Stratified Kähler spaces, costratified Hilbert spaces, and singular holomorphic quantization

J. Huebschmann

USTL, UFR de Mathématiques
CNRS-UMR 8524
59655 Villeneuve d'Ascq Cédex, France
Johannes.Huebschmann@math.univ-lille1.fr

Varna, June 20, 2007

Abstract

The appearance of singularities in classical phase spaces is the rule rather than the exception. Within canonical quantization the implementation of singularities is far from being clear. To explore the possible impact of classical phase space singularities on quantum problems, I have developed the notions of stratified Kähler space and that of costratified Hilbert space. A costratified Hilbert space (over a stratified space) is a Hilbert space together with a system which consists of (closed) subspaces associated with the strata and their orthoprojectors. Holomorphic quantization on a stratified Kähler space leads to a costratified Hilbert space. The latter structure is thus, perhaps, the quantum structure which has the classical singularities as its shadow. In collaboration with G. Rudolph and M. Schmidt, within this approach, on a single spatial plaquette, a quantum lattice gauge theory relative to a compact structure group has been constructed which incorporates the classical singularities. For the special case where the structure group is the group SU(2), we discovered among other things non-trivial tunneling probabilities among strata. The approach involves a holomorphic Peter-Weyl theorem. A consequence thereof is the existence of a uniquely determined unitary isomorphism, established already by B. Hall, between a corresponding Schrödinger Hilbert space and a certain Hilbert space of holomorphic functions, and this isomorphism coincides with the corresponding Blattner-Kostant-Sternberg pairing map. The spectral decomposition of the energy operator then arises as a refinement of a Peter-Weyl decomposition. The purpose of the talk is to discuss some of the foundational issues of this program.

These results establish the first steps of a program aimed at developing lattice gauge models for quantum mechanics on spaces with singularities.

Contents

1 Correspondence principle

$$(M, C^\infty(M), \{\,\cdot\,,\,\cdot\,\}) \leftrightarrow (\mathcal{H}, [\,\cdot\,,\,\cdot\,])$$

DIRAC: Comparison mathematical structures $(M = \mathrm{T}^*Q, \sigma = -d\vartheta)$:
SCHRÖDINGER $\mathcal{H} = L^2(Q, dq)$
Internal degrees of freedom not necessarily separated into configuration and momentum variables: (M, σ) general *symplectic* or $(M, C^\infty(M), \{\,\cdot\,,\,\cdot\,\})$ general POISSON
polarization: $(\mathcal{H}, [\,\cdot\,,\,\cdot\,])$
vertical polarization of $M = \mathrm{T}^*Q$:
$\mathcal{H} = L^2(Q, dq)$
holomorphic polarization:
M necessarily KÄHLER
$\mathcal{H} = \mathcal{H}L^2(M, d\nu)$, a Hilbert space of holomorphic functions which are square integrable relative to a suitable measure.
Side remark, crucial later: pre-Hilbert space of holomorphic functions usually a Hilbert space unlike for a Hilbert space of the kind $L^2(Q, dq)$, the elements of a Hilbert space of the kind

$\mathcal{H}L^2(M, d\nu)$ are ordinary functions, not classes of functions.

Example: $Q = \mathbb{R}^n$, $\mathrm{T}^*Q \cong \mathbb{C}^n$ with ordinary complex structure having $z = q + ip$ as holomorphic variables, endowed with ordinary hermitian structure

Segal-Bargmann :

ε Liouville measure, $\kappa(q + ip) = p^2$

$$\mathcal{H}L^2(\mathbb{C}^n, \mathrm{e}^{-\kappa/\hbar}\varepsilon) \leftrightarrow L^2(\mathbb{R}^n, dq)$$

$\mathrm{e}^{-\kappa/\hbar}\varepsilon$ Gaussian

Singularities rule rather than the exception. Simple mecanical systems and solution spaces of field theories come with singularities.

Examples:

— ℓ particles in $\mathbb{R}^s$ with total angular momentum zero;

reduced classical phase space: space of complex symmetric $(\ell \times \ell)$-matrices of rank at most equal to $\min(s, \ell)$

special case $s = 3$ our solar system.

— ℓ harmonic oscillators in $\mathbb{R}^s$ with total angular momentum zero and constant energy: exotic projective variety

— Lattice gauge theory: K compact Lie

Lie algebra $\mathfrak{k}$ with invariant inner product

$$\mathrm{T}^*K \cong \mathrm{T}K \longrightarrow K \times \mathfrak{k} \longrightarrow K^{\mathbb{C}}$$

— complex structure on $K^{\mathbb{C}}$

— cotangent bundle symplectic structure on T^*K

combine to K-bi-invariant KÄHLER structure

$$Q = K^\ell,\ \mathrm{T}^*Q = \mathrm{T}^*K^\ell \cong (K^{\mathbb{C}})^\ell$$

K-symmetry by conjugation

reduced space

$$\mathrm{T}^*K^\ell//K \cong (K^{\mathbb{C}})^\ell//K^{\mathbb{C}}$$

special case $\ell = 1$, a single spatial plaquette: maximal torus $T \subseteq K$, rank r, Weyl group W

$$\mathrm{T}^*T \cong T^{\mathbb{C}} \cong (\mathbb{C}^*)^r$$

$$\mathrm{T}^*K//K \cong \mathrm{T}^*T/W \cong (\mathbb{C}^*)^r/W$$

even more special case:

$$K = \mathrm{SU}(2),\ K^{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C}),\ W \cong \mathbb{Z}/2$$

maximal torus $T \cong S^1$

$$\mathrm{T}^*T \cong T^{\mathbb{C}} \cong \mathbb{C}^*$$

$$\mathcal{P} = \mathrm{T}^*K//K \cong \mathrm{T}^*T/W \cong \mathbb{C}^*/W$$

W-invariant holomorphic map

$$f \colon C^* \to \mathbb{C},\ f(z) = z + z^{-1}$$

induces a complex analytic isomorphism

$$\mathbb{C}^*/W \longrightarrow \mathbb{C}$$

More generally: $K = \mathrm{SU}(n)$;
maximal torus $(\mathbb{C}^*)^{n-1}$; realize this torus as the subspace of $(\mathbb{C}^*)^n$ which consists of all $(z_1, \ldots, z_n)$ such that $z_1 \ldots z_n = 1$
$\sigma_1, \ldots, \sigma_{n-1}$ elementary symmetric functions

$$(\sigma_1, \ldots, \sigma_{n-1})\colon (\mathbb{C}^*)^{n-1} \longrightarrow \mathbb{C}^{n-1},$$
$$\mathbf{z} = (z_1, \ldots, z_n) \longmapsto (\sigma_1(\mathbf{z}), \ldots, \sigma_{n-1}(\mathbf{z}))$$

realizes the complex analytic quotient
As a *stratified Kähler space*, quotient $\mathcal{P}$ considerably more complicated.
Special case $n = 2$, $K = \mathrm{SU}(2)$:
COMPLEX ANALYTIC STRUCTURE:

$$S^1 \cong T \subseteq K, \ \mathbb{C}^* \cong T^{\mathbb{C}} \subseteq K^{\mathbb{C}}$$

quotient $\mathcal{P} = \mathrm{T}^*K//K$ amounts to

$$T^{\mathbb{C}}/(\mathbb{Z}/2) \cong \mathbb{C}$$

realized via the holomorphic map

$$f\colon \mathbb{C}^* \longrightarrow \mathbb{C}, \ f(z) = z + z^{-1}$$

as explained above, $Z = z + z^{-1}$ holomorphic coordinate on the quotient.

REAL STRUCTURE:

$$z = x + iy,\ Z = X + iY,\ r^2 = x^2 + y^2$$

$$X = x + \frac{x}{r^2},\ Y = y - \frac{y}{r^2},\ \tau = \frac{y^2}{r^2}$$

$C^\infty(\mathcal{P})$: smooth functions in the variables X, Y, τ, subject to the relation

$$Y^2 = (X^2 + Y^2 + 4(\tau - 1))\tau$$

POISSON BRACKETS

$$\{X, Y\} = X^2 + Y^2 + 4(2\tau - 1)$$
$$\{X, \tau\} = 2(1 - \tau)Y$$
$$\{Y, \tau\} = 2\tau X$$

Resulting *stratified Kähler* structure on $\mathcal{P} \cong \mathbb{C}$ *singular* at $-2 \in \mathbb{C}$ and $2 \in \mathbb{C}$, that is, the POISSON structure *vanishes*; at these two points, the function τ *not* an ordinary smooth function of the variables X and Y

$$\tau = \frac{1}{2}\sqrt{Y^2 + \frac{(X^2 + Y^2 - 4)^2}{16}} - \frac{X^2 + Y^2 - 4}{8}$$

Away from $-2 \in \mathbb{C}$ and $2 \in \mathbb{C}$, POISSON structure symplectic.

Definition. A *stratified Kähler space* consists of a complex analytic space N, with
(i) a stratification, finer than complex analytic
(ii) a stratified symplectic structure

$$(C^\infty N, \{\cdot,\cdot\})$$

compatible with complex analytic structure.
That is: $(C^\infty N, \{\cdot,\cdot\})$ is a Poisson algebra of continuous functions on N; on each stratum, an ordinary symplectic Poisson algebra which combines with the complex analytic structure to a Kähler structure.
$C^\infty N$ an algebra of *continuous* functions, not necessarily ordinary smooth functions, referred to as a *smooth* structure on N.

3 Holomorphic half-form quantization on the complexified group

General compact Lie group K
global Kähler potential κ

$$\kappa(x\,\mathrm{e}^{iY}) = |Y|^2,\ x \in K,\ Y \in \mathfrak{k}$$

that is, symplectic structure on $\mathrm{T}^*K \cong K^{\mathbb{C}}$ given by

$$i\partial\overline{\partial}\kappa$$

ε symplectic (or Liouville) volume form on

$$\mathrm{T}^*K \cong K^{\mathbb{C}}$$

η the real K-bi-invariant function on $K^{\mathbb{C}}$

$$\eta(x\,\mathrm{e}^{iY}) = \sqrt{\left|\frac{\sin(\mathrm{ad}(Y))}{\mathrm{ad}(Y)}\right|},\ x \in K,\ Y \in \mathfrak{k}$$

half-form Kähler quantization on $K^{\mathbb{C}}$:
Hilbert space $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)$ holomorphic functions on $K^{\mathbb{C}}$ square-integrable relative to $\mathrm{e}^{-\kappa/\hbar}\eta\varepsilon$

scalar product given by

$$\langle \psi_1, \psi_2 \rangle = \frac{1}{\mathrm{vol}(K)} \int_{K^{\mathbb{C}}} \overline{\psi_1} \psi_2 \mathrm{e}^{-\kappa/\hbar} \eta \varepsilon$$

left and right translation:
$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)$ unitary $(K \times K)$-representation
Hilbert space associated with $\mathcal{P}$ by reduction after quantization: subspace

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K$$

of K-invariants relative to conjugation

4 Singular quantum structure: costratified Hilbert space

"Singular points in a quantum problem are a set of measure zero so cannot possibly be important."

EMMRICH-RÖMER: *Wave functions may congregate near singular points.*

Ignoring singularities: inconsistencies

(1) Can we unveil a *structure on the quantum level which has the classical phase space singularities as its shadow*?

(2) Do the strata carry physical information?

Answer to (2): for $K = \mathrm{SU}(2)$ *tunneling probabilities among strata* [HRS]

Concerning (1):

Attempt SCHRÖDINGER quantization:

$\mathrm{T}^*Q//K$ a stratified symplectic space but

— strata are *not* cotangent bundles

— $\mathrm{T}^*Q//K \longrightarrow Q/K$ not a stratified map

KÄHLER quantization: each stratum of reduced space a KÄHLER manifold even better, structures combine to a *stratified Kähler structure*

Costratified Hilbert spaces

Let N be a stratified space. Let $\mathcal{C}_N$ be the category whose objects are the strata of N and whose morphisms are the inclusions $Y' \subseteq \overline{Y}$ where Y and Y' range over strata.

DEFINITION. A *costratified Hilbert space* relative to N assigns a Hilbert space $\mathcal{C}_Y$ to each stratum Y, together with a bounded linear map $\mathcal{C}_{Y_2} \to \mathcal{C}_{Y_1}$ for each inclusion $Y_1 \subseteq \overline{Y_2}$ such that, whenever $Y_1 \subseteq \overline{Y_2}$ and $Y_2 \subseteq \overline{Y_3}$, the composite of $\mathcal{C}_{Y_3} \to \mathcal{C}_{Y_2}$ with $\mathcal{C}_{Y_2} \to \mathcal{C}_{Y_1}$ coincides with the bounded linear map $\mathcal{C}_{Y_3} \to \mathcal{C}_{Y_1}$ associated with the inclusion $Y_1 \subseteq \overline{Y_3}$.

Construction of costratified Hilbert space relative to the reduced phase space $\mathcal{P}$ as **quantum analogue** of *orbit type stratification*

Start: Hilbert space

$$\mathcal{H} = \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K$$

single out subspaces associated with strata
special case:

$$K = \mathrm{SU}(2),\ \mathcal{P} = \mathrm{T}^*K//K \cong \mathbb{C}$$

elements of $\mathcal{H}$: ordinary functions on $K^{\mathbb{C}}$
being K-invariant, they induce functions on

$$\mathcal{P} = K^{\mathbb{C}}//K^{\mathbb{C}} \cong T^{\mathbb{C}}/W \cong \mathbb{C}$$

strata

$$\mathcal{P}_+ = \{2\} \subseteq \mathbb{C},\ \mathcal{P}_- = \{-2\} \subseteq \mathbb{C}$$
$$\mathcal{P}_1 = \mathbb{C} \setminus \mathcal{P}_0 \subseteq \mathbb{C} = \mathbb{C} \setminus \{2, -2\}$$

consider the closed subspaces

$$\mathcal{V}_+ = \{f \in \mathcal{H}; f\big|_{\mathcal{P}_+} = 0\} \subseteq \mathcal{H}$$
$$\mathcal{V}_- = \{f \in \mathcal{H}; f\big|_{\mathcal{P}_-} = 0\} \subseteq \mathcal{H}$$
$$\mathcal{V}_1 = \{f \in \mathcal{H}; f\big|_{\mathcal{P}_1} = 0\} \subseteq \mathcal{H}$$

define the Hilbert spaces $\mathcal{H}_+$, $\mathcal{H}_-$, and $\mathcal{H}_1$ to
be the orthogonal complements in $\mathcal{H}$ so that

$$\mathcal{H} = \mathcal{V}_+ \oplus \mathcal{H}_+ = \mathcal{V}_- \oplus \mathcal{H}_- = \mathcal{V}_1 \oplus \mathcal{H}_1$$

resulting system $\{\mathcal{H};\ \mathcal{H}_1,\ \mathcal{H}_+,\ \mathcal{H}_-\}$, together
with the corresponding orthogonal projections:
costratified Hilbert space relative to $\mathcal{P}$

5 Energy eigenvalues and eigenstates; holomorphic Peter-Weyl theorem

Choose a dominant WEYL chamber in $\mathfrak{t}$
basis of $\mathcal{H} = \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K$?
Notation: highest weight λ: $\chi^{\mathbb{C}}_\lambda$ the irreducible character of $K^{\mathbb{C}}$ associated with λ

Theorem. [Holomorphic Peter-Weyl] *The Hilbert space $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)$ contains the vector space $\mathbb{C}[K^{\mathbb{C}}]$ of representative functions on $K^{\mathbb{C}}$ as a dense subspace and, as a unitary $(K \times K)$-representation, this Hilbert space decomposes as the direct sum*

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon) \cong \widehat{\bigoplus}_{\lambda \in \widehat{K^{\mathbb{C}}}} V^*_\lambda \otimes V_\lambda$$

into $(K \times K)$-isotypical summands.

Consequence of *holomorphic* PETER-WEYL: irreducible characters $\chi^{\mathbb{C}}_\lambda$ of $K^{\mathbb{C}}$ a basis of

$$\mathcal{H} = \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K$$

Notation: highest weight λ:
— χ_λ the corresponding irreducible character of K, restriction of $\chi_\lambda^{\mathbb{C}}$ to K
— $\rho := \frac{1}{2}\sum_{\alpha\in R^+}\alpha$ half sum positive roots
— $C_\lambda := (\hbar\pi)^{\dim(K)/2}\mathrm{e}^{\hbar|\lambda+\rho|^2}$, where $|\lambda+\rho|$ norm relative to inner product on $\mathfrak{k}$
consequence of ordinary Peter-Weyl: $\{\chi_\lambda\}$ an *orthonormal* basis of $L^2(K, \mathrm{d}x)^K$
Theorem. *The assignment to* χ_λ *of*

$$C_\lambda^{-1/2}\chi_\lambda^{\mathbb{C}},$$

as λ *ranges over the highest weights, yields a unitary isomorphism*

$$L^2(K, \mathrm{d}x)^K \longrightarrow \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K$$

Thus costratified Hilbert space structure, arising from *stratified Kähler* quantization, carries over to Schrödinger quantization.

Kähler: only constants quantizable
Schrödinger: functions at most quadratic in generalized momenta quantizable
(classical) Hamiltonian of model quantizable
associated quantum Hamiltonian

$$H = -\frac{\hbar^2}{2}\Delta_K + \frac{\nu}{2}(3 - \chi_1)\,.$$

The operator Δ_K arises from the non-positive Laplace-Beltrami operator associated with bi-invariant Riemannian metric on K.
Since metric bi-invariant, so is Δ_K, whence Δ_K restricts to self-adjoint operator on $L^2(K, \mathrm{d}x)^K$, still written as Δ_K.
By means of $L^2(K, \mathrm{d}x)^K \longrightarrow \mathcal{H}$, transfer Hamiltonian, in particular, the operator Δ_K, to self-adjoint operators on $\mathcal{H}$

SCHUR's lemma:

— each isotypical $(K \times K)$-summand $L^2(K, dx)_\lambda$ of $L^2(K, dx)$ in the PETER-WEYL decomposition an eigenspace

— representative functions eigenfunctions for Δ_K

— eigenvalue $-\varepsilon_\lambda$ of Δ_K corresponding to the highest weight λ

$$\varepsilon_\lambda = (|\lambda + \rho|^2 - |\rho|^2),$$

— in holomorphic quantization on $\mathrm{T}^*K \cong K^{\mathbb{C}}$, energy operator arises as the unique extension of the operator $-\frac{1}{2}\Delta_K$ on $\mathcal{H}$ to an unbounded self-adjoint operator

— spectral decomposition thereof refines to holomorphic PETER-WEYL decomposition of $\mathcal{H}$