A step towards Lie's dream

Johannes Huebschmann
Université des Sciences et Technologies de Lille
Johannes.Huebschmann@math.univ-lille1.fr

Darmstadt, October 6, 2012

Abstract

The origins of Lie theory are well known: Galois theory had clarified the relationship between the solutions of polynomial equations and their symmetries. Lie had attended lectures by Sylow on Galois theory and came up with the idea to develop a similar theory for differential equations and their symmetries which he and coworkers then successfully built. At a certain stage, they noticed that "transformations groups" with finite-dimensional Lie algebra was a very tractable area. This resulted in a brilliant and complete theory, that of Lie groups, but the connection with the origins gets somewhat lost.

The idea of a Galois theory for differential equations prompted as well what has come to be nowadays known as differential Galois theory. We will present a kind of generalized gauge theory that encompasses ordinary Galois extensions (of commutative rings), differential Galois theory, and principal bundles (in differential geometry and algebraic geometry). The new notion that we introduce for that purpose is that of *principal comorphism of Lie-Rinehart algebras*. This approach can be seen as an attempt to go back to the origins of Lie theory.

Contents

Upshot

In the framework of Lie-Rinehart algebras,
foliated principal bundle
Picard-Vessiot extension (in diff Galois thy)
same mathematical structure
foliated principal G-bundle:

- $\xi\colon P \to M$ principal G-bundle
- $D_M \subseteq \mathrm{T}M$ integrable distribution
- $D_P \subseteq \mathrm{T}P$ a G-equivariant integrable distribution on P transverse to $P \times \mathfrak{g} \subseteq \mathrm{T}P$

  ξ induces iso $D_P \longrightarrow P \times_M D_M$

$$\begin{array}{ccccc}
0 & \to & P \times_M D_M & = & P \times_M D_M \\
\downarrow & & \downarrow & & \downarrow \\
P \times \mathfrak{g} & \to & \mathrm{T}P & \to & P \times_M \mathrm{T}M \\
\| & & \downarrow & & \downarrow \\
P \times \mathfrak{g} & \to & Q_P & \to & P \times_M Q_M
\end{array}$$

Q_M, Q_P normal bundles

1 Lie's dream

Develop a theory of groups of transformations of differential equations

Lie, Killing, E. Cartan noticed that such transformation groups with finite-dimensional Lie algebra are very tractable; this resulted in a brilliant theory, nowadays known as that of finite-dimensional Lie groups, independent of any space on which such a group acts.

On the other hand, this is no longer true of Lie pseudo groups, "groups" of transformations with "infinite-dimensional" Lie algebra.

2 Goal

Single theory which eventually comprises

1. Galois extensions of rings
2. more generally: Hopf Galois extension
3. Picard-Vessiont theory or linear differential Galois theory
4. inseparable Galois extensions
5. (foliated) principal bundles
6. general differential Galois theory for modules over a Lie-Rinehart algebra
7. D-modules

3 Galois theory of linear differential equations

Picard, Vessiot, Ritt, Kolchin, Kolchin-Lang, Kaplansky, Białynicki-Birula, Seidenberg, ..., Umemura, Malgrange, ...

Differential field: Field with a derivation D more generally, family of derivations,

Constants: Members of the kernel of D

Differential field extension : field extension $E \supseteq F$, together with extension $D\colon E \to E$

$G(E|F)$ *differential Galois group*, relative autos compatible with differential structure

Fund thm of Diff Galois theory:

Given differential field extension, 1-1 corresp differential subfield extensions and subgroups of the group of differential autos

kind of extension that fits in the corresp., playing the role of Galois extensions of fields:

Picard-Vessiot extension with algebraically closed field of constants

L diff operator over the diff field F:

$L(Y) = Y^{(n)} + a_{n-1}Y^{(n-1)} + \ldots + a_1 Y' + a_0$.

given diff extension $E \supseteq F$, *solutions* in E

$$V = \{y \in E; L(y) = 0\}$$

differential field extension $E \supset F$ a *Picard-Vessiot* extension of F for L if:

1. The constants of E are those of F.
2. E contains a full set V of solutions of $L = 0$.
3. E smallest diff subfield of E containing both V and F - "E generated over F as a diff field by solutions of $L = 0$".

$E \supset F$ is a *Picard-Vessiot* extension of F if it is a Picard-Vessiot extension for some L.

Theorem. *Let $E \supseteq F$ be a diff field ext alg'y closed common field of consts C Suppose there are*

- *finite-dimensional C-subspace V of E generating E over F differentially*
- *a group G of diff autos of E over F such that $G(V) \subseteq V$ and $E^G = F$*

Then

- *$E \supseteq F$ Picard-Vessiot extension*
- *$G = G(E \big| F)$ algebraic grp in $\mathrm{GL}_C(V)$,*
- *E field of fractions of coordinate ring over F of an affine variety T*
- *Lie algebra of G coincides with the Lie subalgebra of $\mathrm{Der}(E \big| F)$ which consists of the derivations of E over F that commute with the differentiation.*

coordinate of affine variety T in theorem: *Picard-Vessiot* ring

standard trick carries lin diff eq $L(Y) = 0$ to matrix diff eq $Y' = AY$.
Picard-Vessiot theory for matrix diff eq's
Given diff ext $E \supseteq F$, *solutions* in E^n

$$V = \{y \in E^n; y' = Ay\}$$

Guiding question, or Ariadne's thread today:
Interpretation of the above in geometry?

- ring of f's on total space P of pr'l bundle $\xi\colon P \to M$ subst'd for Picard-Vessiot ring
- structure gp substituted for diff Galois gp
- foliation substituted for diff structure
- functions on base M subst'd for field F
- functions constant on the leaves substituted for field of constants

Examples of diff Galois

1. $F = \mathbb{C}(t), Y' = \frac{1}{t}, E = F(f), f(t) = \log(t)$
   Diff Galois gp copy of $\mathbb{C}$, action $f \longmapsto f + c$
   $E = \mathbb{C}(t)(f)$ many subfields but no intermediate *differential* subfields
2. $E = F(u_1, u_2, u_3),$
   $u_1' = \frac{1}{t},\ u_2' = \frac{1}{t+1},\ u_3' = \frac{1}{t}u_2$
   diff Galois integral Heisenberg group.
3. $k = \mathbb{C}(x),\ x' = 1,\ \alpha \in \mathbb{C},$
   $Y' = \frac{\alpha}{x}Y,\ S = k[Y, \frac{1}{Y}]$
   - $\alpha = \frac{n}{m}$, $(n, m) = 1$: $G \cong \mathbb{Z}/m$
     Pic.-Vess. rg $R = S/(Y^m - x^n) = k(x^{\frac{m}{n}})$
   - $\alpha \notin \mathbb{Q}$: $R = S$, $G \cong \mathrm{GL}(1, \mathbb{C})$
4. $k = \mathbb{C}(x),\ x' = 1,\ Y' = Y,\ G \cong \mathrm{GL}(1, \mathbb{C})$
   $S = k[Y, \frac{1}{Y}]:\ R = S$ Picard-Vessiot ring

Liouville field extension of diff field K
diff field ext $L \supseteq K$ together with

$$K = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n = L$$

each field K_{j+1} either algebraic over K_j or generated by an indefinite integral or the exponential of an indefinite integral of a member of K_j
normal series of Galois groups

$$\{1\} = G_n \subseteq G_{n-1} \subseteq \ldots \subseteq G_1 \subseteq G_0 \subseteq G_{-1}$$

each G_j/G_{j+1} iso to $\mathbb{C}$, or $\mathbb{C}^*$, or finite
One can then prove that, e.g.,

$$u' = t - u^2$$

has no solution which belongs to a Liouville extension of $\mathbb{C}(t)$. This means that this differential equation has no solution which can be written in terms of elementary functions, indefinite integrals of elementary functions, or exponentials of indefinite integrals of elementary functions, or indefinite integrals of those functions, etc.

4 Lie-Rinehart algebras

4.1 The definition

R commutative ring with 1
A commutative R-algebra
(R, A)-*Lie algebra* [Rinehart]
Lie algebra L over R
$L \otimes A \to A$ left action by derivations
$A \otimes L \to L$ left A-module structure
compatibility conditions
generalize Lie algebra vector fields on manifold as a module over its ring of functions

$$[\alpha, a\beta] = \alpha(a)\beta + a[\alpha, \beta]$$
$$(a\alpha)(b) = a(\alpha(b))$$

for $a, b \in A$ and $\alpha, \beta \in L$

4.2 Modules

A-module M and left L-module structure

$$L \otimes_R M \to M, \ (\alpha, x) \mapsto \alpha(x)$$

(A, L)-*module*:

$$\alpha(ax) = \alpha(a)x + a\alpha(x)$$
$$(a\alpha)(x) = a(\alpha(x))$$

for $a \in A, \ x \in M, \ \alpha \in L$
a *flat* (A, L)-*connection*

4.3 Examples of Lie-Rinehart algebras

(i) M manifold, $(A, L) = (C^\infty(M), \mathrm{Vect}(M))$
(ii) A algebra, $(A, L) = (A, \mathrm{Der}(A))$
(iii) $\vartheta \colon E \to B$ Lie algebroid: $(A, L) = (C^\infty(B), \Gamma(\vartheta))$
special case: foliation
(iv) K a field, together with a family Δ of (not necessarily commuting) differential operators
C the field of constants
$L \subseteq \mathrm{Der}(K|C)$ Lie algebra generated by Δ
(K, L) Lie-Rinehart algebra

4.4 Induced Lie-Rinehart algebra

(A, L_A) Lie-Rinehart algebra, $B \supseteq A$ an extension of algebras
suppose the L_A-action on A extends to an action

$$L_A \otimes B \longrightarrow B$$

by derivations; then
$(B, B \otimes_A L_A)$ acquires Lie-Rinehart structure
induced Lie-Rinehart algebra $(B, B \odot_A L_A)$

5 Picard-Vessiot problem for Lie-Rinehart algebras

(A, L_A) Lie-Rinehart algebra, N_0 an R-module;
split (A, L_A)-module: iso to $A \otimes N_0$
provisionally:
Picard-Vessiot problem for (A, L)-module N
"generalized differential system"
construct induced Lie-Rinehart algebra
$(B, L_B) = (B, B \odot_A L_A)$ such that

1. obvious map $A^{L_A} \to B^{L_B}$ isomorphism
2. induced (B, L_B)-module $B \otimes_A N$ is (B, L_B)-split: can morphism of (B, L_B)-modules

$$B \otimes_{B^{L_B}} (B \otimes_A N)^{L_B} \longrightarrow B \otimes_A N$$

   isomorphism of (B, L_B)-modules;
3. relative to 1. and 2., $(B, B \odot_A L_A)$ minimal

notation $-^{L_A}$ L_A-invariants
$(B \otimes_A N)^{L_B}$ "space of solutions"

Define an induced Lie-Rinehart algebra $(B, B \odot_A L_A)$ to be a *weak Picard-Vessiot Lie-Rinehart algebra* when

1. the induced map $A^{L_A} \to B^{L_A}$ is an iso
2. the canonical map

$$B \otimes_{B^{L_A}} (B \otimes_A B)^{L_A} \longrightarrow B \otimes_A B$$

   an isomorphism of $(B, B \odot_A L_A)$-modules.

$(B, B \odot_A L_A)$ *Picard-Vessiot L-R algebra*: given a non-zero $(B, B \odot_A L_A)$-ideal J of B, the (A, L_A)-ideal $J \cap A$ of A is non-trivial.
extends Picard-Vessiot ring for fields
does not naively apply to principal bundles

6 Galois extension

G group, $\mathfrak{h}$ (R,G)-*Lie algebra*, R-algebra B
B a $(G,\mathfrak{h})$-*algebra*
category of $(B,G,\mathfrak{h})$-*modules* N
submodule $N^{G,\mathfrak{h}} = (N^{\mathfrak{h}})^G$ of $(G,\mathfrak{h})$-invariants
we say that a $(B,G,\mathfrak{h})$-module N is $(B,G,\mathfrak{h})$-*induced* when

$$\phi_N \colon B \otimes_{B^{G,\mathfrak{h}}} N^{G,\mathfrak{h}} \longrightarrow N$$

isomorphism of $(B,G,\mathfrak{h})$-modules
let $\mathcal{M}$ be a class of $(B,G,\mathfrak{h})$-modules
define $(A = B^{\mathfrak{h}})^G \subseteq B, G, \mathfrak{h})$ to be a *Galois* extension with respect to $\mathcal{M}$, with *Galois group* G and *Galois Lie algebra* $\mathfrak{h}$: any member of $\mathcal{M}$ $(B,G,\mathfrak{h})$-*induced*

- Galois extension of rings, in particular fields [HKR]
- principal G-bundle $\xi \colon P \to M$

  $\mathcal{M}$: finitely gen G-vector bundles on P

7 Principal comorphism

(B, L_B) and (A, L_A) Lie-Rinehart algebras
comorphism

$$(\varphi, \Phi)\colon (B, L_B) \longrightarrow (A, L_A)$$

of Lie-Rinehart algebras consists of
— morphism $\varphi\colon A \to B$ of algebras
— morphism $\Phi\colon L_B \to B \otimes_A L_A$ of B-mod

$$\begin{array}{ccc} L_B & \stackrel{\Phi}{\longrightarrow} & B \otimes_A L_A \\ \big\downarrow & & \big\downarrow \\ \mathrm{Der}(B) & \stackrel{\varphi^*}{\longrightarrow} & \mathrm{Der}(A, B) \end{array}$$

commutative
guiding example: smooth map $f\colon M_1 \to M_2$

$$(C^\infty(M_2), \mathrm{Vect}(M_2)) \longrightarrow (C^\infty(M_1), \mathrm{Vect}(M_1))$$

comorphism (j, Φ) *principal* relative to $\mathcal{M}$, *with structure group* G and *structure Lie algebra* $\mathfrak{h}$

— j Galois extension with Galois group G and Galois Lie algebra $\mathfrak{h}$, with respect to $\mathcal{M}$

— $\Phi\colon L_B \to B \otimes_A L_A$ compatible:

exact sequence

$$0 \longrightarrow B \odot \mathfrak{h} \stackrel{\iota}{\longrightarrow} L_B \stackrel{\Phi}{\longrightarrow} B \otimes_A L_A \longrightarrow 0$$

of $(B, G, \mathfrak{h})$-modules in $\mathcal{M}$

special case: principal bundle $\xi\colon P \to M$

$$0 \longrightarrow P \times \mathfrak{g} \longrightarrow \mathrm{T}P \longrightarrow P \times_M \mathrm{T}M \longrightarrow 0$$

8 Picard-Vessiot problem revisited

Atiyah sequence

$$0 \longrightarrow (B \odot \mathfrak{h})^{G,\mathfrak{h}} \stackrel{\iota}{\longrightarrow} L_B^{G,\mathfrak{h}} \xrightarrow{\Phi^{G,\mathfrak{h}}} L_A \longrightarrow 0$$

write $L = L_B^{G,\mathfrak{h}}$
(A, L) a Lie-Rinehart algebra
$(B, B \odot_A L) \cong (B, L_B)$ induced
principal bundle case:
L "G-invariant vector fields on the total space"
we say a principal comorphism (j, Φ) is a *Picard-Vessiot comorphism*:
$(B, B \odot_A L)$ Picard-Vessiot Lie-Rinehart algebra in the sense defined earlier
given (A, L_A)-module N, Picard-Vessiot problem that of finding Picard-Vessiot comorphism

$$(j, \Phi)\colon (A, L_A) \longrightarrow (B, L_B)$$

that splits N
in this language, a foliated principal bundle and a Picard-Vessiot extension the same mathematical structure

9 Galois theory relative to pseudogroups

if we can ever push through the above with pseudogroups rather than groups, this will presumably settle Lie's dream
E. Cartan, Singer-Sternberg, Guillemin-Sternberg, Malgrange, Olver, ...