Equivariant cohomology via relative homological algebra

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Abstract

We will explain how the appropriate categorical framework involving (co)monads and standard constructions provides categorical definitions of various relative derived functors including equivariant de Rham cohomology, Lie-Rinehart cohomology, Poisson cohomology, etc. This leads, in particular, to a description of equivariant de Rham theory as a suitable differential Ext, in the sense of Eilenberg and Moore, over a category of modules relative to the group and the cone on its Lie algebra. Extending a decomposition lemma of Bott's, we obtain a decomposition of the functor defining equivariant de Rham cohomology into two constituents, one constituent being a kind of Lie algebra cohomology functor and the other one the differentiable cohomology functor. For the case of a compact group, standard comparison arguments then lead to the familiar Weil and Cartan models and in particular explain why these models calculate the equivariant cohomology initially defined via a Borel construction. Pushing further the approach we arrive at a construction defining equivariant Lie algebroid or, somewhat more generally, equivariant Lie-Rinehart cohomology. This kind of construction provides, perhaps, a framework to explore constrained hamiltonian systems, the variational bicomplex, the Noether theorems and related topics. Interesting issues arise, e.g. how to define the cone in the category of Lie-Rinehart algebras or Lie algebroids. The question whether and how these constructions extend to Lie groupoids is momentarily open as is the question of existence of injective modules over Lie groupoids.

The talk will illustrate the formal approach, with an emphasis on the development of these ideas (Cartan, Weil, Cartan-Chevalley-Eilenberg, Cartan-Eilenberg, Mac Lane, Hochschild and collaborators, Bott and collaborators, ...).

Contents

1 Origins and goals

Group G, G-space X, orbit space X/G
Basic question: Relationship between geometry of X, G and X/G?
often X/G bad
Replacement: h'quotient or Borel $EG \times_G X$
Action good: $EG \times_G X \to X/G$ h'eq
Developments:
Group coho, Chern-Weil, Lie alg coho, etc.
Riddle for audience:
[CE] Ex. XIII.14: Lie algebra $\mathfrak{g}$
CCE cx written as $V(\mathfrak{g})$
alternate construction: $\mathrm{U}[C\mathfrak{g}]$ written as $W(\mathfrak{g})$
Why the notation $W(\mathfrak{g})$?
Nowadays: Equiv coho via Borel construction
Earlier: Weil model, Cartan model
Relationship between Borel and Cartan-Weil?
Algebraic topology: group G, G-space X
EM: equivariant coho $\mathrm{Ext}_{C_*G}(R, C^*(X))$
Hochschild: coho algebraic groups coefficients rational modules: suitable CoExt

breaks down for de Rham

de Rham $\mathcal{A}(G)$, Lie group G, not a coalgebra

Today: Completely formal explanation in the framework of relative homological algebra

Not only formal: possible generalization relative to Lie algebroids, perhaps Lie groupoids

Crucial step: Extension of a Decomposition Lemma due to Bott

Question: Is there an equivariant Lie algebroid cohomology and what does it signify?

What about Lie groupoids?

How to establish existence of injective modules over Lie groupoids? What do they signify?

Why equivariant cohomology at all?

Momentum mapping same thing as an equivariantly closed extension of a closed equivariant 2-form (not equivariantly closed)

Kirwan: Smooth compact hamiltonian G-manifold for a compact group G equivariantly formal

In general: equivariant formality stronger than just momentum mapping

My favorite subject: group cohomology

Adjunction determines a (monad and a) comonad
Categories $\mathcal{C}$ and $\mathcal{M}$

functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{M}$

$\Box\colon \mathcal{M} \to \mathcal{C}$ right-adjoint to $\mathcal{F}$

$\mathcal{L} = \mathcal{F}\Box\colon \mathcal{M} \longrightarrow \mathcal{M}$

$\eta\colon \mathcal{I} \to \Box\mathcal{F}$ *unit*:

$$\mathrm{Hom}_{\mathcal{M}}(\mathcal{F}X, \mathcal{F}X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, \Box\mathcal{F}X)$$

$\varepsilon\colon \mathcal{L} \to \mathcal{I}$ *counit*:

$$\mathrm{Hom}_{\mathcal{M}}(\mathcal{F}\Box Y, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\Box Y, \Box Y)$$

δ natural transformation:

$$\delta = \mathcal{F}\eta\Box\colon \mathcal{L} = \mathcal{F}\Box \longrightarrow \mathcal{F}\Box\mathcal{F}\Box = \mathcal{L}^2.$$

$(\mathcal{L}, \epsilon, \delta)$ *comonad* over $\mathcal{M}$:
CD's

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\delta} & \mathcal{L}^2 \\ \delta\downarrow & & \downarrow\mathcal{L}\delta \\ \mathcal{L}^2 & \xrightarrow{\delta\mathcal{L}} & \mathcal{L}^3 \end{array}$$

$$\begin{array}{ccccc} \mathcal{L} & \xleftarrow{=} & \mathcal{L} & \xrightarrow{=} & \mathcal{L} \\ {\scriptstyle =}\downarrow & & {\scriptstyle \delta}\downarrow & & {\scriptstyle =}\downarrow \\ \mathcal{I}\mathcal{L} & \xleftarrow{\varepsilon\mathcal{L}} & \mathcal{L}^2 & \xrightarrow{\mathcal{L}\varepsilon} & \mathcal{L}\mathcal{I} \end{array}$$

Standard construction simplicial object

$\Big(\mathcal{L}^{n+1}, d_j \colon \mathcal{L}^{n+2} \to \mathcal{L}^{n+1}, s_j \colon \mathcal{L}^{n+1} \to \mathcal{L}^{n+2}\Big)_{n\in\mathbb{N}}$;

$n \geq 1$:

$d^n_j = \mathcal{L}^j \varepsilon \mathcal{L}^{n-j} \colon \mathcal{L}^{n+1} \to \mathcal{L}^n, \ j = 0, \ldots, n,$

$s^n_j = \mathcal{L}^j \delta \mathcal{L}^{n-j-1} \colon \mathcal{L}^n \to \mathcal{L}^{n+1}, \ j = 0, \ldots, n-1.$

Object W of $\mathcal{M}$,

$$\mathbf{L}(W) = \Big(\mathcal{L}^{n+1}(W), d_j, s_j\Big)_{n\in\mathbb{N}}$$

a simplicial object in $\mathcal{M}$, the *standard object associated with* W *and the comonad*;

Suitable circumstances:

chain complex $|\mathbf{L}(W)|$

a relatively projective resolution of W,

leads to relative differential

$$\mathrm{Tor}^{(\mathcal{M},\mathcal{C})}, \ \mathrm{Ext}_{(\mathcal{M},\mathcal{C})}$$

3 Examples

3.1 Borel construction

Reminder:
$\mathcal{C}$ symmetric monoidal category with cocommutative diagonal
e.g. spaces, smooth manifolds, groups, vector spaces, Lie algebras, etc.,
object Y of $\mathcal{C}$: two simplicial objects in $\mathcal{C}$
trivial object Y: copy of Y in each degree and all simplicial operations the identity
total object EY: $p \geq 0$,

$$EY_p = Y \times Y \ldots \times Y \ (p + 1 \text{ copies of } Y)$$

face operations by omission
degeneracy operations by insertion
Y a group G: simplicial group EG
diagonal injection $G \to EG$:
EG a simplicial principal G-space
X left G-spaces: $N(G, X) = EG \times_G X$
homogeneous Borel-construction

Comonadic description:
G group, $\mathcal{C} =$ Smooth, $\mathcal{M} =$ Smooth$_G$

$$\mathcal{F}\colon \text{Smooth} \longrightarrow \text{Smooth}_G,\ \mathcal{F}(Z) = Z \times G$$

obvious right G-action
left adjoint to forgetful
$\Box\colon$ Smooth$_G \rightarrow$ Smooth
Standard construction applied to resulting comonad and Z a point o:

$$(EG)^{\text{nonhomog}}$$

nonhomogeneous version of the total simplicial right G-object for G (universal G-bundle)
X left G-manifold

$$\mathcal{N}(G, X) = (EG)^{\text{nonhomog}} \times_G X$$

non-homogeneous Borel construction
isomorphism

$$(EG)^{\text{nonhomog}} \longrightarrow EG$$

of simplicial G-mfolds
induces simplicial mfd iso

$$\mathcal{N}(G, X) \longrightarrow N(G, X) = EG \times_G X$$

onto ordinary Borel $N(G, X)$

3.2 Infinitesimal equivariant (co)homology

$\mathfrak{g}$ Lie algebra, $C\mathfrak{g}$ cone in dgL
$\mathcal{C}_{\mathfrak{g}}$ category of right $\mathfrak{g}$-chain complexes

$$\mathcal{F}\colon \mathcal{C}_{\mathfrak{g}} \longrightarrow \mathrm{Mod}_{C\mathfrak{g}}$$
$$\mathcal{F}N = N \otimes_{\mathfrak{g}} \mathrm{U}[C\mathfrak{g}] \cong N \otimes_{\tau_{\mathfrak{g}}} \Lambda'_{\partial}[s\mathfrak{g}]$$

$\mathcal{F}N$ the CCE cx calculating $\mathrm{H}_*(\mathfrak{g}, N)$
Result: $(C\mathfrak{g})$-modules V, W:

$$\mathrm{Tor}^{(C\mathfrak{g},\mathfrak{g})}(V, W) = \mathrm{Tor}^{(\mathrm{U}C\mathfrak{g},\mathrm{U}\mathfrak{g})}(V, W)$$
$$\mathrm{Ext}_{(C\mathfrak{g},\mathfrak{g})}(V, W) = \mathrm{Ext}_{(\mathrm{U}C\mathfrak{g},\mathrm{U}\mathfrak{g})}(V, W)$$

relative derived in the sense of Hochschild and Eilenberg-Moore
$\mathrm{Ext}_{(C\mathfrak{g},\mathfrak{g})}(R, W)$:
Infinitesimal $\mathfrak{g}$*-equivariant cohomology of* W
Justification for this terminology later

4 Relative Lie algebra cohomology

Reminder: Lie algebra $\mathfrak{g}$, Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$
$\mathfrak{h}$-representation $\mathfrak{q} = \mathfrak{g}/\mathfrak{h}$

Hochschild: Relative Lie algebra (co)homology of the pair $(\mathfrak{g}, \mathfrak{h})$ defined via

$$\left(\Lambda'[s\mathfrak{q}]) \otimes_{\mathfrak{h}} \mathrm{U}[\mathfrak{g}], \partial\right), \ \partial \text{ CCE}$$

Now: Same construction, applied to $\mathfrak{g} \subseteq C\mathfrak{g}$ formally, yields object of the kind

$$\mathrm{S}'[s^2\mathfrak{g}] \otimes_{\mathfrak{g}} \mathrm{U}[C\mathfrak{g}] \cong \mathrm{S}'[s^2\mathfrak{g}] \otimes_{\partial} \Lambda'[s\mathfrak{g}] = W'[\mathfrak{g}]$$

Theorem 4.1. *The relative Hochschild resolution, carried out formally for the pair* $(C\mathfrak{g}, \mathfrak{g})$ *of dgL, yields a dgc* $W'[\mathfrak{g}]$ *whose dual is the ordinary Weil algebra for* $\mathfrak{g}$.

Corollary 4.2. *For any right* $(C\mathfrak{g})$*-module* N*, the relative* $\mathrm{Ext}_{(C\mathfrak{g},\mathfrak{g})}(R, N)$ *is the homology of the chain complex*

$$\mathrm{Hom}(W'[\mathfrak{g}], N)^{C\mathfrak{g}}.$$

N.B. $\mathrm{Hom}(W'[\mathfrak{g}], N)^{C\mathfrak{g}} \cong (W[\mathfrak{g}] \otimes N)^{C\mathfrak{g}}$ looks like a Weil model:
$W[\mathfrak{g}] = (W'[\mathfrak{g}])^*$ ordinary Weil algebra
$C\mathfrak{g}$-invariants: basic elements, i. e. invariant under contraction, that is, horizontal

and under Lie-derivative, that is, invariant in the usual sense

Special case: $R = \mathbb{R}$, $\mathfrak{g}$ reductive (e. g. cpt)

$$\mathrm{Ext}_{(C\mathfrak{g},\mathfrak{g})}(\mathbb{R},\mathbb{R}) = W[\mathfrak{g}]^{C\mathfrak{g}} = (\mathrm{S}^*[s^2\mathfrak{g}])^{\mathfrak{g}}$$

algebra of invariants of algebra of symmetric functions on double suspension $s^2\mathfrak{g}$ of $\mathfrak{g}$

Indeed: subalgebra of $W[\mathfrak{g}]$ invariant under contraction is the symmetric algebra $\mathrm{S}^*[s^2\mathfrak{g}]$

5 Monads and dual standard constructions

Functor $\mathcal{G} \colon \mathcal{C} \to \mathcal{M}$

functor $\Box \colon \mathcal{M} \to \mathcal{C}$ left-adjoint to $\mathcal{G}$

$$\mathcal{T} = \mathcal{G}\Box \colon \mathcal{M} \longrightarrow \mathcal{M}$$

$\eta \colon \mathcal{I} \to \mathcal{T}$ *unit*

$\varepsilon \colon \Box\mathcal{G} \to \mathcal{I}$ *counit* of adjunction

μ the natural transformation

$$\mu = \mathcal{G}\varepsilon\Box \colon \mathcal{G}\Box\mathcal{G}\Box = \mathcal{T}^2 \longrightarrow \mathcal{T} = \mathcal{G}\Box$$

$(\mathcal{T}, \eta, \mu)$ a *monad* over $\mathcal{M}$

CD's

$$
\begin{array}{ccc}
\mathcal{T} & \xleftarrow{\ \mu\ } & \mathcal{T}^2 \\
{\scriptstyle \mu}\uparrow & & \uparrow{\scriptstyle \mathcal{T}\mu} \\
\mathcal{T}^2 & \xleftarrow{\ \mu\mathcal{T}\ } & \mathcal{T}^3
\end{array}
$$

$$
\begin{array}{ccccc}
\mathcal{T} & \xrightarrow{\ =\ } & \mathcal{T} & \xleftarrow{\ =\ } & \mathcal{T} \\
{\scriptstyle =}\uparrow & & {\scriptstyle \mu}\uparrow & & {\scriptstyle =}\uparrow \\
\mathcal{I}\mathcal{T} & \xrightarrow{\ \eta\mathcal{T}\ } & \mathcal{T}^2 & \xleftarrow{\ \mathcal{T}\eta\ } & \mathcal{T}\mathcal{I}
\end{array}
$$

Dual standard construction:
cosimplicial object

$$\left(\mathcal{T}^{n+1}, \varepsilon^j \colon \mathcal{T}^{n+1} \to \mathcal{T}^{n+2}, \eta^j \colon \mathcal{T}^{n+2} \to \mathcal{T}^{n+1}\right)_{n\in\mathbb{N}}$$

$$\varepsilon^j = \mathcal{T}^j \eta \mathcal{T}^{n-j+1} \colon \mathcal{T}^{n+1} \to \mathcal{T}^{n+2},\ j = 0, \ldots, n+1,$$
$$\eta^j = \mathcal{T}^j \mu \mathcal{T}^{n-j} \colon \mathcal{T}^{n+2} \to \mathcal{T}^{n+1},\ j = 0, \ldots, n.$$

Given object V of $\mathcal{M}$

$$\mathbf{T}(V) = \left(\mathcal{T}^{n+1}(V), \varepsilon^j, \eta^j\right)_{n\in\mathbb{N}}$$

cosimplicial object in $\mathcal{M}$
associated chain complex $|\mathbf{T}(V)|$
a relatively injective resolution of V: $\mathrm{Ext}_{(\mathcal{M},\mathcal{C})}$

6 Examples

Very many examples: Ext for modules, group cohomology, Lie algebra cohomology, Hochschild cohomology, André-Quillen cohomology, ...

6.1 Sheaves

Forerunner Godement 1958: sheaf $\mathcal{F}$ on X
$\mathcal{T}(X) = C(X, \mathcal{F})$, sheaf of germs of sections of $\mathcal{F}$ (not necessarily continuous)
dual stand const the can flabby resolution

6.2 Differentiable cohomology

$\mathcal{G}_G\colon \mathrm{Vect} \to \mathrm{Mod}_G,\ \mathcal{G}_G V = \mathcal{A}^0(G, V)$,
$\mathcal{A}^0(G, V)$ right G-module via left transl on G
G-module V, chain complex arising from *dual standard construction* $\mathbf{T}(V)$:
standard differentiably injective resolution of V in Mod_G defining differentiable cohomology $\mathrm{H}^*_{\mathrm{cont}}(G, V)$ (Hochschild-Mostow)

in a cosimplicial degree $n \geq 0$:

$$\varepsilon^j \colon \mathrm{Map}(G^{\times(n+1)}, V) \longrightarrow \mathrm{Map}(G^{\times(n+2)}, V)$$
$$(\varepsilon^j(\alpha))(x_0, \ldots, x_{n+1}) = \alpha(\ldots, x_j x_{j+1}, \ldots)$$
$$(\varepsilon^{n+1}(\alpha))(x_0, \ldots, x_{n+1}) = \alpha(x_0, \ldots, x_n) x_{n+1}$$
$$\eta^j \colon \mathrm{Map}(G^{\times(n+2)}, V) \longrightarrow \mathrm{Map}(G^{\times(n+1)}, V)$$
$$(\eta^j(\alpha))(x_0, \ldots, x_n) = \alpha(\ldots x_j, e, x_{j+1} \ldots)$$

6.3 Infinitesimal equivariant cohomology revisited

$$\mathcal{G}^{\mathfrak{g}}_{C\mathfrak{g}} \colon \mathcal{C}_{\mathfrak{g}} \to \mathrm{Mod}_{C\mathfrak{g}}$$

$\mathcal{G}^{\mathfrak{g}}_{C\mathfrak{g}}(V) = \mathrm{Hom}_{\mathfrak{g}}(\mathrm{U}[C\mathfrak{g}], V) \cong (\mathrm{Alt}(\mathfrak{g}, V), d)$

$(\mathrm{Alt}(\mathfrak{g}, V), d)$ CCE cx calculating $\mathrm{H}^*(\mathfrak{g}, V)$

$(\mathrm{Alt}(\mathfrak{g}, V), d)$ right $(C\mathfrak{g})$-module

via contraction and Lie derivative

alternate characterization of $\mathrm{Ext}_{(C\mathfrak{g}, \mathfrak{g})}$

Functor

$$\mathcal{G}_{(G,C\mathfrak{g})}V = \mathcal{A}(G,V)$$

Which theory do we get?
On the category $\mathcal{C}$ of chain complexes

$$\mathcal{G}_{(G,C\mathfrak{g})} \colon \mathcal{C} \longrightarrow \mathrm{Mod}_{(G,C\mathfrak{g})}$$

G a left G-manifold via left translation
de Rham cx $\mathcal{A}(G,V)$ $(G,C\mathfrak{g})$-module via left translation, contraction and Lie derivative
$\mathrm{Mod}_{(G,C\mathfrak{g})}$ category of right $(G,C\mathfrak{g})$-modules
G-and $C\mathfrak{g}$-actions interwine
$\mathfrak{g}$-action derivative of G-action

METATHEOREM: *This category is a replacement for the missing category of $\mathcal{A}(G)$-comodules.*
Result: relative $\mathrm{Ext}_{((G,C\mathfrak{g});\mathcal{C})}$, relative to category $\mathcal{C}$ of chain complexes
Key observation: X left G-manifold
$\mathcal{A}(X)$ belongs to $\mathrm{Mod}_{(G,C\mathfrak{g})}$
$C\mathfrak{g}$-action by contraction and Lie derivative

Theorem 6.1. *The cosimplicial chain complex* $\mathcal{A}(\mathcal{N}(G,X))$ *associated with the non-homogeneous simplicial Borel construction* $\mathcal{N}(G,X)$ *is canonically isomorphic to the cosimplicial chain complex* $|\mathbf{T}(\mathcal{A}(X))|^{(G,C\mathfrak{g})}$*, the dual standard construction relative to the monad* $(\mathcal{T},\eta,\mu)$ *over the category* $\mathrm{Mod}_{(G,C\mathfrak{g})}$ *and the* $(G,C\mathfrak{g})$*-module* $\mathcal{A}(X)$*. Consequently the* G*-equivariant de Rham cohomology* $\mathrm{H}^*_G(X)$ *of* X *is canonically isomorphic to the differential* $\mathrm{Ext}_{((G,C\mathfrak{g});\mathcal{C})}(\mathbb{R},\mathcal{A}(X))$*.*

7 Extended decomposition Lemma

$$\mathcal{G}^G_{(G,C\mathfrak{g})}\colon \mathrm{Mod}_G \longrightarrow \mathrm{Mod}_{(G,C\mathfrak{g})}$$
$$\mathcal{G}^G_{(G,C\mathfrak{g})}(V) = (\mathrm{Alt}(\mathfrak{g},V),d)$$

$(\mathrm{Alt}(\mathfrak{g},V),d)$ CCE cx calculating $\mathrm{H}^*(\mathfrak{g},V)$
$\mathcal{G}^G_{(G,C\mathfrak{g})}$ variant of functor considered before, but now on $\mathcal{C}_G$ rather than $\mathcal{C}_\mathfrak{g}$

Proposition 7.1. *Decomposition*

$$\mathcal{G}_{(G,C\mathfrak{g})}\colon \mathcal{C} \xrightarrow{\mathcal{G}_G} \mathcal{C}_G \xrightarrow{\mathcal{G}^G_{(G,C\mathfrak{g})}} \mathrm{Mod}_{(G,C\mathfrak{g})}$$

Decomposition translates to *decomposition* for corresponding standard constructions.

Lemma 7.1. *(Extended decomposition lemma) The degreewise left trivialization of the tangent bundle of the simplicial group EG induces a $(G,C\mathfrak{g})$-equivariant isomorphism*

$$\mathrm{Hom}^{\tau_{E\mathfrak{g}}}(\Lambda'_\partial[sE\mathfrak{g}],\mathcal{A}^0(EG,\mathbf{V})) \to \mathcal{A}(EG,\mathbf{V})$$

from the differential graded cosimplicial diagonal object on the left-hand side onto the cosimplicial chain complex $\mathcal{A}(EG,\mathbf{V})$.

$\mathbf{V} = \mathbb{R}$: Bott's decomposition Lemma (1973)
X left G-manifold
$\mathbf{V} = \mathcal{A}(X)$, right $(G, C\mathfrak{g})$-module
$\mathcal{A}(EG, \mathbf{V})^{(G,C\mathfrak{g})}$ equivariant de Rham complex replaced with

$$\mathrm{Hom}^{\tau_E \mathfrak{g}}(\Lambda'_\partial[sE\mathfrak{g}], \mathcal{A}^0(EG, \mathbf{V}))^{(G,C\mathfrak{g})}$$

Roughly speaking G compact:
$\mathcal{A}^0(EG, \mathbf{V})$ equivalent to $\mathbf{V}$
cut the object to size: replace

$$\mathrm{Hom}^{\tau_E \mathfrak{g}}(\Lambda'_\partial[sE\mathfrak{g}], \mathcal{A}^0(EG, \mathbf{V}))^{(G,C\mathfrak{g})}$$

with

$$\mathrm{Hom}^{\tau_E \mathfrak{g}}(\Lambda'_\partial[sE\mathfrak{g}], \mathbf{V})^{(G,C\mathfrak{g})}$$

$\Lambda'_\partial[sE\mathfrak{g}]$ a *simplicial Weil coalgebra*
comparison: replace simplicial Weil coalgebra with ordinary Weil coalgebra $W'[\mathfrak{g}]$
arrive at Weil model

$$\mathrm{Hom}(W'[\mathfrak{g}], \mathbf{V})^{(G,C\mathfrak{g})}$$

Reformulation: Cartan model

8 Lie-Rinehart algebras

(A, L) Lie-Rinehart algebra
$\mathrm{U}(A, L)$ universal algebra

$$\mathcal{F}\colon {}_A\mathrm{Mod} \to {}_{\mathrm{U}(A,L)}\mathrm{Mod}$$
$$\mathcal{F}(M) = \mathrm{U}(A, L) \otimes_A M$$

Resulting relative $\mathrm{Ext}_{(\mathrm{U}(A,L),A)}$ the Palais-Rinehart coho defined via generalized CCE complex adapted to Lie-Rinehart algebras
L the (R, A)-Lie algebra $D_{\{\,\cdot\,,\,\cdot\,\}}$ associated with a Poisson structure $\{\,\cdot\,,\,\cdot\,\}$ on A:
relative $\mathrm{Ext}_{(\mathrm{U}(A,L),A)}(A, A)$:
Poisson cohomology of A
Question: Does $\mathrm{Ext}_{(C\mathfrak{g},\mathfrak{g})}$ generalize and, if so, what does the generalization signify?
Cone on a Lie-Rinehart algebra is ill-defined:
(A, L)-module M, for $a \in A$ and $\alpha \in L$, the operations i of contraction and λ of Lie derivative satisfy the familiar identity

$$\lambda_{a\alpha}(\omega) = a\lambda_\alpha(\omega) + da \cup i_\alpha(\omega) \qquad (8.1)$$

$(a \in A,\ \alpha \in L,\ \omega \in \mathrm{Alt}_A(L, M))$ involving

the term $da \cup i_\alpha(\omega)$ which does not arise for an ordinary Lie algebra
values of functor $\mathcal{G}$ on the category ${}_{(A,L)}\mathrm{Mod}$ of (A, L)-modules which assigns the *Rinehart complex* (generalized de Rham complex)

$$\mathcal{G}(V) = (\mathrm{Alt}_A(L, V), d)$$

to the (A, L)-module V lie in a certain category $\mathcal{M}$ of (A, L)-modules which are also endowed with an action of the *ordinary cone* CL on L in the category of Lie algebras
(beware: not Lie-Rinehart algebras),
subject to certain identities including (8.1)
resulting adjunction defines a monad
corresponding dual standard construction yields a relative differential graded Ext
Perhaps formally correct approach:
BRST-complex
variational bicomplex,
Noether identities.
How can the constructions for Lie algebroids then be globalized via Lie groupoids?

reinforce relevant ideas a bit:
With comorphism as morphism
Lie-Rinehart algebras a symm monoidal cat
Lie-Rinehart diagonal a diagonal morphism in this category
Given (A, L), the *total* object

$$E(A, L) = (EA, EL) \tag{8.2}$$

is defined as a *simplicial* Lie-Rinehart algebra.
EL a simplicial R-Lie algebra,
EA a cosimplicial R-algebra,
structure encoded by two pairings

$$EA \otimes EL \longrightarrow EL, \ EL \otimes EA \longrightarrow EA$$

satisfying appropriate Lie-Rinehart axioms
Application of CCE $\mathrm{Alt}_{(\,\cdot\,)}(\,\cdot\,,\,\cdot\,)$ to (EA, EL)
cosimplicial differential graded algebra

$$\mathrm{Alt}_{EA}(EL, EA)$$

(CL)-action on $\mathrm{Alt}_A(L, A)$ by contraction and
Lie-derivative extends to (CL)-action on $\mathrm{Alt}_{EA}(EL, EA$
compatible with the cosimplicial structure
$\mathcal{N}$ an (A, CL)-module

view it as the trivial cosimplicial (A, CL)-module product $(EA) \otimes \mathcal{N}$ of cosimpl objects defined inherits a canonical (A, CL)-module structure compatible with the cosimplicial structure application of CCE $\mathrm{Alt}_{(\,\cdot\,)}(\,\cdot\,,\,\cdot\,)$ to simplicial Lie-Rinehart algebra (EA, EL) yields cosimplicial differential graded $\mathrm{Alt}_{EA}(EL, EA)$-module

$$\mathrm{Alt}_{EA}(EL, (EA) \otimes \mathcal{N}).$$

define the (A, L)-equivariant cohomology $\mathrm{H}^*_{(A,L)}(\mathcal{N})$ of $\mathcal{N}$ to be the homology

$$\mathrm{H}^*_{(A,L)}(\mathcal{N}) = \mathrm{H}\left(\mathrm{Alt}_{EA}(EL, (EA) \otimes \mathcal{N})^{CL}\right).$$

can be viewed as the relative

$$\mathrm{Ext}_{(A,CL,L)}(EA, \mathcal{N})$$

Illustration 1. $\mathcal{F}$ a foliation of V,
$A = C^\infty(V)$, $\tau_{\mathcal{F}}$ tangent bundle of $\mathcal{F}$,
L the $(\mathbb{R}, A)$-Lie algebra
vector fields tangent to $\mathcal{F}$
injection $L \to L_V = \mathrm{Vect}(V)$
morphism of $\mathbb{R}, A)$-Lie algebras

de Rham complex $\mathcal{N} = \mathcal{A}(V) = \operatorname{Alt}_A(L_V, A)$ of V an (A, L, CL)-module via the A-module structure and contraction and Lie derivative $\mathrm{H}^*_{(A,L)}(\mathcal{N})$ the *de Rham cohomology of V that is equivariant rel to foliation $\mathcal{F}$*
foliation from a principal bundle with compact connected structure group:
this equiv coho comes down to ordinary equiv de Rham coho of V

Illustration 2. $\lambda \colon \Lambda \to N$ Lie algebroid
simplicial Lie algebroid $E\lambda$ has the form

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \Lambda^{\times(n+1)} & \longrightarrow & \ldots & \longrightarrow & \Lambda \times \Lambda & \longrightarrow & \Lambda \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & N^{\times(n+1)} & \longrightarrow & \ldots & \longrightarrow & N \times N & \longrightarrow & N
\end{array}$$

associated simplicial Lie algebroid $E\lambda \times_\lambda \tau_N$ has the form

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \Lambda^{\times n} \times \mathrm{T}N & \longrightarrow & \ldots & \longrightarrow & \Lambda \times \mathrm{T}N & \longrightarrow & \mathrm{T}N \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & N^{\times(n+1)} & \longrightarrow & \ldots & \longrightarrow & N \times N & \longrightarrow & N
\end{array}$$

References

[1] J. Huebschmann, *Homological perturbations, equivariant cohomology, and Koszul duality*, `math.AT/0401160`.

[2] J. Huebschmann, *Relative homological algebra, equivariant de Rham cohomology, and Koszul duality*, `math.AT/0401161`.