A survey on homological perturbation theory

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Ariadne's thread of the talk

Whenever a spectral sequence arises, a strong homotopy structure is lurking behind, and the spectral sequence is an invariant thereof. The higher homotopy structure is then finer than the spectral sequence itself.
S. Mac Lane: To do effective calculations in group cohomology, one needs tools that are stronger than spectral sequences
HPT is, perhaps, such a tool.

Origins of homotopy and higher homotopies

GAUSS 1833: linking number
GAUSS 1828: idea of a classifying space
HILBERT: exploration of syzygies, generating function for the number of invariants of each degree a rational function
homogeneous ideal I of a polynomial ring S, "number of independent linear conditions" for a form of degree d in S to lie in I" a polynomial function of d
problem of counting the number of conditions already considered in projective geometry and in invariant theory
CAYLEY 1848: general statement of the problem, clear understanding of the role of syzygies–without the word, introduced by SYLVESTER 1853
CAYLEY somewhat develops what is nowadays referred to as the KOSZUL resolution

Origins of homotopy and higher homotopies

POINCARÉ 1895: terminology *homotopy*, loop composition
failure of Alexander-Whitney multiplication of cocycles to be commutative:
STEENROD $\cup_i$-products, prompted
s(trongly)h(omotopy)c(ommutative) structures
STEENROD operations
A_∞-structure system of higher homotopies
together with suitable coherence conditions
MASSEY products invariants of certain A_∞-structures
elementary example Borromean rings
a MASSEY product of three variables detects the simultaneous linking of all three circles
SUGAWARA recognition principle for characterizing loop spaces up to homotopy type, associativity problem of loop multiplications

Origins of homotopy and higher homotopies

STASHEFF
— complete understanding of h'tpy invariancence of associativity
— clean recognition principle for loop spaces
— classifying space
— nested sequence of homotopy associativity conditions
— A_n-spaces
any space an A_1-space, H-space an A_2-space
every homotopy associative H-space is A_3
A_∞-space homotopy type of a loop space
algebraic analogue of A_n-space in category of algebras
A_n-algebra, the case $n = \infty$ being included
motivating example sing. chains on based loop space of a space
variant L_∞-algebras
key observation: A_∞-structures behave correctly under homotopy
YURI RAINICH one of Stasheff's mentors as an undergraduate:
in projective geometry, associativity was an option

Homological perturbations — HPT

Fundamental means to handle higher homotopies

EILENBERG AND MACLANE [1953]

contraction

$$(M \underset{\nabla}{\overset{\pi}{\rightleftarrows}} N, h)$$

— chain complexes M and N

— chain maps $\nabla\colon M \to N$ and $\pi\colon N \to M$

— degree 1 morphism $h\colon N \to N$

requirements

— deformation retraction

$$\pi\nabla = \mathrm{Id},\ \nabla\pi - \mathrm{Id} = dh + hd$$

— annihilation properties

$$h\nabla = 0,\ \pi h = 0,\ hh = 0$$

Homological perturbations — HPT

EILENBERG AND MAC LANE [1953]:
comparison between reduced bar and W-constructions a reduction, conjectured that it is a contraction

— contraction *filtered*: N and M filtered chain complexes, π, ∇ and h filtration preserving

— *perturbation* of the differential d of a chain complex X: $\partial\colon X \longrightarrow X$ same degree as d such that

— ∂ lowers filtration

— $(d+\partial)^2 = 0$ or, equivalently, $[d,\partial] + \partial\partial = 0$

— sum $d+\partial$, the *perturbed differential*, again differential

— *coalgebra perturbation*: X a graded coalgebra structure such that (X, d) a differential graded coalgebra *perturbed differential* $d+\partial$ comp. with graded coalg. structure

— similarly *algebra perturbation*

HELLER [1954] *recursive structure of triangular complex*

Perturbation lemma

Filtered contraction

$$(M \underset{\nabla}{\overset{g}{\rightleftarrows}} N, h)$$

∂ perturbation of the differential on M, filtrations complete

$$\mathcal{D} = \sum_{n\geq 0} g\partial(h\partial)^n\nabla = \sum_{n\geq 0} g(\partial h)^n\partial\nabla$$

$$\nabla_\partial = \sum_{n\geq 0} (h\partial)^n\nabla, \quad g_\partial = \sum_{n\geq 0} g(\partial h)^n,$$

$$h_\partial = \sum_{n\geq 0} (h\partial)^n h = \sum_{n\geq 0} h(\partial h)^n.$$

Infinite series converge, $\mathcal{D}$ perturbation of the differential on N new chain complexes constitute new filtered contraction

$$(M_{\mathcal{D}} \underset{\nabla_\partial}{\overset{g_\partial}{\rightleftarrows}} N_\partial, h_\partial)$$

natural in terms of the given data.

Homological perturbations — some history

SHI [1962] lurking behind formulas
M. BARRAT made explicit (unpublished)
BROWN [1964] in print
GUGENHEIM [1972] *twisted* EILENBERG- ZILBER theorem
CHEN [1977] $\Omega_\partial[\mathrm{H}_*(X)]$
model for real chains on loop space of smooth manifold X
formal power series connection involving de Rham complex of X
GUGENHEIM [1982] perturbation theory for homology of loop space, ordinary singular setting, reworks and extends CHEN's con'n
WONG [1986] Heidelberg Diplomarbeit confirmed EML-conjecture: comparison between reduced bar and W-constructions a contraction
MARKL [2006?] operadic perturbation lemma
perturbation lemma under weaker hypotheses
BERGER AND H. [1996] geometric comparison between bar and W-constructions
ingredient for rigorous lattice gauge theory [H. Topology 1999]

Perturbation theory for chain equivalences

H. and Kadeishvili [1991]
Labeled rooted trees lurking behind, details below

Iterative perturbations

H. [1986]
X connected simplicial set, simple π_1, π_2, etc. homotopy groups

$$k^j\colon X_{j-2} \longrightarrow \mathrm{K}(\pi_{j-1}, j), j \geq 3$$

maps representing the k-invariants of X

Theorem. *These maps determine a differential d on the tensor product*

$$\mathrm{K}(\pi_1, 1) \otimes \mathrm{K}(\pi_2, 2) \otimes \dots$$

and a chain equivalence between X and the resulting chain complex

$$(\mathrm{K}(\pi_1, 1) \otimes \mathrm{K}(\pi_2, 2) \otimes \dots, d)$$

Main ingredient: Iteration of perturbation lemma
Application: Interpret. of k-inv's of algebraic K-th'y of finite field
Construction taken up in Lambe-Stasheff [1987] models for iterated fibrations

Compatibility of HPT with algebraic structure

H. [1989], [1991]
compatibility of HPT- constructions with algebraic structure
suitable algebraic HPT-constructions to exploit A_∞-modules
arising in group cohomology
construction of suitable free resolutions
explicit numerical calculations in group cohomology
until today still not doable by other methods
spectral seq's show up not collapsing from E_2
illustrate a typical phenomenon:
Whenever a spectral sequence arises
a strong homotopy structure lurking behind
spectral sequence invariant thereof
higher homotopy structure finer than spectral sequence itself

Explicit illustration

Metacyclic group G

$$\langle x, y; y^r = 1, x^s = y^f, xyx^{-1} = y^t \rangle,$$
$$s > 1, r > 1, t^s \equiv 1 \mod r, tf \equiv f \mod r,$$

in particular, $\frac{t^s - 1}{r}$ and $\frac{(t-1)f}{r}$ integers
A HPT construction yields an explicit resolution of the ground ring over the group ring. The resolution is an instance of a higher homotopies structure that reflects the structure of the group as an extension of $\mathbb{Z}/r$ by $\mathbb{Z}/s$. The spectral sequence of the group extension is an invariant of that higher homotopies structure.
SAMPLE RESULT. p a prime which divides r and s, and suppose

$$t \not\equiv 1 \mod p.$$

Further, let j_0 be the order of t modulo p, i. e. j_0 the smallest number j so that

$$t^j \equiv 1 \mod p.$$

Explicit illustration, particular case

Suppose that p does not divide the number $\frac{t^s-1}{r}$. Then the cohomology spectral sequence of the group extension collapses from E_3, and the multiplicative extension problem is trivial. Moreover, $\mathrm{H}^*(G,\mathbb{Z}/p)$ has classes

$$\omega_{2j_0-1}, \omega_{4j_0-1}, \ldots, \omega_{2pj_0-1}, c_{2pj_0},$$

of filtration zero which restrict to the classes $\omega_y c_y^{ij_0-1}$ and $c_y^{pj_0}$ in

$$\mathrm{H}^*(N,\mathbb{Z}/p) \cong \Lambda[\omega_y] \otimes \mathrm{P}[c_y]$$

so that, as gr. comm. algebra, $\mathrm{H}^*(G,\mathbb{Z}/p)$ generated by

$$\omega_x, c_x, \omega_{2j_0-1}, \omega_{4j_0-1}, \ldots, \omega_{2pj_0-1}, c_{2pj_0},$$

subject to the relations

$$\omega_a\omega_b = 0,\ a,\ b \in \{2j_0-1, 4j_0-1, \ldots, 2pj_0-1\},$$
$$c_x\omega_a = 0,\ a \in \{2j_0-1, 4j_0-1, \ldots, 2(p-1)j_0-1\}.$$

Rooted planar trees

Kontsevich-Soibelman [2000]
Description of A_∞-algebra structures in terms of sums over *oriented rooted planar trees* endowed with suitable labels
these sums over oriented rooted planar trees *behind* the HPT-constructions that establish
(co)algebra perturbation lemma in H. and Kadeishvili [1991]
at the time no need to spell out oriented rooted planar trees exp'ly
Details: H. [2008] (Kadeishvili Festschrift)
One application:
Minimality theorem for A_∞-structures via the perturbation lemma
Original minimality theorem:
Chen [1977] over the de Rham algebra of smooth manifold
Kadeishvili [1980] for the rational cochain algebra of a space

Master equation and HPT

Given: coaugmented differential graded cocommutative coalgebra C and differential graded Lie algebra $\mathfrak{g}$, *Lie algebra twisting cochain* $t\colon C \to \mathfrak{g}$ a homogeneous morphism of degree -1

$$Dt = \frac{1}{2}[t, t], \; t\eta = 0.$$

H. and Stasheff [2002]: given chain complex contraction

$$(\mathrm{H}(\mathfrak{g}) \underset{\pi}{\overset{\nabla}{\rightleftarrows}} \mathfrak{g}, h)$$

Master equation and HPT

Theorem. *The data determine, via HPT,*
(i) *a coalgebra perturbation* $\mathcal{D}$ *on* $\mathrm{S^c}[s\mathrm{H}(\mathfrak{g})]$ *of zero differential turning* $\mathrm{S^c}[s\mathrm{H}(\mathfrak{g})]$ *into a coaugmented differential graded coalgebra — $\mathcal{D}$ an sh-Lie algebra or L_∞-algebra structure on* $\mathrm{H}(\mathfrak{g})$
(ii) *a Lie algebra twisting cochain* $\tau\colon \mathrm{S^c_{\mathcal{D}}}[s\mathrm{H}(\mathfrak{g})] \to \mathfrak{g}$ *whose adjoint*

$$\overline{\tau} = (\mathrm{S^c}\nabla)_\partial\colon \mathrm{S^c_{\mathcal{D}}}[s\mathrm{H}(\mathfrak{g})] \to \mathcal{C}[\mathfrak{g}],$$

induces an isomorphism on homology. Furthermore, $(\mathrm{S^c}\nabla)_\partial$ *admits, via HPT, an extension to a new contraction*

$$(\mathrm{S^c_{\mathcal{D}}}[s\mathrm{H}(\mathfrak{g})] \underset{(\mathrm{S^c}\pi)_\partial}{\overset{(\mathrm{S^c}\nabla)_\partial}{\rightleftarrows}} \mathrm{S^c_\partial}[s\mathfrak{g}], (\mathrm{S^c}h)_\partial)$$

of filtered chain complexes (not nececessarily of coalgebras).
Application: formal solutions of the master equation

Higher homotopies, HPT, and the working mathematician

Higher homotopies and homological perturbations may be used to solve problems phrased in language entirely different from that of higher homotopies and HPT. Higher homotopies and HPT-constructions occur implicitly in various other situations in ordinary mathematics where they are at first not even visible.

Higher homotopies, HPT, and the working mathematician

List (not exhaustive)

- Kodaira-Nirenberg-Spencer: def's of cx structures
- Frölicher spectral sequence of a complex manifold
- Toledo-Tong: Parametrix
- Fedosov: Deformation quantization
- Whitney, Gugenheim: Extension of geometric integration to contraction
- Whitney ground work for Sullivan's theory of rational differential forms
- Gugenheim integration map in de Rham theory sh-multiplicative
- H.: Foliations; requisite higher homotopiss described in terms of generalized Maurer-Cartan algebra
- H.: Equiv. coho and Koszul duality
- Operads

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