Line bundles on moduli and related spaces

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Abstract

Let G be a Lie goup, let M and N be smooth connected G-manifolds, let $f\colon M \to N$ be a smooth G-map, and let P_f denote the *fiber* of f. Given a closed and equivariantly closed relative 2-form for f with integral periods, we construct the principal G-circle bundles with connection on P_f having the given relative 2-form as curvature. Given a compact Lie group K, a biinvariant Riemannian metric on K, and a closed Riemann surface Σ of genus ℓ, when we apply the construction to the particular case where f is the familiar relator map from $K^{2\ell}$ to K, which sends the 2ℓ-tuple $(a_1, b_1, \ldots, a_\ell, b_\ell)$ of elements a_j, b_j of K to $\prod[a_j, b_j]$, we obtain the principal K-circle bundles on the associated extended moduli spaces which, via reduction, then pass to the corresponding line bundles on possibly twisted moduli spaces of representations of $\pi_1(\Sigma)$ in K, in particular, on moduli spaces of semistable holomorphic vector bundles or, more precisely, on a smooth open stratum when the moduli space is not smooth. The construction also yields an alternative geometric object, distinct from the familiar gerbe construction, representing the fundamental class in the third integral cohomology group of K or, equivalenlently, the first Pontrjagin class of the classifying space of K.

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Contents

1 Origins and motivation

G Lie group bi-invariant Riemannian metric
Σ Riemann surface
$\xi\colon P \to \Sigma$ principal bundle
$\pi = \pi_1(\Sigma)$
moduli space $\mathrm{Rep}(\pi, G)$
more generally: twisted moduli space $\mathrm{Rep}_\xi(\pi, G)$
$G = \mathrm{U}(n)$: ms's ss holom. v bundles on Σ
[Narasimhan-Seshadri]
[Atiyah-Bott] gauge theory: ms's projectively flat constant central curvature connections on principal bundle over Σ
structure on these spaces:
sometimes: compact Kähler [NS]
more generally stratified symplectic
[Karshon], [Weinstein]
[Huebschmann-Jeffrey]: Extended ms's:
ms by symplectic reduction from suitable finite-dimensional hamiltonian G-space
[Alexeiev-Meinrenken]: reworked in language of quasi-hamiltonian G-spaces

[Huebschmann] gen. gauge theory situation
application: purely combinatorial construction of Chern-Simons function over 3-manifold
side remark: stratified Kähler
integral: line bundle
symplectic structure (more precisely: on smooth stratum) integral
Weinstein: construct line bundle
or principal S^1-bundle, S^1 circle group
purpose of talk: solution of this problem
line bundle or principal circle bundle not necessarily defined on moduli space itself
we will present construction of
G-equivariant circle bundle
on extended moduli space
we will abstract from particular case
explore more general case:

G-equivariant smooth map $f\colon M \to N$
together with (i) and (ii)
(i) closed G-equivariant relative 2-form (ζ, λ)
with integral periods
ζ a G-invariant 2-form on M
λ a G-invariant 3-form on N
subject to: $d\zeta = f^*\lambda$
integral periods:
given 3-mfold C and commutative diagram

$$\begin{array}{ccc} \partial C & \longrightarrow & C \\ h\big\downarrow & & \big\downarrow H \\ M & \xrightarrow{f} & N \end{array} \qquad h, H \quad \text{(piecewise) smooth:}$$

difference $\displaystyle\int_C H^*(\lambda) - \int_{\partial C} h^*(\zeta)$ integer
(ii) requisite additional technical ingredient
to carry out construction G-equivariantly
encoded in G-equivariant linear map
$\vartheta\colon \mathfrak{g} \longrightarrow \mathcal{A}(N)$, $\mathfrak{g}$ Lie algebra of G
values in space $\mathcal{A}(N)$ of 1-forms on N

aim: principal S^1-bundle on fiber P_f of f
G-equivariantly
having characteristic class represented by (ζ, λ)
ϑ contains information needed to construct
G-momentum mapping from P_f to $\mathfrak{g}^*$:
additional constituent to extend (ζ, λ)
more precisely: associated closed G-invariant
2-form $\zeta_{(f,\lambda,\zeta)}$ on P_f to
equivariantly closed 2-form
special case: ℓ genus of Σ
compact connected Lie group K
choose biinvariant Riemannian metric on K
$M = K^{2\ell}$, $N = K$, f relator map $K^{2\ell} \to K$
$f(a_1, b_1, \ldots, a_\ell, b_\ell) = \prod [a_j, b_j]$
λ fundamental 3-form on K (E. Cartan)
ζ and ϑ forms explored in quoted sources
yields solution of Weinstein's problem:
extended moduli spaces lie K-equivariantly in
fiber P_f of relator map f
K-equivariant principal S^1-bundles on
extended moduli space obtained by restriction

2 Basic idea

o base point of M, $f(o)$ base point of N
I unit interval, I^2 unit square

$$j_1 \colon I \longrightarrow I^2,\ j_1(t) = (t,0) \in I^2$$

construct total space $\widehat{P}_f$ of principal S^1-bundle on fiber P_f of f as a space of equivalence classes of *strings*

$$\begin{array}{ccccc} \{0\} & \longrightarrow & I & \stackrel{j_1}{\longrightarrow} & I^2 \\ \Big\downarrow & & \Big\downarrow w & & \Big\downarrow \phi \\ \{o\} & \longrightarrow & M & \stackrel{f}{\longrightarrow} & N \end{array}$$

suitable equivalence classes of strings of that kind: second relative homotopy group $\pi_2(f)$
in general: principal Γ_f-bundle $\widehat{P}_f \to P_f$
structure group Γ_f split central extension

$$1 \longrightarrow S^1 \longrightarrow \Gamma_f \longrightarrow \pi_1(P_f) \longrightarrow 1$$

construction of principal bundle
sort of "blows up" the construction of second homotopy group $\pi_2(P_f)$ as a second relative homotopy group $\pi_2(f)$

3 Further illustration

via holonomy, fiber of relator map $K^{2\ell} \to K$: based h' equiv to $\mathrm{Map}^o(\Sigma, BK)$, space of based maps from Σ to classifying space BK of K

$$\pi_0(\mathrm{Map}^o(\Sigma, BK)) \leftrightarrow K\text{-bundles on } \Sigma$$

each path comp of $\mathrm{Map}^o(\Sigma, BK)$ classifying space of associated group of based gauge trafos

K-equiv. princip. S^1-bundle on P_f yields K-equiv. princip. S^1-bundle on $\mathrm{Map}^o(\Sigma, BK)$

association can be made functorial in terms of geometric presentations of surface variable Σ

geometric object which thereby results represents cohomology class given by Cartan 3-form

alternative to equivariant gerbe representing first Pontrjagin class of classifying space of K

present approach can be extended to construction of principal S^1-bundles in more general situation of *equivariant plots* for arbitrary gauge theory situation

extended moduli space special case of equivariant plot

4 Reconstruction of circle bundle from curvature

4.1 Goal

$f\colon M \to N$ smooth, P_f *fiber* of f
closed relative 2-form for f integral periods
P_f and "integral periods" precise below
principal S^1-bundles with connection on P_f
having given relative 2-form as curvature
construct total space from space E_f of strings

$$\begin{array}{ccccc} \{0\} & \longrightarrow & I & \xrightarrow{j_1} & I^2 \\ \big\downarrow & & \big\downarrow w & & \big\downarrow \phi \\ \{o\} & \longrightarrow & M & \xrightarrow{f} & N \end{array}$$

identification of strings to classes (points of the total space) involves relative 2-form
construction complicated
explain analogous much simpler construction of equivariant S^1-bundle on ordinary space
conclude with hints over P_f
possible connections with stringy world remain yet to be explored !

4.2 Lifting functions, "topological horizontal lift"

N space
o *base point* point of N
$P_o(N)$ space of paths in N, starting at o
$p_o \colon P_o(N) \to N$ path to its end point
fibration onto path component of o
fiber $p_o^{-1}(o)$:
$\Omega_o(N)$ closed based loops in N based at o
familiar construction:
universal cover $\widetilde{N}$ of N from $P_o(N)$

variant yields principal S^1-bundles on N:
$B^I = \mathrm{Map}(I, B)$
$p\colon E \to B$ map

$$p_0\colon B^I \longrightarrow B,\ (u\colon I \to B) \longmapsto u(0)$$

$E \times_B B^I$ associated fiber product

$$p^I\colon E^I \longrightarrow E \times_B B^I,\ w \longmapsto (w(0), p \circ w)$$

lifting function for p, "horizontal lift"

$$\lambda\colon E \times_B B^I \longrightarrow E^I$$

right-inverse for p^I, i. e.

$$p^I \circ \lambda = \mathrm{Id}\colon E \times_B B^I \longrightarrow E \times_B B^I$$

$p\colon E \to B$ fibration $\Leftrightarrow$ p admits lifting fn
pick base points $o \in B$, $o \in E$ with $p(o) = o$
choice of base points induces injection
$j_o\colon P_o(B) \to E \times_B B^I$
given lifting function $\lambda\colon E \times_B B^I \longrightarrow E^I$ for
$p\colon E \to B$, composite

$$\gamma\colon P_o(B) \xrightarrow{j_o} E \times_B B^I \xrightarrow{\lambda} E^I \xrightarrow{p_o} E$$

map over B hence morphism of fibrations

4.3 Circle bundles

$\tau\colon S \to N$ topological principal S^1-bundle
choose lifting function for τ
pick pre-image o in S of o
restrict $\gamma\colon P_o(B) \to E$ to $\Omega_o(N)$:

$$\gamma_o\colon \Omega_o(N) \longrightarrow S^1$$

homomorphism relative to composition of loops
"topological holonomy" of τ
determined by the lifting function
under transgression (here iso) $[\gamma_o] \in \mathrm{H}^1(\Omega_o(N))$
goes to characteristic class $[\tau] \in \mathrm{H}^2(N)$ of τ
reconstruct S^1-bundle τ:
identify two paths w_1, w_2 in N having o as starting point and having the same end point provided composite $w_2^{-1}w_1$, which is a closed path in $\Omega_o(N)$, has value $1 \in S^1$ under γ_0
map γ from $P_o(N)$ to S identifies space of equivalence classes in $P_o(N)$ with S

4.4 The differential-geometric construction

4.4.1 Warm-up: construction of integral cohomology classes

N smooth manifold
α closed 1-form on N: real cohomology class
α integral periods: recover integral class:
pick base point o of N
given point x of N, w_x path joining o to x

$$F\colon N \longrightarrow S^1,\ F(x) = \int_{w_x} \alpha \mod \mathbb{Z}$$

well defined since α integral periods
F represents class in $\mathrm{H}^1(N, \mathbb{Z})$
below exploit variants of this construction

4.4.2 Differential-geometric variant of topological construction

N smooth manifold
lifting functions provided by horizontal lift relative to a connection
aim: reconstruct S^1-bundle from holonomy
c closed 2-form on N with integral periods
$P_o(N)$ space of *piecewise smooth* paths in N starting at o
$\Omega_o(N)$ *piecewise smooth* closed based loops in N, based at o
$\Omega_o(N)_0$ piecewise smooth closed loops homotopic to zero relative o
identify piecewise smooth paths w_1 and w_2 homotopic under piecewise smooth homotopy h from w_1 to w_2 relative to endpoints such that $\int_{I\times I} h^*c$ an integer
c integral periods: condition independent of choice of homotopy h
$\overline{S}$ space of equivalence classes
obvious projection maps

$$\overline{\tau}\colon \overline{S} \to \widetilde{N},\ \widehat{\tau}\colon \overline{S} \to N$$

Γ: equivalence classes closed loops at o
composition closed loops: Γ a group
surjective maps

$$\Omega_o(N) \longrightarrow \Gamma,\ u \mapsto [u]$$
$$\Omega_o(N)_0 \longrightarrow S^1, u \mapsto \int_{I\times I} h^*c \mod \mathbb{Z}$$

(when c represents non-trivial S^1-bundle)
h a null homotopy of u rel to o
commutative diagram

$$\begin{array}{ccccc}
\Omega_o(N)_0 & \longrightarrow & \Omega_o(N) & \longrightarrow & \pi_1(N) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \mathrm{Id} \\
S^1 & \longrightarrow & \Gamma & \longrightarrow & \pi_1(N)
\end{array}$$

composition of paths

$$P_o(N) \times \Omega_o(N) \longrightarrow P_o(N)$$

principal Γ-bundle $\widehat{\tau}$: $\overline{S} \longrightarrow N$
principal S^1-bundle $\overline{\tau}$: $\overline{S} \to \widetilde{N}$

$\tau\colon S \to N$ principal S^1-bundle
connection 1-form ω having curvature c
horizontal lift rel. to ω: commutative diagram

$$\begin{array}{ccccc} \Gamma & \longrightarrow & \overline{S} & \stackrel{\overline{\tau}}{\longrightarrow} & N \\ \downarrow & & \downarrow & & \downarrow \mathrm{Id} \\ S^1 & \longrightarrow & S & \stackrel{\tau}{\longrightarrow} & N \end{array}$$

left-hand unlabelled vertical homomorphism:
from holonomy $\Omega_o(N) \to S^1$
splits $1 \to S^1 \longrightarrow \Gamma \to \pi_1(N) \to 1$:
splittings $\Gamma \cong S^1 \times \pi_1(N)$ correspond to choices of principal S^1-bundles on N
with connection having curvature c
homotopy operator:

$$\eta\colon \mathcal{A}^*(P_o(N)) \to \mathcal{A}^{*-1}(P_o(N))$$

by integration along the paths which constitute the points of $P_o(N)$

$$d\eta + \eta d = \mathrm{Id}$$

integration of c along the paths which constitute the points of $P_o(N)$:

1-form $\vartheta_c = \eta(p_o^* c)$ on $P_o(N)$ such that

$$p_o^*(c) = d\vartheta_c$$

1-form ϑ_c descends via $P_o(N) \to \overline{S}$ to Γ-conn'n $\overline{\omega}_c\colon \mathrm{T}\overline{S} \longrightarrow \mathbb{R}$ on $\overline{S}$ having curv. c connection form $\overline{\omega}_c$ descends via $\overline{S} \to S$ to ω *Given the splitting* $\sigma\colon \Gamma \to S^1$, *the induced principal* S^1*-bundle*

$$\tau_\sigma = \sigma_*(\widehat{\tau})\colon S_\sigma \longrightarrow N$$

with conn'n $\omega_\sigma = \sigma_*(\omega_c)$ *has curvature* c.
The group $\mathrm{H}^1(\pi_1(N), S^1) = \mathrm{Hom}(\pi_1(N), S^1)$ *acts simply transitively on the isomorphism classes of principal* S^1*-bundles with connection on* N *having curvature* c.
Two such principal S^1*-bundles with connection topologically equivalent* $\Leftrightarrow$ *"difference" in* $\mathrm{Hom}(\pi_1(N), S^1)$ *lifts to homomorphism from* $\pi_1(N)$ *to* $\mathbb{R}$.
N *simply connected: up to gauge trafo, there is a unique principal* S^1*-bundle with connection on* N *having curvature* c.

5 The equivariant extension

G Lie group, $\mathfrak{g}$ Lie algebra, N a G-manifold
infinitesimal $\mathfrak{g}$-action $\mathfrak{g} \to \mathrm{Vect}(N)$
$C^\infty(N)$ a $\mathfrak{g}$-module

$$d_{\mathfrak{g}} \colon \mathrm{Alt}(\mathfrak{g}, C^\infty(N)) \longrightarrow \mathrm{Alt}(\mathfrak{g}, C^\infty(N))$$

CCE Lie algebra cohomology operator
c a G-invariant 2-form on N
$\tau \colon S \to N$ principal S^1-bundle on N
with connection ∇ having curvature c
G_τ group of pairs (ϕ, x)
$\phi \colon S \to S$ bundle auto
on base N: diffeo x_N induced from $x \in G$
c G-invariant: extension

$$1 \longrightarrow \mathcal{G}(\tau) \longrightarrow G_\tau \longrightarrow G \longrightarrow 1$$

$\mathcal{G}(\tau) \cong \mathrm{Map}(N, S^1)$ ab grp gauge trafos of τ
conjugation in G_τ induces G-action on $\mathcal{G}(\tau)$
same as coming from G-action on N

$$\mathfrak{g}(\tau) \cong \mathrm{Map}(N, \mathbb{R}) = C^\infty(N)$$

abelian Lie alg. infinitesimal gauge trafos of τ
G- and hence $\mathfrak{g}$-module

associated Lie algebra extension:

$$0 \longrightarrow \mathfrak{g}(\tau) \longrightarrow \mathfrak{g}_\mathcal{T} \longrightarrow \mathfrak{g} \longrightarrow 0$$

via infinitesimal $\mathfrak{g}$-action on N:
connection ∇ induces section $\nabla_\mathfrak{g} \colon \mathfrak{g} \to \mathfrak{g}_\mathcal{T}$ in category of vector spaces
$c_\mathfrak{g} \in \mathrm{Alt}^2(\mathfrak{g}, C^\infty(N))$:
$C^\infty(N)$-valued Lie algebra 2-cocycle on $\mathfrak{g}$ determined by $\nabla_\mathfrak{g}$ and Lie algebra extension
$X \in \mathfrak{g}$: X_N fundamental vector field on N
momentum mapping for c: G-equivariant map $\mu \colon N \to \mathfrak{g}^*$ such that *comomentum* (adjoint) $\mu^\sharp \colon \mathfrak{g} \to C^\infty(N)$ satisfies

$$d(\mu^\sharp(X)) = c(X_N, \,\cdot\,), \ X \in \mathfrak{g};$$

connection ∇ and 2-form c being fixed comomenta are precisely the G-equivariant $C^\infty(N)$-valued 1-cochains δ on $\mathfrak g$ such that

$$d_{\mathfrak g}(\delta) = c_{\mathfrak g} \in \mathrm{Alt}(\mathfrak g, C^\infty(N))$$

in particular, each such comomentum

$$\delta\colon \mathfrak g \longrightarrow \mathfrak g(\tau) \cong \mathrm{Map}(N, \mathbb R) = C^\infty(N)$$

yields Lie algebra section

$$\nabla_{\mathfrak g} + \delta\colon \mathfrak g \longrightarrow \mathfrak g_\tau$$

for Lie algebra extension
well known and classical:

Proposition. *When G is connected, a momentum mapping $\mu\colon N \to \mathfrak g^*$ for c induces a lift of the G-action on N to an action of a suitable covering group $\widetilde G$ on the total space S compatible with the S^1-bundle structure and thus turning τ into a $\widetilde G$-equivariant principal S^1-bundle, and every such lift induces a momentum mapping for c.*
The connection ∇ on τ is then $\widetilde G$-invariant.

6 Circle bundles on the fiber of a map

M, N path connected spaces, $f\colon M \to N$
ultimate goal: relator map $K^{2\ell} \to K$

6.1 Fiber of a map

o base point of M, $f(o)$ base point of N
P_f *fiber* of f:

$$\begin{aligned} P_f &= M \times_N P_{f(o)}(N) \subseteq M \times P_{f(o)}(N) \\ &= \{(q,u); u(0) = f(o), u(1) = f(q)\} \end{aligned}$$

projection $\pi_f\colon P_f \longrightarrow M,\ (q,u) \longmapsto q \in M$
$j_f\colon P_f \to P_{f(o)}(N),\ (q,u) \longmapsto u$

$$\begin{array}{ccc} P_f & \xrightarrow{j_f} & P_{f(o)}(N) \\ {\scriptstyle \pi_f}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{f(o)}} \\ M & \underset{f}{\longrightarrow} & N \end{array} \qquad \text{pull back}$$

π_f fibration, fiber $\pi_f^{-1}(o) = \Omega_{f(o)}(N)$

6.2 The construction in a nutshell

substitute for space of paths $P_o(N)$:
space E_f of strings of the kind

$$\begin{array}{ccccc} \{0\} & \longrightarrow & I & \xrightarrow{j_1} & I^2 \\ \Big\downarrow & & \Big\downarrow w & & \Big\downarrow \phi \\ \{o\} & \longrightarrow & M & \xrightarrow{f} & N \end{array}$$

such that, for $0 \leq s \leq 1$,
(i) $\phi(0,s) = f(o)$
(ii) $\phi(1,s)$ independent of s
E_f contractible

additional ingredient that corresponds to
2-form c with integral periods:
pair (ζ,λ), ζ 2-form on M, λ 3-form on N
$d\zeta = f^*\lambda$
(ζ,λ) integral periods: given 3-mfold C and

$$\begin{array}{ccc} \partial C & \longrightarrow & C \\ h\big\downarrow & & \big\downarrow H \\ M & \stackrel{f}{\longrightarrow} & N \end{array} \quad \text{CD, } h, H \quad \text{(piecewise) smooth:}$$

difference $\displaystyle\int_C H^*(\lambda) - \int_{\partial C} h^*(\zeta)$ integer
imposing on contracting homotopies a constraint defined in terms of (ζ,λ) formally of
the same kind as the constraint imposed on homotopies among paths via c
we obtain space $\widehat{P}_f$ such that
obvious projection map $\widehat{\tau}_f \colon \widehat{P}_f \to P_f$
principal bundle whose structure group Γ_f (say)
S^1 path component of identity
data determine closed 2-form $\zeta_{(f,\lambda,\zeta)}$ on P_f
and connection on $\widehat{\tau}_f$ with curvature $\zeta_{(f,\lambda,\zeta)}$

Theorem. *The projection* $\widehat{\tau}_f \colon \widehat{P}_f \to P_f$ *is a principal* Γ_f*-bundle, and the data determine a* Γ_f*-connection, with connection form* $\omega_{(f,\lambda,\zeta)}$ *on* $\widehat{P}_f$*, having curvature* $\zeta_{(f,\lambda,\zeta)}$.

$\sigma \colon \Gamma_f \to S^1$ splitting of

$$1 \longrightarrow S^1 \longrightarrow \Gamma_f \longrightarrow \pi_1(P_f) \longrightarrow 1$$

induced principal S^1-bundle

$$\tau_\sigma = \sigma_*(\widehat{\tau}_f) \colon S_{\sigma,f} \longrightarrow P_f$$

with connection $\omega_{\sigma,f,\lambda,\zeta} = \sigma_*(\omega_{(f,\lambda,\zeta)})$ has curvature $\zeta_{(f,\lambda,\zeta)}$

7 The case where the target is a Lie group

7.1 Equivariant Maurer-Cartan calculus

H Lie group, $\mathfrak{h}$ its Lie algebra
H an H-group via conjugation
$\cdot$ invariant symmetric bilinear form on $\mathfrak{h}$
ω_H ($\overline{\omega}_H$) left-(right)-invariant MC
triple pr. $(x, y, z) \mapsto [x, y] \cdot z$ 3-form on $\mathfrak{h}$
left translate closed biinvariant 3-form λ on H
α any form on H: α_j pullback to $H \times H$ by projection p_j to j'th component
$\Omega = \frac{1}{2}\omega_1 \cdot \overline{\omega}_2 :\quad$ 2-form on $H \times H$
$\vartheta \in \mathcal{A}^{2,1}_H(H)$: H-invariant map $\vartheta\colon \mathfrak{h} \to \mathcal{A}^1(H)$
adjoint $\vartheta^\flat \in \mathcal{A}^1(H, \mathfrak{h}^*)$: $\frac{1}{2}(\omega + \overline{\omega})$, combined with adjoint $\mathfrak{h} \to \mathfrak{h}^*$ of 2-form on $\mathfrak{h}$; when we view $X \in \mathfrak{h}$ as constant $\mathfrak{h}$-valued 0-form on H

$$\vartheta(X) = \frac{1}{2} X \cdot (\omega + \overline{\omega})$$

Ω crucial ingredient symplectic structures on moduli spaces equivariant Maurer-Cartan calculus:

$$\begin{aligned} d\Omega &= \delta\lambda \\ \delta\Omega &= 0 \\ \delta_H\Omega &= -\delta\vartheta \\ d\lambda &= 0 \\ \delta_H\lambda &= d\vartheta \\ \delta_H\vartheta &= 0 \end{aligned}$$

(i) $\Omega - \lambda$ equivariant closed form (*not* equivariantly closed) of (total) degree 4
(ii) form $Q_4 = \Omega - \lambda + \vartheta$ equivariantly closed element of (total) degree 4 in total complex $(\mathcal{A}_H^{*,*}(H^*); d, \delta, \delta_H)$ of equivariant bar de Rham

$\lambda = \frac{1}{12}[\omega_H, \omega_H] \cdot \omega_H$ Cartan 3-form on H

$\vartheta^\flat = \frac{1}{2}(\omega_H + \overline{\omega}_H) \in \mathcal{A}^1(H, \mathfrak{h})$

$\vartheta \in \mathcal{A}^1(H, \mathfrak{h}^*)$: $\mathfrak{h}^*$-valued 1-form on H
composite of $\vartheta^\flat$ with adjoint $\mathfrak{h} \to \mathfrak{h}^*$ of form

$$\delta_H\vartheta = 0, \quad \delta_H\lambda = -d\vartheta$$

H compact: inv. inner product on $\mathfrak{h}$ exists
Cartan 3-form λ integral periods

7.2 Application

$f\colon M \to H$: more structure available
$P_e(H)$ and $\Omega_e(H)$ groups
projection $\pi_e\colon P_e(H) \to H$ homomorphism
principal $\Omega_e(H)$-fiber bundle
$\pi_f\colon P_f \to M$ principal $\Omega_e(H)$-fiber bundle
H connected, $\pi_2(H)$ is zero
Γ simply S^1, earlier diagram now

$$\begin{array}{ccccc}
S^1 & \longrightarrow & \widehat{\Omega_e(H)} & \longrightarrow & \Omega_e(H) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\Gamma_f & \longrightarrow & \widehat{P}_f & \longrightarrow & P_f \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\pi_1(M) & \longrightarrow & \widetilde{M} & \longrightarrow & M
\end{array}$$

when $\widehat{\Omega_e(H)}$ acquires grp structure, e. g. H simply conn, $\widehat{P}_f \to \widetilde{M}$ principal $\widehat{\Omega_e(H)}$-bundle

$$0 \longrightarrow S^1 \longrightarrow \widehat{\Omega_e(H)} \longrightarrow \Omega_e(H) \longrightarrow 1$$

universal central extension loop gp
connections with stringy world?

8 Application to moduli spaces

Σ closed surface, genus ℓ
K compact connected Lie group
$\cdot$ inv. inner product on Lie algebra $\mathfrak{k}$ of K

$$\mathcal{P} = \langle x_1, y_1, \dots, x_\ell, y_\ell; r\rangle,\ r = \prod [x_j, y_j],$$

presentation $\pi_1(\Sigma)$
K and $K^{2\ell}$: K-action by conjugation
relator map $r\colon K^{2\ell} \longrightarrow K$ K-equivariant
equivariant Maurer-Cartan calculus:
K-invariant 2-form ζ on $M = K^{2\ell}$ such that

$$d\zeta = r^*\lambda$$

(ζ, λ) arises from form of total degree 4 on simplicial model for class. space BK of K
represents *Pontrjagin* class: integral periods
(ζ, λ) integral periods
$r_*\colon \pi_1(K^{2\ell}) \longrightarrow \pi_1(K)$ trivial
choice central element $z \in \widetilde{K}$ determines lift

$$r_z\colon K^{2\ell} \longrightarrow \widetilde{K}$$

fiber P_{r_z} connected, even simply connected $(\pi_2(K) = 0)$

as z ranges over center of $\widetilde{K}$ or, equivalently, over fundamental group of K:
spaces P_{r_z} range over path components of fiber P_r of relator map r
P_r a K-space
take f to be any of the maps r_z as z ranges over center of $\widetilde{K}$
apply previous constructions with this f
$\zeta_{(r,\lambda,\zeta)}$ the closed K-invariant 2-form constructed separately on each path component of P_f of the kind P_{r_z}
integral periods, necessarily K-invariant
previous Theorem yields principal S^1-bundle $\tau_r \colon S \to P_r$ with conn $\omega_{(r,\lambda,\zeta)}$ and curv $\zeta_{(r,\lambda,\zeta)}$
each path component of P_r simply connected
S^1-bundle with connection uniquely determined by data up to gauge transformations

$$\vartheta \in \mathcal{A}^1(K, \mathfrak{k}^*) \cong \mathcal{A}^{2,1}(K)$$

form introduced before
K substituted for H

$$\delta_K(\zeta) = r^*(\vartheta) \in \mathcal{A}^{2,1}(K^{2\ell})$$

previous result yields momap $\mu_{f,\vartheta}\colon P_r \to \mathfrak{k}^*$
lift of K-action to $\widetilde{K}$: $\tau_r\colon S \to P_r$
$\widetilde{K}$-equivariant principal S^1-bdle with conn
construction natural in terms of data
extended moduli space $\mathcal{H}$ embeds
K-equivariantly into P_r
composite of injection with the momap $\mu_{f,\vartheta}$:
momentum mapping $\mu^\sharp$ on extended ms $\mathcal{H}$
2-form $\zeta_{(r,\lambda,\zeta)}$ restricts to 2-form on ext. ms
present construction recovers extended ms
S^1-bundle τ_r restricts to $\widetilde{K}$-equivariant
principal S^1-bndle on $\mathcal{H}$, Chern class $[\omega_{\zeta,\mathcal{P}}]$
symplectic reduction: moduli spaces of
(possibly) twisted reps of $\pi_1(\Sigma)$ in K
reduction carries principal S^1-bundle to
replacement for (in general missing) principal
S^1-bundle on moduli space

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