Duality for Lie-Rinehart algebras and Batalin-Vilkovisky algebras

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Abstract

We will unveil the structure of a special kind of Batalin-Vilkovisky algebra in terms of Lie-Rinehart algebras: For any Lie-Rinehart algebra (A, L), Batalin-Vilkovisky algebra structures ∂ on the exterior A-algebra $\Lambda_A L$ correspond bijectively to right (A, L)-module structures on A; likewise, generators for the Gerstenhaber algebra $\Lambda_A L$ correspond bijectively to right (A, L)-connections on A. When L is projective as an A-module, given a Batalin-Vilkovisky algebra structure ∂ on $\Lambda_A L$, the Batalin-Vilkovisky algebra $(\Lambda_A L, \partial)$ coincides with the standard complex computing the homology of L with coefficients in A with reference to the right (A, L)-module structure determined by ∂. When L is also of finite rank n, via duality, there are bijective correspondences between (A, L)-connections on $\Lambda_A^n L$ and right (A, L)-connections on A and between left (A, L)-module structures on $\Lambda_A^n L$ and right (A, L)-module structures on A. Hence there are bijective correspondences between (A, L)-connections on $\Lambda_A^n L$ and generators for the Gerstenhaber bracket on $\Lambda_A L$ and between (A, L)-module structures on $\Lambda_A^n L$ and Batalin-Vilkovisky algebra structures on $\Lambda_A L$. The homology of such a Batalin-Vilkovisky algebra $(\Lambda_A L, \partial)$ coincides with the cohomology of L with coefficients in $\Lambda_A^n L$, with reference to the left (A, L)-module structure determined by ∂. These observations have various applications to Poisson structures and to differential geometry.

The generalization of the mutual structure of interaction to the differential graded context leads to twilled Lie-Rinehart algebras. These generalize, in the Lie-Rinehart context, complex structures on smooth manifolds. An almost complex manifold determines an "almost twilled pre-Lie-Rinehart algebra", which is a true twilled Lie-Rinehart algebra if and only if the almost complex structure is integrable. The Gerstenhaber algebra arising from an almost complex structure is a differential Gerstenhaber algebra if and only if the almost complex structure is integrable. This kind of Gerstenhaber algebra, endowed with a generator turning it into a Batalin-Vilkovisky algebra, arises from a Calabi-Yau manifold and explains in particular the relationship between holomorphic

volume forms and exact generators for the corresponding differential Gerstenhaber algebra. This observation yields in particular a conceptual proof of the Tian-Todorov lemma. The notion of generator admits thus an interpretation in terms of homological duality for differential graded Lie-Rinehart algebras.

Contents

1 Lie-Rinehart algebras

1.1 The definition

R commutative ring with 1
A commutative R-algebra
(R, A)-*Lie algebra* [Rinehart]
Lie algebra L over R
$L \otimes A \longrightarrow A$ left action by derivations
$A \otimes L \longrightarrow L$ left A-module structure
compatibility conditions
generalize Lie algebra vector fields on manifold
as a module over its ring of functions

$$[\alpha, a\beta] = \alpha(a)\beta + a[\alpha, \beta]$$
$$(a\alpha)(b) = a(\alpha(b))$$

for $a, b \in A$ and $\alpha, \beta \in L$
when emphasis on pair (A, L)
with mutual structure of interaction
pair (A, L) : *Lie-Rinehart* algebra

1.2 Modules and connections

A-module M
association

$$L \otimes_R M \to M,\ (\alpha, x) \mapsto \alpha(x)$$

not necessarily a left L-module structure, an (A, L)-*connection*:

$$\alpha(ax) = \alpha(a)x + a\alpha(x) \qquad (1.1)$$
$$(a\alpha)(x) = a(\alpha(x)) \qquad (1.2)$$

for $a \in A,\ x \in M,\ \alpha \in L$
left (A, L)-*module structure* on M
connection on M that is also a left L-module structure
a left (A, L)-module structure:
a *flat* (A, L)-*connection*.

N an A-module
an association

$$N \otimes_R L \to N,\ (x, \alpha) \mapsto x \circ \alpha$$

not necessarily right L-module
somewhat simpler, written as $(x, \alpha) \mapsto x\alpha$
rather than $(x, \alpha) \mapsto x \circ \alpha$
a *right* (A, L)-*connection*:

$$(ax)\alpha = a(x\alpha) - (\alpha(a))x \qquad (1.3)$$

$$x(a\alpha) = a(x\alpha) - (\alpha(a))x \qquad (1.4)$$

for $a \in A,\ x \in N,\ \alpha \in L$
a right (A, L)-connection:
a *right* (A, L)-module structure:
pairing a right L-module structure
a right (A, L)-module structure:
flat right (A, L)-*connection*
(1.4) not replica of (1.2):
associativity: one consistent way to interpret
$x(a\alpha) = (xa)\alpha$ whence

$$x(a\alpha) = (xa)\alpha = (ax)\alpha = a(x\alpha) - (\alpha(a))x$$

1.3 Examples of Lie-Rinehart algebras

(i) M manifold, $(A, L) = (C^\infty(M), \mathrm{Vect}(M))$
(ii) A algebra, $(A, L) = (A, \mathrm{Der}(A))$
(iii) $\vartheta\colon E \to B$ Lie algebroid: $(A, L) = (C^\infty(B), \Gamma(\vartheta))$
(iv) $(A, \{\,\cdot\,,\,\cdot\,\})$ Poisson algebra
A-module D_A of formal differentials on A
$[\,\cdot\,,\,\cdot\,]\colon D_A \otimes D_A \to D_A, [du, dv] = d\{u, v\}$,
$D_A \otimes A \to A, du(a) = \{u, a\}$
(A, D_A) with structure just explained a Lie-Rinehart algebra
(v) *twilled Lie-Rinehart algebra*
(A, L) Lie-Rinehart
$L = L' \oplus L''$ decomposition of A-modules,
(A, L') and (A, L'') both Lie-Rinehart
Example: M smooth manifold
$(A, L) = (C^\infty(M, \mathbb{C}), \mathrm{Vect}(M, \mathbb{C}))$
$J\colon \mathrm{T}M \to \mathrm{T}M$ almost complex, induces
$L = L' \oplus L''$, decomposition of A-modules
decomposition being twilled signifies that the almost complex structure is integrable

1.4 Salient features of Lie-Rinehart algebras

(R, A)-Lie algebra L,
universal algebra $(U(A, L), \iota_L, \iota_A)$
R-algebra $U(A, L)$ together with
$\iota_A \colon A \longrightarrow U(A, L)$ morphism of R-algebras
$\iota_L \colon L \longrightarrow U(A, L)$ morphism R-Lie algebras
such that

$$\begin{aligned} \iota_A(a)\iota_L(\alpha) &= \iota_L(a\,\alpha) \\ \iota_L(\alpha)\iota_A(a) - \iota_A(a)\iota_L(\alpha) &= \iota_A(\alpha(a)) \end{aligned}$$

$(U(A, L), \iota_L, \iota_A)$ *universal* among
triples (B, ϕ_L, ϕ_A) having these properties
M manifold, $(A, L) = (C^\infty(M), \mathrm{Vect}(M))$
$U(A, L)$ *algebra of (globally defined) differential operators on* M

1.5 Rinehart complex

$\Lambda_A(sL)$ exterior A-Hopf alg. on susp. sL
CE operator d on $U(A,L) \otimes_A \Lambda_A(sL)$

$$K(A,L) = (U(A,L) \otimes_A \Lambda_A(sL), d)$$

$\mathrm{Hom}_{U(A,L)}(K(A,L),A) = \mathrm{Alt}_A(L,A)$
R-chain cx A-valued A-multilin. fn's on L
L projective as left A-module:
$K(A,L)$ projective resolution of A over $U(A,L)$
$(\mathrm{Alt}_A(L,A),d)$ computes $\mathrm{Ext}^*_{U(A,L)}(A,A)$
N manifold, $(A,L) = (C^\infty(M), \mathrm{Vect}(M))$
$(\mathrm{Alt}_A(L,A),d)(= \mathrm{Hom}_{U(A,L)}(K(A,L),A))$
ordinary *de Rham complex of* N
de Rham cohomology of N:
$\mathrm{Ext}^*_{U(A,L)}(A,A)$ over algebra $U(A,L)$
of differential operators on N
L ordinary Lie algebra over R
$K(R,L)$ the ordinary Koszul complex
L projective as an R-module:
$K(R,L)$ Koszul resolution of ground ring R
Coho Lie-Rinehart algebras comprises de Rham- as well as Lie algebra coho

1.6 Maurer-Cartan algebra

(A,L) Lie-Rinehart: $(\mathrm{Alt}_A(L,A),d)$:
Maurer-Cartan algebra associated with (A,L)
Converse:
(A,L), L finitely generated projective A-module

Theorem 1.1. *Let d be an R-linear operator on $\mathrm{Alt}_A(L,A)$ such that $\mathrm{Alt}_A(L,A)$ differential graded R-algebra. Then $(\mathrm{Alt}_A(L,A),d)$ is the Maurer-Cartan algebra of a uniquely determined Lie-Rinehart structure on (A,L).*

1.7 Generalization

structures make sense in other categories
(A,L',L'') *almost twilled*:
$\mathcal{A}'' = (\mathrm{Alt}_A(L'',A),d'')$ dg algebra
$\mathcal{L}' = (\mathrm{Alt}_A(L'',L'),d'')$ dg Lie algebra
$(\mathcal{A}'',\mathcal{L}')$ graded Lie-Rinehart

Theorem 1.2. *(A,L',L'') twilled if and only if $(\mathcal{A}'',\mathcal{L}';d'')$ a differential graded Lie-Rinehart algebra.*

almost complex case: integrability condition

2 Gerstenhaber and Batalin-Vilkovisky algebras

2.1 Gerstenhaber algebra

graded commutative algebra $\mathcal{A}$
degree -1 Lie bracket

$$[\,\cdot\,,\,\cdot\,]\colon \mathcal{A} \otimes_R \mathcal{A} \longrightarrow \mathcal{A}$$

ordinary graded Lie bracket when degrees of elements of $\mathcal{A}$ are lowered by 1
$a \in \mathcal{A}$ homogeneous: $[a,\,\cdot\,]\colon \mathcal{A} \to \mathcal{A}$
derivation of degree $|a| - 1$

2.2 Example of Gerstenhaber algebra

(A, L) Lie-Rinehart
graded exterior A-algebra $\Lambda_A L$ over L
L is taken concentrated in degree 1
typical elements of $\Lambda_A L$ in the form

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n, \quad \alpha_1, \alpha_2, \ldots, \alpha_n \in L$$

bracket $[\cdot, \cdot]$ on L induces Gerstenhaber bracket

$$[\cdot, \cdot] \colon \Lambda_A L \otimes_R \Lambda_A L \longrightarrow \Lambda_A L$$

on $\Lambda_A L$
explicit formula: $\alpha_1, \ldots, \alpha_n \in L$
$u = \alpha_1 \wedge \cdots \wedge \alpha_\ell \in \Lambda_A^\ell L$
$v = \alpha_{\ell+1} \wedge \cdots \wedge \alpha_n \in \Lambda_A^{n-\ell} L$
value $[u, v]$ given by $(-1)^{|u|}$ times

$$\sum_{j \leq \ell < k} (-1)^{(j+k)} [\alpha_j, \alpha_k] \wedge \alpha_1 \wedge \ldots \widehat{\alpha_j} \ldots \widehat{\alpha_k} \cdots \wedge \alpha_n$$

A ring of functions on manifold
L the Lie algebra of vector fields
$\Lambda_A L$ is the algebra of multivector fields
Schouten bracket

2.3 Batalin-Vilkovisky algebra

Gerstenhaber algebra $\mathcal{A}$
R-linear operator D on $\mathcal{A}$ of degree -1
generates the Gerstenhaber bracket:
for every homogeneous $a, b \in \mathcal{A}$

$$[a, b] = (-1)^{|a|} \Big(D(ab) - (Da)b - (-1)^{|a|} a(Db) \Big)$$

D referred to as a *generator*
special case considered by Koszul:
M smooth mfold
$A = C^{\infty}(M)$, $L = \mathrm{Vect}(M)$, $\mathcal{A} = \Lambda_A L$,
generator D of Gerstenhaber algebra $\mathcal{A}$ *exact*:
$D^2 = 0$
write exact generator as ∂
Gerstenhaber algebra $\mathcal{A}$ with exact gen. ∂
Batalin-Vilkovisky algebra $(\mathcal{A}, \partial)$
since $\partial^2 = 0$, homology $\mathrm{H}(\mathcal{A}, \partial)$ is defined
signification of $\mathrm{H}(\mathcal{A}, \partial)$?

3 Relationship among these notions

Gerstenhaber algebra $\mathcal{A}$:
A_j degree j constituent

Theorem 3.1. (i) *Assignment to Gerstenhaber algebra* $\mathcal{A}$ *of* (A_0, A_1) *yields functor*

$$\text{Gerstenhaber} \longrightarrow \text{Lie} - \text{Rinehart}$$

(ii) *This functor has*

$$(A, L) \mapsto \Lambda_A L$$

as left-adjoint.

Gerstenhaber algebra $\mathcal{A}$
canonical morphism of Gerstenhaber algebras

$$\Lambda_{A_0} A_1 \longrightarrow \mathcal{A}$$

Theorem 3.2. *Given the commutative algebra* A *and the* A*-module* L*:*
bijective correspondence between
(R, A)*-Lie algebra structures on* L
and
Gerstenhaber algebra structures on $\Lambda_A L$*.*

Theorem 3.3. *Bijective correspondence*
right (A, L)-connections on A
and
generators of Gerstenhaber bracket on $\Lambda_A L$
right (A, L)-module structures on A
correspond to
generators of square zero.

More precisely: Given generator D of the Gerstenhaber bracket on $\Lambda_A L$ formula

$$a \circ \alpha = a(D\alpha) - \alpha(a), \quad a \in A, \ \alpha \in L,$$

defines right (A, L)-connection on A.
Conversely,
given the right (A, L)-connection $\circ$ on A,
the operator D on $\Lambda_A L$ defined by means of

$$D(\alpha_1 \wedge \cdots \wedge \alpha_n) =$$
$$\sum_{i=1}^{n} (-1)^{(i-1)} (1 \circ \alpha_i)(\alpha_1 \wedge \ldots \widehat{\alpha_i} \cdots \wedge \alpha_n) \quad +$$
$$\sum_{j<k} (-1)^{(j+k)} [\alpha_j, \alpha_k] \wedge \alpha_1 \ldots \widehat{\alpha_j} \ldots \widehat{\alpha_k} \ldots \alpha_n$$

generator of Gerstenhaber bracket on $\Lambda_A L$.

Theorem 3.4. *Given an exact generator ∂ for the Gerstenhaber algebra $\Lambda_A L$, the Batalin-Vilkovisky algebra $(\Lambda_A L, \partial)$ coincides as a chain complex with $(A_\partial \otimes_{U(A,L)} K(A,L), d)$. In particular, when L is projective as an A-module, the Batalin-Vilkovisky algebra $(\Lambda_A L, \partial)$ computes*

$$\mathrm{H}_*(L, A_\partial) \left(= \mathrm{Tor}_*^{U(A,L)}(A_\partial, A)\right),$$

the homology of L with coefficients in A_∂

3.1 Illustration

—(A,L) – $(C^\infty(M),\Gamma(\zeta))$, $\zeta\colon E \to M$ Lie algebroid

—$L = \mathrm{Vect}(M)$,

right (A,L)-module structure on A:

$(\Lambda_A L, \partial)$ "de Rham currents"

notion of "de Rham homology"

— $(A,\{\,\cdot\,,\,\cdot\,\})$ Poisson algebra

write D_A with associated (R,A)-Lie algebra structure as $D_{\{\,\cdot\,,\cdot\,\}}$

right $(A, D_{\{\,\cdot\,,\cdot\,\}})$-module structure on A:

$$a(bdu) = \{ab, u\}$$

$\partial_{\{\,\cdot\,,\cdot\,\}}$ associated exact generator of Gerstenhaber bracket on $\Lambda_A D_{\{\,\cdot\,,\cdot\,\}}$

$$\Big(\Lambda_A D_{\{\,\cdot\,,\cdot\,\}}, \partial_{\{\,\cdot\,,\cdot\,\}}\Big) = A \otimes_{U(A, D_{\{\,\cdot\,,\cdot\,\}})} K(A, D_{\{\,\cdot\,,\cdot\,\}})$$

the complex defining Poisson homology

4 Duality

(A, L) Lie-Rinehart, general assumption:
as A-module, L projective of finite rank n
$\Lambda_A^n L$ highest exterior power of L over A
fact: A and $U(A, L)$ *satisfy duality and inverse duality in dimension* n, with *dualizing module* $C_L = \mathrm{H}^n(L, U(A, L))$
with induced right (A, L)-module structure:
natural isomorphisms

$$\Phi \colon \mathrm{H}^k(L, M) \to \mathrm{H}_{n-k}(L, C_L \otimes_A M)$$
$$\Psi \colon \mathrm{H}_k(L, N) \to \mathrm{H}^{n-k}(L, \mathrm{Hom}_A(C_L, N))$$

Theorem 4.1. *Bijective correspondence*
(A, L)-connections on $\Lambda_A^n L$
and
right (A, L)-connections on A
Under this correspondence,
left (A, L)-module structures on $\Lambda_A^n L$
(i. e. flat connections)
correspond to right
(A, L)-module structures on A.

More precisely:
Given (A,L)-connection ∇ on $\Lambda^n_A L$,
the negative of the (generalized) Lie-derivative
on $A \cong \mathrm{Hom}_A(\Lambda^n_A L, M)$
with reference to connection ∇ on $M = \Lambda^n_A L$,
that is, the formula

$$(\phi\alpha)x = \phi(\alpha x) - \nabla_\alpha(\phi(x))$$

$x \in \Lambda^n_A L$,$\alpha \in L$,$\phi \in \mathrm{Hom}_A(\Lambda^n_A L, \Lambda^n_A L) \cong A$
yields right (A,L)-connection on A
Conversely,
given right (A,L)-connection on A
(written as $(a,\alpha) \mapsto a\alpha$),
on $\Lambda^n_A L \cong \mathrm{Hom}_A(C_L, A)$,
the formula

$$(\nabla_\alpha \psi)x = \psi(x\alpha) - (\psi x)\alpha$$

for $x \in C_L$, $\alpha \in L$, $\psi \in \mathrm{Hom}_A(C_L, A)$,
yields an (A,L)-connection ∇

$$L \otimes_R \Lambda^n_A L \to \Lambda^n_A L, \quad (\alpha, \psi) \mapsto \nabla_\alpha \psi$$

Corollary 4.2. *Bijective correspondence*
(A, L)*-connections on* $\Lambda^n_A L$
and
generators of Gerstenhaber bracket on $\Lambda_A L$
flat connections ∇ *on* $\Lambda^n_A L$ *correspond to differentials* ∂ *on* $\Lambda_A L$

Theorem 4.3. *Pair* (∂, ∇) *from bijective correspondence in* (4.2) *entails isomorphism*

$$(\Lambda_A L, \partial) \to \big(\mathrm{Alt}_A(L, \Lambda^n_A L), d_\nabla\big) \qquad (4.1)$$

whence $\mathrm{H}(\Lambda_A L, \partial)$ *naturally isomorphic to*

$$\mathrm{H}^*(L, \Lambda^n_A L_\nabla) \quad \Big(= \mathrm{Ext}^*_{U(A,L)}(A, \Lambda^n_A L_\nabla)\Big)$$

special case: $\Lambda^n_A L_\nabla$ free A-module

$$\omega\colon \Lambda^n_A L_\nabla \longrightarrow A$$

iso (A, L)-modules, "volume form"
iso (4.1) becomes iso

$$(\Lambda_A L, \partial) \to (\mathrm{Alt}_A(L, A), d) \qquad (4.2)$$

5 The differential graded case

Theory extends to differential graded case
cf. earlier remark about other categories
Illustration:
complex manifold M of complex dimension n
$(A, L) = (C^\infty(M, \mathbb{C}), \mathrm{Vect}(M, \mathbb{C}))$
$J\colon \mathrm{T}M \to \mathrm{T}M$ associated almost complex
$L = L' \oplus L''$, decomposition of A-modules
(A, L', L'') associated twilled Lie-Rinehart
associated differential graded Lie-Rinehart
$\big(\mathcal{A}'', \mathcal{L}'\big) = \big((\mathrm{Alt}_A(L'', A), d''), (\mathrm{Alt}_A(L'', L'), d'')\big)$
earlier Gerstenhaber algebra $\Lambda_A L$ gets replaced with differential graded Gerstenhaber algebra

$$\Lambda_{\mathcal{A}''}\mathcal{L}' = \big(\mathrm{Alt}_A(L'', \Lambda_A L'), d''\big)$$

iso (4.1) becomes iso
$\big(\Lambda_{\mathcal{A}''}\mathcal{L}', \partial\big) \to \big(\mathrm{Alt}_A(L'', \big(\mathrm{Alt}_A(L', \Lambda^n_A L'), d''\big)\ , d_{\nabla'}\big)$
holomorphic volume form $\Omega\colon \Lambda^n_A L' \to A$
iso (4.2) becomes iso
$(\Gamma(\Lambda^*\overline{\tau}^*_M \otimes \Lambda^*\tau_M), \overline{\partial}, \partial_\Omega) \to (\Gamma(\Lambda^*\overline{\tau}^*_M \otimes \Lambda^*\tau^*_M), \overline{\partial}, \partial)$
This is the statement of the Tian-Todorov lemma.

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