A survey on homological perturbation theory

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Abstract

Higher homotopies are nowadays playing a prominent role in mathematics as well as in certain branches of theoretical physics. Homological perturbation theory (HPT), in a simple form first isolated by Eilenberg and Mac Lane in the early 1950's, is nowadays a standard tool to handle algebraic incarnations of higher homotopies. A basic observation is that higher homotopy structures behave much better relative to homotopy than strict structures, and HPT enables one to exploit this observation in various concrete situations. In particular, this leads to the effective calculation of various invariants which are otherwise intractable.

Higher homotopies and HPT-constructions abound but they are rarely recognized explicitly and their significance is hardly understood; at times, their appearance might at first glance even come as a surprise, for example in the Kodaira-Spencer approach to deformations of complex manifolds or in the theory of foliations.

The talk will illustrate, with a special emphasis on the compatibility of perturbations with algebraic structure, how HPT may be successfully applied to various mathematical problems arising in group cohomology, algebraic K-theory, and deformation theory.

Contents

1 Origins of homotopy and higher homotopies

GAUSS 1833: linking number
GAUSS 1828: idea of a classifying space
HILBERT: exploration of syzygies
generating function for the number of invariants of each degree a rational function
homogeneous ideal I of a polynomial ring S, "number of independent linear conditions" for a form of degree d in S to lie in I" a polynomial function of d
problem of counting the number of conditions already considered in projective geometry and in invariant theory
CAYLEY 1848: general statement of the problem, clear understanding of the role of syzygies–without the word, introduced by SYLVESTER 1853
CAYLEY somewhat develops what is nowadays referred to as the KOSZUL resolution
POINCARÉ 1895: terminology *homotopy*
loop composition

2 Some 20'th century higher homotopies

Failure of Alexander-Whitney multiplication of cocycles to be commutative
Steenrod $\cup_i$-products, prompted
s(trongly)h(omotopy)c(ommutative) structures
Steenrod operations
A_∞-structure system of higher homotopies together with suitable coherence conditions
Massey products invariants of certain A_∞-structures
elementary example Borromean rings
a Massey product of three variables detects the simultaneous linking of all three circles
Sugawara recognition principle for characterizing loop spaces up to homotopy type, associativity problem of loop multiplications

STASHEFF — complete understanding of homotopy invariance of associativity
— clean recognition principle for loop spaces
— classifying space
— nested sequence of homotopy associativity conditions
— A_n-spaces
any space an A_1-space, H-space an A_2-space
every homotopy associative H-space is A_3
A_∞-space homotopy type of a loop space
algebraic analogue of A_n-space in category of algebras
A_n-algebra, the case $n = \infty$ being included
original and motivating example singular chains on the based loop space of a space
variant L_∞-algebras
key observation: A_∞-structures behave correctly with respect to homotopy
at Michigan, YURI RAINICH was one of Stasheff's mentors as an undergraduate
here Stasheff learned that, in projective geometry, associativity was an option

3 Homological perturbations — HPT

Fundamental means to handle
higher homotopies
EILENBERG AND MACLANE [1953]
contraction

$$(M \underset{\nabla}{\overset{\pi}{\leftrightarrows}} N, h)$$

— chain complexes M and N
— chain maps $\nabla\colon M \to N$ and $\pi\colon N \to M$
— degree 1 morphism $h\colon N \to N$

requirements

— deformation retraction

$$\pi\nabla = \mathrm{Id},\ \nabla\pi - \mathrm{Id} = dh + hd$$

— annihilation properties

$$h\nabla = 0,\ \pi h = 0,\ hh = 0$$

EILENBERG AND MAC LANE [1953]
comparison between reduced bar and
W-constructions a reduction
conjectured that it is a contraction

Some notions

— contraction *filtered*: N and M filtered chain complexes, π, ∇ and h filtration preserving
— *perturbation* of the differential d of a chain complex X: (homogeneous) morphism

$$\partial \colon X \longrightarrow X$$

of the same degree as d such that
— ∂ lowers the filtration
— $(d+\partial)^2 = 0$ or, equivalently,

$$[d, \partial] + \partial\partial = 0$$

— sum $d + \partial$, referred to as the *perturbed differential*, again differential
— *coalgebra perturbation*: X a graded coalgebra structure such that (X, d) a differential graded coalgebra
perturbed differential $d + \partial$ compatible with graded coalgebra structure
— similarly *algebra perturbation*
Heller [1954] *recursive structure of triangular complex* later identified as a perturbation

Perturbation lemma.

$$(M \underset{\nabla}{\overset{g}{\leftrightarrows}} N, h)$$

filtered contraction
∂ *perturbation of the differential on* N

$$\mathcal{D} = \sum_{n \geq 0} g\partial(h\partial)^n \nabla = \sum_{n \geq 0} g(\partial h)^n \partial \nabla$$

$$\nabla_\partial = \sum_{n \geq 0} (h\partial)^n \nabla$$

$$g_\partial = \sum_{n \geq 0} g(\partial h)^n$$

$$h_\partial = \sum_{n \geq 0} (h\partial)^n h = \sum_{n \geq 0} h(\partial h)^n.$$

Suppose filtrations on M *and* N *complete.*
Then: Infinite series converge,
$\mathcal{D}$ *a perturbation of the differential on* M
notation N_∂ *and* $M_\mathcal{D}$ *new chain complexes,*

$$(M_\mathcal{D} \underset{\nabla_\partial}{\overset{g_\partial}{\leftrightarrows}} N_\partial, h_\partial)$$

constitute new filtered contraction
natural in terms of the given data.

Some history

SHI [1962] lurking behind formulas
M. BARRAT made explicit (unpublished)
BROWN [1964] in print
GUGENHEIM [1972] *twisted* EILENBERG-ZILBER theorem via perturbation lemma
CHEN [1977] $\Omega_{\partial}[\mathrm{H}_*(X)]$ model for real chains on loop space of a smooth manifold X by means of formal power series connection involving de Rham complex of X
GUGENHEIM [1982] perturbation theory for homology of loop space, ordinary singular setting, reworks and extends CHEN's con'n
WONG [1986] Heidelberg diploma thesis (Diplomarbeit) supervised by H.
using perturbation lemma, confirmed EML-conjecture: comparison between reduced bar and W-constructions a contraction
MARKL [2006?] operadic perturbation lemma
perturbation lemma under weaker hypotheses
BERGER AND H. [1996]
geometric comparison between bar and W-con's

H. and Kadeishvili [1991]

Theorem. *Let*

$$(\mu, X \underset{\beta}{\overset{\alpha}{\leftrightarrows}} Y, \nu)$$

be a filtered chain equivalence, let ∂ be a perturbation of the differential d on Y, and suppose that the filtrations on X and Y are complete. Then HPT yields formulas for

- *a perturbation $\mathcal{D}\colon X \longrightarrow X$ of the differential on X, and for*
- *new morphisms*

$$\alpha_\partial\colon X \longrightarrow Y, \quad \beta_\partial\colon Y \longrightarrow X,$$
$$\mu_\partial\colon X \longrightarrow X, \quad \nu_\partial\colon Y \longrightarrow Y,$$

which yield a new filtered chain equivalence

$$(\mu_\partial, X_{\mathcal{D}} \underset{\beta_\partial}{\overset{\alpha_\partial}{\leftrightarrows}} Y_\partial, \nu_\partial)$$

which is natural in terms of the given data.

Remark about proof: Reduce the constructions to ordinary perturbation lemma by means of mapping cyclinder

5 Iterative perturbations

H. [1986]
X a connected simplicial set, simple
π_1, π_2, etc. its homotopy groups

$$k^j \colon X_{j-2} \longrightarrow \mathrm{K}(\pi_{j-1}, j), j \geq 3$$

maps representing the k-invariants of X

Theorem. *These maps determine a differential d on the tensor product*

$$\mathrm{K}(\pi_1, 1) \otimes \mathrm{K}(\pi_2, 2) \otimes \ldots$$

and a chain equivalence between X and the resulting chain complex

$$(\mathrm{K}(\pi_1, 1) \otimes \mathrm{K}(\pi_2, 2) \otimes \ldots, d)$$

Main ingredient: Iteration of perturbation lemma, applied to corresponding iterated fibrations

Application: Interpretation of the k-invariants of the algebraic K-theory of a finite field

Construction taken up in
Lambe-Stasheff [1987]
models for iterated fibrations

6 Compatibility of HPT with algebraic structure

H. [1989], [1991]
compatibility of HPT- constructions
with algebraic structure
suitable algebraic HPT-constructions to exploit
A_∞-modules arising in group cohomology
construction of suitable free resolutions
explicit numerical calculations in group coho
until today still not doable by other methods
spectral seq's show up not collapsing from E_2
illustrate a typical phenomenon:
Whenever a spectral sequence arises
a strong homotopy structure lurking behind
spectral sequence invariant thereof
higher homotopy structure finer than spectral
sequence itself

Visit the workshop

EXAMPLE 1. Group extension

$$\mathbf{e}\colon 1 \longrightarrow N \longrightarrow G \longrightarrow K \longrightarrow 1$$

free resolutions

$$\mathrm{M}(K) = \mathrm{M}^\sharp(K) \otimes_d RK$$
$$\mathrm{M}(N) = \mathrm{M}^\sharp(N) \otimes_d RN$$

of the ground ring R in the categories of right RK– and RN–modules

augmentation maps

$$\varepsilon\colon \mathrm{M}^\sharp(K) \longrightarrow R, \quad \varepsilon\colon \mathrm{M}^\sharp(N) \longrightarrow R$$

$\mathrm{M}^\sharp(K)$ and $\mathrm{M}^\sharp(N)$ connected as graded modules, i.e. in degree zero a single copy of the ground ring R

explicit HPT-*construction of a free resolution* of R in the category of right RG–modules from the given data:

$$\mathrm{M}(N) \otimes_{RN} RG = \mathrm{M}^\sharp(N) \otimes_d RG$$

a free resolution of RK in the category of right RG–modules write

$$d^0 = \mathrm{M}^\sharp(K) \otimes d \quad (= \mathrm{Id}_{\mathrm{M}^\sharp(K)} \otimes d),$$

where d the differential on $\mathrm{M}^\sharp(N) \otimes_d RG$
an RG–linear differential on

$$\mathrm{M}^\sharp(K) \otimes \mathrm{M}^\sharp(N) \otimes RG$$

vertical in an obvious sense
projection map

$$g\colon \mathrm{M}^\sharp(K) \otimes \mathrm{M}^\sharp(N) \otimes RG \longrightarrow \mathrm{M}^\sharp(K) \otimes RK$$

induces an isomorphism on homology relative to differential d^0 on source, zero differential on target
object $\mathrm{M}^\sharp(K) \otimes RK$ filtered by degrees
degree filtration of $\mathrm{M}^\sharp(K)$ induces a filtration

$$\{\mathrm{F}_i = \mathrm{F}_i(\mathrm{M}^\sharp(K) \otimes \mathrm{M}^\sharp(N) \otimes RG)\}_{i \geq 0}$$

for $\mathrm{M}^\sharp(K) \otimes \mathrm{M}^\sharp(N) \otimes RG$ in the obvious way

Theorem. *There is a differential d on*

$$\mathrm{M}^\sharp(K) \otimes \mathrm{M}^\sharp(N) \otimes RG$$

having the following properties:

1. *It can be written as*

   $$d = d^0 + d^1 + d^2 + \cdots$$

   where, for $i \geq 1$, the operator d^i lowers filtration by i.

2. *The projection map is a chain map*

   $$(\mathrm{M}^\sharp(K) \otimes \mathrm{M}^\sharp(N)) \otimes_d RG \to \mathrm{M}^\sharp(K) \otimes_d RK$$

   that is compatible with the differentials, the filtrations, and the right RG–module structures.

Moreover, the resulting chain complex

$$(\mathrm{M}^\sharp(K) \otimes \mathrm{M}^\sharp(N)) \otimes_d RG$$

is acyclic and hence a free resolution of the ground ring R in the category of right RG–modules. Finally, the higher terms d^i, $i \geq 1$, can be constructed explicitly via HPT.

Under these circumstances, the series

$$d^1 + d^2 + \cdots$$

is a *perturbation* of d^0.

ILLUSTRATION. Metacyclic groups
such a group G has a presentation of the form

$$\langle x, y; y^r = 1, x^s = y^f, xyx^{-1} = y^t \rangle$$

where x and y are generators of $K = \mathbb{Z}/s$ and $N = \mathbb{Z}/r$ respectively, and where

$$s > 1, r > 1, t^s \equiv 1 \mod r, tf \equiv f \mod r,$$

in particular, $\frac{t^s-1}{r}$ and $\frac{(t-1)f}{r}$ integers

SAMPLE RESULTS. p a prime which divides r and s, and suppose

$$t \not\equiv 1 \mod p.$$

Further, let j_0 be the order of t modulo p, i. e. j_0 the smallest number j so that

$$t^j \equiv 1 \mod p.$$

Theorem 1. *Suppose that p divides the number $\frac{t^s-1}{r}$. Then the mod p cohomology spectral sequence of the group extension collapses from E_2, and the multiplicative extension problem from*

$$E_2 = E_\infty$$

to $\mathrm{H}^*(G, \mathbb{Z}/p)$ is trivial. Moreover,

$$\mathrm{H}^*(G, \mathbb{Z}/p)$$

has classes

$$\omega_{2j_0-1}, c_{2j_0}, |\omega_{2j_0-1}| = 2j_0 - 1, |c_{2j_0}| = 2j_0,$$

of filtration zero which restrict to the classes $\omega_y c_y^{j_0-1}$ and $c_y^{j_0}$ in

$$\mathrm{H}^*(N, \mathbb{Z}/p) \cong \Lambda[\omega_y] \otimes \mathrm{P}[c_y]$$

respectively, so that, as a graded commutative algebra, $\mathrm{H}^*(G, \mathbb{Z}/p)$ can be written as

$$\Lambda[\omega_x] \otimes \mathrm{P}[c_x] \otimes \Lambda[\omega_{2j_0-1}] \otimes \mathrm{P}[c_{2j_0}].$$

Theorem 2. *Suppose that p does not divide the number $\frac{t^s-1}{r}$. Then the cohomology spectral sequence of the group extension collapses from E_3, and the multiplicative extension problem is trivial. Moreover,* $\mathrm{H}^*(G, \mathbb{Z}/p)$ *has classes*

$$\omega_{2j_0-1}, \omega_{4j_0-1}, \ldots, \omega_{2pj_0-1}, c_{2pj_0},$$

of filtration zero which restrict to the classes $\omega_y c_y^{ij_0-1}$ and $c_y^{pj_0}$ in

$$\mathrm{H}^*(N, \mathbb{Z}/p) \cong \Lambda[\omega_y] \otimes \mathrm{P}[c_y]$$

respectively so that, as a graded commutative algebra,

$$\mathrm{H}^*(G, \mathbb{Z}/p)$$

is generated by

$$\omega_x, c_x, \omega_{2j_0-1}, \omega_{4j_0-1}, \ldots, \omega_{2pj_0-1}, c_{2pj_0},$$

subject to the relations

$$\omega_a \omega_b = 0,$$
$$a,\, b \in \{2j_0-1, 4j_0-1, \ldots, 2pj_0-1\},$$
$$c_x \omega_a = 0,$$
$$a \in \{2j_0-1, 4j_0-1, \ldots, 2(p-1)j_0-1\}.$$

Example 2.
H. and Kadeishvili [1991]
H. [2004]
Small models for chain algebras
algebra and coalgebra perturbation lemmata
Sample result

Theorem. *Let*

$$(\mu, X \underset{\beta}{\overset{\alpha}{\rightleftarrows}} Y, \nu)$$

be a chain equivalence, let

$$(\mathrm{T}\mu, \mathrm{T}[X] \underset{\mathrm{T}\beta}{\overset{\mathrm{T}\alpha}{\rightleftarrows}} \mathrm{T}[Y], \mathrm{T}\nu)$$

be the corresponding filtered chain equivalence of augmented differential graded algebras, and let ∂ *be a multiplicative perturbation of the differential on* $\mathrm{T}[Y]$ *with respect to the augmentation filtration. Suppose further that* X *and* Y *are connected. Then HPT yields formulas for*

- *a multiplicative perturbation*

$$\mathcal{D} \colon \mathrm{T}[X] \longrightarrow \mathrm{T}[X]$$

of the differential on $\mathrm{T}[X]$*, and for*

- *new morphisms*

$$\begin{aligned} \mathrm{T}_\partial\alpha &\colon \mathrm{T}[X] \longrightarrow \mathrm{T}[Y] \\ \mathrm{T}_\partial\beta &\colon \mathrm{T}[Y] \longrightarrow \mathrm{T}[X] \\ \mathrm{T}_\partial\mu &\colon \mathrm{T}[X] \longrightarrow \mathrm{T}[X] \\ \mathrm{T}_\partial\nu &\colon \mathrm{T}[Y] \longrightarrow \mathrm{T}[Y], \end{aligned}$$

which combine to a new filtered chain equivalence

$$(\mathrm{T}_\partial\mu, \mathrm{T}_{\mathcal{D}}[X] \underset{\mathrm{T}_\partial\beta}{\overset{\mathrm{T}_\partial\alpha}{\rightleftarrows}} \mathrm{T}_\partial[Y], \mathrm{T}_\partial\nu)$$

of augmented differential graded algebras where $\mathrm{T}_{\mathcal{D}}[X]$ *and* $\mathrm{T}_\partial[Y]$ *refer to the new chain complexes.*

In particular, the new filtered chain equivalence is natural in terms of the given data.

Application: models for differential graded algebras, in particular for chain algebras

7 Rooted planar trees

KONTSEVICH-SOIBELMAN [2000]
Description of A_∞-algebra structures in terms of sums over *oriented rooted planar trees* endowed with suitable labels
these sums over oriented rooted planar trees *behind* the HPT-constructions that establish *(co)algebra perturbation lemma* in
H. AND KADEISHVILI [1991]
At the time no need to spell out the oriented rooted planar trees explicitly
Details: H. [2008] (Kadeishvili Festschrift)
One application:
Minimality theorem for A_∞-structures via the perturbation lemma
Original minimality theorem:
CHEN [1977] over the de Rham algebra of smooth manifold
KADEISHVILI [1980] for the rational cochain algebra of a space

8 Master equation and HPT

Given a coaugmented differential graded cocommutative coalgebra C and a differential graded Lie algebra $\mathfrak{h}$, a *Lie algebra twisting cochain* $t\colon C \to \mathfrak{h}$ is a homogeneous morphism of degree -1 whose composite with the coaugmentation map is zero and which satisfies

$$Dt = \frac{1}{2}[t, t].$$

H. and Stasheff [2002]

Data: differential graded Lie algebra $\mathfrak{g}$ together with contraction

$$(\mathrm{H}(\mathfrak{g}) \underset{\pi}{\overset{\nabla}{\rightleftarrows}} \mathfrak{g}, h)$$

of chain complexes, the corresponding coalgebra perturbation of the differential on $\mathcal{S}^{\mathrm{c}}[s\mathfrak{g}]$ being written as ∂

Theorem. *The data determine, via HPT,* (i) *a differential* $\mathcal{D}$ *on* $\mathcal{S}^{\mathrm{c}}[s\mathrm{H}(\mathfrak{g})]$ *turning the latter into a coaugmented differential graded coalgebra (i. e.* $\mathcal{D}$ *is a coalgebra perturbation of the zero differential) and hence endowing* $\mathrm{H}(\mathfrak{g})$ *with an sh-Lie algebra structure and*
(ii) *a Lie algebra twisting cochain*

$$\tau\colon \mathcal{S}^{\mathrm{c}}_{\mathcal{D}}[s\mathrm{H}(\mathfrak{g})] \to \mathfrak{g}$$

whose adjoint $\overline{\tau}$*, written as*

$$(\mathcal{S}^{\mathrm{c}}\nabla)_{\partial}\colon \mathcal{S}^{\mathrm{c}}_{\mathcal{D}}[s\mathrm{H}(\mathfrak{g})] \to \mathcal{C}[\mathfrak{g}],$$

induces an isomorphism on homology. Furthermore, $(\mathcal{S}^{\mathrm{c}}\nabla)_{\partial}$ *admits, via HPT, an extension to a new contraction*

$$(\mathcal{S}^{\mathrm{c}}_{\mathcal{D}}[s\mathrm{H}(\mathfrak{g})] \ \underset{(\mathcal{S}^{\mathrm{c}}\pi)_{\partial}}{\overset{(\mathcal{S}^{\mathrm{c}}\nabla)_{\partial}}{\rightleftarrows}} \ \mathcal{S}^{\mathrm{c}}_{\partial}[s\mathfrak{g}], (\mathcal{S}^{\mathrm{c}}h)_{\partial})$$

of filtered chain complexes (not nececessarily of coalgebras).
Application: formal solutions of the master equation

9 Higher homotopies, homological perturbations, and the working mathematician

Higher homotopies and homological perturbations may be used to solve problems phrased in language entirely different from that of higher homotopies and HPT. Higher homotopies and HPT-constructions occur implicitly in a number of other situations in ordinary mathematics where they are at first not even visible.

List (not exhaustive)

- Kodaira-Nirenberg-Spencer: Deformations of complex structures
- Frölicher spectral sequence of a complex manifold
- Toledo-Tong: Parametrix
- Fedosov: Deformation quantization
- Whitney, Gugenheim: Extension of geometric integration to contraction
- Whitney ground work for Sullivan's theory of rational differential forms
- Gugenheim integration map in de Rham theory sh-multiplicative
- H.: Foliations; requisite higher homotopiss described in terms of generalized Maurer-Cartan algebra
- H.: Equiv. coho and Koszul duality
- Operads

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