Kirillov's character formula, the holomorphic Peter-Weyl theorem, and the Blattner-Kostant-Sternberg pairing

J. Huebschmann

USTL, UFR de Mathématiques
CNRS-UMR 8524
59655 Villeneuve d'Ascq Cédex, France
Johannes.Huebschmann@math.univ-lille1.fr

Krakow, July 3, 2008

Abstract

By means of the orbit method we show that, for a compact Lie group, the Blattner-Kostant-Sternberg pairing map, with the constants being appropriately fixed, is unitary. Along the way we establish a holomorphic Peter-Weyl theorem for the complexification of a compact Lie group. Among our crucial tools is Kirillov's character formula. The basic observation is that the Weyl vector is lurking behind the Kirillov character formula as well as behind the requisite half-form correction on which the Blatter-Kostant-Sternberg-pairing for the compact Lie group relies and thus produces the appropriate shift which, in turn, controls the unitarity of the BKS-pairing map. Our methods are independent of heat kernel harmonic analysis, which has been used by B. C. Hall to obtain a number of these results. The heat kernel analysis comes out of our approach as well.

These results yield the first steps within a joint research program with G. Rudolph and M. Schmidt aimed at developing lattice gauge models for singular quantum mechanics. Within this program, we have already discovered a tunneling effect between singular strata.

Contents

1 Motivation

$Q = \mathbb{R}^n,\ \mathrm{T}^*Q \cong \mathbb{C}^n$
$z = q + ip$ holomorphic variables
SEGAL-BARGMANN :
ε LIOUVILLE measure, $\kappa(q+ip) = p^2$

$$\mathcal{H}L^2(\mathbb{C}^n, \mathrm{e}^{-\kappa/\hbar}\varepsilon) \leftrightarrow L^2(\mathbb{R}^n, dq)$$

— Lattice gauge theory: K compact Lie
Lie algebra $\mathfrak{k}$ with invariant inner product

$$\mathrm{T}^*K \cong \mathrm{T}K \longrightarrow K \times \mathfrak{k} \longrightarrow K^{\mathbb{C}}$$

— complex structure on $K^{\mathbb{C}}$
— cotg. bdle symplectic structure on T^*K
combine to K-bi-invariant KÄHLER structure

$$Q = K^\ell,\ \mathrm{T}^*Q = \mathrm{T}^*K^\ell \cong (K^{\mathbb{C}})^\ell$$

K-symmetry by conjugation
reduced space

$$\mathrm{T}^*K^\ell//K \cong (K^{\mathbb{C}})^\ell//K^{\mathbb{C}}$$

Quantization:
costratified Hilbert space, defined in terms of Hilbert space $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)$ holomorphic functions on $K^{\mathbb{C}}$ square-integrable relative to $\mathrm{e}^{-\kappa/\hbar}\eta\varepsilon$
κ and η to be defined shortly
Hilbert space $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)$ "sees the strata"
We do not know how ordinary Hilbert space $L^2(K, dx)$ could possibly see the strata.
Need $L^2(K, dx)$ to understand observables
B. Hall: coherent state transform

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon) \leftrightarrow L^2(K, dx)$$

comes down to BKS pairing map and in particular yields unitarity of BKS pairing map
relies on heat kernel harmonic analysis

Today: Holomorphic Peter-Weyl theorem for the Hilbert space

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)$$

Unitarity of BKS pairing map

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon) \leftrightarrow L^2(K, dx)$$

a direct consequence
new approach to coherent state transform
independent of heat kernels
heat kernel analysis comes out for free

2 Polar decomposition and Kähler structure

General compact Lie group K

Lie algebra $\mathfrak{k}$ with invariant inner product

$$\mathrm{T}^*K \cong \mathrm{T}K \longrightarrow K \times \mathfrak{k} \longrightarrow K^{\mathbb{C}}$$

global Kähler potential κ

$$\kappa(x\,\mathrm{e}^{iY}) = |Y|^2,\ x \in K,\ Y \in \mathfrak{k}:$$

symplectic structure on $\mathrm{T}^*K \cong K^{\mathbb{C}}$: $i\partial\overline{\partial}\kappa$

ε symplectic volume form on $\mathrm{T}^*K \cong K^{\mathbb{C}}$

η the real K-bi-invariant function on $K^{\mathbb{C}}$

$$\eta(x\,\mathrm{e}^{iY}) = \sqrt{\left|\frac{\sin(\mathrm{ad}(Y))}{\mathrm{ad}(Y)}\right|},\ x \in K,\ Y \in \mathfrak{k}$$

half-form Kähler quantization on $K^{\mathbb{C}}$:

holomorphic Hilbert space $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)$

functions square-integrable relative to $\mathrm{e}^{-\kappa/\hbar}\eta\varepsilon$

scalar product

$$\langle \psi_1, \psi_2 \rangle = \frac{1}{\mathrm{vol}(K)} \int_{K^{\mathbb{C}}} \overline{\psi_1}\psi_2 \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon$$

left and right translation: $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)$

unitary $(K \times K)$-representation

3 Cartan-Weyl decomposition

$\mathbb{C}[K^{\mathbb{C}}]$ representative functions on $K^{\mathbb{C}}$
algebra via convolution product
$\widehat{K^{\mathbb{C}}}$ isomorphism classes of irred reps of $K^{\mathbb{C}}$
maximal torus in K
dominant Weyl chamber C^+
highest weight λ: $T_\lambda \colon K^{\mathbb{C}} \to \mathrm{End}(V_\lambda)$
representation in the class of λ
$\psi \in V_\lambda^*$, $w \in V_\lambda$
representative function $\Phi_{\psi,w}$ given by

$$\Phi_{\psi,w}(q) = \psi(qw),\ q \in K^{\mathbb{C}}$$

association

$$V_\lambda^* \otimes V_\lambda \ni \psi \otimes w \longmapsto \Phi_{\psi,w}$$

morphism of $(K^{\mathbb{C}} \times K^{\mathbb{C}})$-reps

$$\iota_\lambda \colon V_\lambda^* \otimes V_\lambda \longrightarrow \mathbb{C}[K^{\mathbb{C}}]$$

irreducible representation $T_\lambda\colon K \to \mathrm{End}(W_\lambda)$
Fourier coefficient $\widehat{f}_\lambda \in \mathrm{End}(W_\lambda)$ of
L^2-function f on K relative to λ

$$\widehat{f}_\lambda = \frac{1}{\mathrm{vol}(K)} \int_K f(x) T_\lambda(x^{-1}) dx$$

irreducible rational representation
$T_\lambda\colon K^{\mathbb{C}} \to \mathrm{End}(V_\lambda)$ of $K^{\mathbb{C}}$
Fourier coefficient $\widehat{\Phi}_\lambda \in \mathrm{End}(V_\lambda)$ of
holomorphic function Φ on $K^{\mathbb{C}}$:
Fourier coefficient of the restriction of Φ to K

$$F_\lambda\colon \mathbb{C}[K^{\mathbb{C}}] \longrightarrow \mathrm{End}(V_\lambda),\ \Phi \mapsto \widehat{\Phi}_\lambda$$

Theorem 3.1. [Cartan-Weyl]
(i) *As* $(K^{\mathbb{C}} \times K^{\mathbb{C}})$*-representations*

$$\mathbb{C}[K^{\mathbb{C}}] = \oplus_\lambda V_\lambda^* \otimes V_\lambda$$

(ii) $V_\lambda^* \otimes V_\lambda$ *isotypical summand of* $\mathbb{C}[K^{\mathbb{C}}]$
(iii) *relative to convolution product on* $\mathbb{C}[K^{\mathbb{C}}]$,

$$(F_\lambda)\colon \mathbb{C}[K^{\mathbb{C}}] \longrightarrow \oplus_\lambda \mathrm{End}(V_\lambda)$$

algebra isomorphism
isomorphism of $(K^{\mathbb{C}} \times K^{\mathbb{C}})$*-representations*

4 Holomorphic Peter-Weyl theorem

R^+ positive roots
Weyl vector: $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$

$$C_{t,\lambda} = (t\pi)^{\dim(K)/2}\mathrm{e}^{t|\lambda+\rho|^2}$$

on $\mathrm{End}(V_\lambda)$ inner product $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ given by

$$\langle A, B\rangle_\lambda = \mathrm{tr}(A^*B),\ A, B \in \mathrm{End}(V_\lambda)$$

adjoint A^* of A being computed as usual with respect to a K-invariant inner product on V_λ
d_λ dimension of V_λ
endow $\oplus_{\lambda\in\widehat{K^{\mathbb{C}}}}\mathrm{End}(V_\lambda)$ with the inner product which, on the summand $\mathrm{End}(V_\lambda)$, is given by

$$\frac{d_\lambda}{C_{t,\lambda}}\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$$

up to a constant familiar *Hilbert-Schmidt* norm

$$\widehat{\bigoplus}_{\lambda\in\widehat{K^{\mathbb{C}}}}\mathrm{End}(V_\lambda)$$

completion relative to inner product

Theorem 4.1. [Holomorphic Peter-Weyl thm] (i) $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/t}\eta\varepsilon)$ *contains vs* $\mathbb{C}[K^{\mathbb{C}}]$ *of represent. fts on* $K^{\mathbb{C}}$ *as dense subspace as unitary* $(K \times K)$*-representation,* $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/t}\eta\varepsilon)$ *decomposes as*

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/t}\eta\varepsilon) = \widehat{\bigoplus}_{\lambda \in \widehat{K^{\mathbb{C}}}} V_\lambda^* \otimes V_\lambda$$

into $(K \times K)$*-isotypical summands*
(ii) *convolution induces a convolution product* $*$ *on* $\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/t}\eta\varepsilon)$
(iii) *relative to convolution, as* λ *ranges over the irreducible rational representations of* $K^{\mathbb{C}}$*, the assignment to a holomorphic function* Φ *on* $K^{\mathbb{C}}$ *of its Fourier coefficients* $\widehat{\Phi}_\lambda \in \mathrm{End}(V_\lambda)$ *yields an isomorphism*

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/t}\eta\varepsilon) \longrightarrow \widehat{\bigoplus}_{\lambda \in \widehat{K^{\mathbb{C}}}} \mathrm{End}(V_\lambda)$$

of Hilbert algebras

Consequence of *holomorphic* Peter-Weyl: irreducible characters $\chi_\lambda^{\mathbb{C}}$ of $K^{\mathbb{C}}$ a basis of

$$\mathcal{H} = \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/\hbar}\eta\varepsilon)^K$$

5 Abstract identification of the two Hilbert spaces

$R(K)$ algebra of representative fn's on K
W_λ unitary K-representation arising from V_λ by restriction to K

$$\iota_\lambda \colon W_\lambda^* \otimes W_\lambda \longrightarrow R(K) = \mathbb{C}[K^{\mathbb{C}}]$$

Theorem 5.1. *The association*

$$\begin{aligned} V_\lambda^* \otimes V_\lambda &\longrightarrow W_\lambda^* \otimes W_\lambda \\ \varphi^{\mathbb{C}} &\longmapsto C_{t,\lambda}^{1/2} \varphi = (t\pi)^{\dim(K)/4} \mathrm{e}^{t|\lambda+\rho|^2/2} \varphi \end{aligned}$$

as λ ranges over the highest weights induces a unitary isomorphism

$$H_t \colon \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/t} \eta\varepsilon) \longrightarrow L^2(K, dx)$$

of unitary $(K \times K)$-representations.

6 BKS pairing

Theorem 6.1. *Above iso coincides with BKS-pairing map*

$$\begin{aligned} H_t\colon \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/t}\eta\varepsilon) &\longrightarrow L^2(K, dx) \\ \Phi &\longmapsto F_\Phi \end{aligned}$$

where, for $x \in K$*,*

$$(F_\Phi)(x) = \int_{\mathfrak{k}} \Phi(x \exp(iY))\mathrm{e}^{-\frac{|Y|^2}{2t}}\eta(Y/2)dY.$$

K abelian: measure on $K^{\mathbb{C}}$ Haar measure on K times Gaussian in imaginary directions decomposition of the Hilbert spaces into irreducible (1-dimensional) constituents: inspection of characters establishes unitarity of BKS-pairing map

our argument for general K the same, with *shift* by *Weyl vector* incorporated, points in dual of abelian Lie algebra that correspond to characters replaced with entire integral coadjoint orbits

7 Kirillov's character formula

Real analytic function $j\colon K^{\mathbb{C}} \cong K \times \mathfrak{k} \longrightarrow \mathbb{R}$

$$j(x,Y) = j(Y) = \det\left(\frac{\sinh(\mathrm{ad}(Y/2))}{\mathrm{ad}(Y/2)}\right)^{\frac{1}{2}}$$

obvious identity

$$j(iY) = \eta(Y/2)$$

highest weight λ

ϑ variable on $\Omega_{\lambda+\rho} = \mathrm{Ad}^*(K)(\lambda+\rho) \subseteq \mathfrak{k}^*$

Kirillov's character formula:

$$\mathrm{vol}(\Omega_\rho) j(X)\chi_\lambda(\exp X) = \int_{\Omega_{\lambda+\rho}} \mathrm{e}^{i\vartheta(X)} d\sigma$$

In our approach, this formula is used to evaluate integrals, e. g.:

Lemma 7.1. $\varphi^{\mathbb{C}}$ *in* $V_\lambda^* \otimes V_\lambda$

$$\int_{K^{\mathbb{C}}} \overline{\varphi}^{\mathbb{C}} \varphi^{\mathbb{C}} \mathrm{e}^{-\kappa/t} \eta \varepsilon = C_{t,\lambda} \int_K \overline{\varphi}\varphi dx$$

$$C_{t,\lambda} = (t\pi)^{\dim(K)/2} \mathrm{e}^{t|\lambda+\rho|^2}$$

8 Energy quantization

Kähler: only constants quantizable
Schrödinger: functions at most quadratic in generalized momenta quantizable
(classical) Hamiltonian of model quantizable
associated quantum Hamiltonian

$$H = -\frac{\hbar^2}{2}\Delta_K + \text{potential}$$

The operator Δ_K arises from the non-positive Laplace-Beltrami operator associated with bi-invariant Riemannian metric on K.
Since metric bi-invariant, so is Δ_K, whence Δ_K restricts to self-adjoint operator on $L^2(K, \mathrm{d}x)^K$
By means of

$$L^2(K, \mathrm{d}x) \longrightarrow \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/t}\eta\varepsilon)$$

transfer Hamiltonian, in particular, the operator Δ_K, to self-adjoint operators on

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/t}\eta\varepsilon)$$

SCHUR's lemma:
— each isotypical $(K \times K)$-summand $L^2(K, dx)_\lambda$ of $L^2(K, dx)$ in the *Peter-Weyl* decomposition an eigenspace
— representative functions eigenfunctions for Δ_K
— eigenvalue $-\varepsilon_\lambda$ of Δ_K corresponding to the highest weight λ

$$\varepsilon_\lambda = (|\lambda + \rho|^2 - |\rho|^2),$$

— in holomorphic quantization on $\mathrm{T}^*K \cong K^{\mathbb{C}}$, energy operator arises as the unique extension of the operator $-\frac{1}{2}\Delta_K$ on

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/t}\eta\varepsilon)$$

to an unbounded self-adjoint operator
— spectral decomposition thereof refines to holomorphic PETER-WEYL decomposition of

$$\mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/t}\eta\varepsilon)$$

9 Relationship with heat equation

$$p_t(x) = u(t,x) = \sum_\lambda d_\lambda \mathrm{e}^{-\frac{\varepsilon_\lambda}{2}t}\chi_\lambda(x)$$

fundamental solution of heat equation

$$\frac{du}{dt} = \frac{1}{2}\Delta_K u$$

on K, subject to initial condition p_0 Dirac distribution supported at identity of K

Under inverse transformation

$$L^2(K,dx) \longrightarrow \mathcal{H}L^2(K^{\mathbb{C}}, \mathrm{e}^{-\kappa/t}\eta\varepsilon)$$

up to a constant, a smooth function f on K goes to unique holomorphic function on $K^{\mathbb{C}}$ whose restriction to K given by

$$\mathrm{e}^{-t|\rho|^2/2}\mathrm{e}^{t\Delta_K/2}f$$

for $y \in K$ value $(\mathrm{e}^{t\Delta_K/2}f)(y)$ given by

$$(\mathrm{e}^{t\Delta_K/2}f)(y) = \int_K p_t(yx^{-1})f(x)dx = (p_t * f)(y),$$

cf. [Nelson]. This establishes the relation between the BKS-pairing and the heat equation observed by B. Hall.

References

[1] B.C. Hall: Phase space bounds for quantum mechanics on a compact Lie group. Commun. Math. Phys. **184**, 233–250 (1997)

[2] B.C. Hall: Geometric quantization and the generalized Segal-Bargmann transform for Lie groups of compact type. Commun. Math. Phys. **226**, 233–268 (2002), `quant-ph/0012015`

[3] B. C. Hall, The Segal-Bargmann "Coherent State" Transform for compact Lie groups, Journal of Functional Analysis **122** (1994), 103–151

[4] B. C. Hall, The inverse Segal-Bargmann transform for compact Lie groups, J. Funct. Anal. **143** (1997), 98–116

[5] J. Huebschmann: Kähler spaces, nilpotent orbits, and singular reduction. Memoirs of the AMS **172 (814)**, Amer. Math. Soc., Providence R.I., 2004, `math.DG/0104213`

[6] J. Huebschmann: Kähler quantization and reduction. J. reine angew. Math. **591** (2006), 75–109, `math.SG/0207166`

[7] J. Huebschmann: Stratified Kähler structures on adjoint quotients, Differential Geom. Applications (to appear), `math.DG/0404141`

[8] J. Huebschmann: Singular Poisson-Kähler geometry of certain adjoint quotients. In: *The mathematical legacy of C. Ehresmann*, Proceedings, Bedlewo, 2005, Banach Center Publications **76** (2007), 325–347, `math.SG/0610614`

[9] J. Huebschmann: Kirillov's character formula, the holomorphic Peter-Weyl theorem and the Blattner-Kostant-Sternberg pairing, J. Geom. Phys. **58** (2008), 833–848, `math.DG/0610613`

[10] J. Huebschmann, Lie-Rinehart algebras, descent, and quantization, in: Galois theory, Hopf algebras, and semiabelian categories, Fields Institute Communications **43** (2004),295–316, Amer. Math. Society, Providence, R. I., `math.SG/0303016`

[11] J. Huebschmann, Classical phase space singularities and quantization, in: Quantum Theory and Symmetries. IV. V. Dobrev, ed., Heron Press, Sofia (2006), 51–65, `math-ph/0610047`

[12] J. Huebschmann, G. Rudolph, and M. Schmidt, A gauge model for quantum mechanics on a stratified space, `hep-th/0702017`

[13] J. Huebschmann, G. Rudolph, and M. Schmidt, Quantum mechanical tunneling on a space with singularities, in: Lie Theory and its Applications in Physics. VII. Proceedings of the VII-th international workshop, Varna, Bulgaria, 18-24 June, 2007, suppl. to Bulg. J. of physics, vol. 35, H.-D. Doebner and V. K. Dobrev, eds., Heron Press, Sofia (2008), 190–210