Multi derivation Maurer-Cartan algebras and sh-Lie-Rinehart algebras

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Abstract

A finite-dimensional Lie algebra $\mathfrak{g}$ can be characterized in terms of its Maurer-Cartan algebra—the familiar differential graded algebra of alternating forms on $\mathfrak{g}$ with values in the ground field, endowed with the standard Lie algebra cohomology operator. We will explain how this characterization can be extended to sh Lie-Rinehart algebras. Our approach avoids any higher brackets but reproduces these brackets in a conceptual manner. The new technical tool we develop is a notion of filtered multi derivation chain algebra, somewhat more general than the standard notion of a multicomplex endowed with a compatible algebra structure.

Abstract continued

The crucial observation, just as for ordinary Lie-Rinehart algebras, is this: For a general sh Lie-Rinehart algebra, the generalized Cartan-Chevalley-Eilenberg operator on the corresponding graded algebra involves two operators, one coming from the sh Lie algebra structure and the other from the generalized action on the corresponding algebra; the sum of the operators is defined on the algebra while the operators are individually defined only on a larger ambient algebra. A special case of a multi derivation chain algebra arises from a quasi Lie-Rinehart algebra; quasi Lie-Rinehart algebras arise in the theory of foliations as a tool to capture the higher homotopies structure of a foliation [Hue05].

Abstract continued

The general notion of multi derivation chain algebra is intended to be applied to diffieties arising from jet bundles, e.g., in the study of Noether's theorems. These ideas are part of a program aimed at developing a higher Lie theory that applies, in particular, to partial differential equations.

Origins and motivation

Noether theorems
Constrained systems
Batalin-Fradkin-Vilkovisky formalism
BRST
In the 1980's Stasheff started a research program aimed at developing or isolating the higher homotopies behind the formalism

Some literature

Kjesth 2001: Ph. d thesis supervised by J. Stasheff
develops notion of sh Lie-Rinehart
published as [Kje01a], [Kje01b]
Huebschmann 2003: Quasi Lie-Rinehart algebras:
higher homotopies arising from a foliation [Hue05]
Vitagliano 2012 [Vit12]
Huebschmann 2013 [Hue13]

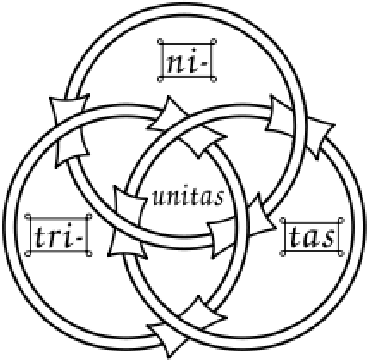
perhaps related with
Fredenhagen-Rejzner arxiv:1208.1428
Paugam arxiv:1106.4955

Structure of the talk

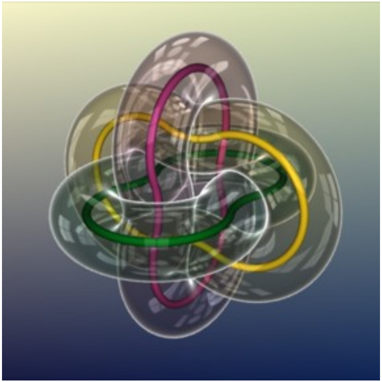
Higher homotopies in a nutshell
Maurer-Cartan algebras
Lie algebras
Lie-Rinehart algebras
Higher homotopies generalization
Quasi Lie-Rinehart algebras
Multi derivation Maurer-Cartan algebras and sh-Lie-Rinehart algebras

Higher homotopies in a nutshell

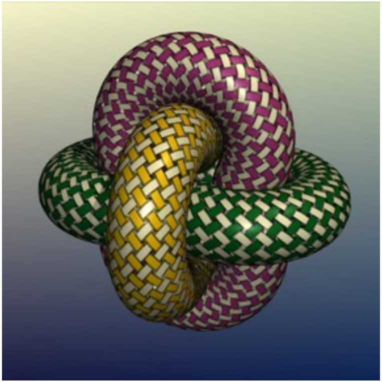
Borromean rings as a symbol of the Christian Trinity, from a 13th-century manuscript



Higher homotopies in a nutshell



Higher homotopies in a nutshell



Higher homotopies in a nutshell continued

$L \subseteq \mathbb{R}^3$ Borromean rings
$M = \mathbb{R}^3 \setminus L$ a 3-manifold
$\mathrm{H}^*_{\mathrm{deRham}}(M)$?
$\mathrm{H}^1_{\mathrm{deRham}}(M)$ three generators α, β, γ
$\alpha\beta = 0$, $\beta\gamma = 0$, $\alpha\gamma = 0$: rings are pairwise unlinked
Abusing notation: $\alpha\beta = du$, $\beta\gamma = dv$
$v\gamma - \alpha u$ a 2-cocycle
$\langle\alpha, \beta, \gamma\rangle = [v\gamma - \alpha u] \neq 0 \in \mathrm{H}^2_{\mathrm{deRham}}(M)$
non-trivial Massey product
recovers the fact that the three rings are non-trivially linked

Maurer-Cartan algebras

R commutative ring with 1, A commutative R-algebra

A *Maurer-Cartan algebra* is the graded A-algebra $\mathrm{Alt}_A(L, A)$ of A-multilinear alternating forms on an A-module L, together with a differential d turning $\mathrm{Alt}_A(L, A)$ into a differential graded algebra over the ground ring R

beware: *not* in general a differential graded A-algebra

Special case $A = R$: In [VE89], van Est uses terminology *Maurer-Cartan* algebra

goes on to notice that, in [Car53] ("La structure des groupes infinis", pp. 1335-1384, published in 1936), E. Cartan explored Maurer-Cartan algebras as a technical device for the structure theory of finite- and infinite-dimensional Lie groups, where "infinite-dimensional" means Lie pseudogroup.

Lie algebras

Lie algebra $\mathfrak{g}$
$(\mathrm{Alt}(\mathfrak{g}, \mathbb{R}), d)$ ordinary Cartan-Chevalley-Eilenberg complex
$\mathfrak{g}$ finite-dimensional:
$\mathrm{Alt}(\mathfrak{g}, \mathbb{R}) \cong \Lambda\mathfrak{g}^*$, exterior algebra on the dual $\mathfrak{g}^*$,
structure encoded in *cobracket* $\mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$:
Lie *coalgebra*
Lie coalgebras were exploited in differential Galois theory
(Ritt, etc.)

Lie-Rinehart algebras

R commutative ring with 1, A commutative R-algebra
(R, A)-*Lie algebra* [Rin63] [Rinehart]
Lie algebra L over R
$L \otimes A \to A$ left action by derivations
$A \otimes L \to L$ left A-module structure
compatibility conditions
generalize Lie algebra vector fields on manifold
as a module over its ring of functions

$$[\alpha, a\beta] = \alpha(a)\beta + a[\alpha, \beta]$$
$$(a\alpha)(b) = a(\alpha(b))$$

for $a, b \in A$ and $\alpha, \beta \in L$

when emphasis on pair (A, L) with mutual structure of interaction
pair (A, L) : *Lie-Rinehart* algebra

Examples of Lie-Rinehart algebras

(i) M manifold, $(A, L) = (C^\infty(M), \mathrm{Vect}(M))$
(ii) A algebra, $(A, L) = (A, \mathrm{Der}(A))$
(iii) $\vartheta\colon E \to B$ Lie algebroid: $(A, L) = (C^\infty(B), \Gamma(\vartheta))$
(iv) $(A, \{\,\cdot\,,\,\cdot\,\})$ Poisson algebra
A-module D_A of formal differentials on A
$[\,\cdot\,,\,\cdot\,]\colon D_A \otimes D_A \to D_A, [du, dv] = d\{u, v\}$,
$D_A \otimes A \to A, du(a) = \{u, a\}$
(A, D_A) with structure just explained a Lie-Rinehart algebra
(v) *twilled Lie-Rinehart algebra*
(A, L) Lie-Rinehart
$L = L' \oplus L''$ decomposition of A-modules,
(A, L') and (A, L'') both Lie-Rinehart
Example: M smooth manifold
$(A, L) = (C^\infty(M, \mathbb{C}), \mathrm{Vect}(M, \mathbb{C}))$
$J\colon \mathrm{T}M \to \mathrm{T}M$ almost complex, induces
$L = L' \oplus L''$, decomposition of A-modules
decomposition being twilled signifies that the almost complex structure is integrable

Lie-Rinehart algebras continued

Theorem

Given a pair that consists of a commutative algebra A and an A-module L, under suitable mild hypotheses (e. g. L finitely generated and projective as an A-module), Lie-Rinehart algebra structures on the pair (A, L) correspond bijectively to operators d turning the graded A-algebra $\mathrm{Alt}_A(L, A)$ into a differential graded algebra over the ground ring R (beware: not over A)

Lie-Rinehart algebras continued

Recall: ordinary Lie algebra $\mathfrak{g}$, structure encoded in *cobracket* $\mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$: Lie *coalgebra*

Beware: No naive notion of Lie-Rinehart coalgebra unless L-action on A trivial, i. e. ordinary Lie algebra, since L an A-module but Lie algebra only over the ground ring

Maurer-Cartan algebra structure serves as replacement for non-existing Lie-Rinehart coalgebra

diff Galois theory: Lie coalgebras used by [Ritt]

Maurer-Cartan algebra structure explored by Malgrange, e.g., in [Mal10], under the name $(\mathbb{C}, S)$-cogèbre de Lie

Some remarks on the proof

Given: L an A-module, two pieces of structure:
$[\cdot\,,\,\cdot]\colon L \times L \to L$ skew-symmetric pairing
$t\colon L \longrightarrow \mathrm{Der}(A|R)$ R-linear;
for $\alpha \in L$ and $a \in A$ write $\alpha(a) = (t(\alpha))(a)$
R-algebra $\mathrm{Alt}(L, A)$ of A-valued R-multilin. altern. forms on L
R-linear derivations ∂^t and $\partial^{[\cdot\,,\,\cdot]}$ on $\mathrm{Alt}(L, A)$ familiar expressions

$$(\partial^t f)(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} (-1)^{(i-1)} \alpha_i (f(\alpha_1, \ldots \widehat{\alpha}_i \ldots, \alpha_n))$$

$$(\partial^{[\cdot\,,\,\cdot]} f)(\alpha_1, \ldots, \alpha_n) = \sum_{1 \leq j < k \leq n} (-1)^{(j+k)} f([\alpha_j, \alpha_k], \alpha_1, \ldots \widehat{\alpha}_j \ldots)$$

$$\mathcal{D} = \partial^t + \partial^{[\cdot\,,\,\cdot]} \quad \text{derivation}$$

Some remarks on the proof, continued

Recall A-module L, pieces of structure:
$[\,\cdot\,,\,\cdot\,]\colon L \times L \to L$ skew-symmetric pairing
$t\colon L \longrightarrow \mathrm{Der}(A|R)$ R-linear

$$\mathcal{D} = \partial^t + \partial^{[\,\cdot\,,\,\cdot\,]}\colon \mathrm{Alt}(L, A) \to \mathrm{Alt}(L, A) \quad \text{derivation}$$

Proposition

When $[\,\cdot\,,\,\cdot\,]$ Lie bracket and t a morphism of R-Lie algebras, the derivation $\mathcal{D} = \partial^t + \partial^{[\,\cdot\,,\,\cdot\,]}$ a differential, classical CCE operator, computes CCE cohomology of L with coefficients in A.

Concerning Lie-Rinehart algebras, a crucial observation

Proposition

When (A, L) is a Lie-Rinehart algebra, derivation $\mathcal{D} = \partial^t + \partial^{[\,\cdot\,,\,\cdot\,]}$ descends to R-linear differential on $\mathrm{Alt}_A(L, A) \subseteq \mathrm{Alt}(L, A)$, even though this is not true of the individual operators ∂^t and $\partial^{[\,\cdot\,,\,\cdot\,]}$ unless $A = R$ and ∂^t trivial.

Upshot

A single theory having ordinary Lie algebra cohomology and
ordinary de Rham cohomology as its offspring
both arise as the derived functor of the operation of taking
invariants with respect to an algebra of differential operators

Broader perspective: general gauge theory for Lie-Rinehart
algebras that encompasses
classical gauge theory
differential Galois theory, in particular ordinary Galois theory
Lie theory for differential equations

Higher homotopies generalization

N graded vector space or, more generally, graded module over the ground ring R

suspension sN: $(sN)_{j+1} = N_j$

$\mathfrak{g}$ ungraded (concentrated in degree zero), $s\mathfrak{g}$ concentrated in degree 1

$\mathrm{S}^{\mathrm{c}}[s\mathfrak{g}]$ graded symmetric coalgebra on suspension $s\mathfrak{g}$

Lie brackets on $\mathfrak{g}$ in bijection with coalgebra diff's d on $\mathrm{S}^{\mathrm{c}}[s\mathfrak{g}]$

Question to audience: significance of dual $\mathrm{Hom}((\mathrm{S}^{\mathrm{c}}[s\mathfrak{g}], d), R)$?

$$
\begin{array}{ccc}
\mathrm{S}^{\mathrm{c}}_2[s\mathfrak{g}] & \xrightarrow{\ d\ } & s\mathfrak{g} \\
\big\uparrow & & \big\uparrow \\
\mathfrak{g} \otimes \mathfrak{g} & \xrightarrow{[\,\cdot\,,\,\cdot\,]} & \mathfrak{g}
\end{array}
$$

Higher homotopies generalization, continued

$\mathfrak{g}$ graded module over the ground ring R

sh-Lie structure or L_∞*-structure on* $\mathfrak{g}$:

coalgebra differential d on $\mathrm{S}^{\mathrm{c}}[s\mathfrak{g}]$: $d = d^1 + d^2 + \dots$

brackets $[\,\cdot\,,\,\cdot\,]_{j+1}$

$$\begin{array}{ccc} \mathrm{S}^{\mathrm{c}}_{j+1}[s\mathfrak{g}] & \xrightarrow{d^j} & s\mathfrak{g} \\ \uparrow & & \uparrow \\ \mathfrak{g}^{\otimes(j+1)} & \xrightarrow{[\,\cdot\,,\,\cdot\,]_{j+1}} & \mathfrak{g} \end{array}$$

dual $\mathrm{Hom}((\mathrm{S}^{\mathrm{c}}[s\mathfrak{g}], d), R)$: generalized *Maurer-Cartan algebra*

Can we phrase the notion of (A, L) *sh Lie-Rinehart algebra?*

If so, *what does it signify and does it occur in nature?*

dichotomy Lie-Rinehart structure (A, L):

L an A-module

R-Lie algebra that acts on A by derivations

prevents us from phrasing a naive sh Lie algebra characterization in terms of coalgebra differential d on $\mathrm{S}^{\mathrm{c}}[sL]$

Quasi Lie-Rinehart algebras [Hue05]

$(M, \mathcal{F})$ foliated manifold, $A = C^\infty(M)$
$\tau_{\mathcal{F}} \colon \mathrm{T}\mathcal{F} \to M$ tangent bundle to $\mathcal{F}$
$H = \Gamma(\tau_{\mathcal{F}})$: (A, H) Lie-Rinehart algebra (Frobenius)
Q a complement of H (space of sections of normal bundle)

$$\mathrm{Vect}(M) = H \oplus Q = \Gamma(\tau_{\mathcal{F}}) \oplus Q$$

$$(A, H, Q) = (C^\infty(M), L_{\mathcal{F}}, Q) \quad \textit{Lie-Rinehart triple}$$

$$(\mathcal{A}, \mathcal{Q}) = (\mathrm{Alt}_A(H, A), \mathrm{Alt}_A(H, Q)) \quad \textit{quasi-Lie-Rinehart algebra}$$

$\mathcal{A}$ "algebra of generalized functions"
$\mathcal{Q}$ "generalized Lie algebra of vector fields"
plus structure of mutual interaction, made precise shortly

$$(\mathrm{H}^*(\mathcal{A}), \mathrm{H}^*(\mathcal{Q})) \colon \quad \text{graded Lie-Rinehart algebra}$$

$$(A^H, Q^H) = (\mathrm{H}^0(\mathcal{A}), \mathrm{H}^0(\mathcal{Q})) \quad \text{Lie-Rinehart algebra}$$

$$\mathrm{H}^0(\mathcal{A}) \colon \quad \text{functions on space of leaves}$$

$$\mathrm{H}^0(\mathcal{Q}) \colon \quad \text{vector fields on space of leaves}$$

but $(\mathcal{A}, \mathcal{Q})$ contains more information: history

Quasi Lie-Rinehart algebras continued

Special case: foliation from smooth fiber bundle $\pi\colon E \to B$

$$(\mathrm{H}^0(\mathcal{A}), \mathrm{H}^0(\mathcal{Q})) = (C^\infty(B), \mathrm{Vect}(B))$$

General case: structure of graded Lie-Rinehart algebra $(\mathrm{H}(\mathcal{A}), \mathrm{H}(\mathcal{Q}))$ more complicated than $(\mathrm{H}^0(\mathcal{A}), \mathrm{H}^0(\mathcal{Q}))$
$(\mathcal{A}, \mathcal{Q}) = (\mathrm{Alt}_A(H, A), \mathrm{Alt}_A(H, Q))$ even more complicated

Quasi Lie-Rinehart algebras continued

$$\begin{aligned}
\mathrm{Vect}(M) &= H \oplus Q = \Gamma(\tau_{\mathcal{F}}) \oplus Q \\
(A, H, Q) &= (C^\infty(M), L_{\mathcal{F}}, Q) \quad \textit{Lie-Rinehart triple} \\
(\mathcal{A}, \mathcal{Q}) &= (\mathrm{Alt}_A(H, A), \mathrm{Alt}_A(H, Q)) \quad \textit{quasi-Lie-Rinehart algebra}
\end{aligned}$$

True invariant structure:
filtration of $\mathrm{Alt}_A(Q, \mathrm{Alt}_A(H, A))$ by Q-degree leads to spectral sequence $(\mathrm{E}_r^{*,*}, d_r)$ having

$$\begin{aligned}
(\mathrm{E}_0, d_0) &= (\mathrm{Alt}_A(Q, \mathrm{Alt}_A(H, A)), d_0) \\
\mathrm{E}_1^{p,q} &= \mathrm{H}^q(H, \mathrm{Alt}_A^p(Q, A)), \quad \mathrm{E}_1^{*,0} = \mathrm{H}(\mathcal{A})
\end{aligned}$$

and E_2 can be seen as an instance of Rinehart cohomology of the graded Lie-Rinehart algebra $(\mathrm{H}(\mathcal{A}), \mathrm{H}(\mathcal{Q}))$

filtration is intrinsic, not just a technical tool

spectral sequence an invariant of the entire structure

fiber bundle case: spectral sequence ordinary fiber bundle ss

Quasi Lie-Rinehart algebras continued

Promised definition of quasi-Lie-Rinehart algebra:
$\mathcal{A}$ differential graded commutative algebra; notation $A = \mathcal{A}^0$
$\mathcal{Q}$ differential graded $\mathcal{A}$-module, $\mathcal{Q} = \mathcal{A} \otimes_A Q$ as graded $\mathcal{A}$-module
— a graded skew-symmetric R-bilinear pairing of degree zero

$$[\,\cdot\,,\cdot\,]_{\mathcal{Q}} \colon \mathcal{Q} \otimes_R \mathcal{Q} \to \mathcal{Q},$$

— an R-bilinear pairing of degree zero

$$\mathcal{Q} \otimes_R \mathcal{A} \to \mathcal{A}$$

— an A-trilinear operation of degree -1

$$\langle \cdot, \cdot\,; \cdot \rangle_{\mathcal{Q}} \colon Q \otimes_A Q \otimes_A \mathcal{A} \to \mathcal{A}$$

$(\mathcal{A}, \mathcal{Q})$ *quasi Lie-Rinehart algebra*: axioms imposed on these pieces of structure; enable us to define

$$(\mathrm{Alt}_{\mathcal{A}}(\mathcal{Q}, \mathcal{A}), d_0, d_1, d_2) \quad \text{Maurer-Cartan algebra}$$

Quasi Lie-Rinehart algebras continued

$\partial^1_{[\,\cdot\,,\,\cdot\,]}\colon \mathrm{S}[s\mathcal{Q}] \to \mathrm{S}[s\mathcal{Q}]$ coderivation induced by

$$[\,\cdot\,,\,\cdot\,]_{\mathcal{Q}}\colon \mathcal{Q} \otimes_R \mathcal{Q} \to \mathcal{Q}$$

$t_1\colon \mathrm{S}^1[s\mathcal{Q}] = s\mathcal{Q} \to \mathrm{Der}(\mathcal{A}|R)$ the degree -1 morphism of R-modules composite of desuspension with adjoint of

$$\mathcal{Q} \otimes_R \mathcal{A} \to \mathcal{A}$$

$\partial_1^{[\,\cdot\,,\,\cdot\,]}$ derivation on $\mathrm{Hom}(\mathrm{S}[s\mathcal{Q}], \mathcal{A})$ induced by $\partial^1_{[\,\cdot\,,\,\cdot\,]}$
∂^{t_1} derivation on $\mathrm{Hom}(\mathrm{S}[s\mathcal{Q}], \mathcal{A})$ induced by t_1,
compatibility conditions imply that $\mathcal{D}_1 = \partial_1^{[\,\cdot\,,\,\cdot\,]} + \partial^{t_1}$ passes to derivation $\mathcal{D}_1$ on $\mathrm{Alt}_A(Q, \mathcal{A})$, formally CCE operator; recall

Proposition

When (A, L) is a Lie-Rinehart algebra, derivation $\mathcal{D} = \partial^{[\,\cdot\,,\,\cdot\,]} + \partial^t$ descends to R-linear differential on $\mathrm{Alt}_A(L, A)$, even though this is not true of the individual operators ∂^t and $\partial^{[\,\cdot\,,\,\cdot\,]}$ unless $A = R$ and ∂^t trivial.

Multi derivation Maurer-Cartan algebras and sh-Lie-Rinehart algebras [Hue13]

A differential graded commutative algebra, L an A-module
$\partial_{[\,\cdot\,,\,\cdot\,]} = \partial^1_{[\,\cdot\,,\,\cdot\,]} + \partial^2_{[\,\cdot\,,\,\cdot\,]} + \ldots$ degree -1 coderivation on $\mathrm{S}[sL]$
$t = t^1 + t^2 + \ldots : \mathrm{S}[sL] \longrightarrow \mathrm{Der}(A|R)$ "Lie algebra twisting cochain"
(A, L) *sh Lie-Rinehart algebra* compatibility conditions
$\mathrm{Sym}_A(sL, A)$: A-multilinear A-valued graded symmetric maps on sL
$\mathcal{D}_0$ algebra diff'l on $\mathrm{Sym}_A(sL, A)$ induced from diff's on A and L
induced derivations $\partial_j^{[\,\cdot\,,\,\cdot\,]}$ and ∂^{t_j} on $\mathrm{Sym}_A(sL, A)$
$\mathcal{D}_j = \partial_j^{[\,\cdot\,,\,\cdot\,]} + \partial^{t_j}$ derivation on $\mathrm{Sym}_A(sL, A)$

Theorem (Main result)

The data $(A, L, \partial_{[\,\cdot\,,\,\cdot\,]}, t)$ constitute an sh Lie-Rinehart algebra if and only if $(\mathrm{Sym}_A(sL, A), \mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \ldots)$ is a multi derivation chain algebra, necessarily the multi derivation Maurer-Cartan algebra associated to $(A, L, \partial_{[\,\cdot\,,\,\cdot\,]}, t)$.

Recall from section on ordinary Lie-Rinehart algebras

Theorem

Given a pair that consists of a commutative algebra A and an A-module L, under suitable mild hypotheses (e. g. L finitely generated and projective as an A-module), Lie-Rinehart algebra structures on the pair (A, L) correspond bijectively to operators d turning the graded A-algebra $\mathrm{Alt}_A(L, A)$ into a differential graded algebra over the ground ring R (beware: not over A)

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