

Singular Poisson-Kähler geometry of stratified Kähler spaces and quantum theory

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Some pictures of stratified Kähler spaces aimed at illustrating the talk follow.

The entire holomorphic function $w \mapsto \cosh(w)$, written out in terms of the ordinary real coordinates s and t in the plane that are related with the holomorphic coordinate w via $w = s + i t$, takes the form

$$s + i t \mapsto \cosh(s) \cos(t) + i \sinh(s) \sin(t).$$

For $s = a$, the image curve is an ellipse whereas for $t = b$, the image curve is a hyperbola. The two resulting families of curves are the two orthogonal families that show up in the subsequent picture.

The canoe or plane with two vertices

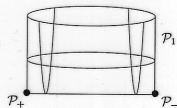
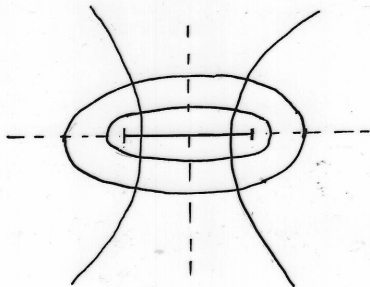
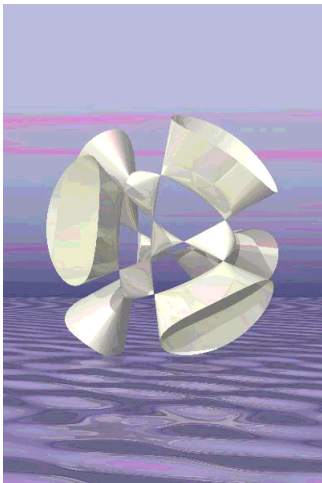


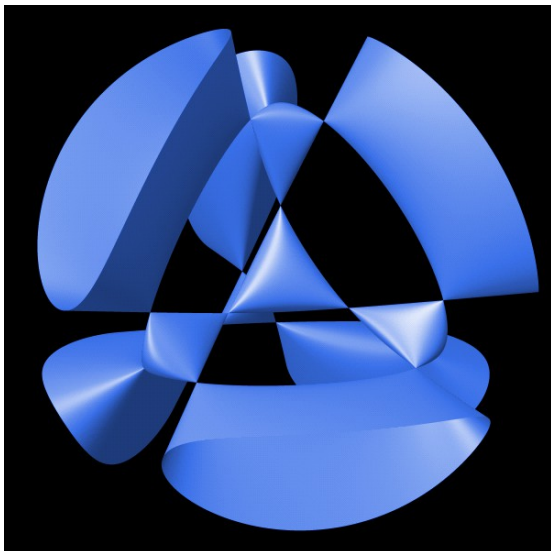
Figure 1: The reduced phase space \mathcal{P} for $K = \text{SU}(2)$.



Kummer's surface 1



The Kummer surface again



The moduli space of semi-stable holomorphic vector bundles of rank 2 and degree zero with trivial determinant on a Riemann surface of genus 2 is ordinary complex projective 3-space. The points that come from semi-stable vector bundles that are not stable constitute a Kummer surface, a degree 4 projective surface with 16 double points.

Guiding question

The exotic plane with one vertex

The canoe via hyperbolic cosine

Stratified spaces

Stratified symplectic spaces

Stratified Kähler spaces

Examples of stratified Kähler spaces

Costratified Hilbert spaces

The canoe revisited

The canoe costratified Hilbert space

Tunneling

Guiding question

Is there a quantum structure that has classical phase space singularities as its shadow?

ordinary Schrödinger quantization does not see singularities:

Hilbert space of L^2 -functions or of L^2 -sections

hard to talk about singularities in terms of L^2 -functions

holomorphic quantization: quantum Hilbert space a Hilbert space of holomorphic functions or, more generally, holomorphic sections on a line bundle

can understand singularities in terms of holomorphic functions

one of the results: new quantum effect: *tunneling among strata*

The exotic plane with one vertex

In \mathbb{R}^3 with coordinates x, y, r , semicone

$$N: x^2 + y^2 = r^2, r \geq 0$$

consider the algebra $C^\infty N$ of smooth functions in the variables x, y, r subject to the relation $x^2 + y^2 = r^2$. Define the Poisson bracket $\{\cdot, \cdot\}$ on this algebra by

$$\{x, y\} = 2r, \quad \{x, r\} = 2y, \quad \{y, r\} = -2x,$$

complex structure on N : $z = x + iy$ holomorphic coordinate

Poisson bracket *defined at the vertex* as well

away from the vertex Poisson structure symplectic

complex structure does *not* “see” the singularity at the vertex

radius function r is *not* a smooth function of x and y

N the classical reduced phase space of a single particle in affine space of dimension ≥ 2 with angular momentum zero

N a nilpotent orbit in $\mathfrak{sp}(1, \mathbb{R})$

general theory: same kind of structure on closure of any

holomorphic nilpotent orbit

The canoe via hyperbolic cosine

On \mathbb{C} , with holomorphic coordinate w , consider the holomorphic function

$$Z = 2 \cosh(w) = z + z^{-1}, \quad z = e^w.$$

Z the composite of the exponential function

$$\mathbb{C} \longrightarrow \mathbb{C}^*, \quad w \mapsto z = e^w$$

with the function

$$Z: \mathbb{C}^* \longrightarrow \mathbb{C}, \quad z \longmapsto z + z^{-1}.$$

The group C_2 with two elements acts on \mathbb{C}^* in the obvious manner and the function Z induces a holomorphic isomorphism $\mathbb{C}^*/C_2 \longrightarrow \mathbb{C}$ from the orbit space \mathbb{C}^*/C_2 onto the complex line \mathbb{C} ; this identifies the orbit space, as a complex analytic space, with a copy of \mathbb{C} .

Stratified spaces

Stratified space X : X decomposed into smooth manifolds that fit together in a certain precise way

X a Hausdorff paracompact topological space. A collection \mathcal{A} of subsets of X is said to be *locally finite* provided every $x \in X$ has a neighborhood U_x in X such that $U_x \cap A \neq \emptyset$ for at most finitely many A in \mathcal{A} . Let \mathcal{I} be a partially ordered set with order relation denoted by \leq . An \mathcal{I} -decomposition of X is a locally finite collection of disjoint locally closed manifolds $S_i \subseteq X$ called *pieces* such that

- ▶ $X = \cup S_i$
- ▶ $S_i \cap \bar{S}_j \neq \emptyset \iff S_i \subseteq \bar{S}_j \iff i \leq j$ (frontier condition).

An \mathcal{I} -decomposed space X is said to be a *stratified space* when the pieces of X , referred to as *strata*, satisfy certain additional conditions.

Stratified symplectic spaces

A *stratified symplectic space* is a stratified space X endowed with the following additional structure:

- ▶ a Poisson algebra $(C^\infty(X), \{ , \})$ of continuous functions on X ;
- ▶ an ordinary smooth symplectic structure on each piece;
- ▶ for each piece, each function in $C^\infty(X)$ restricts to an ordinary smooth function on that piece and, furthermore, relative to the ordinary smooth symplectic Poisson structure on that piece, the restriction mapping is a Poisson mapping.

Stratified Kähler spaces

A *stratified Kähler space* consists of a complex analytic space N , together with

- (i) a complex analytic stratification
- (ii) a stratified symplectic structure $(C^\infty N, \{ \cdot, \cdot \})$.

The two structures are required to be *compatible*:

- (i) For each point q of N and each holomorphic function f defined on an open neighborhood U of q , there is an open neighborhood V of q with $V \subset U$ such that, on V , f is the restriction of a function in $C^\infty(N)$;
- (ii) on each stratum, the symplectic structure determined by the symplectic Poisson structure (on that stratum) combines with the complex analytic structure to a Kähler structure.

Examples of stratified Kähler spaces

EXAMPLE 1: The *exotic plane*

more generally: *closure of a holomorphic nilpotent orbit*

Projectivization of the closure of a holomorphic nilpotent orbit yields an *exotic projective variety*

includes complex quadrics, SEVERI and SCORZA varieties

in physics, reduced classical phase spaces for systems of harmonic oscillators with zero angular momentum and constant energy

EXAMPLE 2: Moduli spaces of semistable holomorphic vector

bundles or, more generally, moduli spaces of semistable principal bundles on a non-singular complex projective curve

in conformal field theory: spaces of *conformal blocks*

special case moduli space of semistable rank 2 degree zero vector bundles with trivial determinant on a curve of genus 2:

as a space, ordinary complex projective 3-space

stratified symplectic structure involves more functions than just ordinary smooth functions

complement of space of stable v' bundles a *Kummer surface*

Examples of stratified Kähler spaces

EXAMPLE 3:

Theorem

Let N be a Kähler manifold, acted upon holomorphically by a complex Lie group G such that the action, restricted to a compact real form K of G , preserves the Kähler structure and is hamiltonian, with momentum mapping $\mu: N \rightarrow \mathfrak{k}^$. Then the reduced space $N_0 = \mu^{-1}(0)/K$ inherits a stratified Kähler structure.*

EXAMPLE 4: Lattice gauge theory

particular case: K a compact Lie group, $T^*K \cong K^{\mathbb{C}}$ viewed as a Kähler manifold

reduced space (Kähler quotient)

$$T^*K // K \cong K^{\mathbb{C}} // K^{\mathbb{C}}$$

special case thereof: canoe, details below

Costratified Hilbert spaces

N a stratified space

costratified Hilbert space relative to N :

a Hilbert space \mathcal{C}_Y for each stratum Y

a bounded linear map $\mathcal{C}_{Y_2} \rightarrow \mathcal{C}_{Y_1}$ for each inclusion $Y_1 \subseteq \overline{Y_2}$

whenever $Y_1 \subseteq \overline{Y_2}$ and $Y_2 \subseteq \overline{Y_3}$, the composite of $\mathcal{C}_{Y_3} \rightarrow \mathcal{C}_{Y_2}$ with $\mathcal{C}_{Y_2} \rightarrow \mathcal{C}_{Y_1}$ coincides with the bounded linear map $\mathcal{C}_{Y_3} \rightarrow \mathcal{C}_{Y_1}$ associated to the inclusion $Y_1 \subseteq \overline{Y_3}$

Costratified Hilbert spaces

Typically, a costratified Hilbert space arises as follows:

$\zeta: E \rightarrow M$ prequantum bundle on stratified Kähler space

closure \overline{O} of a stratum O in M is likewise a stratified Kähler space

ζ restricts to line bundle $\zeta_{\overline{O}}: E_{\overline{O}} \rightarrow \overline{O}$

restriction linear map of complex vector spaces

$$\text{res}_O: \Gamma^{\text{hol}}(\zeta) \longrightarrow \Gamma^{\text{hol}}(\zeta_{\overline{O}})$$

suppose $\Gamma^{\text{hol}}(\zeta)$ endowed with inner product: Hilbert space \mathcal{H}

$\mathcal{N}_O \subseteq \mathcal{H}$ closure of $\ker(\text{res}_O)$

\mathcal{H}_O orthogonal complement of \mathcal{N}_O in \mathcal{H} , so that

$$\mathcal{H} = \mathcal{N}_O \oplus \mathcal{H}_O$$

$p_O: \mathcal{H} \rightarrow \mathcal{H}_O$ orthogonal projection

as O ranges over the strata, construction yields costratified Hilbert

space

$$[\mathcal{H}, \mathcal{H}_O, p_O]$$

The canoe revisited

K compact connected Lie group, $K^{\mathbb{C}}$ complexification
choice of inner product and polar map induce diffeomorphism

$$T^*K \longrightarrow K^{\mathbb{C}}$$

yields Kähler structure on $K^{\mathbb{C}}$

denote the Kähler reduced space by \mathcal{P} ; combines
symplectic quotient $T^*K//K$ and complex analytic quotient
 $K^{\mathbb{C}}//K^{\mathbb{C}}$

special case $K = \mathrm{SU}(2)$, $K^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$

maximal torus T in $\mathrm{SU}(2)$ copy of circle group S^1

$T^*T \cong T^{\mathbb{C}}$ a copy of the space \mathbb{C}^* of non-zero complex numbers
the W -invariant holomorphic map

$$f: \mathbb{C}^* \longrightarrow \mathbb{C}, \quad f(z) = z + z^{-1} \quad (10.1)$$

induces complex analytic iso $\mathcal{P} \longrightarrow \mathbb{C}$ from the reduced space

$$\mathcal{P} = T^*K//K \cong T^*T/W \cong \mathbb{C}^*/W$$

onto \mathbb{C}

Stratified Kähler structure of the canoe

as a space, \mathcal{P} real semi-algebraic set in \mathbb{R}^3 given by

$$Y^2 = (X^2 + Y^2 + 4(\tau - 1))\tau, \quad \tau \geq 0,$$

\mathcal{P} homeomorphic to the canoe

algebra $C^\infty(\mathcal{P})$: Whitney-smooth functions, i.e., continuous functions that are restrictions of smooth functions on ambient \mathbb{R}^3
two functions are identified whenever they coincide on \mathcal{P}

Poisson bracket on $C^\infty(\mathcal{P})$ determined by

$$\{X, Y\} = X^2 + Y^2 + 4(2\tau - 1),$$

$$\{X, \tau\} = 2(1 - \tau)Y,$$

$$\{Y, \tau\} = 2\tau X$$

Stratified Kähler structure of the canoe

on the subalgebra of $C^\infty(\mathcal{P})$ which consists of real polynomial functions in the variables X , Y , τ , the relation

$$Y^2 = (X^2 + Y^2 + 4(\tau - 1))\tau$$

is defining. The resulting *stratified Kähler* structure on $\mathcal{P} \cong \mathbb{C}$ is *singular* at $-2 \in \mathbb{C}$ and $2 \in \mathbb{C}$, that is, the Poisson structure *vanishes* at either of these two points. Further, at $-2 \in \mathbb{C}$ and $2 \in \mathbb{C}$, the function τ is *not* an ordinary smooth function of the variables X and Y , viz.

$$\tau = \frac{1}{2} \sqrt{Y^2 + \frac{(X^2 + Y^2 - 4)^2}{16}} - \frac{X^2 + Y^2 - 4}{8},$$

whereas away from $-2 \in \mathbb{C}$ and $2 \in \mathbb{C}$, the Poisson structure is an ordinary symplectic Poisson structure. This makes explicit, in the case at hand, the singular character of the reduced space \mathcal{P} as a stratified Kähler space which, as a complex analytic space, is just \mathbb{C}

The canoe costratified Hilbert space

$$\mathcal{H} = \mathcal{H}L^2(K^{\mathbb{C}}, e^{-\kappa/\hbar}\eta_{\varepsilon})^K \cong \mathcal{H}L^2(T^{\mathbb{C}}, e^{-\kappa/\hbar}\gamma_{\varepsilon_T})^W,$$

we single out subspaces associated to the strata
special case canoe

$$K = \mathrm{SU}(2), \mathcal{P} = T^*K//K \cong \mathbb{C},$$

elements of \mathcal{H} ordinary K -invariant holomorphic functions on $K^{\mathbb{C}}$
 W -invariant holomorphic functions on $T^{\mathbb{C}}$,
determined by the holomorphic functions on

$$\mathcal{P} = K^{\mathbb{C}}//K^{\mathbb{C}} \cong T^{\mathbb{C}}/W \cong \mathbb{C}$$

in terms of realization of \mathcal{P} as \mathbb{C} , stratification $\mathbb{C} = \mathcal{P}_+ \cup \mathcal{P}_- \cup \mathcal{P}_1$

$$\mathcal{P}_+ = \{2\} \subseteq \mathbb{C}, \mathcal{P}_- = \{-2\} \subseteq \mathbb{C}, \mathcal{P}_1 = \mathbb{C} \setminus \mathcal{P}_0 = \mathbb{C} \setminus \{2, -2\}$$

The canoe costratified Hilbert space

closed subspaces

$$\mathcal{V}_+ = \{f \in \mathcal{H}; f|_{\mathcal{P}_+} = 0\} \subseteq \mathcal{H}$$

$$\mathcal{V}_- = \{f \in \mathcal{H}; f|_{\mathcal{P}_-} = 0\} \subseteq \mathcal{H}$$

define the Hilbert spaces \mathcal{H}_+ and \mathcal{H}_- by

$$\mathcal{H} = \mathcal{V}_+ \oplus \mathcal{H}_+ = \mathcal{V}_- \oplus \mathcal{H}_-;$$

take \mathcal{H}_1 to be the entire space \mathcal{H}

$$\{\mathcal{H}; \mathcal{H}_1, \mathcal{H}_+, \mathcal{H}_-\},$$

together with the corresponding orthogonal projections, is the *costratified Hilbert space* associated to the stratification of \mathcal{P} . By construction, this costratified Hilbert space structure is a *quantum analogue* of the *orbit type stratification* of \mathcal{P} .

Tunneling







$$\dim \mathcal{H}_+ = \dim \mathcal{H}_- = 1$$

bases ψ_+ and ψ_- , respectively

inner product $\langle \psi_+, \psi_- \rangle$ non-zero, can be calculated

tunneling probability between the strata \mathcal{P}_+ and \mathcal{P}_- , i. e., the probability for state prepared at \mathcal{P}_+ to be measured at \mathcal{P}_- , and vice versa.

Some literature

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Some literature cont'd



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