

A survey on homological perturbation theory

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Abstract

Higher homotopies are nowadays playing a prominent role in mathematics as well as in certain branches of theoretical physics. Homological perturbation theory (HPT), in a simple form first isolated by Eilenberg and Mac Lane in the early 1950's, is nowadays a standard tool to handle algebraic incarnations of higher homotopies. A basic observation is that higher homotopy structures behave much better relative to homotopy than strict structures, and HPT enables one to exploit this observation in various concrete situations. In particular, this leads to the effective calculation of various invariants which are otherwise intractable.

Higher homotopies and HPT-constructions abound but they are rarely recognized explicitly and their significance is hardly understood; at times, their appearance might at first glance even come as a surprise, for example in the Kodaira-Spencer approach to deformations of complex manifolds or in the theory of foliations.

The talk will illustrate, with a special emphasis on the compatibility of perturbations with algebraic structure, how HPT may be successfully applied to various mathematical problems arising in group cohomology, algebraic K -theory, and deformation theory.

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1 Origins of homotopy and higher homotopies

GAUSS 1833: linking number

GAUSS 1828: idea of a classifying space

HILBERT: exploration of syzygies

generating function for the number of invariants of each degree a rational function

homogeneous ideal I of a polynomial ring S , “number of independent linear conditions” for a form of degree d in S to lie in I ” a polynomial function of d

problem of counting the number of conditions already considered in projective geometry and in invariant theory

CAYLEY 1848: general statement of the problem, clear understanding of the role of syzygies—without the word, introduced by

SYLVESTER 1853

CAYLEY somewhat develops what is nowadays referred to as the KOSZUL resolution

POINCARÉ 1895: terminology *homotopy*

loop composition

2 Some 20'th century higher homotopies

Failure of Alexander-Whitney multiplication of cocycles to be commutative

STEENROD U_i -products, prompted s(trongly)h(omotopy)c(ommutative) structures

STEENROD operations

A_∞ -structure system of higher homotopies together with suitable coherence conditions

MASSEY products invariants of certain A_∞ -structures

elementary example Borromean rings

a MASSEY product of three variables detects the simultaneous linking of all three circles

SUGAWARA recognition principle for characterizing loop spaces up to homotopy type, associativity problem of loop multiplications

STASHEFF

— complete understanding of the homotopy invariance properties of associativity

— clean recognition principle for loop spaces

— classifying space

— nested sequence of homotopy associativity conditions

— A_n -spaces

any space an A_1 -space, H-space an A_2 -space

every homotopy associative H-space is A_3

A_∞ -space homotopy type of a loop space

algebraic analogue of an A_n -space in the category of algebras

A_n -algebra, the case $n = \infty$ being included

original and motivating example singular chains

on the based loop space of a space

variant L_∞ -algebras

key observation: A_∞ -structures behave correctly with respect to homotopy

3 Homological perturbations — HPT

Fundamental means to handle higher homotopies

EILENBERG AND MACLANE [1953]

contraction

$$(M \begin{array}{c} \xleftarrow{\pi} \\ \xrightarrow{\nabla} \end{array} N, h)$$

— chain complexes M and N

— chain maps $\nabla: M \rightarrow N$ and $\pi: N \rightarrow M$

— degree 1 morphism $h: N \rightarrow N$

requirements

— deformation retraction

$$\pi\nabla = \text{Id}, \quad \nabla\pi - \text{Id} = dh + hd$$

— annihilation properties

$$h\nabla = 0, \quad \pi h = 0, \quad hh = 0$$

EILENBERG AND MACLANE [1953]

comparison between reduced bar and W-constructions a reduction

conjectured that it is a contraction

Some notions

- contraction *filtered*: N and M filtered chain complexes, π , ∇ and h filtration preserving
- *perturbation* of the differential d of a chain complex X : (homogeneous) morphism

$$\partial: X \longrightarrow X$$

of the same degree as d such that

- ∂ lowers the filtration
- $(d + \partial)^2 = 0$ or, equivalently,

$$[d, \partial] + \partial\partial = 0$$

- sum $d + \partial$, referred to as the *perturbed differential*, again differential

- *coalgebra perturbation*: X a graded coalgebra structure such that (X, d) a differential graded coalgebra

perturbed differential $d + \partial$ compatible with graded coalgebra structure

- similarly *algebra perturbation*

HELLER [1954] *recursive structure of triangular complexes* later identified as a perturbation

Perturbation lemma. *Let*

$$(M \xleftrightarrow[\nabla]{g} N, h)$$

be a filtered contraction, let ∂ be a perturbation of the differential on N , and let

$$\mathcal{D} = \sum_{n \geq 0} g \partial (h \partial)^n \nabla = \sum_{n \geq 0} g (\partial h)^n \partial \nabla$$

$$\nabla_{\partial} = \sum_{n \geq 0} (h \partial)^n \nabla$$

$$g_{\partial} = \sum_{n \geq 0} g (\partial h)^n$$

$$h_{\partial} = \sum_{n \geq 0} (h \partial)^n h = \sum_{n \geq 0} h (\partial h)^n.$$

If the filtrations on M and N are complete, these infinite series converge, \mathcal{D} is a perturbation of the differential on M and, if we write N_{∂} and $M_{\mathcal{D}}$ for the new chain complexes,

$$(M_{\mathcal{D}} \xleftrightarrow[\nabla_{\partial}]{g_{\partial}} N_{\partial}, h_{\partial})$$

constitute a new filtered contraction that is natural in terms of the given data.

Some history

SHI [1962] lurking behind formulas

M. BARRAT made explicit (unpublished)

BROWN [1964] in print

GUGENHEIM [1972] *twisted* EILENBERG-ZILBER theorem via perturbation lemma

CHEN [1977] $\Omega_{\partial}[H_*(X)]$ model for the real chains on the loop space of a smooth manifold X by means of a formal power series connection involving the de Rham complex of X

GUGENHEIM [1982] perturbation theory for homology of loop space, reworks and extends CHEN's construction for the ordinary singular setting

WONG [1986] Heidelberg diploma thesis (Diplomarbeit) supervised by J. H.

using perturbation lemma, verified EML-conjecture that comparison between reduced bar and W -constructions a contraction

BERGER AND HUEBSCHMANN [1996] [2]

geometric comparison between bar and W -constructions

4 Perturbation theory for chain equivalences

HUEBSCHMANN AND KADEISHVILI [1991]

Theorem. *Let*

$$(\mu, X \begin{array}{c} \xleftarrow{\alpha} \\ \xrightarrow{\beta} \end{array} Y, \nu)$$

be a filtered chain equivalence, let ∂ be a perturbation of the differential d on Y , and suppose that the filtrations on X and Y are complete. Then HPT yields formulas for

- *a perturbation $\mathcal{D}: X \longrightarrow X$ of the differential on X , and for*
- *new morphisms*

$$\begin{aligned} \alpha_{\partial}: X &\longrightarrow Y, & \beta_{\partial}: Y &\longrightarrow X, \\ \mu_{\partial}: X &\longrightarrow X, & \nu_{\partial}: Y &\longrightarrow Y, \end{aligned}$$

which yield a new filtered chain equivalence

$$(\mu_{\partial}, X_{\mathcal{D}} \begin{array}{c} \xleftarrow{\alpha_{\partial}} \\ \xrightarrow{\beta_{\partial}} \end{array} Y_{\partial}, \nu_{\partial})$$

which is natural in terms of the given data.

Remark about proof: Reduce the constructions to ordinary perturbation lemma by means of mapping cylinder

5 Iterative perturbations

HUEBSCHMANN [1986] [18]

X a connected simplicial set, simple
 π_1, π_2 , etc. its homotopy groups

$$k^j : X_{j-2} \longrightarrow K(\pi_{j-1}, j), j \geq 3$$

maps representing the k -invariants of X

Theorem. *These maps determine a differential d on the tensor product*

$$K(\pi_1, 1) \otimes K(\pi_2, 2) \otimes \dots$$

and a chain equivalence between X and the resulting chain complex

$$(K(\pi_1, 1) \otimes K(\pi_2, 2) \otimes \dots, d)$$

Main ingredient: Iteration of the perturbation lemma, applied to the corresponding iterated fibrations

Application: Interpretation of the k -invariants of the algebraic K -theory of a finite field

Construction taken up in

LAMBE-STASHEFF [1987] [40]

models for iterated fibrations

6 Compatibility of HPT with algebraic structure

HUEBSCHMANN [1989], [1991]

compatibility of HPT- constructions with algebraic structure

suitable algebraic HPT-constructions to exploit

A_∞ -modules arising in group cohomology

construction of suitable free resolutions

explicit numerical calculations in group cohomology

until today still not doable by other methods

spectral sequences show up which do not collapse from E_2

illustrate a typical phenomenon:

Whenever a spectral sequence arises from a certain mathematical structure

a certain strong homotopy structure lurking behind

spectral sequence is an invariant thereof

higher homotopy structure finer than spectral sequence itself

ALVAREZ ET AL [2006] [1]

Comparison between bar and W-constructions (commutative case): compatibility properties relative to algebra structures of map from W-construction to bar construction

Visit the workshop

EXAMPLE 1. Group extension

$$\mathbf{e}: 1 \longrightarrow N \longrightarrow G \longrightarrow K \longrightarrow 1$$

free resolutions

$$M(K) = M^\sharp(K) \otimes_d RK$$

$$M(N) = M^\sharp(N) \otimes_d RN$$

of the ground ring R in the categories of right RK - and RN -modules

augmentation maps

$$\varepsilon: M^\sharp(K) \longrightarrow R, \quad \varepsilon: M^\sharp(N) \longrightarrow R$$

$M^\sharp(K)$ and $M^\sharp(N)$ connected as graded modules, i.e. in degree zero a single copy of the ground ring R

explicit HPT-construction of a free resolution of R in the category of right RG -modules

from the given data:

$$M(N) \otimes_{RN} RG = M^\sharp(N) \otimes_d RG$$

a free resolution of RK in the category of right RG -modules

write

$$d^0 = M^\sharp(K) \otimes d \quad (= \text{Id}_{M^\sharp(K)} \otimes d),$$

where d the differential on $M^\sharp(N) \otimes_d RG$
an RG -linear differential on

$$M^\sharp(K) \otimes M^\sharp(N) \otimes RG$$

vertical in an obvious sense

projection map

$$g: M^\sharp(K) \otimes M^\sharp(N) \otimes RG \longrightarrow M^\sharp(K) \otimes RK$$

induces an isomorphism on homology relative
to differential d^0 on source, zero differential on
target

object $M^\sharp(K) \otimes RK$ filtered by degrees
degree filtration of $M^\sharp(K)$ induces a filtration

$$\{F_i = F_i(M^\sharp(K) \otimes M^\sharp(N) \otimes RG)\}_{i \geq 0}$$

for $M^\sharp(K) \otimes M^\sharp(N) \otimes RG$ in the obvious way

Theorem. *There is a differential d on*

$$M^\sharp(K) \otimes M^\sharp(N) \otimes RG$$

having the following properties:

1. *It can be written as*

$$d = d^0 + d^1 + d^2 + \dots$$

where, for $i \geq 1$, the operator d^i lowers filtration by i .

2. *The projection map is a chain map*

$$(M^\sharp(K) \otimes M^\sharp(N)) \otimes_d RG \rightarrow M^\sharp(K) \otimes_d RK$$

that is compatible with the differentials, the filtrations, and the right RG -module structures.

Moreover, the resulting chain complex

$$(M^\sharp(K) \otimes M^\sharp(N)) \otimes_d RG$$

is acyclic and hence a free resolution of the ground ring R in the category of right RG -modules. Finally, the higher terms d^i , $i \geq 1$, can be constructed explicitly via HPT.

Under these circumstances, the series

$$d^1 + d^2 + \dots$$

is a *perturbation* of d^0 .

ILLUSTRATION. Metacyclic groups such a group G has a presentation of the form

$$\langle x, y; y^r = 1, x^s = y^f, xyx^{-1} = y^t \rangle$$

where x and y are generators of $K = \mathbb{Z}/s$ and $N = \mathbb{Z}/r$ respectively, and where

$$s > 1, r > 1, t^s \equiv 1 \pmod{r}, tf \equiv f \pmod{r},$$

so that, in particular, $\frac{t^s-1}{r}$ and $\frac{(t-1)f}{r}$ are integers

SAMPLE RESULTS. Let p be a prime which divides the numbers r and s , and suppose that

$$t \not\equiv 1 \pmod{p}.$$

Further, let j_0 be the order of t modulo p , i.e. j_0 is the smallest number j so that

$$t^j \equiv 1 \pmod{p}.$$

Theorem 1. *Suppose that p divides the number $\frac{t^s-1}{r}$. Then the mod p cohomology spectral sequence of the group extension collapses from E_2 , and the multiplicative extension problem from*

$$E_2 = E_\infty$$

to $H^(G, \mathbb{Z}/p)$ is trivial. Moreover,*

$$H^*(G, \mathbb{Z}/p)$$

has classes

$$\omega_{2j_0-1}, c_{2j_0}, |\omega_{2j_0-1}| = 2j_0 - 1, |c_{2j_0}| = 2j_0,$$

of filtration zero which restrict to the classes $\omega_y c_y^{j_0-1}$ and $c_y^{j_0}$ in $H^(N, \mathbb{Z}/p)$ respectively, so that, as a graded commutative algebra, $H^*(G, \mathbb{Z}/p)$ may be written as*

$$\Lambda[\omega_x] \otimes P[c_x] \otimes \Lambda[\omega_{2j_0-1}] \otimes P[c_{2j_0}].$$

Theorem 2. *Suppose that p does not divide the number $\frac{t^s-1}{r}$. Then the cohomology spectral sequence of the group extension collapses from E_3 , and the multiplicative extension problem is trivial. Moreover, $H^*(G, \mathbb{Z}/p)$ has classes*

$$\omega_{2j_0-1}, \omega_{4j_0-1}, \dots, \omega_{2pj_0-1}, c_{2pj_0},$$

of filtration zero which restrict to the classes $\omega_y c_y^{ij_0-1}$ and $c_y^{pj_0}$ in $H^(N, \mathbb{Z}/p)$ respectively so that, as a graded commutative algebra,*

$$H^*(G, \mathbb{Z}/p)$$

is generated by

$$\omega_x, c_x, \omega_{2j_0-1}, \omega_{4j_0-1}, \dots, \omega_{2pj_0-1}, c_{2pj_0},$$

subject to the relations

$$\omega_a \omega_b = 0,$$

$$a, b \in \{2j_0 - 1, 4j_0 - 1, \dots, 2pj_0 - 1\},$$

$$c_x \omega_a = 0,$$

$$a \in \{2j_0 - 1, 4j_0 - 1, \dots, 2(p-1)j_0 - 1\}.$$

EXAMPLE 2.

HUEBSCHMANN AND KADEISHVILI [1991]

HUEBSCHMANN [2004] [26]

Small models for chain algebras

algebra and coalgebra perturbation lemmata

Sample result

Theorem. *Let*

$$(\mu, X \begin{array}{c} \xleftarrow{\alpha} \\ \xrightarrow{\beta} \end{array} Y, \nu)$$

be a chain equivalence, let

$$(\mathrm{T}\mu, \mathrm{T}[X] \begin{array}{c} \xleftarrow{\mathrm{T}\alpha} \\ \xrightarrow{\mathrm{T}\beta} \end{array} \mathrm{T}[Y], \mathrm{T}\nu)$$

be the corresponding filtered chain equivalence of augmented differential graded algebras, and let ∂ be a multiplicative perturbation of the differential on $\mathrm{T}[Y]$ with respect to the augmentation filtration. Assume further that X and Y are connected. Then HPT yields formulas for

- *a multiplicative perturbation*

$$\mathcal{D}: \mathrm{T}[X] \longrightarrow \mathrm{T}[X]$$

of the differential on $\mathrm{T}[X]$, and for

- *new morphisms*

$$\begin{aligned}
T_{\partial\alpha}: T[X] &\longrightarrow T[Y] \\
T_{\partial\beta}: T[Y] &\longrightarrow T[X] \\
T_{\partial\mu}: T[X] &\longrightarrow T[X] \\
T_{\partial\nu}: T[Y] &\longrightarrow T[Y],
\end{aligned}$$

which combine to a new filtered chain equivalence

$$(T_{\partial\mu}, T_{\mathcal{D}}[X] \begin{array}{c} \xleftarrow{T_{\partial\alpha}} \\ \xrightarrow{T_{\partial\beta}} \end{array} T_{\partial}[Y], T_{\partial\nu})$$

of augmented differential graded algebras where $T_{\mathcal{D}}[X]$ and $T_{\partial}[Y]$ refer to the new chain complexes.

In particular, the new filtered chain equivalence is natural in terms of the given data.

Application: models for differential graded algebras, in particular for chain algebras

7 Rooted planar trees

KONTSEVICH-SOIBELMAN [2000]

Description of A_∞ -algebra structures in terms of sums over *oriented rooted planar trees* endowed with suitable labels

these sums over oriented rooted planar trees *behind* the HPT-constructions that establish *(co)algebra perturbation lemma* in

HUEBSCHMANN AND KADEISHVILI [1991]

At the time no need to spell out the oriented rooted planar trees explicitly

Details: HUEBSCHMANN [34]

One application:

Minimality theorem for A_∞ -structures via the perturbation lemma

Original minimality theorem:

CHEN [1977] over the de Rham algebra of smooth manifold

KADEISHVILI [1980] for the rational cochain algebra of a space

Details explained in HUEBSCHMANN [34]

8 Master equation and HPT

Given a coaugmented differential graded co-commutative coalgebra C and a differential graded Lie algebra \mathfrak{h} , a *Lie algebra twisting cochain* $t: C \rightarrow \mathfrak{h}$ is a homogeneous morphism of degree -1 whose composite with the coaugmentation map is zero and which satisfies

$$Dt = \frac{1}{2}[t, t].$$

HUEBSCHMANN AND STASHEFF [2002] [36]
 Data: differential graded Lie algebra \mathfrak{g} together with contraction

$$(H(\mathfrak{g}) \begin{array}{c} \xleftarrow{\nabla} \\ \xrightarrow{\pi} \end{array} \mathfrak{g}, h)$$

of chain complexes, the corresponding coalgebra perturbation of the differential on $\mathcal{S}^c[s\mathfrak{g}]$ being written as ∂

Theorem. *The data determine, via HPT,*
(i) *a differential \mathcal{D} on $\mathcal{S}^c[sH(\mathfrak{g})]$ turning the latter into a coaugmented differential graded coalgebra (i. e. \mathcal{D} is a coalgebra perturbation of the zero differential) and hence endowing $H(\mathfrak{g})$ with an sh-Lie algebra structure and*
(ii) *a Lie algebra twisting cochain*

$$\tau: \mathcal{S}_{\mathcal{D}}^c[sH(\mathfrak{g})] \rightarrow \mathfrak{g}$$

whose adjoint $\bar{\tau}$, written as

$$(\mathcal{S}^c\nabla)_{\partial}: \mathcal{S}_{\mathcal{D}}^c[sH(\mathfrak{g})] \rightarrow \mathcal{C}[\mathfrak{g}],$$

induces an isomorphism on homology.

Furthermore, $(\mathcal{S}^c\nabla)_{\partial}$ admits via HPT an extension to a new contraction

$$(\mathcal{S}_{\mathcal{D}}^c[sH(\mathfrak{g})]) \begin{array}{c} \xleftarrow{(\mathcal{S}^c\nabla)_{\partial}} \\ \xrightarrow{(\mathcal{S}^c\pi)_{\partial}} \end{array} \mathcal{S}_{\partial}^c[s\mathfrak{g}], (\mathcal{S}^ch)_{\partial}$$

of filtered chain complexes (not necessarily of coalgebras).

Application: formal solutions of the master equation

9 Higher homotopies, homological perturbations, and the working mathematician

Higher homotopies and homological perturbations may be used to solve problems phrased in language entirely different from that of higher homotopies and HPT. Higher homotopies and HPT-constructions occur implicitly in a number of other situations in ordinary mathematics where they are at first not even visible.

List (not exhaustive)

- **KODAIRA-NIRENBERG-SPENCER**: Deformations of complex structures [38]
- **FRÖLICHER** spectral sequence of a complex manifold [23], [25]
- **TOLEDO-TONG**: Parametrix [48]
- **FEDOSOV**: Deformation quantization [8]
- **WHITNEY, GUGENHEIM**: Extension of geometric integration to contraction [10], [49]
- **WHITNEY** ground work for **SULLIVAN**'s theory of rational differential forms [49]
- **GUGENHEIM** integration map in de **RHAM** theory sh-multiplicative [10]
- **HUEBSCHMANN**: Foliations; requisite higher homotopies described in terms of generalized **MAURER-CARTAN** algebra [26]
- **HUEBSCHMANN**: Equivariant cohomology and **KOSZUL** duality [27], [28]
- Operads

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