularities.

Examples:

— ℓ particles in $\mathbb{R}^s$ with total angular momentum zero;
reduced classical phase space: space of complex symmetric $(\ell \times \ell)$-matrices of rank at most equal to $\min(s, \ell)$
special case $s = 3$ our solar system.

— ℓ harmonic oscillators in $\mathbb{R}^s$ with total angular momentum zero and constant energy: exotic projective variety

— Lattice gauge theory

3 Physical systems with classical phase space singularities

3.1 Example of classical phase space singularity: Exotic plane

$\mathbb{R}^3$ coordinates x, y, r
semicone N: $x^2 + y^2 = r^2$, $r \geq 0$
exotic plane with a single vertex
classical reduced *phase space* single particle in affine space angular momentum zero
reduced Poisson algebra $(C^\infty N, \{\,\cdot\,,\,\cdot\,\})$
algebra $C^\infty N$ of smooth functions in x, y, r subject to

$$x^2 + y^2 = r^2$$

Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$

$$\{x, y\} = 2r,\ \{x, r\} = 2y,\ \{y, r\} = -2x,$$

complex structure $z = x + iy$
Poisson bracket *defined at vertex*
away from vertex Poisson structure symplectic
complex structure does *not* "see" the vertex
at vertex radius r *not* smooth in x and y
vertex singular point for Poisson structure
not singular point for complex structure
Poisson and complex structure combine to "stratified Kähler structure"

3.2 Lattice gauge theory

K compact Lie, $\mathfrak{k}$ its Lie algebra,
$K^{\mathbb{C}}$ complexification of K
$\mathfrak{k}$ invariant inner product
diffeomorphism

$$\mathrm{T}^*K \cong \mathrm{T}K \longrightarrow K \times \mathfrak{k} \longrightarrow K^{\mathbb{C}}$$

complex structure on $K^{\mathbb{C}}$ and cotangent bundle symplectic structure on T^*K:
K-bi-invariant Kähler structure
lattice gauge thy from config. space $Q = K^\ell$
unreduced momentum phase space

$$\mathrm{T}^*Q = \mathrm{T}^*K^\ell \cong (K^{\mathbb{C}})^\ell$$

reduction modulo conjugation
reduced phase space

$$\mathrm{T}^*K^\ell//K \cong (K^{\mathbb{C}})^\ell//K^{\mathbb{C}}$$

singularites
special case $\ell = 1$: *adjoint quotient* $K^{\mathbb{C}}//K^{\mathbb{C}}$
maximal torus T of K, r its rank
W Weyl group of T in K
as a space, T^*T the complexification $T^{\mathbb{C}}$ of T

$T^{\mathbb{C}}$ a product $(\mathbb{C}^*)^r$ of r copies of $\mathbb{C}^*$
reduced phase space

$$\mathcal{P} \cong \mathrm{T}^* T\big/W \cong (\mathbb{C}^*)^r\big/W$$

space of W-orbits in $(\mathbb{C}^*)^r$ relative to W
viewed as $\mathrm{T}^* T\big/W$: $\mathcal{P}$ inherits stratified symplectic structure by reduction
(i) algebra $C^\infty(T^{\mathbb{C}})^W$ of smooth W-invariant functions on $T^{\mathbb{C}}$ inherits Poisson bracket: Poisson algebra of continuous functions on $\mathcal{P}$
(ii) for each stratum, Poisson structure ordinary symplectic one on that stratum
(iii) restriction mapping from $C^\infty(T^{\mathbb{C}})^W$ to the algebra of ordinary smooth functions on that stratum Poisson map
viewed as $T^{\mathbb{C}}\big/W$: $\mathcal{P}$ complex structure
complex and Poisson structurea combine to *stratified Kähler structure* on $\mathcal{P}$:
Poisson structure satisfies (ii), (iii) above and
(iv) for each stratum, necessarily complex manifold, symplectic and complex structures combine to Kähler structure

3.3 The canoe

special case:

$$K = \mathrm{SU}(2),\ K^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C}),\ W \cong \mathbb{Z}/2$$

maximal torus $T \cong S^1$

$$\mathrm{T}^*T \cong T^{\mathbb{C}} \cong \mathbb{C}^*$$
$$\mathcal{P} = \mathrm{T}^*K//K \cong \mathrm{T}^*T/W \cong \mathbb{C}^*/W$$

W-invariant holomorphic map

$$f\colon C^* \to \mathbb{C},\ f(z) = z + z^{-1}$$

induces a complex analytic isomorphism

$$\mathbb{C}^*/W \longrightarrow \mathbb{C}$$

More generally: $K = \mathrm{SU}(n)$;
maximal torus $(\mathbb{C}^*)^{n-1}$; realize this torus as the subspace of $(\mathbb{C}^*)^n$ which consists of all $(z_1,\ldots,z_n)$ such that $z_1 \ldots z_n = 1$
$\sigma_1,\ldots,\sigma_{n-1}$ elementary symmetric functions

$$(\sigma_1,\ldots,\sigma_{n-1})\colon (\mathbb{C}^*)^{n-1} \longrightarrow \mathbb{C}^{n-1},$$
$$\mathbf{z} = (z_1,\ldots,z_n) \longmapsto (\sigma_1(\mathbf{z}),\ldots,\sigma_{n-1}(\mathbf{z}))$$

realizes the complex analytic quotient

As a *stratified Kähler space*, quotient $\mathcal{P}$ considerably more complicated.
Special case $n = 2$, $K = \mathrm{SU}(2)$:
COMPLEX ANALYTIC STRUCTURE:

$$S^1 \cong T \subseteq K,\ \mathbb{C}^* \cong T^{\mathbb{C}} \subseteq K^{\mathbb{C}}$$

quotient $\mathcal{P} = \mathrm{T}^*K//K$ amounts to

$$T^{\mathbb{C}}/(\mathbb{Z}/2) \cong \mathbb{C}$$

realized via the holomorphic map

$$f\colon \mathbb{C}^* \longrightarrow \mathbb{C},\ f(z) = z + z^{-1}$$

as explained above, $Z = z + z^{-1}$ holomorphic coordinate on the quotient.
REAL STRUCTURE:

$$z = x + iy,\ Z = X + iY,\ r^2 = x^2 + y^2$$

$$X = x + \frac{x}{r^2},\ Y = y - \frac{y}{r^2},\ \tau = \frac{y^2}{r^2}$$

$C^\infty(\mathcal{P})$: smooth functions in the variables X, Y, τ, subject to the relation

$$Y^2 = (X^2 + Y^2 + 4(\tau - 1))\tau$$

POISSON BRACKETS

$$\{X,Y\} = X^2 + Y^2 + 4(2\tau - 1)$$
$$\{X,\tau\} = 2(1-\tau)Y$$
$$\{Y,\tau\} = 2\tau X$$

Resulting *stratified Kähler* structure on $\mathcal{P} \cong \mathbb{C}$ *singular* at $-2 \in \mathbb{C}$ and $2 \in \mathbb{C}$, that is, the POISSON structure *vanishes*; at these two points, τ *not* smooth in X and Y

$$\tau = \frac{1}{2}\sqrt{Y^2 + \frac{(X^2+Y^2-4)^2}{16}} - \frac{X^2+Y^2-4}{8}$$

away from $-2 \in \mathbb{C}$ and $2 \in \mathbb{C}$, POISSON structure symplectic

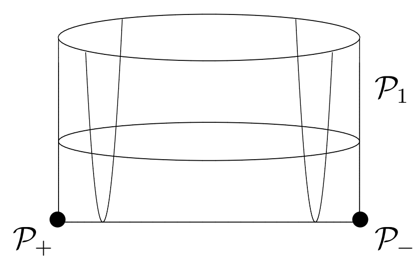


Figure 1: The reduced phase space $\mathcal{P}$ for $K = \mathrm{SU}(2)$

complex analytic space:
topological space X, together with
a sheaf of rings $\mathcal{O}_X$ such that:
X can be covered by open sets Y,
each Y embeds into polydisc U in some $\mathbb{C}^n$
(the number n may vary as U varies) as zero
set of finite system of holomorphic functions
$f_1, \ldots, f_q$ on U
restriction $\mathcal{O}_Y$ of sheaf $\mathcal{O}_X$ to Y isomorphic
as a sheaf to quotient sheaf $\mathcal{O}_U/(f_1, \ldots, f_q)$
$\mathcal{O}_U$ sheaf of germs of hol fn's on U
sheaf $\mathcal{O}_X$ defined to be *sheaf of holomorphic functions on* X

DEFINITION. A *stratified Kähler space*
consists of complex analytic space N, with
(i) complex analytic stratification, finer than
complex analytic
(ii) stratified symplectic structure

$$(C^\infty N, \{\cdot, \cdot\})$$

compatible with complex analytic structure.

Compatibility of the two structures:
(i) For each point q of N and each holomorphic function f defined on an open neighborhood U of q, there is an open neighborhood V of q with $V \subset U$ such that, on V, f is the restriction of a function in $C^\infty(N)$;
(ii) on each stratum, the symplectic structure determined by the symplectic Poisson structure (on that stratum) combines with the complex analytic structure to a Kähler structure
EXAMPLE 1: *exotic plane*
generalizes to class of examples:
closure of a holomorphic nilpotent orbit
angular momentum zero reduced spaces
solar system
Projectivization: *exotic projective variety*
complex quadrics
SEVERI and SCORZA varieties and their *secant* varieties
reduced classical phase spaces for systems of harmonic oscillators with zero angular momentum and constant energy

EXAMPLE 2: Moduli spaces of semistable holomorphic vector bundles or, more generally, moduli spaces of semistable principal bundles on a non-singular complex projective curve
special case moduli space of semistable rank 2 degree zero vector bundles with trivial determinant on a curve of genus 2
as a space just $\mathbb{C}P_3$
stratified symplectic structure involves more functions than ordinary smooth functions
complement of space of stable vector bundles
a *Kummer surface*
ATIYAH AND BOTT: *singularities* ?
THEOREM. *Let N be a Kähler manifold, acted upon holomorphically by a complex Lie group G such that the action, restricted to a compact real form K of G, preserves the Kähler structure and is hamiltonian, with momentum mapping $\mu\colon N \to \mathfrak{k}^*$. Then the reduced space $N_0 = \mu^{-1}(0)\big/K$ inherits a stratified Kähler structure.*
EXAMPLE 3: adjoint quotient

5 Correspondence principle

$$(M, C^\infty(M), \{\,\cdot\,,\,\cdot\,\}) \leftrightarrow (\mathcal{H}, [\,\cdot\,,\,\cdot\,])$$

DIRAC: Comparison mathematical structures
$(M = \mathrm{T}^*Q, \sigma = -d\vartheta)$:
SCHRÖDINGER $\mathcal{H} = L^2(Q, dq)$
Internal degrees of freedom not necessarily separated into configuration and momentum variables: (M, σ) general *symplectic* or
$(M, C^\infty(M), \{\,\cdot\,,\,\cdot\,\})$ general POISSON
polarization: $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$
vertical polarization of $M = \mathrm{T}^*Q$:
$\mathcal{H} = L^2(Q, dq)$
holomorphic polarization:
M necessarily KÄHLER
$\mathcal{H} = \mathcal{H}L^2(M, d\nu)$ Hilbert space holomorphic functions square integrable relative to suitable measure.
Example: $Q = \mathbb{R}^n$, $\mathrm{T}^*Q \cong \mathbb{C}^n$ with ordinary complex structure having $z = q + ip$ as holomorphic variables, endowed with ordinary hermitian structure

correspondence principle leads to *quantization*
albeit that notion not well defined
HILBERT SPACE $\mathcal{H}$:
$\mathcal{L}(\mathcal{H})$ vector space of linear operators on $\mathcal{H}$
$(N, \{\,\cdot\,,\,\cdot\,\})$ POISSON
B LIE subalgebra of $C^{\infty}(N)$
quantization of B: representation

$$B \longrightarrow \mathcal{L}(\mathcal{H}), \quad f \mapsto \widehat{f}$$

of B by self-adjoint operators $\widehat{f}$ on complex HILBERT space $\mathcal{H}$ such that :
(i) constants in B act by *multiplication*
(ii) given $f, h \in B$, DIRAC *condition*

$$\widehat{\{f, h\}} = i\,[\widehat{f}, \widehat{h}]$$

(iii) representation *irreducible*
$B = C^{\infty}(N)$: no quantization
way out: restrict to LIE subalgebras B

7 Quantum singularities; costratified Hilbert space

“Singular points in a quantum problem set of measure zero so cannot be important.”
EMMRICH-RÖMER: *Wave functions may congregate near singular points.*
Ignoring singularities: inconsistencies
(1) Can we unveil a *structure on the quantum level which has the classical phase space singularities as its shadow*?
(2) Do the strata carry physical information?
Answer to (2): for $K = \mathrm{SU}(2)$ *tunneling probabilities among strata* [HRS]
Concerning (1):
Attempt SCHRÖDINGER quantization:
$\mathrm{T}^*Q//K$ a stratified symplectic space but
— strata are *not* cotangent bundles
— $\mathrm{T}^*Q//K \longrightarrow Q/K$ not a stratified map
KÄHLER quantization: each stratum of reduced space a KÄHLER manifold
even better, structures combine to a *stratified Kähler structure*

Costratified Hilbert space

Let N be a stratified space. Let $\mathcal{C}_N$ be the category whose objects are the strata of N and whose morphisms are the inclusions $Y' \subseteq \overline{Y}$ where Y and Y' range over strata.

DEFINITION. A *costratified Hilbert space* relative to N assigns a Hilbert space $\mathcal{C}_Y$ to each stratum Y, together with a bounded linear map $\mathcal{C}_{Y_2} \to \mathcal{C}_{Y_1}$ for each inclusion $Y_1 \subseteq \overline{Y_2}$ such that, whenever $Y_1 \subseteq \overline{Y_2}$ and $Y_2 \subseteq \overline{Y_3}$, the composite of $\mathcal{C}_{Y_3} \to \mathcal{C}_{Y_2}$ with $\mathcal{C}_{Y_2} \to \mathcal{C}_{Y_1}$ coincides with the bounded linear map $\mathcal{C}_{Y_3} \to \mathcal{C}_{Y_1}$ associated with the inclusion $Y_1 \subseteq \overline{Y_3}$.

Illustration: situation of canoe

Construction of costratified Hilbert space relative to the reduced phase space $\mathcal{P}$ as **quantum analogue** of *orbit type stratification*

Start: Hilbert space

$$\mathcal{H} = \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K$$

single out subspaces associated with strata special case:

$$K = \mathrm{SU}(2),\ \mathcal{P} = \mathrm{T}^*K//K \cong \mathbb{C}$$

elements of $\mathcal{H}$: ordinary functions on $K^{\mathbb{C}}$ being K-invariant, they induce functions on

$$\mathcal{P} = K^{\mathbb{C}}//K^{\mathbb{C}} \cong T^{\mathbb{C}}/W \cong \mathbb{C}$$

strata

$$\mathcal{P}_+ = \{2\} \subseteq \mathbb{C},\ \mathcal{P}_- = \{-2\} \subseteq \mathbb{C}$$
$$\mathcal{P}_1 = \mathbb{C} \setminus \{2, -2\} \subseteq \mathbb{C}$$

consider the closed subspaces

$$\mathcal{V}_+ = \{f \in \mathcal{H}; f\big|_{\mathcal{P}_+} = 0\} \subseteq \mathcal{H}$$
$$\mathcal{V}_- = \{f \in \mathcal{H}; f\big|_{\mathcal{P}_-} = 0\} \subseteq \mathcal{H}$$
$$\mathcal{V}_1 = \{f \in \mathcal{H}; f\big|_{\mathcal{P}_1} = 0\} = \{0\} \subseteq \mathcal{H}$$

let $\mathcal{H}_1 = \mathcal{H}$, *define* Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ to be the orthogonal complements in $\mathcal{H}$

$$\mathcal{H} = \mathcal{V}_+ \oplus \mathcal{H}_+ = \mathcal{V}_- \oplus \mathcal{H}_-$$

resulting system $\{\mathcal{H};\ \mathcal{H}_1,\ \mathcal{H}_+,\ \mathcal{H}_-\}$, together with the corresponding orthogonal projections: *costratified Hilbert space* relative to $\mathcal{P}$

8 Correspondence principle, Lie-Rinehart algebras

To make sense of *correspondence principle* for stratified symplectic structure
need *replacement* for the tangent bundle of a smooth symplectic manifold
Lie-Rinehart algebra
satisfactory generalization of the Lie algebra of smooth vector fields in the smooth case
This enables us to put *flesh on the bones of Dirac's correspondence principle in certain singular situations*

A *Lie-Rinehart algebra* consists of a commutative algebra and a Lie algebra with additional structure which generalizes the mutual structure of interaction between the algebra of smooth functions and the Lie algebra of smooth vector fields on a smooth manifold:

Definition. *Lie-Rinehart* algebra (A, L)
commutative algebra A
Lie-algebra L
$L \otimes A \to A$ action on A by derivations

A-module structure $A \otimes L \to L$
subject to

$$[\alpha, a\beta] = \alpha(a)\beta + a[\alpha, \beta],$$
$$(a\alpha)(b) = a(\alpha(b)),$$

where $a, b \in A$ and $\alpha, \beta \in L$.
$(A, \{\,\cdot\,,\,\cdot\,\})$ Poisson algebra
D_A the the A-module of formal differentials of A the elements of which we write as du, for $u \in A$
For $u, v \in A$, association

$$(du, dv) \longmapsto \pi(du, dv) = \{u, v\}$$

A-valued A-bilinear skew-symmetric 2-form $\pi = \pi_{\{\,\cdot\,,\,\cdot\,\}}$ on D_A
adjoint

$$\pi^\sharp \colon D_A \longrightarrow \mathrm{Der}(A) = \mathrm{Hom}_A(D_A, A)$$

morphism of A-modules

$$[adu, bdv] = a\{u, b\}dv + b\{a, v\}du + abd\{u, v\}$$

Lie bracket $[\,\cdot\,,\,\cdot\,]$ on D_A

THEOREM. *The A-module structure on D_A, the bracket $[\cdot,\cdot]$, and the morphism $\pi^\sharp$ of A-modules turn the pair (A, D_A) into a Lie-Rinehart algebra.*

write resulting Lie-Rinehart algebra as $(A, D_{\{\,\cdot\,,\,\cdot\,\}})$

$A = C^\infty(M)$, symplectic manifold M,

A-dual $\mathrm{Der}(A) = \mathrm{Hom}_A(D_A, A)$ of D_A the A-module $\mathrm{Vect}(M)$ of smooth vector fields

$$(\pi^\sharp, \mathrm{Id})\colon (D_A, A) \longrightarrow (\mathrm{Vect}(M), C^\infty(M))$$

morphism of Lie-Rinehart algebras,

A-module morphism $\pi^\sharp$ surjective kernel consists of those formal differentials which "vanish at each point of" M

general Poisson algebra $(A, \{\,\cdot\,,\,\cdot\,\})$

notion of *prequantum module*, defined in terms of $(A, D_{\{\,\cdot\,,\,\cdot\,\}})$, yields *replacement for prequantum bundle*

9 Prequantum modules

Poisson 2-form $\pi_{\{\,\cdot\,,\,\cdot\,\}}$: central *extension*

$$0 \longrightarrow A \longrightarrow \overline{L}_{\{\,\cdot\,,\,\cdot\,\}} \longrightarrow D_{\{\,\cdot\,,\,\cdot\,\}} \longrightarrow 0$$

as A-modules

$$\overline{L}_{\{\,\cdot\,,\,\cdot\,\}} = A \oplus D_{\{\,\cdot\,,\,\cdot\,\}}$$

Lie bracket on $\overline{L}_{\{\,\cdot\,,\,\cdot\,\}}$: value $[(a, du), (b, dv)]$:

$$(\{u, b\} + \{a, v\} - \{u, v\}, d\{u, v\})$$

Definition. An $(A \otimes \mathbb{C})$-module M, endowed with $(A, \overline{L}_{\{\,\cdot\,,\,\cdot\,\}})$-module structure

$$\chi \colon \overline{L}_{\{\,\cdot\,,\,\cdot\,\}} \longrightarrow \mathrm{End}_{\mathbb{R}}(M) :$$

a *prequantum module for* $(A, \{\,\cdot\,,\,\cdot\,\})$ if
(i) values of χ lie in $\mathrm{End}_{\mathbb{C}}(M)$, i. e. for $a \in A$ and $\alpha \in D_{\{\,\cdot\,,\,\cdot\,\}}$, the operators $\chi(a, \alpha)$ complex linear transformations, and
(ii) for every $a \in A$

$$\chi(a, 0) = i\, a\, \mathrm{Id}_M .$$

Prequantization now proceeds via

$$A \ni a \longmapsto (a, da) \in \overline{L}_{\{\,\cdot\,,\,\cdot\,\}}$$

holomorphic quantization scheme extends to stratified Kähler spaces
main steps:
1) Kähler polarization generalizes to *stratified Kähler polarization*,
defined in terms of appropriate Lie-Rinehart algebra
specifies *polarizations on the strata* and, encapsulates *mutual positions of polarizations on the strata*
Under the circumstances of Kähler reduction, *symplectic reduction carries a Kähler polarization preserved by the symmetries into stratified Kähler polarization.*
2) prequantum bundle generalizes to *stratified prequantum module*
a system of prequantum modules, one for the closure of each stratum,
together with appropriate morphisms among them which reflect the stratification

3) quantum Hilbert space generalizes to
costratified quantum Hilbert space
costratified structure reflects stratification on the classical level
Thus the costratified Hilbert space structure is a quantum structure which has the classical singularities as its shadow.
4) The main result says that $[Q, R] = 0$:
quantization commutes with reduction.

11 Holomorphic half-form quantization on the complexification of a compact Lie group

General compact Lie group K
global KÄHLER potential κ

$$\kappa(x\,\mathrm{e}^{iY}) = |Y|^2,\ x \in K,\ Y \in \mathfrak{k}$$

that is, symplectic structure on $\mathrm{T}^*K \cong K^{\mathbb{C}}$ given by

$$i\partial\overline{\partial}\kappa$$

ε symplectic (or Liouville) volume form on

$$\mathrm{T}^*K \cong K^{\mathbb{C}}$$

η the real K-bi-invariant function on $K^{\mathbb{C}}$

$$\eta(x\,\mathrm{e}^{iY}) = \sqrt{\left|\frac{\sin(\mathrm{ad}(Y))}{\mathrm{ad}(Y)}\right|},\ x \in K,\ Y \in \mathfrak{k}$$

half-form KÄHLER quantization on $K^{\mathbb{C}}$:
Hilbert space $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)$ holomorphic functions on $K^{\mathbb{C}}$ square-integrable relative to $\mathrm{e}^{-\kappa/\hbar}\eta\varepsilon$

scalar product given by

$$\langle \psi_1, \psi_2 \rangle = \frac{1}{\mathrm{vol}(K)} \int_{K^{\mathbb{C}}} \overline{\psi_1} \psi_2 \mathrm{e}^{-\kappa/\hbar} \eta \varepsilon$$

left and right translation:
$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)$ unitary
$(K \times K)$-representation
Hilbert space associated with $\mathcal{P}$ by reduction after quantization: subspace

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K$$

of K-invariants relative to conjugation

12 Energy eigenvalues and eigenstates; holomorphic Peter-Weyl theorem

Choose a dominant WEYL chamber in $\mathfrak{t}$
basis of $\mathcal{H} = \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K$?
Notation: highest weight λ: $\chi_\lambda^{\mathbb{C}}$ the irreducible character of $K^{\mathbb{C}}$ associated with λ

Theorem. [Holomorphic Peter-Weyl] *The Hilbert space* $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)$ *contains the vector space* $\mathbb{C}[K^{\mathbb{C}}]$ *of representative functions on* $K^{\mathbb{C}}$ *as a dense subspace and, as a unitary* $(K \times K)$*-representation, this Hilbert space decomposes as the direct sum*

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon) \cong \widehat{\bigoplus}_{\lambda \in \widehat{K^{\mathbb{C}}}} V_\lambda^* \otimes V_\lambda$$

into $(K \times K)$*-isotypical summands.*

Consequence of *holomorphic* PETER-WEYL: irreducible characters $\chi_\lambda^{\mathbb{C}}$ of $K^{\mathbb{C}}$ a basis of

$$\mathcal{H} = \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K$$

Notation: highest weight λ:
— χ_λ the corresponding irreducible character of K, restriction of $\chi_\lambda^{\mathbb{C}}$ to K
— $\rho := \frac{1}{2}\sum_{\alpha\in R^+}\alpha$ half sum positive roots — $C_\lambda := (\hbar\pi)^{\dim(K)/2}\mathrm{e}^{\hbar|\lambda+\rho|^2}$,
$|\lambda+\rho|$ norm relative to inner product on $\mathfrak{k}$
consequence of ordinary PETER-WEYL:
$\{\chi_\lambda\}$ an *orthonormal* basis of $L^2(K,\mathrm{d}x)^K$

Theorem. *The assignment to* χ_λ *of*

$$C_\lambda^{-1/2}\chi_\lambda^{\mathbb{C}},$$

as λ *ranges over the highest weights, yields a unitary isomorphism*

$$L^2(K,\mathrm{d}x)^K \longrightarrow \mathcal{H}L^2(K^{\mathbb{C}},\mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K$$

Thus costratified Hilbert space structure, arising from *stratified Kähler* quantization, carries over to SCHRÖDINGER quantization.

KÄHLER: only constants quantizable
SCHRÖDINGER: functions at most quadratic in generalized momenta quantizable
(classical) Hamiltonian of model quantizable
associated quantum Hamiltonian

$$H = -\frac{\hbar^2}{2}\Delta_K + \frac{\nu}{2}(3 - \chi_1)$$

operator Δ_K arises from non-positive LAPLACE-BELTRAMI operator associated with bi-invariant Riemannian metric on K
metric bi-invariant, hence Δ_K, whence Δ_K restricts to self-adjoint operator on $L^2(K, \mathrm{d}x)^K$, still written as Δ_K
via $L^2(K, \mathrm{d}x)^K \longrightarrow \mathcal{H}$, transfer Hamiltonian, in particular, the operator Δ_K, to self-adjoint operators on $\mathcal{H}$

SCHUR's lemma:
— each isotypical $(K \times K)$-summand $L^2(K, dx)_\lambda$ of $L^2(K, dx)$ in the PETER-WEYL decomposition an eigenspace
— representative functions eigenfunctions for Δ_K
— eigenvalue $-\varepsilon_\lambda$ of Δ_K corresponding to the highest weight λ

$$\varepsilon_\lambda = (|\lambda + \rho|^2 - |\rho|^2),$$

— in holomorphic quantization on $\mathrm{T}^*K \cong K^{\mathbb{C}}$, energy operator arises as the unique extension of the operator $-\frac{1}{2}\Delta_K$ on $\mathcal{H}$ to an unbounded self-adjoint operator
— spectral decomposition thereof refines to holomorphic PETER-WEYL decomposition of $\mathcal{H}$

13 The lattice gauge theory model arising from SU(2)

some remarks about physical interpretation
for special case arising from canoe
roots of $K = \mathrm{SU}(2)$: α and $-\alpha$
half sum $\varrho = \frac{1}{2}\alpha$
invariant inner product on Lie algebra $\mathfrak{k}$ of K

$$-\frac{1}{2\beta^2}\mathrm{tr}(Y_1 Y_2)\,, \qquad Y_1, Y_2 \in \mathfrak{k}\,,$$

with scaling factor $\beta > 0$ left unspecified (e.g., $\beta = \frac{1}{\sqrt{8}}$ for the Killing form); then

$$|\alpha|^2 = 4\beta^2\,, \qquad |\varrho|^2 = \beta^2\,.$$

highest weights $\lambda_n = \frac{n}{2}\alpha$, $n = 0, 1, 2, \ldots$ (twice the spin)

$$\varepsilon_n \equiv \varepsilon_{\lambda_n} = \beta^2 n(n+2)$$
$$C_n \equiv C_{\lambda_n} = (\hbar\pi)^{3/2}\mathrm{e}^{\hbar\beta^2(n+1)^2}$$

on $T^{\mathbb{C}}$, complex characters $\chi_n^{\mathbb{C}} \equiv \chi_{\lambda_n}^{\mathbb{C}}$

$$\chi_n^{\mathbb{C}}\big(\mathrm{diag}(z, z^{-1})\big) = z^n + z^{n-2} + \cdots + z^{-n}$$

whereas, on T, corresponding real characters

$$\chi_n\big(\mathrm{diag}(\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}x})\big) = \frac{\sin\big((n+1)x\big)}{\sin(x)}$$

$(x \in \mathbb{R}\,, \quad n \geq 0)$
Weyl group W permutes the two entries of the elements in T
hence reduced configuration space $\mathcal{X} = T/W$ parametrized by $x \in [0, \pi]$ through

$$x \mapsto \mathrm{diag}(\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}x})$$

in this parametrization measure v on T

$$v\,\mathrm{d}t = \frac{\mathrm{vol}(K)}{\pi} \sin^2(x)\,\mathrm{d}x$$

assignment to $\psi \in C^\infty(T)^W$ of function

$$x \mapsto \sqrt{2}\ \sin x\ \psi\big(\mathrm{diag}(\mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}x})\big)$$

$(x \in [0, \pi])$ defines Hilbert space isomorphism

$$L^2(K, \mathrm{d}x)^K \longrightarrow L^2[0, \pi]$$

from $L^2(K, \mathrm{d}x)^K$, realized as Hilbert space of W-invariant L^2-functions on T, onto ordinary $L^2[0, \pi]$

scalar product in $L^2[0,\pi]$ normalized:
constant function with value 1 has norm 1
in particular, for $n \geq 0$, the character χ_n is mapped to the function

$$\chi_n(x) = \sqrt{2}\,\sin((n+1)x)$$

in view of isomorphism

$$L^2(K, \mathrm{d}x)^K \longrightarrow L^2[0,\pi]$$

and Peter-Weyl isomorphism
we work in abstract Hilbert space $\mathcal{H}$
with distinguished orthonormal basis

$$\{|n\rangle \ : \ n = 0, 1, 2, \dots\}$$

passage to holomorphic realization

$$\begin{aligned} \mathcal{H} &\longrightarrow \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K \\ |n\rangle &\longmapsto C_n^{-1/2}\chi_n^{\mathbb{C}} \end{aligned}$$

passage to Schrödinger realization

$$\begin{aligned} \mathcal{H} &\longrightarrow L^2(K, \mathrm{d}x)^K \\ |n\rangle &\longmapsto \chi_n \end{aligned}$$

passage to L^2-realization

$$\begin{aligned} \mathcal{H} &\longrightarrow L^2[0,\pi] \\ |n\rangle &\longmapsto \sqrt{2}\sin(n+1)x \end{aligned}$$

plotting wave functions in realization

$$\mathcal{H} \cong L^2[0,\pi]$$

has advantage that one can read off directly from the graph the corresponding probability densities with respect to Lebesgue measure on parameter space $[0,\pi]$
determine subspaces $\mathcal{H}_\tau$ for special case
$K = \mathrm{SU}(2)$ (canoe)
orbit type strata are $\mathcal{P}_+$, $\mathcal{P}_-$ and $\mathcal{P}_1$
$\mathcal{P}_\pm$ consists of class of $\pm\mathbb{1}$
$\mathcal{P}_1 = \mathcal{P} \setminus (\mathcal{P}_+ \cup \mathcal{P}_-)$
since $\mathcal{P}_1$ dense in $\mathcal{P}$: $\mathcal{V}_1 = \{0\}$ so $\mathcal{H}_1 = \mathcal{H}$ by definition
$\mathcal{V}_+$: the functions $\psi \in \mathcal{H}$ with $\psi(\mathbb{1}) = 0$
$\mathcal{V}_-$: the functions $\psi \in \mathcal{H}$ with $\psi(-\mathbb{1}) = 0$

(i)$\{\chi_n^{\mathbb{C}} - (n+1)\chi_0^{\mathbb{C}} \ : \ n = 1, 2, 3, \dots\}$
basis in $\mathcal{V}_+$
(ii) $\{\chi_n^{\mathbb{C}} + (-1)^n \frac{n+1}{2} \chi_1^{\mathbb{C}} \ : \ n = 0, 2, 3, \dots\}$
basis in $\mathcal{V}_-$
taking orthogonal complements:
THEOREM. *The subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ have dimension 1. They are spanned by the normalized vectors*

$$\psi_+ := \frac{1}{N} \sum_{n=0}^{\infty} (n+1)\, \mathrm{e}^{-\hbar\beta^2 (n+1)^2/2} \, |n\rangle$$

$$\psi_- := \frac{1}{N} \sum_{n=0}^{\infty} (-1)^n \, (n+1)\, \mathrm{e}^{-\hbar\beta^2 (n+1)^2/2} \, |n\rangle$$

respectively. The normalization factor N is determined by the identity

$$N^2 = \sum_{n=1}^{\infty} n^2 \, \mathrm{e}^{-\hbar\beta^2 n^2} \, .$$

Hence, in Dirac notation, orthogonal projections $\Pi_\pm \colon \mathcal{H} \to \mathcal{H}_\pm$ given by

$$\Pi_\pm = |\psi_\pm\rangle \, \langle\psi_\pm|$$

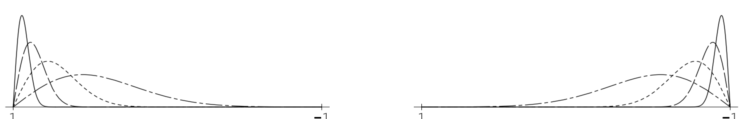
in terms of θ-constant

$$\theta_3(Q) = \sum_{k=-\infty}^{\infty} Q^{k^2}$$

normalization factor N determined by

$$N^2 = \frac{1}{2} \mathrm{e}^{-\hbar\beta^2} \theta_3'(\mathrm{e}^{-\hbar\beta^2})$$

following figure shows plots of $\psi_\pm$ in the realization of $\mathcal{H}$ via $L^2[0,\pi]$ for
$\hbar\beta^2 = 1/128$ (continuous line)
$\hbar\beta^2 = 1/32$ (long dash)
$\hbar\beta^2 = 1/8$ (short dash)
$\hbar\beta^2 = 1/2$ (alternating short-long dash)



ψ_+ ψ_-

scalar product of ψ_+ and ψ_-

$$\langle\psi_+,\psi_-\rangle = \frac{1}{N^2}\sum_{n=1}^{\infty}(-1)^{n+1}\,n^2\,\mathrm{e}^{-\hbar\beta^2\,n^2}$$

$$= -\,\frac{\theta_3'\big(-\mathrm{e}^{-\hbar\beta^2}\big)}{\theta_3'\big(\mathrm{e}^{-\hbar\beta^2}\big)}$$

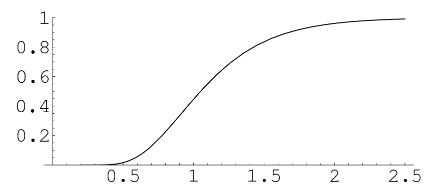
hence: subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ of $\mathcal{H}$ not orthogonal, share a certain overlap

which depends on the combined parameter $\hbar\beta^2$

quantity $|\langle\psi_+,\psi_-\rangle|^2$ yields tunneling probability between strata $\mathcal{P}_+$ and $\mathcal{P}_-$

i. e., the probability for a state prepared at $\mathcal{P}_+$ to be measured at $\mathcal{P}_-$ and vice versa

the following figure shows a plot of the tunneling probability against $\hbar\beta^2$
for large values, this probability tends to 1 whereas for $\hbar\beta^2 \to 0$, i.e., in the semiclassical limit, it vanishes



pass to realization of $\mathcal{H}$ via $L^2[0,\pi]$
apply general formula for radial part of the Laplacian on a compact group
from the description

$$H = -\frac{\hbar^2}{2}\Delta_K + \frac{\nu}{2}(3-\chi_1),$$

of quantum Hamiltonian: formal expression

$$-\frac{\hbar^2\beta^2}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+1\right)+\frac{\nu}{2}(3-\chi_1)$$

for H on $L^2[0,\pi]$
hence stationary Schrödinger equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+2\tilde{\nu}\cos(x)+\left(\frac{2E}{\hbar^2\beta^2}+1-3\tilde{\nu}\right)\right)\psi(x)=0\,,$$

where $\tilde{\nu}=\frac{\nu}{\hbar^2\beta^2}\equiv\frac{1}{\hbar^2\beta^2g^2}$, E eigenvalue
change of variable $y=(x-\pi)/2$ leads to Mathieu equation

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}f(y)+(a-2q\cos(2y))f(y)=0\,,$$

where

$$a = \frac{8E}{\hbar^2\beta^2} + 4 - 12\tilde{\nu}\,, \qquad q = 4\tilde{\nu}\,,$$

f a Whitney smooth function on $[-\pi/2, 0]$
satisfying boundary conditions

$$f(-\pi/2) = f(0) = 0$$

for certain characteristic values of parameter
a depending analytically on q,
usually denoted by $b_{2n+2}(q)$, $n = 0, 1, 2, \ldots$,
solutions satisfying boundary conditions exist
given $a = b_{2n+2}(q)$, corresponding solution
unique up to complex factor
can be chosen to be real-valued
usually denoted by $\mathrm{se}_{2n+2}(y; q)$ "sine elliptic"
thus, in realization of $\mathcal{H}$ via $L^2[0, \pi]$
stationary states are given by

$$\xi_n(x) = (-1)^{n+1}\sqrt{2}\left(\mathrm{se}_{2n+2}\left(\frac{x-\pi}{2}; 4\tilde{\nu}\right)\right)$$

corresponding eigenvalues by

$$E_n = \frac{\hbar^2\beta^2}{2}\left(\frac{b_{2n+2}(4\tilde{\nu})}{4} + 3\tilde{\nu} - 1\right)$$

factor $(-1)^{n+1}$ entails: $\tilde{\nu} = 0$: $\xi_n = \chi_n$
for any value of parameter q, functions

$$\sqrt{2}\,\mathrm{se}_{2n+2}(y;q)\,, \qquad n = 0, 1, 2, , \ldots\ ,$$

orthonormal basis in $L^2[-\pi/2, 0]$
characteristic values satisfy

$$b_2(q) < b_4(q) < b_6(q) < \cdots$$

ξ_n's orthonormal basis in $\mathcal{H}$
eigenvalues E_n nondegenerate

16 Expectation values of the costratification orthoprojectors

on level of observables, costratification: orthoprojectors $\Pi_\pm$ onto the subspaces $\mathcal{H}_\pm$ expectation values in the energy eigenstates

$$P_{\pm,n} := \langle \xi_n | \Pi_\pm \xi_n \rangle \, ,$$

probability that system prepared in stationary state ξ_n is measured in the subspace $\mathcal{H}_\pm$ functions $P_{\pm,n}$ depend on $\hbar$, β^2, ν only via the combinations $\hbar\beta^2$ and $\tilde{\nu} = \nu/(\hbar^2\beta^2)$ Figure 2 displays $P_{\pm,n}$ for $n = 0, \ldots, 5$ as functions of $\tilde{\nu}$ for three specific values of $\hbar\beta^2$, thus treating $\tilde{\nu}$ and $\hbar\beta^2$ as independent parameters. This is appropriate for the discussion of the dependence of $P_{\pm,n}$ on the coupling parameter g for fixed values of $\hbar$ and β^2. The plots have been generated by Mathematica through numerical integration.
For $n = 0$, $P_{+,n}$ has a dominant peak which is enclosed by less prominent maxima of the other $P_{+,n}$'s and moves to higher $\tilde{\nu}$ when $\hbar\beta^2$

decreases. That is to say, for a certain value of the coupling constant, the state ψ_+ which spans $\mathcal{H}_+$ seems to coincide almost perfectly with the ground state.

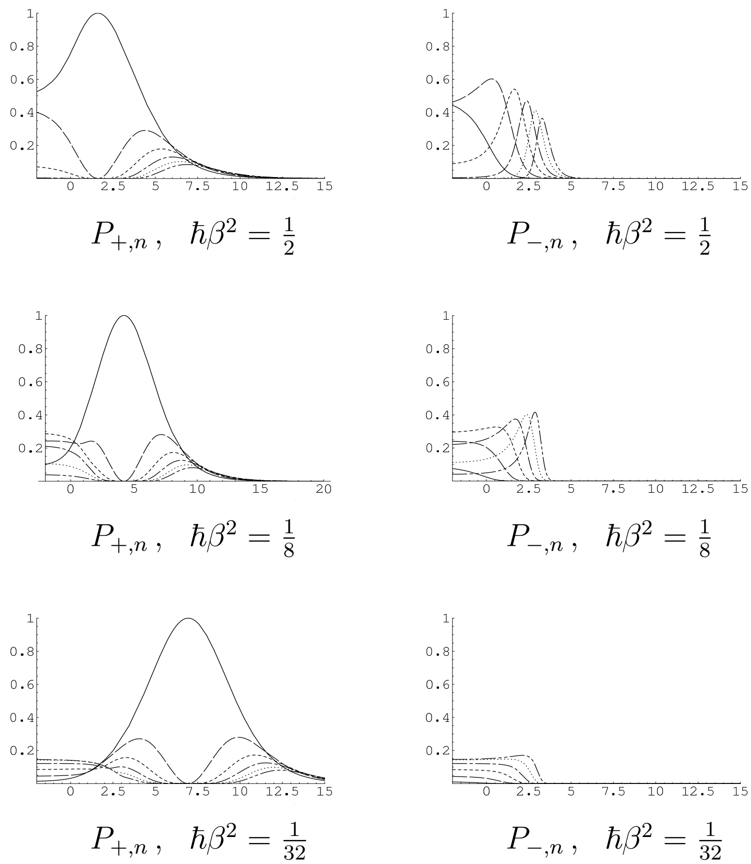


Figure 2: Expectation values $P_{+,n}$ and $P_{-,n}$ for $n = 0$ (continuous line), $n = 1$ (long dash), $n = 2$ (short dash), $n = 3$ (long-short dash), $n = 4$ (dotted line) and $n = 5$ (long-short-short dash), plotted over $\log \tilde{\nu}$ for $\hbar\beta^2 = \frac{1}{2}, \frac{1}{8}, \frac{1}{32}$.