SUPER SYMPLECTIC GEOMETRY AND PREQUANTIZATION

G.M. TUYNMAN

Abstract. We review the prequantization procedure in the context of super symplectic manifolds with a symplectic form which is not necessarily homogeneous. In developing the theory of non homogeneous symplectic forms, there is one surprising result: the Poisson algebra no longer is the set of smooth functions on the manifold, but a subset of functions with values in a super vector space of dimension $1|1$. We show that this has no notable consequences for results concerning coadjoint orbits, momentum maps, and central extensions. Another surprising result is that prequantization in terms of complex line bundles and prequantization in terms of principal circle bundles no longer are equivalent if the symplectic form is not even.

1. Introduction

Prequantization is usually seen as the first step in the geometric quantization procedure. In [Tu1] I have argued that it might be a better idea to interpret prequantization as part of classical mechanics, at least after a small modification. In this paper I will argue that for super symplectic manifolds the interpretation of prequantization as part of a quantization scheme is less obvious than one might think. In order to understand the argument, we have to take a closer look at the standard prequantization procedure.

The key point in the whole argument is that there is not one single prequantization procedure, but that there are two (equivalent) procedures. One of them finds its origin in representation theory (the orbit method) and is usually associated with the name of Kostant [Ko1]. In this approach prequantization of a symplectic manifold $(M, \omega)$ is an answer to the question: find a complex line bundle $L \to M$ with a connection $\nabla$ such that its curvature is $(i/\hbar)$ times the symplectic form $\omega$. And as is well known, such a complex line bundle exists if and only if $\omega/\hbar$ represents an integral cohomology class in the de Rham cohomology group $H^2_{dR}(M)$. The other prequantization procedure has its origin in physics and is associated with the name of Souriau [So]. For him prequantization of a symplectic manifold is an answer to the question: find a principal $S^1$-bundle $\pi : Y \to M$ with an $S^1$-invariant 1-form $\alpha$ such that $\pi^* \omega = d\alpha$ and such that $\int_{\text{fibre}} \alpha = 2\pi \hbar$. And again, such a bundle exists if and only if $\omega/\hbar$ represents an integral cohomology class. The equivalence with the approach by Kostant is given by taking the associated complex line bundle relative to the canonical representation of $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$ on $\mathbb{C}$. As Th. Fried
c

Typeset by AMS-TEX
for the Lie algebra of $S^1$, Souriau’s question can be restated as: find a principal $S^1$-bundle $\pi : Y \to M$ with a connection 1-form $\pi^{-1} \omega$ such that $\pi^{-1} \omega \otimes b = d \pi^{-1} \omega$ under the condition that the basis vector $b$ is the smallest non-zero positively oriented vector such that $\exp(2\pi \hbar b) = 1$, where $\exp$ denotes the exponential map from Lie algebra to Lie group. Apart from the orientation, the condition on $b$ is equivalent to the condition $\exp(t b) = 1 \iff t \in 2\pi \hbar \mathbb{Z}$. For vector bundles we do not have a similar choice: for a vector bundle, the curvature of a connection is a 2-form with values in the endomorphisms of the typical fiber (a vector space). And the endomorphisms of $C$ are canonically isomorphic to $C$.

My approach in [Tu1] was to drop, in Souriau’s question, the condition $\exp(\hbar b) = 1$ or equivalently the condition $\exp(t b) = 1 \iff t \in 2\pi \hbar \mathbb{Z}$ on the basis vector $b$. In this paper my argument will be that for super symplectic manifolds Souriau’s question has an answer provided $\omega / \hbar$ represents an integral cohomology class, whereas Kostant’s question will in general not have an answer. The argument is that the curvature of a connection on a vector bundle must be an even 2-form, whereas I will consider mixed symplectic forms on supermanifolds. The reason not to restrict attention to even symplectic forms on supermanifolds is that coadjoint orbits of super Lie groups will in general carry a natural mixed symplectic form, not necessarily an even one.

In sections 3–7 we will explain the theory of super symplectic manifolds with mixed symplectic forms, the consequences for the Poisson algebra, momentum maps, and central extensions, as well as the theory of super coadjoint orbits. In sections 8–9 we will give a detailed review of the two prequantization procedures in the context of super symplectic manifolds.

2. Conventions, notation and useful results

I will work with the geometric $H^\infty$ version of DeWitt supermanifolds, which is equivalent to the theory of graded manifolds of Leites and Kostant (see [DW], [Ko2], [Le], [Tu2]). Any reader using a (slightly) different version of supermanifolds should be able to translate the results to her/his version of supermanifolds.

- The basic graded ring will be denoted as $\mathcal{A}$ and we will think of it as the exterior algebra $\Lambda V$ of an infinite dimensional real vector space $V$.
- Any element $x$ in a graded space splits into an even and an odd part $x = x_0 + x_1$. The parity function $\varepsilon$ is defined on homogeneous elements, i.e., elements $x$ for which either the even part $x_0$ or the odd part $x_1$ is zero. More precisely, if $x = x_\alpha$, then $\varepsilon(x) = \alpha$.
- On any graded space we define an involution $\mathcal{C}$ by $\mathcal{C} : x = x_0 + x_1 \mapsto x_0 - x_1$, where $x_\alpha$ denotes the homogeneous part of $x$ of parity $\alpha$. If $a$ and $b$ are two elements of $\mathcal{A}$ of which $\alpha$ is homogeneous, we then can write $a \cdot b = \mathcal{C}^{\varepsilon(a)}(b) \cdot a$, meaning that if $a$ is even, $ab = ba$ and if $a$ is odd, $ab = \mathcal{C}(b) \cdot a = (b_0 - b_1) \cdot a$.
- All (graded) objects over the basic ring $\mathcal{A}$ have an underlying real structure, called their body, in which all nilpotent elements in $\mathcal{A}$ are ignored/killed. This forgetful map is called the body map, denoted by the symbol $B$. For the ring $\mathcal{A}$ this is the map/projection $B : \mathcal{A} = \Lambda V \to \Lambda^0 V = \mathbb{R}$.
- If $\omega$ is a $k$-form and $X$ a vector field, we denote the contraction of the vector field $X$ with the $k$-form $\omega$ by $\iota(X) \omega$, which yields a $k-1$-form. If $X_1, \ldots, X_\ell$ are $\ell \leq k$ vector fields, we denote the repeated contraction of $\omega$ by $\iota(X_1, \ldots, X_\ell) \omega$. 

\[ \alpha \Rightarrow \alpha \]
More precisely:

\[ \iota(X_1, \ldots, X_\ell) \omega = \left( \iota(X_1) \circ \cdots \circ \iota(X_\ell) \right) \omega . \]

In the special case \( \ell = k \) this definition differs by a factor \((-1)^{k(k-1)/2}\) from the usual definition of the evaluation of a \( k \)-form on \( k \) vector fields. This difference is due to the fact that in ordinary differential geometry repeated contraction with \( k \) vector fields corresponds to the direct evaluation in the reverse order. And indeed, \((-1)^{k(k-1)/2}\) is the signature of the permutation changing 1, 2, \ldots, \( k \) in \( k, k-1, \ldots, 2, 1 \). However, in graded differential geometry this permutation not only introduces this signature, but also signs depending upon the parities of the vector fields. These additional signs are avoided by our definition.

- The evaluation of a left linear map \( \mu \) on a vector \( v \) is denoted as \( \langle v | \mu \rangle \). For the contraction of a multi-linear form with a vector we will use the same notation as for the contraction of a differential form with a vector field. In particular, we denote the evaluation of a left bilinear map \( \Omega \) on a vector \( v \) by \( \iota(v) \Omega \), which yields a left linear map \( w \mapsto (w | \iota(v) \Omega) \equiv \iota(w, v) \Omega \).

- If \( E \) is an \( \mathcal{A} \)-vector space, \( E^\ast \) will denote the left dual of \( E \), i.e., the space of all left linear maps from \( E \) to \( \mathcal{A} \).

- The general formula for evaluating the exterior derivative of a \( k \)-form \( \omega \) on \( k + 1 \) vector fields \( X_0, \ldots, X_k \) is given by the formula

\[
(2.1) \quad (-1)^k \iota(X_0, \ldots, X_k) d\omega = \\
\sum_{0 \leq i \leq k} (-1)^i \sum_{p<i} \epsilon(X_p) \epsilon(X_i) X_i \iota(X_0, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k) \omega \\
+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \sum_{p<i<j} \epsilon(X_p) \epsilon(X_j) \iota(X_0, \ldots, X_{i-1}, [X_i, X_j], X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_k) \omega
\]

The factor \((-1)^k\) is conventional and (again) is a consequence of our convention that \( \iota(X_0, \ldots, X_k) \) denotes the repeated contraction \( \iota(X_0) \circ \cdots \circ \iota(X_k) \).

The special cases of a 1-form and a 2-form are sufficiently interesting to write the definition explicitly. For a 1-form \( \omega \) and homogeneous vector fields \( X \) and \( Y \) we have

\[
(2.2) \quad -\iota(X, Y) d\omega = X \iota(Y) \omega - (-1)^{\epsilon(Y)} Y \iota(X) \omega - \iota([X, Y]) \omega.
\]

For a 2-form \( \omega \) and homogeneous vector fields \( X, Y, Z \) we have

\[
(2.3) \quad \iota(X, Y, Z) d\omega = X \iota(Y, Z) \omega \\
- (-1)^{\epsilon(Y)} Y \iota(X, Z) \omega + (-1)^{\epsilon(Z)} Y \iota(X, Y, Z) \omega \\
- \iota([X, Y], Z) \omega + (-1)^{\epsilon(Z)} [X, Z, Y] \omega + \iota([X, Y, Z]) \omega.
\]

- The Lie derivative of a \( k \)-form in the direction of a vector field \( X \) is defined as usual by the formula \( \mathcal{L}(X) = \iota(X) \circ d + d \circ \iota(X) \). It obeys the usual relation with the contraction with the commutator of two vector fields \( X \) and \( Y \):

\[
\iota([X, Y]) = [\mathcal{L}(X), \iota(Y)] = [\iota(X), \mathcal{L}(Y)].
\]
• If $G$ is an $\mathcal{A}$-Lie group, then its $\mathcal{A}$-Lie algebra $\mathfrak{g}$ is $\mathfrak{g} = T_e G$, whose Lie algebra structure is given by the commutator of left-invariant vector fields (who are determined by their value at $e \in G$).

• If $\Phi : G \times M \to M$ denotes the (left) action of an $\mathcal{A}$-Lie group $G$ on an $\mathcal{A}$-manifold $M$, then for all $v \in \mathfrak{g} = T_e G$ the associated fundamental vector field $v^M$ on $M$ is defined as $v^M|_m = -T_{(e,m)}\Phi(v,0)$. The minus sign is conventional and ensures that the map from $\mathfrak{g}$ to vector fields on $M$ is a homomorphism of $\mathcal{A}$-Lie algebras.

Similarly, if $\Phi : M \times G \to M$ is a right action of $G$ on $M$, then the fundamental vector field $v^M$ associated to $v \in \mathfrak{g}$ is defined as $v^M|_m = T_{(m,e)}\Phi(0,v)$. And again the map from $\mathfrak{g}$ to vector fields on $M$ is a morphism of $\mathcal{A}$-Lie algebras. In the special case when $G = M$ with the natural right action on itself, the fundamental vector fields are exactly the left-invariant vector fields on $G$.

3. Super symplectic $\mathcal{A}$-manifolds

If we generalize naively the notion of a symplectic manifold to $\mathcal{A}$-manifolds, we would define it as an $\mathcal{A}$-manifold $M$ with a closed and non-degenerate 2-form $\omega$. As in the ungraded case, we would expect that the commutator of two locally hamiltonian vector fields is globally hamiltonian. The following example shows that this is too naive.

Example. Let $M$ be the even part of an $\mathcal{A}$-vector space of dimension $2|2$ with global even coordinates $x, y$ and global odd coordinates $\xi, \eta$. We define the closed 2-form $\omega$ by

$$\omega = dx \wedge dy + d\xi \wedge d\eta + dx \wedge d\xi .$$

That $\omega$ is non-degenerate follows immediately from the equations $\iota(\partial_x) \omega = dy + d\xi$, $\iota(\partial_y) \omega = -dx$, $\iota(\partial_\xi) \omega = d\eta - dx$, and $\iota(\partial_\eta) \omega = d\xi$.

On $M$ we introduce the vector fields $X$ and $Y$ by

$$X = 2y\partial_x - 2y\partial_\eta \quad \text{and} \quad Y = -\xi\partial_x + \eta\partial_\eta + \xi\partial_y .$$

It is immediate that $\iota(X) \omega = d(y^2)$ and that $\iota(Y) \omega = d(\eta \xi)$, i.e., $X$ and $Y$ are globally hamiltonian in the naive sense. An elementary computation shows that

$$[X, Y] = -2\xi\partial_x - 2y\partial_\eta - 2\xi\partial_\eta ,$$

and then it is immediate to obtain $\iota([X, Y]) \omega = d(y\xi) + 2\xi d\xi$, which is not closed.

In other words, the commutator $[X, Y]$ of two globally hamiltonian vector fields is not even locally hamiltonian in the naive sense.

3.1 Definitions. A 2-form $\omega$ on an $\mathcal{A}$-manifold $M$ is called non-degenerate if for all $m \in M$ we have $\ker(\omega|m) = \{0\}$, where we interpret $\omega|m$ as the map $X \mapsto \iota(X)\omega|m$ from $T_m M$ to $(T_m M)^\ast$. The 2-form $\omega$ is called homogeneously non-degenerate if for all $m \in M$ we have $\ker(\omega_0|m) \cap \ker(\omega_1|m) = \{0\}$. Here $\omega_\alpha$ denotes the homogeneous part of parity $\alpha$ of $\omega$, and $\omega_\alpha|m$ is interpreted as the map $X \mapsto \iota(X)\omega_\alpha|m$ from $T_m M$ to $(T_m M)^\ast$.

A 2-form $\omega$ is called symplectic if it is closed and homogeneously non-degenerate. A symplectic $\mathcal{A}$-manifold is an $\mathcal{A}$-manifold $M$ together with a symplectic form $\omega$.

A smooth vector field $X$ on a symplectic $\mathcal{A}$-manifold $(M, \omega)$ is said to be locally/globally hamiltonian if both $\iota(X)\omega_0$ and $\iota(X)\omega_1$ are closed/exact. A locally
hamiltonian vector field is sometimes called an infinitesimal symmetry, and the set of all locally hamiltonian vector fields is denoted as $\text{Symm}(M, \omega)$. Its subset of all globally hamiltonian vector fields is denoted as $\text{HSymm}(M, \omega)$.

Since $\iota(X)\omega = \iota(X)\omega_0 + \iota(X)\omega_1$, it follows immediately that a non-degenerate 2-form is homogeneously non-degenerate. As a consequence, a homogeneous closed 2-form $\omega$ (meaning that either $\omega_0$ or $\omega_1$ is zero) is symplectic if and only if it is non-degenerate, i.e., the natural definition. There are (at least) two reasons to define a symplectic form as only being homogeneously non-degenerate. The first is that coadjoint orbits have a natural symplectic form in this sense which need not be non-degenerate. This is at the same time a reason to consider non-homogeneous symplectic forms. The second reason is that it is the natural condition that makes the definition of the Poisson algebra possible for non-homogeneous symplectic forms. And in fact, the definition of the Poisson algebra is slightly less straightforward in the graded case because, as we have seen, with the naive definition of globally/locally hamiltonian vector fields, the commutator of two globally hamiltonian vector fields needs not even be locally hamiltonian.

If $x^1, \ldots, x^n$ are local coordinates on a symplectic $\mathcal{A}$-manifold $(M, \omega)$ (even and odd together), the symplectic form can be written as $\omega = \sum_{i,j} \omega_{ij} dx^i \wedge dx^j$ for some graded skew-symmetric matrix of local functions $\omega_{ij}$. The classical Darboux theorem says that around every point there exist local coordinates for which these functions $\omega_{ij}$ are constant and of a special form. In the case of mixed symplectic forms such a theorem is no longer possible. The simple reason is that it would imply that in particular the rank of the even 2-form $\omega_0$ is constant. And the example of $\omega = x dx \wedge dy + dx \wedge d\xi + dy \wedge d\eta$ on the $\mathcal{A}$-manifold of dimension 2|2 with coordinates $x, y, \xi, \eta$ shows that this need not be the case. However, under the right circumstances we can prove an analogue of Darboux’s theorem for mixed symplectic forms. The proof is a close copy of the Moser-Weinstein proof of Darboux’s classical theorem [Wo].

3.2 Lemma. Let $M$ be a $\mathcal{A}$-manifold and let $\omega, \sigma$ be two symplectic forms on $M$. Let $m_o \in \mathcal{B}M$ a point with real coordinates such that $\omega_{m_o} = \sigma_{m_o}$. Suppose $U$ is a neighborhood of $m_o$ with the following properties:

(i) there exists a 1-form $\alpha$ on $U$ such that $d\alpha = \sigma - \omega$ on $U$ and $\alpha_{m_o} = 0$;
(ii) there exists an open neighborhood $\tilde{U}$ of $U \times \{s \in \mathcal{A}_0 \mid 0 \leq B_s \leq 1\}$ in $U \times \mathcal{A}_0$;
(iii) there exists an even vector field $X$ on $\tilde{U}$ satisfying $\iota(X)ds = 1$ and $\iota(X)\Omega = 0$ with $\Omega$ the closed 2-form defined by $\Omega(m,s) = \omega_m + s (\sigma_m - \omega_m) + ds \wedge \alpha_m$, where $s$ is a (global even) coordinate on $\mathcal{A}_0$.

Then there exist neighborhoods $V, W \subset U$ of $m_o$ and a diffeomorphism $\rho : V \to W$ such that $\rho^* \sigma = \omega$.

Proof. Without loss of generality we may assume that $U$ is a coordinate chart with coordinates $x^1, \ldots, x^n$. The condition $\iota(X)ds = 1$ implies that $X$ is of the form $X(x,s) = \partial_s + Y(x,s)$ with $Y(x,s) = \sum_i X^i(x,s) \partial_{x^i}$. Using that $\omega_{m_o} = \sigma_{m_o}$ and that $\alpha_{m_o} = 0$, the condition $\iota(X)\Omega = 0$ gives us $\iota(Y_{m_o}) \omega_{m_o} = 0$. We denote by $\phi_t$ the flow of the even vector field $X$ (see [Tu2] for more details on integrating even vector fields on $\mathcal{A}$-manifolds). Since the coefficient of $\partial_s$ is 1, $\phi_t$ is necessarily of the form $\phi_t(m,s) = (\phi_{s+t}, s + t)$. Since $Y$ is even and $\omega$ homogeneously non-degenerate, it follows that $Y_{(m_o,s)} = 0$ for all $s$. It follows that the integral curve of
X through \((m_o,0)\) is defined at least for all \(t \in A_0\) such that \(0 \leq Bt \leq 1\) (because \(\phi_t(m_o,0) = (m_o,t)\), which remains in \(\tilde{U}\) for these values of \(t\)). Since the domain of the flow is open and the interval \([0,1]\) compact, there exists a neighborhood \(V \subset U\) of \(m_o\) and a neighborhood \(I\) of \(0 \in A_0\) such that \(\phi_t(m,s)\) is defined for all \((m,s) \in V \times I\) and all \(0 \leq Bt \leq 1\). We now define \(\rho : V \to U\) by the equation \(\phi_1(m,0) = (\rho(m),1)\). This is a diffeomorphism onto \(W = \rho(V)\) with inverse given by the equation \(\phi_{-1}(m,1) = (\rho^{-1}(m),0)\).

Since \(\Omega\) is closed and \(\iota(X)\Omega = 0\), we have \(\mathcal{L}(X)\Omega = 0\), and thus the flow \(\phi_t\) preserves \(\Omega\): \(\phi_t^*\Omega = \Omega\). We now denote by \(i_0, i_1 : U \to \tilde{U}\) the canonical injections \(i_j(m) = (m,j), j = 0,1\). By definition we have \(\phi_1 \circ i_0 = i_1 \circ \rho\), but also \(i_1^*\Omega = \sigma\) and \(i_0^*\Omega = \omega\). We then compute:

\[ \rho^*\sigma = (i_1 \circ \rho)^*\Omega = (\phi_1 \circ i_0)^*\Omega = i_0^*(\phi_1^*\Omega) = i_0^*\Omega = \omega. \]  

In general it will not be easy to satisfy the conditions of [3.2]. However, in the special case that \(\omega\) is homogeneous, the conditions can be fulfilled (see also [Ko2]).

3.3 Proposition. Let \(\omega\) be a homogeneous symplectic form on a connected \(\mathsf{A}\)-manifold \(M\) of dimension \(pq\) and let \(m_o \in \mathsf{B}M\) be arbitrary.

If \(\omega\) is even, then there exist \(k, \ell \in \mathbb{N}, p = 2k\) (i.e., \(p\) is even) and a coordinate neighborhood \(U\) of \(m_o\) with coordinates \(x^1, \ldots, x^k, y_1, \ldots, y_k, \xi^1, \ldots, \xi^q\) (\(x, y\) even and \(\xi\) odd) such that \(\omega = \sum_{i=1}^k dx^i \wedge dy_i + \sum_{i=1}^\ell d\xi^i \wedge d\xi^i - \sum_{i=\ell+1}^q d\xi^i \wedge d\xi^i\) on \(U\).

If \(\omega\) is odd, then \(p = q\) and there exists a coordinate neighborhood \(U\) of \(m_o\) with coordinates \(x^1, \ldots, x^k, \xi^1, \ldots, \xi^p\) (\(x\) even and \(\xi\) odd) such that \(\omega = \sum_{i=1}^k dx^i \wedge d\xi^i\) on \(U\).

Proof. Let \(x^1, \ldots, x^p, \xi^1, \ldots, \xi^q\) be a coordinate system on a chart \(O\) around \(m_o\).

- **The even case.** At \(m_o\) \(\omega\) has the form \(\omega_{m_o} = \sum_{ij} A_{ij} dx^i \wedge dx^j + \sum_{ij} S_{ij} d\xi^i \wedge d\xi^j\) for some real matrices \(A\) and \(S\), \(A\) skew-symmetric and \(S\) symmetric (\(\omega\) is even and \(m_o\) has real coordinates, hence mixed terms \(dx^i \wedge d\xi^j\) have zero coefficients).

Since \(\omega\) is homogeneous, it is non-degenerate, hence both \(A\) and \(S\) are invertible. By a linear change of coordinates we thus may assume that \(A\) is the diagonal matrix \(\text{diag}(1, \ldots, 1, -1, \ldots, -1)\) with \(\ell\) plus signs and \(q - \ell\) minus signs, \(\ell\) being the signature of the metric \(S\). Renaming the coordinates \(x^{k+1}, \ldots, x^p, y_1, \ldots, y_k\) we thus can define the symplectic form \(\sigma\) on \(O\) by \(\sigma = \sum_{i=1}^k dx^i \wedge dy_i + \sum_{i=1}^\ell d\xi^i \wedge d\xi^i - \sum_{i=\ell+1}^q d\xi^i \wedge d\xi^i\). We then have by construction \(\omega_{m_o} = \sigma_{m_o}\).

Since both \(\omega\) and \(\sigma\) are closed, we may assume (by taking a smaller \(O\) if necessary) that there exists an even-1-form \(\alpha\) on \(O\) such that \(\sigma - \omega = d\alpha\) on \(O\). By changing \(\alpha\) by the gradient of a function we also may assume that \(\alpha_{m_o} = 0\).

In order to find \(U, \tilde{U}\) and \(X\) satisfying conditions (ii) and (iii) of [3.2], we first note that on \(O \times A_0\) the condition \(\iota(X) ds = 1\) implies that \(X\) is of the form \(X(x,s) = \partial_x + Y(x,s)\) with \(Y(x,s) = \sum_i X_i(m,s) \partial_s\), where the \(X_i\) denote all (even and odd) coordinates on \(O\). The condition \(\iota(X) \Omega = 0\) translates into the two equations \(\iota(Y)(\omega + s(\sigma - \omega)) = -\alpha\) and \(\iota(Y)\alpha = 0\). However, since \(Y\) is supposed to be even, \(\iota(Y)\sigma = \iota(Y)\omega = 0\) by skew-symmetry of 2-forms. Hence the second condition \(\iota(Y)\alpha = 0\) is a consequence of the first. We now introduce the
linear maps $A(m, s) : T_m M \to (T_m M)^*$ by $(Y | A(m, s)) = \iota(Y)(\omega_m + s(\sigma_m - \omega_m))$.

Since $\omega$ and $\sigma$ are even, $A(m, s)$ is even. Since $A(m, s) = \omega_m$ is invertible ($\omega$ is non-degenerate) for all $s$, there exist neighborhoods $W_s$ of $m_o$ and $I_s$ of $s$ such that $A(m, t)$ is invertible for all $(m, t) \in U_s \times I_s$. Hence, by compactness of $[0, 1]$, there exists a neighborhood $U$ of $m_o$ and a neighborhood $I$ of $\{s \in A_0 \mid 0 \leq Bs \leq 1\}$ such that $A(m, t)$ is invertible for all $(m, t) \in U \times I$. Taking $\hat{U} = U \times I$ and $Y_{(m, s)} = -(\alpha_m | A(m, s)^{-1})$ then satisfies the conditions because $\alpha$ and $A(m, s)$ are even and thus this $Y$ is too. Hence by [3.2] there exist neighborhoods $V$ and $W$ of $m_o$ and a diffeomorphism $\rho : V \to W$ such that $\rho^* \sigma = \omega$. Composing the coordinates $z'$ on $O$ with $\rho$ gives us the desired coordinate system.

- **The odd case.** At $m_o \omega$ has the form $\omega_{m_o} = \sum_{i,j} A_{ij} dx^i \wedge d\xi^j$ for a real matrix $A$ ($\omega$ is odd and $m_o$ has real coordinates, hence the terms $dx^i \wedge dx^j$ and $d\xi^i \wedge d\xi^j$ have zero coefficients). Since $\omega$ is homogeneous, it is non-degenerate, hence $A$ must be a square invertible matrix. In particular $p = q$. By a linear change of coordinates we thus may assume that $A$ is the identity. We thus define the symplectic form $\sigma$ on $O$ by $\sigma = \sum_{i=1}^p dx^i \wedge d\xi^i$. We then have by construction $\omega_{m_o} = \sigma_{m_o}$.

Since both $\omega$ and $\sigma$ are closed and odd, we may assume that there exists an odd 1-form $\alpha$ on $O$ such that $\sigma - \omega = d\alpha$ on $O$ (no need to take a smaller $O$ because odd closed forms are always exact). By changing $\alpha$ by the gradient of a function we also may assume that $\alpha_{m_o} = 0$. In order to find $U, \hat{U}$ and $X$ satisfying conditions (ii) and (iii) of [3.2], we proceed exactly as in the even case. The only difference is that here $\alpha$ and $A(m, s)$ are odd. But then again $Y$ is even.

We have defined a symplectic form as a homogeneously non-degenerate closed 2-form. For the sequel it is important to note that a different interpretation is possible. The trick we will use is quite general: it is a way to transform any non-homogeneous graded object in an even object. We fix once and for all an $A$-vector space $C$ of dimension $1|1$ with basis $c_0, c_1$ with (of course) parities $\epsilon(c_0) = \alpha$. For any $k$-form $\omega = \omega_0 + \omega_1$ on an $A$-manifold $M$ we then can define the even $C$-valued $k$-form $\underline{\omega}$ by $\underline{\omega} = \omega_0 \otimes c_0 + \omega_1 \otimes c_1$. The map $\omega \mapsto \underline{\omega}$ establishes a bijection between $k$-forms $\omega$ and even $C$-valued $k$-forms $\underline{\omega}$. We will define the exterior derivative of a $C$-valued $k$-form component wise, as well as the notions of closed/exact $C$-valued $k$-forms. In particular, a $C$-valued 1-form $\Omega$ is exact if and only if there exists a (smooth) function $F : M \to C$ (i.e., a $C$-valued 0-form) such that $\Omega = dF$.

### 3.4 Alternative definitions.

A closed 2-form $\omega$ is symplectic if and only if for all $m \in M$ we have $\ker \underline{\omega}_m = \{0\}$, where we interpret $\underline{\omega}_m$ as the map $X \mapsto \iota(X)\underline{\omega}_m$ from $T_m M$ to $(T_m M)^* \otimes C$. Moreover, a vector field $X$ on a symplectic manifold $(M, \omega)$ is locally/globally hamiltonian if and only if $\iota(X)\underline{\omega}$ is closed/exact.

**Remark.** The construction of $\underline{\omega}$ is not intrinsic but depends upon the choice of the basis $c_0, c_1$ for $C$. One can make it more intrinsic by starting from an $A$-vector space $E$ of dimension $1|0$ and to define $C$ as $E \oplus E^\circ$, where $E^\circ \equiv \Pi E$ denotes $E$ with all its parities reversed. In this way $C$ depends only upon the choice of a single basis vector for $E$. We can even get rid of this last arbitrariness by choosing $E = A$, which has a canonical basis vector $1 \in A$. However, since later on we will interpret $C$ as an $A$-Lie algebra and $c_0, c_1$ as a particular basis of this $A$-Lie algebra, we will keep for the moment the arbitrariness in our construction.

#### 3.5 Lemma.

Let $(M, \omega)$ be a symplectic $A$-manifold and $X$ a vector field on $M$.

(i) $X$ is locally hamiltonian $\iff \forall \alpha : \mathcal{L}(X)\omega_\alpha = 0 \iff \forall \alpha, \beta : \mathcal{L}(X\beta)\omega_\alpha = 0$
The guarantee uniqueness of $X$ non-degenerate shows itself in property (iii) of [3.5]: it is the natural condition to

**Remark.** The usefulness of the condition that a symplectic form be homogeneously non-degenerate shows itself in property (iii) of [3.5]: it is the natural condition to guarantee uniqueness of $X$.

### 3.6 Definitions.

The Poisson algebra $\mathcal{P}$ of a symplectic $\mathcal{A}$-manifold $(M, \omega)$ is defined as a subset of $C^\infty(M, C)$ by

$$\mathcal{P} = \{ f \in C^\infty(M, C) \mid \exists X : \iota(X)\omega = -df \} .$$

For $f \in C^\infty(M, C)$ we denote by $f^\alpha \in C^\infty(M)$ the components of $f$ with respect to the basis $c_0, c_1$ of $C$, i.e., $f = f^0c_0 + f^1c_1$. On the other hand, the homogeneous parts of $f$ are denoted as usual by $f_\alpha \in C^\infty(M, C)$, i.e., $\forall m \in M : f_\alpha(m) \in C_\alpha$.

If we decompose each $f_\alpha$ according to the basis $c_0, c_1$, we get four homogeneous functions $f_\alpha^\beta \in C^\infty(M)$ with

$$f_0 = f_0^0c_0 + f_0^1c_1 \quad \text{and} \quad f_1 = f_1^0c_0 + f_1^1c_1 .$$

Taking the parities of $c_\alpha$ into account, we have $\varepsilon(f_\alpha^\beta) = \alpha + \beta$. In particular, if we decompose the coefficient functions $f^\alpha$ into their homogeneous parts, we get

$$(f^\alpha)_\beta = f^\alpha_{\alpha + \beta} .$$

The Poisson algebra $\mathcal{P}$ becomes an $\mathbb{R}$-vector space when we define addition and multiplication by reals in the natural way.

According to [3.5] there can only be one $X$ satisfying the conditions for $f \in C^\infty(M, C)$ to belong to $\mathcal{P}$. This unique vector field is denoted as $X_f$ and is called the hamiltonian vector field associated to $f \in \mathcal{P}$. Splitting the defining equation for the hamiltonian vector field $X \equiv X_f$ in the homogeneous parts of the components gives us the equations $\iota(X_\beta)\omega_\alpha = -df_\beta^\alpha$, simply because the parity of $\iota(X_\beta)\omega_\alpha$ is $\alpha + \beta$, as is the parity of $f_\beta^\alpha$.

The Poisson bracket $\{ f, g \}$ of two elements $f, g \in \mathcal{P}$ is defined as

$$\{ f, g \} = -\iota(X_f, X_g)\omega = -\iota(X_f)\iota(X_g)\omega = Xfg ,$$

where for the last equality we defined the action of a vector field on a $C$-valued function component wise and where we used the defining equation $\iota(X_g)\omega = -dg$. 

QED
Remark. If the symplectic form $\omega$ is homogeneous and if $M$ is connected, then the Poisson algebra $P$ is isomorphic to $C^\infty(M) \times \mathbb{R}$. For instance, let $\omega$ be even, i.e., $\omega_1 = 0$ and $\omega_0 = \omega$ is non-degenerate in the usual sense. It follows that $f^0 c_0 + f^1 c_1$ belongs to $P$ if and only if there exists a vector field $X$ on $M$ such that $\iota(X)\omega_0 = -df^0$ and $df^1 = 0$. Non-degeneracy of $\omega_0$ implies that such an $X$ exists for all $f^0$ and connectedness of $M$ implies that $f^1$ must be constant.

From the above analysis, the reader (just as the author) might get the idea that the map $P \rightarrow C^\infty(M)$, $f \mapsto f^0 + f^1$ only has constant functions in its kernel (and that it might be surjective). However, the following example (a coadjoint orbit!) shows that such a belief is false.

Example. Let $M$ be the even part of an $\mathcal{A}$-vector space of dimension 2|1 with even coordinates $x, y$, and odd coordinate $\xi$. Then the closed 2-form $\omega = dx \wedge dy + dx \wedge d\xi$ is degenerate but homogeneously non-degenerate, i.e., symplectic. For $f = f^0 c_0 + f^1 c_1 \in C^\infty(M, \mathbb{C})$ and $X = x^2 \partial_x + X^y \partial_y + X^\xi \partial_\xi$, the condition that $f$ belongs to $P$ translates as the conditions

$$X^x dy - X^y dx = -df^0 \quad , \quad X^x d\xi - X^\xi dx = -df^1 .$$

From this it follows that $f^0$ is independent of $\xi$, that $f^1$ is independent of $y$ and that we have

$$X^x = -\frac{\partial f^0}{\partial y} = -\frac{\partial f^1}{\partial \xi} , \quad X^y = \frac{\partial f^0}{\partial x} , \quad X^\xi = \frac{\partial f^1}{\partial x} .$$

Since $f^1$ is independent of $y$, we can write $f^1(x, y, \xi) = f^1_0(x) + \xi f^1_1(x)$ for smooth functions $f^1_0, f^1_1$ of $x$. But then $\partial_y f^0(x, y) = f^1_1(x)$ is independent of $y$, and thus $f^0(x, y) = f^0_0(x) + yf^1_1(x)$ for some function $f^0_0$ of $x$. We conclude that $f \in P$ is of the form

$$f(x, y, \xi) = (f^0_0(x) + yf^1_1(x)) \cdot c_0 + (f^0_1(x) + \xi f^1_1(x)) \cdot c_1 .$$

In other words, $P \cong [C^\infty(\mathcal{A}_0)]^2$. It follows that the kernel of the map $P \rightarrow C^\infty(M)$, $f \mapsto f^0 + f^1$ is isomorphic to $C^\infty(\mathcal{A}_0)$ and that its image is isomorphic to $[C^\infty(\mathcal{A}_0)]^2$. Since a function $g$ on $M$ can be written as $g(x, y, \xi) = g_0(x, y) + \xi g_1(x, y)$ for two smooth functions $g_0, g_1$ of $(x, y)$, we have $C^\infty(M) \cong [C^\infty(\mathcal{A}_0)]^2$, which is much larger than the image of $P$.

3.7 Lemma. On a symplectic $\mathcal{A}$-manifold $(M, \omega)$ the commutator $[X, Y]$ of two locally hamiltonian vector fields $X$ and $Y$ is the (globally) hamiltonian vector field associated to $-\iota(X, Y)\mathcal{W} \in \mathcal{P}$.

Proof. Using [3.5] we compute:

$$\iota([X, Y])\mathcal{W} = [\mathcal{L}(X), \iota(Y)]\mathcal{W}$$

$$= \mathcal{L}(X)\iota(Y)\mathcal{W} - \sum_{\alpha, \beta=0}^1 (-1)^{\alpha\beta}\iota(Y_\beta)\mathcal{L}(X_\alpha)\mathcal{W}$$

$$= d\iota(X)\iota(Y)\mathcal{W} + \iota(X)d\iota(Y)\mathcal{W} = d\iota(X, Y)\mathcal{W} .$$

QED
3.8 Lemma. Let \( P \) be the Poisson algebra of a symplectic \( A \)-manifold \((M, \omega)\). The Poisson bracket on \( P \) is a well defined even bracket which gives \( P \) the structure of an \( R \)-Lie algebra. Moreover, the map \( f \mapsto X_f \) from \( P \) to hamiltonian vector fields is an even morphism of \( R \)-Lie algebras. Explicitly:

(i) For \( f, g \in P \) we have \([X_f, X_g] = X_{\{f, g\}}\).

(ii) The bracket is bilinear;

(iii) For homogeneous \( f, g \in P \) we have \( \{f, g\} = -(-1)^{\varepsilon(f)\varepsilon(g)} \{g, f\} \);

(iv) For homogeneous \( f, g, h \in P \) we have

\[
(-1)^{\varepsilon(f)\varepsilon(h)} \{f, \{g, h\}\} + (-1)^{\varepsilon(g)\varepsilon(f)} \{g, \{h, f\}\} + (-1)^{\varepsilon(h)\varepsilon(g)} \{h, \{f, g\}\} = 0.
\]

Proof. We start by proving that the map \( f \mapsto X_f \) is even. We have \( \iota(X_f) \omega = -df \).

If \( f \) is even, non-degeneracy of \( \omega \) (and the fact that it is even) implies that the odd part of \( X_f \) must be zero. Since the odd case is similar, this proves that the map \( f \mapsto X_f \) is even. Property (i) then is an immediate consequence of [3.7]. Since the map \( f \mapsto X_f \) is even, it is immediate that the bracket on \( P \) is even, i.e., if \( f, g \in P \) are homogeneous, then \( \{f, g\} \) is homogeneous of parity \( \varepsilon(f) + \varepsilon(g) \). Properties (ii) and (iii) follow immediately from the defining equations of the bracket and the fact that a 2-form is graded skew-symmetric.

To prove property (iv) we first establish some useful identities. We start with the fact that by definition of the bracket and by property (i) we have

\[
\{f, \{g, h\}\} = -\iota(X_f, X_{\{g, h\}}) \omega = -\iota(X_f, [X_g, X_h]) \omega.
\]

Since \( d\iota(X_f, X_g) \omega = \iota([X_f, X_g]) \omega \) by [3.7], we have for any vector field \( Z \) on \( M \) the equality

\[
Z(\iota(X_f, X_g) \omega) = \iota(Z) d\iota(X_f, X_g) \omega = \iota(Z, [X_f, X_g]) \omega.
\]

If we apply (3.10) and (2.3) to the closed \( C \)-valued 2-form \( \omega \) and the three homogeneous vector fields \( X_f, X_g, X_h \), and if we use the graded skew-symmetry of a 2-form and of the commutator bracket, we obtain

\[
0 = \frac{1}{2} (-1)^{\varepsilon(f)\varepsilon(h)} \iota(X_f, X_g, X_h) d\omega
= (-1)^{\varepsilon(f)\varepsilon(h)} \iota(X_f, [X_g, X_h]) \omega + (-1)^{\varepsilon(g)\varepsilon(f)} \iota(X_g, [X_h, X_f]) \omega
+ (-1)^{\varepsilon(h)\varepsilon(g)} \iota(X_h, [X_f, X_g]) \omega.
\]

Comparing this with (3.9) gives us property (iv). \( \Box \)

4. Central extensions: general theory

4.1 Definition. Let \( g \) be an \( A \)-Lie algebra and let \( E \) be an \( A \)-vector space. A \( k \)-chain on \( g \) with values in (the trivial \( g \)-module) \( E \) is an even left \( k \)-linear graded skew-symmetric map on \( g \) with values in \( E \), i.e., an even linear map \( \Lambda^k g \rightarrow E \). The
set of all such $k$-chains is denoted by $C^k(g, E)$. On the set of $k$-chains we define a coboundary operator $d : C^k(g, E) \to C^{k+1}(g, E)$ by the formula

$$\iota(v_0, \ldots, v_k) dc = (-1)^k \sum_{0 \leq i < j \leq k} (-1)^{j+\sum_{i < p < j} \varepsilon(X_p)\varepsilon(X_j)} \iota(v_0, \ldots, v_{i-1}, [v_i, v_j], v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_k) c.$$ 

4.2 Remark. The global factor $(-1)^k$ in the definition of the coboundary operator is conventional and comes in because we interpret the contraction $\iota(v_0, \ldots, v_k)$ as repeated contractions with a single vector: $\iota(v_0, \ldots, v_k) = \iota(v_0) \circ \cdots \circ \iota(v_k)$, just as in the case of differential forms.

4.3 Lemma. If $G$ is an $A$-Lie group whose associated $A$-Lie algebra is $T_e G = g$, then $C^k(g, E)$ can be identified with the set of all even left-invariant $k$-forms on $G$ with values in $E$, the identification given by taking the value at $e \in G$. In this identification, the coboundary operator corresponds to the exterior derivative.

Proof. A $k$-form with values in $E$ is for each $g \in G$ a left $k$-linear graded skew-symmetric map on $T_g G$ with values in $E$, and thus in particular at $e \in G$ it is a $k$-chain. Conversely, any $k$-chain, i.e., a left $k$-linear graded skew-symmetric map on $T_e G$ with values in $E$, determines by left translations a left-invariant $k$-form on $G$ with values in $E$.

The relation with the exterior derivative follows immediately when we apply the general formula (2.1) for evaluating the exterior derivative of a $k$-form $\omega$ on $k+1$ vector fields to the case of a left invariant $k$-form $c$ with values in $E$ evaluated on $k+1$ left-invariant vector fields (in which case all terms in the single summation yield zero because vector fields applied to a constant function gives zero).

QED

4.4 Corollary. $d^2 = d \circ d : C^k(g, E) \to C^{k+2}(g, E)$ is the zero map: $d^2 = 0$.

4.5 Definition. Associated to the set of $k$-chains $C^k(g, E)$ we define the set $Z^k(g, E) \subset C^k(g, E)$ as

$$Z^k(g, E) = \ker(d : C^k \to C^{k+1});$$

its elements are called $k$-cocycles. We also define the set $B^k(g, E) \subset Z^k(g, E)$ as

$$B^k(g, E) = \text{im}(d : C^{k-1} \to C^k)$$

(with $B^0(g, E) = \{0\}$), whose elements are called $k$-coboundaries. The quotient

$$H^k(g, E) = Z^k(g, E)/B^k(g, E)$$

is called the Lie algebra cohomology in dimension $k$ of $g$ with values in (the trivial $g$-module) $E$. 


4.6 Definition. Let $\mathfrak{g}$ be an $\mathcal{A}$-Lie algebra and let $E$ be an $\mathcal{A}$-vector space, which we interpret as an abelian $\mathcal{A}$-Lie algebra by taking the trivial (zero) bracket. A central extension of $\mathfrak{g}$ by $E$ is an exact sequence $\{0\} \to E \to \mathfrak{h} \to \mathfrak{g} \to \{0\}$ of $\mathcal{A}$-Lie algebra morphisms such that the image of $E$ lies in the center of $\mathfrak{h}$. If the morphisms are understood, one also calls $\mathfrak{h}$ the central extension of $\mathfrak{g}$ by $E$. Two central extensions $\mathfrak{h}$ and $\hat{\mathfrak{h}}$ are called equivalent if there exists an isomorphism of $\mathcal{A}$-Lie algebras $\phi : \mathfrak{h} \to \hat{\mathfrak{h}}$ such that the following diagram is commutative:

\[
\begin{array}{c}
\mathfrak{h} \\
\downarrow \\
\{0\} \\
\{0\} \rightarrow E \rightarrow \mathfrak{g} \rightarrow \{0\}.
\end{array}
\]

Example. Let $(M, \omega)$ be a connected symplectic manifold with its Poisson algebra $\mathcal{P}$. Obviously the constant functions belong to $\mathcal{P}$, i.e., $BC \equiv \mathbb{R}c_0 + \mathbb{R}c_1 \subset \mathcal{P}$. Apart from being infinite dimensional, the exact sequence $\{0\} \to BC \to \mathcal{P} \xrightarrow{f \mapsto X_f} \text{HSymm}(M, \omega) \to \{0\}$ is a central extension of $\text{HSymm}(M, \omega)$ by $BC$ (in the category of $\mathbb{R}$-Lie algebras). That $BC$ is the kernel of the map $f \mapsto X_f$ is a consequence of the connectedness of $M$.

4.8 Construction. For any 2-cocycle $\Omega \in Z^2(\mathfrak{g}, E)$ we define a central extension $\mathfrak{h}$ of $\mathfrak{g}$ by $E$ as follows. As $\mathcal{A}$-vector space we define $\mathfrak{h} = \mathfrak{g} \times E$, with maps $i : E \to \mathfrak{h}$ and $\pi : \mathfrak{h} \to \mathfrak{g}$ defined by $i(e) = (0, e)$ and $\pi(v, e) = v$. This gives us an exact sequence $\{0\} \to E \to \mathfrak{h} \to \mathfrak{g} \to \{0\}$ of $\mathcal{A}$-vector spaces. On $\mathfrak{h}$ we define the bracket by

\[
[(v, e), (w, f)] = (\lbrack v, w \rbrack, \iota(v, w)\Omega).
\]

4.9 Lemma. The exact sequence defined in [4.8] is a well defined central extension of $\mathfrak{g}$ by $E$. Moreover, this construction induces an isomorphism between $H^2(\mathfrak{g}, E)$ and equivalence classes of central extensions of $\mathfrak{g}$ by $E$.

Proof. Since $E$ is considered to be an abelian $\mathcal{A}$-Lie algebra, it is immediate that $i : E \to \mathfrak{h}$ is a morphism of $\mathcal{A}$-Lie algebras. It is also immediate from the definition that $\pi : \mathfrak{h} \to \mathfrak{g}$ is a morphism of $\mathcal{A}$-Lie algebras with $E$ as kernel. It thus remains to show that the bracket is indeed a well defined bracket of an $\mathcal{A}$-Lie algebra. By construction it is even (because $\Omega$ is even), bilinear and graded skew symmetric. The graded Jacobi identity for three homogeneous elements $(u, e), (v, f), (w, g) \in \mathfrak{h}$ translates to the equation

\[
(-1)^{\varepsilon(u)\varepsilon(w)}\iota(u, \lbrack v, w \rbrack)\Omega + (-1)^{\varepsilon(v)\varepsilon(u)}\iota(v, \lbrack w, u \rbrack)\Omega + (-1)^{\varepsilon(w)\varepsilon(v)}\iota(w, \lbrack u, v \rbrack)\Omega = 0.
\]

Using the graded skew-symmetry of $\Omega$ and of the bracket on $\mathfrak{g}$ this can be rewritten as

\[
-\iota([u, v], w)\Omega + (-1)^{\varepsilon(u)\varepsilon(w)}\iota([u, w], v)\Omega + \iota(u, \lbrack v, w \rbrack)\Omega = 0,
\]

which is exactly the condition $d\Omega = 0$ for $\Omega$ to be a 2-cocycle.
To prove that the map $\Omega \mapsto h$ induces an isomorphism between $H^2(\mathfrak{g}, E)$ and equivalence classes of central extensions of $\mathfrak{g}$ by $E$, we proceed as follows. First suppose that $\Omega'$ and $\Omega$ determine the same cohomology class, i.e., $\Omega' = \Omega + dF$ for some $F \in C^1(\mathfrak{g}, E)$, i.e., $F$ is an even linear map $\mathfrak{g} \to E$. We then define $\phi : h = \mathfrak{g} \times E \to h' = \mathfrak{g} \times E$ by $\phi(v, e) = (v, e + \iota(v)F)$ and we claim that this $\phi$ makes the two central extensions $h$ and $h'$ equivalent. Obviously this $\phi$ is an isomorphism of $A$-vector spaces and makes (4.7) commutative. It thus remains to show that it is a morphism of $A$-Lie algebras. We thus compute:

$$\phi([v, w], \iota(v, w)) = \phi([v, w], \iota(v, w)\Omega) = \iota(v, w)\Omega + \iota([v, w])F = [v, \iota(v, w)]_h .$$

It follows that we indeed have a map defined on $H^2(\mathfrak{g}, E)$.

Next suppose that $\Omega, \Omega' \in Z^2(\mathfrak{g}, E)$ define equivalent central extensions. We thus have an isomorphism $\phi : h \to h'$ making the diagram (4.7) commutative. Commutativity implies immediately that $\phi$ must be of the form $\phi(v, e) = (v, e + \iota(v)F)$ for some even linear map $F : \mathfrak{g} \to E$. Using the fact that $\phi$ is a morphism of $A$-Lie algebras, we compute:

$$\phi((v, e), (w, f))_h = \phi((v, w), \iota(v, w)\Omega) = \iota(v, w)\Omega + \iota([v, w])F = \iota(v, w + \iota(v, w) \Omega + dF) .$$

We conclude that for all $v, w \in \mathfrak{g}$ we have $\iota(v, w)\Omega + dF = \iota(v, w)\Omega'$, i.e., $\Omega + dF = \Omega'$. In other words, if $\Omega$ and $\Omega'$ determine equivalent central extensions, they are cohomologous, and thus the map from $H^2(\mathfrak{g}, E)$ to equivalence classes of central extensions of $\mathfrak{g}$ by $E$ is injective.

To prove surjectivity, let $\{0\} \to E \xrightarrow{i} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \to \{0\}$ be a central extension. We choose an even linear map $\sigma : \mathfrak{g} \to h$ satisfying $\pi \circ \sigma = id(\mathfrak{g})$ (a section for $\pi$). We then define $\Omega : \mathfrak{g} \times \mathfrak{g} \to E$ by

$$\iota(v, w)\Omega = \iota([v, w])\sigma - \iota(v)\sigma, \iota(w)\sigma)_h .$$

And indeed the right hand side lies in the kernel of $\pi$, just because $\pi$ is a morphism of $A$-Lie algebras. Exactness of the sequence then shows that it is indeed in the image of $i$; injectivity of $i$ then shows that $\iota(v, w)\Omega \in E$ is unique. Graded skew-symmetry of the Lie algebra bracket shows that $\Omega$ is graded skew-symmetric, i.e., $\Omega \in C^2(\mathfrak{g}, E)$. The graded Jacobi identity translates to the fact that $d\Omega = 0$:

$$\iota(u, v, w)d\Omega = -\iota([u, v], w)\Omega + \iota(v)\iota(w)\iota([u, v], w)\Omega + \iota(u, [v, w])\Omega$$

$$= -\iota([u, v], w)\sigma + \iota([u, v])\sigma, \iota(w)\sigma$$

$$+ (-1)^{\iota(v)\iota(w)}\iota([u, v], w)\sigma - \iota([u, v])\sigma, \iota(v)\sigma$$

$$+ \iota(u, [v, w])\sigma - \iota(u)\sigma, \iota([v, w])\sigma$$

$$= \iota(-[u, v], w) + (-1)^{\iota(v)\iota(w)}([u, v], w)\sigma + [u, [v, w]]\sigma$$

$$+ [\iota(u)\sigma, \iota(v)\sigma], \iota(w)\sigma - \iota(u)\sigma, [\iota(v)\sigma, \iota(w)\sigma]$$

$$- (-1)^{\iota(v)\iota(w)}[\iota(u)\sigma, \iota(w)\sigma], \iota(v)\sigma] = 0 .$$
where the last equality is a direct consequence of the graded Jacobi identity in \( g \) and \( h \), and where the third equality is a direct consequence of the definition of \( \Omega \) and the fact that the extension is central, meaning that all elements of the form \( i(e), e \in E \) disappear when taking the bracket.

Once we know that \( \Omega \) belongs to \( Z^2(g, E) \), we can form the associated central extension \( g \times E \). Since \( i(E) \) is the kernel of \( \pi \) and since \( \sigma \) is a section of \( \pi \), the map \( \phi : g \times E \to h, (v, e) \mapsto \iota(v) \sigma - i(e) \) is an isomorphism of \( A \)-vector spaces making the diagram (4.7) commutative. To check that it is a morphism of \( A \)-Lie algebras, we compute:

\[
\iota([(v, e), (w, f)]) \phi = \iota([\iota(v, w), \iota(v, w) \Omega]) \phi = \iota([v, w]) \sigma - \iota(v, w) \Omega
\]

\[
= [\iota(v) \sigma, \iota(w) \sigma] = [\iota(v, e) \phi, \iota(w, f) \phi],
\]

where the last equality follows from the fact that \( h \) is a central extension of \( g \). We thus have shown that the equivalence class of this extension is the same as the equivalence class determined by \( \Omega \), and thus we have proven surjectivity.

5. Central extensions and momentum maps

5.1 Proposition. Let \( \Phi : G \times M \to M \) be the (left) action of an \( A \)-Lie group \( G \) on a symplectic \( A \)-manifold \((M, \omega)\). For a fixed \( m \in M \) we denote by \( \Phi_m : G \to M \) the map \( g \mapsto \Phi(g, m) \) and we define \( \omega[m] = \Phi^*_m \omega \). If the \( G \)-action preserves \( \omega \), then \( \omega[m] \) is a closed left-invariant 2-form on \( G \), i.e., \( \pi[m] \in Z^2(g, C) \). If \( M \) is connected, the cohomology class of \( \pi[m] \) is independent of \( m \in M \) and thus defines a unique central extension of \( g \) by \( C \) via the construction [4.8].

Proof. For \( g \in G \) we have \( L_g^* \omega[m] = L_g^* (\Phi_m^* \omega) = (\Phi_m \circ L_g)^* \omega \). If we denote by \( \Phi_g : M \to M \) the map \( m \mapsto \Phi(g, m) \), it follows immediately from the fact that \( \Phi \) is a left action that \( \Phi_m \circ L_g = \Phi_g \circ \Phi_m \). Since the \( G \)-action preserves \( \omega \) we have \( \Phi_g^* \omega = \omega \), and hence \( L_g^* \omega[m] = \omega[m] \), i.e., \( \omega[m] \) is a left-invariant 2-form on \( G \). Since \( d \omega[m] = \Phi_m d \omega = 0 \) it is also closed.

To prove that its cohomology class is independent of \( m \), we consider the function \( \Omega : M \to Z^2(g, C) \) defined by \( \Omega(m) = \pi[m] \). For the projection \( \pi : M \to H^2(g, C) = Z^2(g, C)/B^2(g, C) \) to be constant, it is necessary and sufficient that \( \forall m \in M \ \forall X_m \in T_m M \) we have \( X_m \Omega \in B^2(g, C) \) (because \( M \) is connected and \( Z^2(g, C) \) finite dimensional). It follows directly from the definition of the coboundary operator that \( \lambda \in Z^2(g, C) \) belongs to \( B^2(g, C) \) if and only if there exists \( \mu \in C^1(g, C) \) such that \( \lambda = d \mu \), i.e., \( \forall v, w \in g : \iota([v, w]) \lambda = \iota([v, w]) \mu \) (\( v \) and \( w \) are constants with respect to a derivation in the direction of \( m \)). We thus want to compute \( \iota(v, w) X_m \Omega \). For homogeneous \( X, v, \) and \( w \), this is the same as \(( -1)^{v(X)(\iota(v) + \iota(w))} X_m \iota(v, w) \Omega \). We now compute:

\[
\iota(v, w) \Omega(m) = \iota(v, w) \pi[m] = \iota([v, w]^M, -w^M, w) \pi[m],
\]

and thus \( d \iota(v, w) \Omega = \iota([v, w]^M, w) \pi[m] \) by [3.7]. We thus find (for arbitrary \( v, w, X_m \))

\[
\iota(v, w) X_m \Omega = -\iota([v, w]^M, X_m) \pi[m].
\]

Hence if we define \( \mu \in C^1(g, C) \) by \( \iota(u) \mu = -\iota(u^M, X_m) \pi[m] \), then \( X_m \Omega = d \mu \), i.e., \( X_m \Omega \in B^2(g, C) \) as wanted. This proves that for connected \( M \) the cohomology class of \( \Omega(m) \) is independent of \( m \).
5.2 Definitions. Let $G$ be an $\mathcal{A}$-Lie group with $\mathcal{A}$-Lie algebra $\mathfrak{g} = T_e G$ and suppose that $G$ acts on a symplectic $\mathcal{A}$-manifold $(M, \omega)$. If the action preserves the symplectic form $\omega$, it also preserves the homogeneous parts $\omega_\alpha$ separately because diffeomorphisms are even. If the $G$-action preserves $\omega_\alpha$, then the Lie derivative of $\omega_\alpha$ in the direction of a fundamental vector is zero, i.e., $v \in \mathfrak{g} \Rightarrow \mathcal{L}(v^M)\omega_\alpha = 0$.

It follows that the fundamental vector fields associated to the $G$-action are locally hamiltonian. According to [3.7] we have $\iota([x, y]^M)\omega = d\langle x^M, y^M \rangle$, from which it follows that for all $z \in [\mathfrak{g}, \mathfrak{g}]$ (the commutator subalgebra) the fundamental vector field $z^M$ is globally hamiltonian. The $G$-action is called (weakly) hamiltonian if all fundamental vector fields are globally hamiltonian.

As for $k$-forms, we can transform any map $J : M \to \mathfrak{g}^*$ into an even map $\overline{J} : M \to \mathfrak{g}^* \otimes \mathbb{C}$ by $\overline{J} = J_\alpha \otimes e_\alpha + J_\alpha \otimes e_1$, where $J_\alpha : M \to \mathfrak{g}^*_{\alpha}$ denotes the homogeneous component of parity $\alpha$ of $J$. A map $J : M \to \mathfrak{g}^*$ is called a momentum map for a weakly hamiltonian action if

$$\forall v \in \mathfrak{g} : \iota(v^M)\overline{\omega} = -d\langle v, \overline{J} \rangle,$$

which is equivalent to the condition $\forall v \in \mathfrak{g}, \forall \alpha = 0, 1 : \iota(v^M)\omega_\alpha = -d\langle v, J_\alpha \rangle$. In terms of hamiltonian vector fields we can interpret a momentum map as a map $\mathfrak{g} \to \mathcal{P}$, $v \mapsto \langle v, \overline{J} \rangle$ satisfying the condition

$$\forall v \in \mathfrak{g} : v^M = X_{\langle v, \overline{J} \rangle}.$$

The $G$-action is called strongly hamiltonian if there exists an equivariant momentum map, which means in our context that the map $\mathfrak{g} \to \mathcal{P}$ is a morphism of $\mathcal{A}$-Lie algebras:

$$\forall v, w \in \mathfrak{g} : \{ \langle v, \overline{J} \rangle, \langle w, \overline{J} \rangle \} = \langle [v, w], \overline{J} \rangle.$$

Using the fact that a momentum map always exists on the commutator subalgebra, one can give an alternative proof of [5.1].

Alternative proof of [5.1]. Let $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ denote the commutator subalgebra and let $\mathfrak{t} \subset \mathfrak{g}$ be a supplement for $\mathfrak{s}$, i.e., $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{t}$. We know that for all $z \in \mathfrak{s}$ the associated fundamental vector field $z^M$ is globally hamiltonian, and thus in particular there exists a smooth map $J : M \to \mathfrak{s}^*$ such that for all $z \in \mathfrak{s}$ we have $z^M = X_{\langle z, \overline{J} \rangle}$. Moreover, for $x, y \in \mathfrak{g}$ we have

$$d\langle x^M, y^M \rangle = \iota([x, y]^M)\overline{\omega} = -d\langle [x, y], \overline{J} \rangle.$$

Since $M$ is connected, this implies that the function $\iota([x, y]^M)\overline{\omega} + \langle [x, y], \overline{J} \rangle$ is constant. We now define $\mu \in C^1(\mathfrak{g}, C)$ by

$$\langle u | \mu \rangle = \begin{cases} 0 & u \in \mathfrak{t} \\ \langle u, \overline{J} \rangle(m) - \langle u, \overline{J} \rangle(m') & u \in \mathfrak{s}. \end{cases}$$

We now compute

$$\iota(x, y)\overline{\omega}[m] - \iota(x, y)\overline{\omega}[m'] = \iota(x^M, y^M)\overline{\omega}_m - \iota(x^M, y^M)\overline{\omega}_{m'},$$

$$= \langle [x, y], \overline{J} \rangle(m') - \langle [x, y], \overline{J} \rangle(m) = -\langle [x, y], \mu \rangle.$$

Hence $\overline{\omega}[m] - \overline{\omega}[m'] = d\mu$, i.e., $\overline{\omega}[m]$ and $\overline{\omega}[m']$ determine the same element in $H^2(\mathfrak{g}, C)$. \[Q.E.D.\]
5.3 Lemma. Let \( G \) be a symmetry group of a connected symplectic \( \mathcal{A} \)-manifold \((M, \omega)\) admitting a momentum map \( J \) (i.e., the action is hamiltonian). Then

(i) for all \( v, w \in \mathfrak{g} \) the function \( \iota(v, w) \Omega_J : M \to \mathbb{C} \),

\[
\iota(v, w) \Omega_J : m \mapsto \{\langle v \mid J \rangle, \langle w \mid J \rangle\}(m) - \{\langle v, w \rangle \mid J(m)\}
\]

is a constant function;

(ii) the cohomology class of \( \Xi[m] \) defined in [5.1] is also determined by the 2-cocycle \( (v, w) \mapsto \iota(v, w) \Omega_J \) given by (i).

Proof. The definition of the Poisson bracket gives us the equality \( \{\langle v \mid J \rangle, \langle w \mid J \rangle\} = -\iota(\langle v, w \rangle^M)\Xi \). By [3.7] the exterior derivative of this function is \( -\iota(\langle v, w \rangle^M)\Xi \), which is equal to \( d(\langle v, w \rangle\mid J) \) by definition of the momentum map. This proves (i).

In the proof of [5.1] we have seen the equalities \( \iota(v, w)\Xi[m] = \iota(v^M, w^M)\Xi_m = \{\langle v \mid J \rangle, \langle w \mid J \rangle\}(m) \). For a fixed \( m \in M \) we can see \( \Xi(m) \) as an element of \( C^1(\mathfrak{g}, \mathbb{C}) \) and then

\[
\iota(v, w)\Xi[m] = \{\langle v \mid J \rangle, \langle w \mid J \rangle\}(m) = \iota(v, w) \Omega_J + \{\langle v, w \rangle \mid J(m)\},
\]

and thus \( \Xi[m] = \Omega_J + d(\Xi(m)) \), where here the \( d \) denotes the coboundary operator \( C^1 \to B^2 \subset Z^2 \). This proves that the cohomology classes of \( \Xi[m] \) and \( \Omega_J \) are the same.

Remark. Note that we never explicitly proved that \( \Omega_J \) is an element of \( Z^2(\mathfrak{g}, \mathbb{C}) \). However, the fact that we have the equality \( \Omega_J = \Xi[m] + d(\Xi(m)) \) implies automatically that it indeed is a 2-cocycle.

6. Coadjoint orbits

If \( G \) is an \( \mathcal{A} \)-Lie group of dimension \( p|q \), its Lie algebra \( \mathfrak{g} \), as well as its (left) dual \( \mathfrak{g}^* \) is an \( \mathcal{A} \)-vector space of dimension \( p|q \), meaning that there is a basis with \( p \) even vectors and \( q \) odd vectors. But since the coordinates belong to the full ring \( \mathcal{A} \), the dimension of \( \mathfrak{g}^* \) seen as an \( \mathcal{A} \)-manifold is \( n|n \), where \( n = p + q \). What we are going to show is that the coadjoint orbit \( O_\mu \) through a point \( \mu \in \mathbb{B} \mathfrak{g}^* \) (i.e., \( \mu \) has real coordinates with respect to the basis) has a natural symplectic form. As we will see in an explicit example, this symplectic form need not be homogeneous, nor need it be non-degenerate.

Let \( e_1, \ldots, e_n \) be a homogeneous basis for \( \mathfrak{g} \), and let \( e^1, \ldots, e^n \) be the (left) dual basis for \( \mathfrak{g}^* \). In \( \mathfrak{g}^* \) we introduce \( 2n \) homogeneous coordinates \( \mu_1, \ldots, \mu_n, \bar{\mu}_1, \ldots, \bar{\mu}_n \) with parities \( \varepsilon(\mu_i) = \varepsilon(e^i) \), \( \varepsilon(\bar{\mu}_i) = 1 - \varepsilon(e^i) \) of a point \( \mu \in \mathfrak{g}^* \) according to the formula

\[
\mu = \sum_{i=1}^{n} (\mu_i + \bar{\mu}_i) \cdot e^i.
\]

In order to distinguish the subscript indicating the homogeneous part of a vector from the subscript indicating the coordinate, we will put parentheses around a vector before taking the homogeneous parts. More precisely, \( (\mu)_\alpha \) denotes the homogeneous part of parity \( \alpha \) of the vector \( \mu \). In terms of the coordinates introduced above we have

\[
(\mu)_0 = \sum_{i=1}^{n} \mu_i \cdot e^i \quad \text{and} \quad (\mu)_1 = \sum_{i=1}^{n} \bar{\mu}_i \cdot e^i.
\]
From these equations we learn that we can define the coordinates of $\mu$ also as

$$\mu_i = (-1)^{\varepsilon(e_i)} \langle e_i | (\mu)_0 \rangle \quad \text{and} \quad \bar{\mu}_i = \langle e_i | (\mu)_1 \rangle .$$

The coadjoint action of $G$ on $\mu \in \mathfrak{g}^*$ is defined via the adjoint representation according to the formula $\forall v \in \mathfrak{g} : \langle v | \text{Coad}(g)\mu \rangle = (\text{Ad}(g^{-1})v|\mu)$. Just as the algebraic adjoint representation ad of $\mathfrak{g}$ is the infinitesimal version of Ad in the sense $\text{ad} = T_e \text{Ad}$, so is the algebraic coadjoint representation coad the infinitesimal version of Coad : $\text{coad} = T_e \text{Coad} : \mathfrak{g} \to \text{End}_R(\mathfrak{g}^*)$. More precisely, let $g^1, \ldots, g^n$ be local coordinates on $G$ such that the tangent vectors $\partial_{g^i}$ are the basis vectors $e_i$ of $\mathfrak{g} = T_e G$. Then $\sum_{i=1}^n w^i \partial_{g^i}|_e \text{Coad}(g) = \text{coad}(\sum_{i=1}^n w^i e_i).

6.1 Lemma. For homogeneous $v, w \in \mathfrak{g}$ we have

$$\sum_{i=1}^n w^i \frac{\partial}{\partial g^i}|_e \langle v | \text{Coad}(g)\mu \rangle = (-1)^{\varepsilon(v)\varepsilon(w)} \langle v | \text{coad}(w)\mu \rangle$$

$$= (-1)^{\varepsilon(v)\varepsilon(w)} \langle [v, w]|\mu \rangle = -\langle [w, v]|\mu \rangle .$$

In particular, $\forall v, w \in \mathfrak{g} : \langle v | \text{coad}(w)\mu \rangle = \langle [v, w]|\mu \rangle$.

Proof. Using the defining equation $\langle v | \text{Coad}(g)\mu \rangle = \langle \text{Ad}(g^{-1})v|\mu \rangle$ we compute

$$\sum_{i=1}^n w^i \frac{\partial}{\partial g^i}|_e \langle v | \text{Coad}(g)\mu \rangle = \sum_{i=1}^n w^i \frac{\partial}{\partial g^i}|_e \langle \text{Ad}(g^{-1})v|\mu \rangle$$

$$= (-1)^{\varepsilon(v)\varepsilon(w)} \langle v | \text{Coad}(g)\mu \rangle - \langle \text{ad}(w)v|\mu \rangle = \langle [w, v]|\mu \rangle$$

$$= (-1)^{\varepsilon(v)\varepsilon(w)} \langle v | \text{coad}(w)\mu \rangle = (-1)^{\varepsilon(v)\varepsilon(w)} \langle [v, w]|\mu \rangle .$$

The particular case follows directly by replacing $w$ by $(-1)^{\varepsilon(v)\varepsilon(w)}w$. \(\Box\)

In order to determine the action of Coad($g$) in terms of our coordinates on $\mathfrak{g}^*$, we first note that the matrix elements of Coad($g$) are defined by Coad($g$)$_{ij} = \langle e_i | \text{Coad}(g)e_j \rangle$, which is equivalent to saying that Coad($g$)$_{ij} e^j = \sum_j e_i \text{Coad}(g)_{ij}$. Similarly, the matrix elements of coad are defined by coad($w$)$_{ij} = \langle e_i | \text{coad}(w)e_j \rangle$. Since Coad($g$) is an even map it follows that the parity of these matrix elements is given as $\varepsilon(\text{Coad}(g))_{ij} = \varepsilon(e_i) + \varepsilon(e_j)$. Either from these parity considerations, or from the fact that Coad($g$) is even and thus preserves the parity decomposition in $\mathfrak{g}^*$, i.e., (Coad($g$)$_{\mu\alpha}$ = Coad($g$)$_{\mu\alpha}$, one can deduce that the action of Coad($g$) is given in terms of coordinates as

$$\mu_i = \sum_{j=1}^n (-1)^{\varepsilon(e_i)\varepsilon(e_j) + \varepsilon(e_j)} \mu_j \text{Coad}(g)_{ij} = (-1)^{\varepsilon(e_i)} \langle e_i | \text{Coad}(g)(\mu)_0 \rangle ,$$

$$\bar{\mu}_i = \sum_{j=1}^n (-1)^{\varepsilon(e_i)\varepsilon(e_j) + \varepsilon(e_j)} \bar{\mu}_j \text{Coad}(g)_{ij} = \langle e_i | \text{Coad}(g)(\mu)_1 \rangle .$$

In order to compute the fundamental vector field $v^\mathfrak{g}$ on $\mathfrak{g}^*$ associated to a vector $v \in \mathfrak{g}$, we recall that it is defined as $v^\mathfrak{g}|_\mu = -\sum_{k=1}^n v^k \partial_{g^k}|_e \text{Coad}(g)\mu$. Applying this to the coordinate expression of Coad($g$)$_{\mu\alpha}$ and using [6.1] to compute the derivative of matrix elements, we obtain the following result.
6.2 Lemma. The tangent vector $v^\ast\mid_\mu$ is given by

$$- \sum_{i=1}^n \left\{ \mathcal{C}^{(e_i)} \left( \langle e_i \mid \coad(v)(\mu_0) \rangle \right) \cdot \frac{\partial}{\partial \mu_i} \mid_\mu + \mathcal{C}^{(e_i)} \left( \langle e_i \mid \coad(v)(\mu_1) \rangle \right) \cdot \frac{\partial}{\partial \mu_i} \mid_\mu \right\}.$$ 

In particular, $v^\ast\mid_\mu$ is zero if and only if $\forall \alpha : \coad(v)(\mu)_\alpha = 0$.

Proof. $v^\ast\mid_\mu = - \sum_{i,k=1}^n (-1)^{\varepsilon(e_i)} v^k \frac{\partial}{\partial y^k} \mid_\mu \left( \langle e_i \mid \left( \text{Coad}(g)(\mu_0) \right) \rangle \right) \cdot \frac{\partial}{\partial \mu_i} \mid_\mu$

$$- \sum_{i,k=1}^n v^k \frac{\partial}{\partial y^k} \mid_\mu \left( \langle e_i \mid \left( \text{Coad}(g)(\mu_1) \right) \rangle \right) \cdot \frac{\partial}{\partial \mu_i} \mid_\mu$

$$= - \sum_{i,k=1}^n (-1)^{\varepsilon(e_i)\varepsilon(e_k)} v^k \langle e_i \mid \left( \coad(e_k)(\mu_0) \right) \rangle \cdot \frac{\partial}{\partial \mu_i} \mid_\mu$

$$- \sum_{i,k=1}^n (-1)^{\varepsilon(e_i)\varepsilon(e_k)} v^k \langle e_i \mid \left( \coad(e_k)(\mu_1) \right) \rangle \cdot \frac{\partial}{\partial \mu_i} \mid_\mu$

$$= \sum_{i=1}^n \mathcal{C}^{(e_i)} \left( \langle e_i \mid \left( \coad(v)(\mu_0) \right) \rangle \right) \cdot \frac{\partial}{\partial \mu_i} \mid_\mu$

$$- \sum_{i=1}^n \mathcal{C}^{(e_i)} \left( \langle e_i \mid \left( \coad(v)(\mu_1) \rangle \right) \right) \cdot \frac{\partial}{\partial \mu_i} \mid_\mu.$$

Since $\sum_{i=1}^n e^i x_i = \sum_{i=1}^n \mathcal{C}^{(e_i)} (x_i) e^i$, this means that the coefficients of $\partial_{\mu_i}$ and $\partial_{\bar{\mu}_i}$ are the left coordinates of $\coad(v)(\mu_0)$ and $\coad(v)(\mu_1)$ respectively.

We now have sufficient material to define the symplectic form $\omega$ on a coadjoint orbit $O_{\mu_0} = \{ \text{Coad}(g)\mu_0 \mid g \in G \}$. Since we choose $\mu_0 = Bg^*$, this is indeed an $A$-manifold, immersed in $g^*$. The form $\omega$ is also called the Kirillov-Kostant-Souriau form. For any $\mu \in O_{\mu_0}$, the tangent space $T_\mu O_{\mu_0}$ is given by the set of all fundamental vector fields at $\mu$:

$$T_\mu O_{\mu_0} = \{ v^\ast \mid_\mu \mid v \in g \}.$$

We then define $\omega$ by its action on tangent vectors by

$$-\iota(v^\ast\mid_\mu, w^\ast\mid_\mu) \omega\mid_\mu = \langle [v, w] \mid_\mu \rangle \equiv \langle v \mid \text{coad}(w)\mu \rangle.$$

6.3 Lemma. $\omega$ is a well defined closed and homogeneously non-degenerate 2-form on $O_{\mu_0}$, i.e., $O_{\mu_0}$ is a symplectic $A$-manifold. Moreover, the coadjoint action is strongly hamiltonian with momentum map $J(\mu) = \mu$.

Proof. From the equality $-\iota(v^\ast\mid_\mu, w^\ast\mid_\mu) \omega\mid_\mu = \langle [v, w] \mid_\mu \rangle$ we see immediately that $\omega\mid_\mu$ is graded skew-symmetric, and from $-\iota(v^\ast\mid_\mu, w^\ast\mid_\mu) \omega\mid_\mu = \langle v \mid \text{coad}(w)\mu \rangle$ and [6.2] we see that if $w^\ast\mid_\mu = 0$, then $\text{coad}(w)\mu = 0$ and thus $\iota(v^\ast\mid_\mu, w^\ast\mid_\mu) \omega\mid_\mu = 0$. In other words, $\iota(v^\ast\mid_\mu, w^\ast\mid_\mu) \omega\mid_\mu$ is independent of the choice of $w$ as long as $w^\ast\mid_\mu$ does not change. Combining this with the graded skew-symmetry, we see that $\omega\mid_\mu$ is a well defined graded skew-symmetric form on $T_\mu O_{\mu_0}$. 

Q.E.D
To show that it is homogeneously non-degenerate, we first note that the definitions of $\omega$ and $J$ directly gives the homogeneous parts $\omega_\alpha$ as

\begin{equation}
-\iota(v^\ast |_\mu, w^\ast |_\mu)\omega_\alpha = \langle [v, w] | (\mu)_\alpha \rangle = \langle v | \coad(w)(\mu)_\alpha \rangle \label{6.4}
\end{equation}

\begin{equation}
= \langle [v, w] | J_\alpha(\mu) \rangle . \label{6.5}
\end{equation}

Now suppose that $w \in \mathfrak{g}$ is such that $\forall \alpha : \iota(w^\ast |_\mu)\omega_\alpha = 0$, i.e., $\forall v, w \in \mathfrak{g} : \iota(v^\ast |_\mu, w^\ast |_\mu)\omega_\alpha = 0$. Formula (6.4) directly implies that $\forall \alpha : \coad(w)(\mu)_\alpha = 0$, and thus $w^\ast |_\mu = 0$ according to [6.2]. This proves that $\omega$ is homogeneously non-degenerate.

To show that $\omega$ is closed and $J$ strongly hamiltonian, we make a preliminary computation. We first note that for $v = \sum_{i=1}^n v^i e_i$ and $\mu = \sum_{i=1}^n (\mu_i + \bar{\mu}_i)e_i$ we obtain $\langle v | \mu \rangle = \sum_{i=1}^n v^i((-1)^{\epsilon(e_i)}\mu_i + \bar{\mu}_i)$. Using the explicit expression for $w^\ast$ given in [6.2] we compute:

\[
\begin{align*}
\langle u^\ast |_\mu \langle v | J \rangle 
= & - \sum_{i,j,k=1}^n \left( (-1)^{\epsilon(e_i)\epsilon(e_j)\epsilon(e_k)}u^k \langle e_i | \coad(e_k)(\mu)_0 \rangle \cdot \frac{\partial}{\partial \mu_i} |_\mu \\
& \quad + (-1)^{\epsilon(e_i)\epsilon(e_k)}u^k \langle e_i | \coad(e_k)(\mu)_1 \rangle \cdot \frac{\partial}{\partial \mu_i} |_\mu \right)v^j((-1)^{\epsilon(e_j)}\mu_j + \bar{\mu}_j) \\
= & - \sum_{i,j,k=1}^n \left( (-1)^{\epsilon(e_i)\epsilon(e_j)\epsilon(e_k)}u^k \langle e_i | \coad(e_k)(\mu)_0 \rangle v^j(-1)^{\epsilon(e_j)}v^i(-1)^{\epsilon(e_k)}(e_i + \epsilon(e_k)(v)) \\
& \quad + (-1)^{\epsilon(e_i)\epsilon(e_k)}e^j \langle e_i | \coad(e_k)(\mu)_1 \rangle v^i(-1)^{\epsilon(e_k)}v^j(-1)^{\epsilon(e_i)\epsilon(e_k)}(e_i + \epsilon(e_k)(v)) \right) \\
= & - \sum_{i,j,k=1}^n (-1)^{\epsilon(e_i)\epsilon(e_j)\epsilon(e_k)}u^k \langle [v] | \coad(e_k)((\mu)_0 + (\mu)_1) \rangle \\
= & \langle [u, v] | [\mu] \rangle .
\end{align*}
\]

Tracing what happens with $u^\ast |_\mu \langle v | J \rangle$ we find

\begin{equation}
\langle u^\ast |_\mu \langle v | J \rangle \rangle = \langle [u, v] | J_\alpha(\mu) \rangle = \langle [u, v] | (\mu)_\alpha \rangle . \label{6.6}
\end{equation}

Combining (6.6) with (6.5) we immediately have $\iota(v^\ast)\omega_\alpha = -d\langle v | J_\alpha \rangle$ on $\mathcal{O}_{\mu_\alpha}$, i.e., the action is (weakly) hamiltonian. Once we know that $v^\ast$ is the hamiltonian vector field associated to $\langle v | J \rangle$, we can combine (6.6) with the definition of the Poisson bracket to obtain that the action is strongly hamiltonian.

For the last item on our list, $\omega$ closed, we evaluate $d\omega$ on three homogeneous vector fields according to (2.3). Since the map $v \mapsto v^\ast$ is a morphism of graded Lie algebras, the terms like $\iota([u^\ast, v^\ast], w^\ast)_\mu$ become $-\langle [ [u, v], w ] | \mu \rangle$. Combining (6.5) with (6.6) shows that the terms like $u^\ast |_\mu \iota(v^\ast, w^\ast)_\mu$ are also of the form $-\langle [ [u, v], w ] | \mu \rangle$. We thus find:

\[
\begin{align*}
-\iota(u^\ast, v^\ast, w^\ast)_\mu d\omega |_\mu 
= & \langle u^\ast |_\mu \iota(v^\ast, w^\ast)_\omega \rangle \\
& + (-1)^{\epsilon(u)\epsilon(v)\epsilon(w)}u^\ast |_\mu \iota((u^\ast, v^\ast)_\omega \rangle \\
& + (-1)^{\epsilon(w)\epsilon(u)\epsilon(v)}w^\ast |_\mu \iota((u^\ast, v^\ast)_\omega \rangle \\
& + (-1)^{\epsilon(v)\epsilon(u)\epsilon(w)}v^\ast |_\mu \iota((u^\ast, w^\ast)_\omega \rangle \\
& + (-1)^{\epsilon(w)\epsilon(v)\epsilon(u)}w^\ast |_\mu \iota((v^\ast, w^\ast)_\omega \rangle \\
& + (-1)^{\epsilon(v)\epsilon(w)\epsilon(u)}v^\ast |_\mu \iota((w^\ast, v^\ast)_\omega \rangle \\
& + (-1)^{\epsilon(w)\epsilon(v)\epsilon(u)}w^\ast |_\mu \iota((v^\ast, w^\ast)_\omega \rangle \\
& + (-1)^{\epsilon(v)\epsilon(w)\epsilon(u)}v^\ast |_\mu \iota((w^\ast, v^\ast)_\omega \rangle \\
& + (-1)^{\epsilon(w)\epsilon(v)\epsilon(u)}w^\ast |_\mu \iota((v^\ast, w^\ast)_\omega \rangle.
\end{align*}
\]
\[\text{Therefore, } \omega \text{ evaluated on three homogeneous vectors is always zero. But then by trilinearity } d\omega \text{ is zero.} \]

**7. Super Heisenberg-like groups**

The general case.

Let \( E \) be an \( \mathcal{A} \)-vector space of dimension \( p|q \) with homogeneous basis \( e_1, \ldots, e_n \), \( n = p + q \) and let \( \Omega : E \times E \to C \) be an even graded skew-symmetric bilinear form. The (left) coordinates of the element \( \iota(v, \hat{v})\Omega \in C \) with respect to the basis \( a_0, c_1 \) determine an even and an odd graded skew-symmetric \( C \)-valued bilinear form \( \Omega^0 \) and \( \Omega^1 \) on \( E \) with values in \( \mathcal{A} \) by the formula \( \iota(v, \hat{v})\Omega = \iota(v, \hat{v})\Omega^0 \cdot c_0 + \iota(v, \hat{v})\Omega^1 \cdot c_1 \). With these ingredients we define the \( \mathcal{A} \)-Lie group \( G \) as follows. As set \( G \) is \( (E \times C)_0 \), and the group structure is given by

\[(a, b) \cdot (\hat{a}, \hat{b}) = (a + \hat{a}, b + \hat{b} + \frac{1}{2} \iota(a, \hat{a})\Omega) . \]

The neutral element is \((0,0)\), and the inverse of \((a, b)\) is \((a, b)^{-1} = (-a, -b)\).

On \( G \) we introduce \( n + 2 \) homogeneous coordinates \( a^1, \ldots, a^n, b^0, b^1 \) according to the formula \((a, b) = (\sum_i a^i e_i, \sum_o b^o c_o)\). Their parities are given by \( \varepsilon(a^i) = \varepsilon(e_i) \) and \( \varepsilon(b^0) = \varepsilon(c_0) \). In terms of these coordinates a basis of the left invariant vector fields is given as

\[
\frac{\partial}{\partial a^a}|_{(a, b)} + \frac{1}{2} \iota(a, e_i)\Omega^a \frac{\partial}{\partial b^a}|_{(a, b)} \quad \text{and} \quad \frac{\partial}{\partial b^a}|_{(a, b)} .
\]

Identifying the tangent space \( T_{(a,0)}G \) with \( E \times C \), we can identify the above basis for the \( \mathcal{A} \)-Lie algebra \( \mathfrak{g} \) with the basis for \( E \times C \), and in particular we can identify \( e_j \) with the left-invariant vector field \( \partial_{a^j}|_{(a, b)} + \frac{1}{2} \iota(a, e_j)\Omega^a \partial_{b^a}|_{(a, b)} \) and \( c_a \) with \( \partial_{b^a}|_{(a, b)} \). The only non-zero commutators among these left-invariant vector fields are

\[
\left[ \frac{\partial}{\partial a^a}|_{(a, b)} + \frac{1}{2} \iota(a, e_i)\Omega^a \frac{\partial}{\partial b^a}|_{(a, b)} , \frac{\partial}{\partial a^b}|_{(a, b)} + \frac{1}{2} \iota(a, e_j)\Omega^b \frac{\partial}{\partial b^b}|_{(a, b)} \right] = \left( \frac{1}{2} \iota(e_i, j)\Omega^j - \frac{1}{2} (-1)^{\varepsilon(e_i)\varepsilon(e_j)} \iota(e_j, e_i)\Omega^j \frac{\partial}{\partial b^b}|_{(a, b)} \right) = \iota(e_i, e_j)\Omega^j \frac{\partial}{\partial b^b}|_{(a, b)} ,
\]

in other words, \([e_i, e_j] = \iota(e_i, e_j)\Omega^a c_a \). For general \((v, z), (\hat{v}, \hat{z}) \in E \times C = \mathfrak{g}\) this gives the commutator \([v, z], (\hat{v}, \hat{z}) = (0, \iota(v, \hat{v})\Omega)\).

For \( g = (a, b) \in G \) and \( h = (\hat{a}, \hat{b}) \) we find \( ghg^{-1} = (\hat{a}, \hat{b} + \iota(a, \hat{a})\Omega) \). This gives us for the Adjoint representation the formulæ

\[
\text{Ad}(g)e_i = e_i + \iota(a, e_i)\Omega^a c_a \quad \text{and} \quad \text{Ad}(g)c_a = c_a .
\]
We now consider the left dual algebra $g^* = E^* \times C^*$, i.e., the space of all left linear maps from $g$ to $A$ and we denote the dual basis by $e^1, \cdots, e^n, e^0, e^1$. By definition of the coadjoint representation, we thus have the equalities $\langle e_i | \text{Coad}(g)e^j \rangle = \delta_i^j, \langle c_\alpha | \text{Coad}(g)e^j \rangle = 0$, $(e_i | \text{Coad}(g)e^j) = -\iota(a, e_i)\Omega^\beta, \langle c_\alpha | \text{Coad}(g)e^j \rangle = \delta^j_\alpha$, from which we deduce that the coadjoint representation is given by the formulæ

$$\text{Coad}(g)e^i = e^i \quad \text{and} \quad \text{Coad}(g)e^\alpha = e^\alpha + e^1 \cdot \iota(e_i, a)\Omega^\alpha = e^\alpha + \iota(a)\Omega^\alpha,$$

where $\iota(a)\Omega^\alpha \in E^*$ denotes the left linear map $v \mapsto \iota(v, a)\Omega^\alpha$.

When introducing coordinates on $g^*$, we should remember that we look at the full dual and thus that the coefficient of a basis vector takes its values in $\mathfrak{g}$. We thus introduce $2n + 4$ homogeneous coordinates $x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n, y_0, y_1, \bar{y}_0, \bar{y}_1$ with parities $\varepsilon(x_i) = \varepsilon(e^i), \varepsilon(\bar{x}_i) = 1 - \varepsilon(e^i), \varepsilon(y_0) = \varepsilon(c_\alpha), \varepsilon(\bar{y}_0) = 1 - \varepsilon(c_\alpha)$ of a point $\mu = (x, y) \in g^*$ according to the formula

$$(x, y) = (y_0 + \bar{y}_0) \cdot e^0 + (y_1 + \bar{y}_1) \cdot e^1 + \sum_{i=1}^n (x_i + \bar{x}_i) \cdot e^i.$$  

It follows that the coadjoint action of $g = (a, b) \in G$ on an element $(x, y) \in g^*$ is given by

$$\text{Coad}(g)(x, y) = (x + (y_0 + \bar{y}_0) \cdot \iota(a)\Omega^0 + (y_1 + \bar{y}_1) \cdot \iota(a)\Omega^1, y).$$

In order to have genuine submanifolds, we now specialize to the case of an orbit through a point with real coordinates (and thus in particular $y_1 = \bar{y}_1 = 0$). The $y$-coordinates do not change under the action of Coad$(g)$; the $x$-coordinates change according to

$$x_i \mapsto x_i - (-1)^{\varepsilon(e^i)} y_0 \cdot \iota(a, e_i)\Omega^0 \quad \text{and} \quad \bar{x}_i \mapsto \bar{x}_i - \bar{y}_1 \cdot \iota(a, e_i)\Omega^1,$$

where the sign $(-1)^{\varepsilon(e^i)}$ comes from interchanging $e^i$ and $\iota(a, e_i)\Omega^\alpha$. It follows that there are three different types of orbit depending on whether $y_0$ or $\bar{y}_1$ is zero, their dimension depending upon the dimension of the image of the maps $\Omega^\alpha : E \to E^*$; the fourth case $y_0 = \bar{y}_1 = 0$ yields the trivial orbit $\{(x, 0)\}$ of dimension 0/0.

Since the action is linear, it is straightforward to compute the fundamental vector field $(v, z)^\Omega^\alpha$ associated to the element $(v, z) \in g$:

$$(v, z)^\Omega^\alpha |_{(x, y)} = \sum_i \left( (-1)^{\varepsilon(e^i)} y_0 \cdot \iota(v, e_i)\Omega^0 \cdot \frac{\partial}{\partial x_i} |_{(x, y)} + \bar{y}_1 \cdot \iota(v, e_i)\Omega^1 \cdot \frac{\partial}{\partial \bar{x}_i} |_{(x, y)} \right).$$

For the symplectic form $\omega$ on an orbit we obtain the formula

$$-\iota((v, z)^\Omega^\alpha, (\hat{v}, \hat{z})^\Omega^\alpha)\omega_{(x, y)} = \iota\left( \left[ (v, z), (\hat{v}, \hat{z}) \right] \right)(x, y) = y_0 \cdot \iota(v, \hat{v})\Omega^0 + \bar{y}_1 \cdot \iota(v, \hat{v})\Omega^1.$$
An explicit example.

Let $E$ be an $A$-vector space of dimension $3|3$ for which we order the basis vectors $e_1, \ldots, e_6$ such that $e_1, e_2, e_3$ are even; let $\Omega$ be the graded skew-symmetric form given by the matrix ($j$ is the row index)

$$
(l(e_i, e_j)\Omega)^6_{i,j=1} = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} .
$$

For the coadjoint action of $g = (a, b)$ we thus obtain

$$
x_1 \mapsto x_1 - y_0 \cdot a^2 , \quad x_2 \mapsto x_2 + y_0 \cdot a^1 , \quad x_5 \mapsto x_5 + y_0 \cdot a^5 , \quad x_6 \mapsto x_6 - y_0 \cdot a^6 \\
\bar{x}_1 \mapsto \bar{x}_1 - \bar{y}_1 \cdot a^4 , \quad \bar{x}_3 \mapsto \bar{x}_3 - \bar{y}_1 \cdot a^5 , \quad \bar{x}_4 \mapsto \bar{x}_4 + \bar{y}_1 \cdot a^1 , \quad \bar{x}_5 \mapsto \bar{x}_5 + \bar{y}_1 \cdot a^3 ,
$$

while all other coordinates remain unchanged. For the fundamental vector fields we obtain

$$
(v, z)^\Omega = y_0 \cdot \left( v^2 \cdot \frac{\partial}{\partial x_1} - v^3 \cdot \frac{\partial}{\partial x_2} - v^5 \cdot \frac{\partial}{\partial x_5} + v^6 \cdot \frac{\partial}{\partial x_6} \right) + \bar{y}_1 \cdot \left( v^4 \cdot \frac{\partial}{\partial \bar{x}_1} + v^5 \cdot \frac{\partial}{\partial \bar{x}_3} - v^1 \cdot \frac{\partial}{\partial \bar{x}_4} - v^3 \cdot \frac{\partial}{\partial \bar{x}_5} \right).
$$

We now distinguish three cases: (i) $y_0 \neq 0$ (but real!) and $\bar{y}_1 = 0$, (ii) $\bar{y}_1 \neq 0$ and $y_0 = 0$, and (iii) $y_0 \cdot \bar{y}_1 \neq 0$. In the first case the orbit has dimension $2|2$ with even coordinates $x_1, x_2$ and odd coordinates $x_5, x_6$. In order to better distinguish the even from the odd coordinates, we will change, for the odd coordinates only, the letter $x$ to $\xi$. Explicitly this means that we change the names of $x^4, x^5, x^6, \bar{x}^1, \bar{x}^2, \bar{x}^3$ to $\xi^4, \xi^5, \xi^6, \bar{\xi}^1, \bar{\xi}^2, \bar{\xi}^3$. Substituting the fundamental vector fields associated to basis elements $e_i$ in the formula for the symplectic form gives us the following identities

$$
l(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})\omega = \frac{1}{y_0} , \quad l(\frac{\partial}{\partial \xi_5}, \frac{\partial}{\partial \xi_5})\omega = \frac{1}{y_0} , \quad l(\frac{\partial}{\partial \xi_6}, \frac{\partial}{\partial \xi_6})\omega = \frac{1}{y_0} ,
$$

all others being either zero or determined by graded skew-symmetry. From these identities we deduce that the even symplectic form is given as

$$
\omega = (y_0)^{-1} \cdot \left( dx_2 \wedge dx_1 - \frac{1}{2} d\xi_5 \wedge d\xi_5 + \frac{1}{2} d\xi_6 \wedge d\xi_6 \right).
$$

In the second case the orbit dimension is still $2|2$, but now with even coordinates $\bar{x}_1, \bar{x}_5$ and odd coordinates $\xi_1, \xi_3$. Here we obtain for the symplectic form the identities

$$
l(\frac{\partial}{\partial \bar{x}_1}, \frac{\partial}{\partial \bar{x}_4})\omega = \frac{1}{\bar{y}_1} , \quad l(\frac{\partial}{\partial \bar{\xi}_3}, \frac{\partial}{\partial \bar{x}_5})\omega = \frac{1}{\bar{y}_1} ,
$$

from which we deduce that the odd symplectic form is given as

$$
\omega = (\bar{y}_1)^{-1} \cdot \left( d\bar{x}_4 \wedge d\bar{\xi}_1 + d\bar{x}_5 \wedge d\bar{\xi}_3 \right).
$$
In the third case we have to be slightly more careful. We introduce the coordinate change \( \hat{x}_2 = x_2, \ z_0 = y_0 \hat{x}_4 - \bar{y}_1 \hat{x}_2, \ \hat{\xi}_5 = \xi_5, \ z_1 = y_0 \hat{\xi}_3 + \bar{y}_1 \xi_5, \) and then the \( z_i \) do not change under the coadjoint action. It follows that the orbit has dimension 3|3 with even coordinates \( x_1, \hat{x}_2, \hat{\xi}_5 \) and odd coordinates \( \hat{\xi}_1, \hat{\xi}_5, \xi_6. \) In terms of these coordinates the fundamental vector field is given as

\[
(v, z)^* = y_0 \cdot \left( v^2 \cdot \frac{\partial}{\partial x_1} - v^3 \cdot \frac{\partial}{\partial \hat{x}_2} - v^5 \cdot \frac{\partial}{\partial \hat{\xi}_5} + v^6 \cdot \frac{\partial}{\partial \xi_6} \right) \\
+ \bar{y}_1 \cdot \left( v^4 \cdot \frac{\partial}{\partial \hat{\xi}_1} - v^3 \cdot \frac{\partial}{\partial \hat{x}_5} \right)
\]

As before, substituting suitable basis vectors for \( v \) in the formula for the symplectic form gives us the following identities

\[
\iota(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial \hat{x}_2})\omega = \frac{1}{y_0}, \ \iota(\frac{\partial}{\partial \hat{\xi}_1}, \frac{\partial}{\partial \hat{x}_2})\omega = \frac{1}{y_0}, \ \iota(\frac{\partial}{\partial \hat{x}_5}, \frac{\partial}{\partial \hat{\xi}_5})\omega = \frac{1}{y_0} \\
\iota(\frac{\partial}{\partial \hat{\xi}_5}, \frac{\partial}{\partial \xi_6})\omega = -\frac{1}{y_0}, \ \iota(\frac{\partial}{\partial \xi_6}, \frac{\partial}{\partial \xi_6})\omega = \frac{1}{y_0}.
\]

This results in the mixed (degenerate but homogeneously non-degenerate) symplectic form

\[
\omega = (y_0)^{-1} \cdot (d\hat{x}_2 \wedge dx_1 + d\hat{x}_2 \wedge d\hat{\xi}_1 + d\hat{\xi}_5 \wedge dx_5 - \frac{1}{2} d\hat{\xi}_5 \wedge d\hat{\xi}_5 + \frac{1}{2} d\xi_6 \wedge d\xi_6).
\]

If we see the full set of coordinates on \( g^* \) as functions on the orbit, then this mixed symplectic form can be written as

\[
\omega = (y_0)^{-1} \cdot (d\hat{x}_2 \wedge dx_1 - \frac{1}{2} d\hat{\xi}_5 \wedge d\xi_5 + \frac{1}{2} d\xi_6 \wedge d\xi_6) + (\bar{y}_1)^{-1} \cdot (d\hat{x}_4 \wedge d\hat{\xi}_1 + d\hat{x}_5 \wedge d\bar{\xi}_3),
\]

in which form it looks like we take the sum of the symplectic forms on the even and odd orbits separately (which of course doesn’t make sense because the dimensions don’t match).

8. Prequantization

Let us now turn our attention to prequantization of symplectic \( A \)-manifolds. Since the symplectic form can be seen as an even 2-form with values in \( C \), it seems natural that in terms of principal fiber bundles we should look at structure groups whose associated \( A \)-Lie algebra is \(|1| \) dimensional. A simple analysis shows that (up to scaling) there are only three different \( A \)-Lie algebra structures on \( C \) possible: (i) an abelian one, (ii) \([c_0, c_1] = c_1 \) and \([c_1, c_1] = 0 \), and (iii) \([c_0, c_1] = 0 \) and \([c_1, c_1] = c_0 \). If we impose that the even part of the group should be the circle, the second possibility drops out because the simply connected \( A \)-Lie group with this \( A \)-Lie algebra does not have non-trivial discrete normal subgroups (it is the \( a \xi + \alpha \) group of affine transformations of the odd affine line \( A_1 \)). The remaining two possibilities both have \( GS^1 \times A_1 \) as underlying \( A \)-manifold. Here \( GS^1 \) is the circle augmented to even nilpotent elements, which we can write as \( GS^1 = \{e^{ix} \mid x \in A_0\} \) or as \( GS^1 = \{x \mod 2\pi \mathbb{Z} \mid x \in A_0\} \). In the abelian case the group structure is (obviously) given by

\[
(e^{ix}, \xi) \cdot (e^{iy}, \eta) = (e^{i(x+y)}, \xi + \eta).
\]
In the non-abelian case the group structure is given by

\[(e^{ix}, \xi) \cdot (e^{iy}, \eta) = (e^{i(x+y+\xi\eta)}, \xi + \eta).\]

This group can be seen as a 1|1-dimensional Lie subgroup of the multiplicative subgroup of the ring \(A \oplus iA\). As such it is the 1|1-dimensional equivalent of \(S^1\) as the 1-dimensional Lie subgroup of the multiplicative subgroup of \(C = R \oplus iR\).

If we think of prequantization in terms of complex line bundles, it is natural to prefer the non-abelian group because it is the natural group in which transition functions can take their values if the line in question is \(A \otimes \mathbb{C} = A \oplus iA\). However, I could not find any reasonable (nor unreasonable) way to disguise a mixed symplectic form as an even matrix valued 2-form such that it can be the curvature of a linear connection on a line bundle over \(M\). Since the curvature form of a linear connection is necessarily even, this seems to exclude the non-abelian case. Remains the abelian case, for which we have two additional arguments in favor. In the first place: the center of the Poisson algebra is the 1|1-dimensional abelian subalgebra of constant functions. And secondly, the curvature 2-form of a connection on a principal fiber bundle is a 2-form on the base space if and only if the structure group is abelian.

We thus concentrate our efforts to answer the question: does there exist a principal fiber bundle \(\pi : Y \to M\) with abelian structure group \(GS^1 \times A\) and connection \(\alpha\) whose curvature is \(\omega\)?

To answer the above question, we follow very closely the corresponding analysis in [TW]. This involves standard techniques in algebraic topology to show the equivalence between de Rham cohomology and Čech cohomology.

8bis. Intermezzo on cohomology and symplectic \(A\)-manifolds

We start by choosing an open cover \(U = \{U_i \mid i \in I\}\) of the symplectic \(A\)-manifold \((M, \omega)\) such that all finite intersections of elements of \(U\) are contractible or empty.

8.1 Definition. The nerve of the cover \(U\), denoted \(\mathcal{N}(U)\), is defined as

\[\mathcal{N}(U) = \{ (i_0, \ldots, i_k) \in I^{k+1} \mid k \in \mathbb{N}, U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset \},\]

and an element \((i_0, \ldots, i_k)\) is called an \(k\) simplex. The abelian group \(C_k(U)\) of \(k\)-chains is defined to be the free \(\mathbb{Z}\)-module with basis the \(k\)-simplices, i.e., \(C_k(U)\) consists of all finite formal sums \(\sum c_{i_0,\ldots,i_k} (i_0, \ldots, i_k)\) with \((i_0, \ldots, i_k) \in \mathcal{N}(U)\) and \(c_{i_0,\ldots,i_k} \in \mathbb{Z}\). The boundary operator \(\partial_k : C_k(U) \to C_{k-1}(U)\) is the homomorphism defined on the basis of \(C_k(U)\) by

\[\partial_k(i_0, \ldots, i_k) = \sum_{j=0}^k (-1)^j (i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_k).\]

It is an elementary exercise to prove that \(\partial_{k-1} \circ \partial_k = 0\).

Proof.

\[\partial_{k-1} \left(\partial_k(i_0, \ldots, i_k)\right) = \sum_{j=0}^k (-1)^j \partial_{k-1}(i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_k)\]
that with our restrictions (contractible intersections) it is actually independent of
\[ H^k(M, A) \]
cohomology group of
\[ \sum_{\ell=0}^{k-1} (-1)^{\ell} (i_0, \ldots, i_{\ell-1}, i_{\ell+1}, \ldots, i_{j-1}, i_{j+1}, \ldots, i_k) \]

\[ + (-1)^{j-1} (i_0, \ldots, i_{j-2}, i_{j+1}, \ldots, i_k) + (-1)^j (i_0, \ldots, i_{j-1}, i_{j+2}, \ldots, i_k) \]

\[ + \sum_{\ell=j+2}^{k} (-1)^{\ell-1} (i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{\ell-1}, i_{\ell+1}, \ldots, i_k) \]

\[ = \sum_{j=0}^{k-j-2} \sum_{\ell=0}^{j-1} (-1)^{j+\ell} (i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{\ell-1}, i_{\ell+1}, \ldots, i_k) \]

\[ - \sum_{j=1}^{k} (i_0, \ldots, i_{j-2}, i_{j+1}, \ldots, i_k) + \sum_{j=0}^{k-1} (i_0, \ldots, i_{j-1}, i_{j+2}, \ldots, i_k) \]

\[ + \sum_{\ell=0}^{j-2} \sum_{j=\ell+1}^{k} (-1)^{j+\ell-1} (i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{\ell-1}, i_{\ell+1}, \ldots, i_k) \]

\[ = 0 , \]

because the two single sums in the second line cancel and because the two double
sums also cancel.

For any abelian group \( A \), a homomorphism \( h : C_k(\mathcal{U}) \to A \) is completely determined by its values on the basis vectors \((i_0, \ldots, i_k) \in N(\mathcal{U})\); such a homomorphism is called a \( k \)-cochain if it is totally skew-symmetric on these basis vectors, i.e., \( h \) changes sign when one interchanges two entries \( i_p \) and \( i_q \) in \((i_0, \ldots, i_k)\). The set of all \( k \)-cochains with values in the abelian group \( A \) is denoted by \( C^k(\mathcal{U}, A) \); equipped with pointwise addition of functions this is an abelian group. By duality one defines the coboundary operator \( \delta_k : C^k(\mathcal{U}, A) \to C^{k+1}(\mathcal{U}, A) : (\delta_k h)(i_0, \ldots, i_k) = h(\partial_{k+1}(i_0, \ldots, i_k)) \). Since \( \partial_k \circ \partial_{k+1} = 0 \), we have \( \delta_k \circ \partial_{k-1} = 0 \). It follows that \( B^k(\mathcal{U}, A) = \ker(\delta_k) \) is contained in \( Z^k(\mathcal{U}, A) = \ker(\delta_{k-1}) \), so their quotient \( H^k(\mathcal{U}, A) = Z^k(\mathcal{U}, A)/B^k(\mathcal{U}, A) \) is a well defined abelian group. Elements of \( B^k(\mathcal{U}, A) \) are called \( k \)-coboundaries, elements of \( Z^k(\mathcal{U}, A) \) are called \( k \)-cocycles, and \( H^k(\mathcal{U}, A) \) is called the \( k \)-th Čech cohomology group of \( \mathcal{U} \) with values in the abelian group \( A \).

The construction of \( H^k(\mathcal{U}, A) \) can be done for any cover \( \mathcal{U} \), but it can be shown that with our restrictions (contractible intersections) it is actually independent of the cover. It is thus customary to denote these groups by \( H^k_c(M, A) \) and to call them the \( k \)-th Čech cohomology group of \( M \) (with values in \( A \)).

We now take a closer look at the symplectic form \( \omega \). Since the Poincaré lemma also holds for \( \mathcal{A} \)-manifolds, there exist on each \( U_i \) a 1-form \( \theta_i \) such that \( d\theta_i = \omega \). On each (non empty) intersection \( U_i \cap U_j \) we have \( d(\theta_i - \theta_j) = 0 \) and hence there exist smooth functions \( f_{ij} \) on \( U_i \cap U_j \) such that \( \theta_i - \theta_j = df_{ij} \). By choosing a total order on the set of indices \( I \) we even may assume that \( f_{ji} = -f_{ij} \). Hence on a triple intersection \( U_i \cap U_j \cap U_k \) we have \( d(f_{ij} + f_{jk} + f_{ki}) = 0 \) and hence there exist constant functions \( a_{ij,k} \) (necessarily real) such that on \( U_i \cap U_j \cap U_k \) we have \( f_{ij} + f_{jk} + f_{ki} = a_{ij,k} \). Since the \( f_{ij} \) are skew-symmetric in \( i, j \), it follows immediately that the map \( a : C_2(\mathcal{U}) \to \mathbb{R} \), \((i, j, k) \in N(\mathcal{U}) \mapsto a_{ij,k} \) is skew symmetric, i.e.,
that if we replace \( \omega \) cocycle \( a \) that Per\((\delta(f)) \) choices that are possible in the construction of the cocycle \( a \).

**8.2 Lemma.** The (sub)group \( \text{Per}(\omega) = \text{im}(a : \ker \partial_2 \to \mathbb{R}) \subset \mathbb{R} \) depends only upon the cohomology class of \( \omega \) in de Rham cohomology, and not upon the various choices that are possible in the construction of the cocycle \( a \).

**Proof.** If \( \theta_i \) is replaced by \( \theta_i + d \phi_i \) with \( \phi_i \) a smooth function on \( U_i \), then we can replace \( f_{ij} \) by \( f_{ij} + \phi_{ij} - \phi_{ji} \), and \( a_{ijk} \) remains unchanged. If \( f_{ij} \) is replaced by \( f_{ij} + c_{ij} \) a (real) constant, then \( a_{ijk} \) is replaced by \( (a + \partial_2 c)_{ijk} \) because \( \delta(f)_{(i, j, k)} = \delta(\partial_2 f_{ij} + (i, k) - (i, j)) = c_{ij} - c_{ik} + c_{kj} = c_{ij} + c_{jk} + c_{ki} \).

Since \( \delta_2 c \) is (by definition) zero on \( \ker \partial_2 \), this does not change \( \text{Per}(\omega) \). This shows that \( \text{Per}(\omega) \) does not depend upon the choices made in the construction of the cocycle \( a \). To show that it only depends upon the cohomology class of \( \omega \), we note that if we replace \( \omega \) by \( \omega + d \theta \), then we can replace \( \theta_i \) by \( \theta_i + \theta \), and then \( f_{ij} \) does not change.

**8.3 Definition.** The group \( \text{Per}(\omega) \subset \mathbb{R} \) is called the group of periods of the (closed) 2-form \( \omega \).

**8.4 Lemma.** The cocycle \( a \) can be chosen such that \( \forall (i, j, k) \in N(\omega) : a_{ijk} \in \text{Per}(\omega) \).

**Proof.** We define the homomorphism \( b : C_1(\mathcal{U}) \to \mathbb{R} / \text{Per}(\omega) \) as follows. On the subspace \( \ker \partial_2 \) it is defined by \( b = \pi \circ a \circ (\partial_2)^{-1} \), where \( \pi \) denotes the canonical projection \( \mathbb{R} \to \mathbb{R} / \text{Per}(\omega) \); this is independent of the choice in \( (\partial_2)^{-1} \) by definition of \( \text{Per}(\omega) \). Since \( \mathbb{R} / \text{Per}(\omega) \) is a divisible \( \mathbb{Z} \)-module, there exists an extension \( b \) to the whole of \( C_1(\mathcal{U}) \) (see [HiSt, §1.7]). Since \( C_1(\mathcal{U}) \) is a free \( \mathbb{Z} \)-module, there exists a homomorphism \( b' : C_1(\mathcal{U}) \to \mathbb{R} \) satisfying \( \pi \circ b' = b \). Finally we replace the functions \( f_{ij} \) by \( f_{ij} - b'_{ij} \), which changes the cocycle \( a \) into \( a - \partial_1 b' \). By construction of \( b' \) it follows that \( \pi(a_{ijk} - (\delta_1 b')_{ijk}) = 0 \), showing that this modified cocycle takes its values in \( \text{Per}(\omega) \).

**Remarks.**

- The above statement is a purely algebraic statement, independent of the topological properties of \( \text{Per}(\omega) \); the latter can be dense or discrete in \( \mathbb{R} \) without affecting the result.
- The simplex \( (i, j, k) \) is clearly not in \( \ker \partial_2 \); nevertheless the cocycle \( a \) can be chosen such that it takes everywhere values in \( \text{Per}(\omega) \).
- If \( \omega \) is exact then obviously \( \text{Per}(\omega) = \{0\} \), but one can prove that the converse is also true: if \( \text{Per}(\omega) = \{0\} \), then \( \omega \) is exact.
- The construction of the 2-cocycle \( a \) associated to the closed 2-form \( \omega \) is part of a larger construction which serves to prove the equivalence between de Rham cohomology and Čech cohomology. This approach can be used at the same time to prove that the de Rham cohomology of an \( \mathcal{A} \)-manifold is the same as the de Rham cohomology of the underlying \( \mathbb{R} \)-manifold (its body).

**Proof.** Suppose \( \text{Per}(\omega) = \{0\} \) and let \( \theta_i \) and \( f_{ij} \) be as in the construction of the cocycle \( a \). According to [8.4] we may assume that the constants \( a_{ijk} \) are zero, i.e., that \( f_{ij} + f_{jk} + f_{ki} = 0 \). Let \( \rho_i \) be a partition of unity subordinated to the cover \( \mathcal{U} \).

Then the local 1-forms \( \hat{\theta}_i = \theta_i + d(\sum_k \rho_k f_{ki}) \) are well defined on \( U_i \) and they coincide on the intersection \( U_i \cap U_j : \hat{\theta}_i - \hat{\theta}_j = df_{ij} + d(\sum_k \rho_k (f_{ki} - f_{kj})) = df_{ij} - d(\sum_k \rho_k f_{ij}) = 0 \).
These local 1-forms thus define a global 1-form \( \hat{\theta} \), which obviously satisfies \( d\hat{\theta} = \omega \).

8. Prequantization continued

We now come to the construction of the principal fiber bundle \( \pi : Y \to M \) with a connection \( \overline{\omega} \) whose curvature is \( \overline{\omega} \). In order to better discuss some particular details of our construction, we choose \( d \in \mathbb{R}^{\geq 0} \) and we define \( D = d\mathbb{Z} \subset \mathbb{R} \subset \mathcal{A}_0 \subset \mathcal{A} \). The abelian group \( \mathcal{A} \) has two global coordinates \( x \in \mathcal{A}_0 \) and \( \xi \in \mathcal{A}_1 \). Since \( D \) is a discrete subgroup of \( \mathcal{A} \) contained in its even part, the quotient \( \mathcal{A} \)-Lie group \( \mathcal{A}/D = \mathcal{A}_0/D \times \mathcal{A}_1 \) inherits a global odd coordinate \( \xi \) and a local even coordinate \( x \) modulo \( d \) (of course, in case \( d = 0 \), \( x \) is a global coordinate). The sum \( x = x + \xi \) of the two coordinate functions can be seen as a (local) \( \mathcal{A} \)-valued function. Its exterior derivative \( dx \) is a globally well defined mixed 1-form on \( \mathcal{A}/D \). The abelian group \( \mathcal{A}/D \) is (isomorphic to) our abelian group \( \mathbb{GS}^1 \times \mathcal{A}_1 \). The two global vector fields \( \partial_{x} \) and \( \partial_{\xi} \) are left (and right) invariant; we thus can take them as a basis for the \( \mathcal{A} \)-Lie algebra of \( \mathcal{A}/D \). This allows us to make our choice for the \( \mathcal{A} \)-vector space \( C \) and its basis \( c_0, c_1 \) explicit: we let \( C \) be the \( \mathcal{A} \)-Lie algebra of \( \mathcal{A}/D \) and we let the basis \( c_0, c_1 \) be the left-invariant vector fields \( c_0 = \partial_{x}, c_1 = \partial_{\xi} \). In particular the even \( C \)-valued 1-form \( \overline{dx} = dx \otimes c_0 + d\xi \otimes c_1 \) is the Maurer-Cartan 1-form of \( \mathcal{A}/D \).

One word of caution is in order: the basis \( c_0, c_1 \) of \( C \) depends upon the choice of \( d \in \mathbb{R}^{\geq 0} \), hence \( \overline{\omega} \) and \( \overline{\alpha} \) also depend upon this choice! Our final assumption for the construction of the principal fiber bundle \( Y \) is that \( \text{Per}(\omega) \) is contained in \( D \).

And then the actual construction. Since we assume that \( a_{ijk} \in \text{Per}(\omega) \), it follows that the functions

\[
g_{ij} : U_i \cap U_j \to \mathcal{A}/D \ , \quad m \mapsto f_{ij}(m) \mod D
\]

satisfy the cocycle condition

\[
g_{ij} + g_{jk} + g_{ki} = 0 \quad \text{on} \ U_i \cap U_j \cap U_k.
\]

Hence these functions define a principal fiber bundle \( \pi : Y \to M \) with structure group \( \mathcal{A}/D \). Its local trivializations are \( U_i \times \mathcal{A}/D \) with projection \( \pi(m, x_i) = m \) and transition functions

\[
U_i \times \mathcal{A}/D \to U_j \times \mathcal{A}/D
\]

\[
(m, x_i) \mapsto (m, x_i + g_{ij}(m)) = (m, x_j).
\]

On \( Y \) we define the 1-form \( \alpha \) by its expression on each local chart \( U_i \times \mathcal{A}/D \) by

\[
\alpha = \pi^*\theta_i + dx_i,
\]

satisfying \( d\alpha = \pi^*\omega \). That \( \alpha \) is well defined follows from the definition of the functions \( g_{ij} \) and the fact that \( dg_{ij} = df_{ij} \) (same argument as in proving that \( dx \) is a global 1-form on \( \mathcal{A}/D \)). With our choice for \( c_0, c_1 \) it is elementary to verify that

\[
\overline{\omega} \equiv ((\theta_i)_0 + dx_i) \otimes c_0 + ((\theta_i)_1 + d\xi_i) \otimes c_1
\]

is a connection 1-form on \( Y \) whose curvature 2-form is \( \overline{\omega} \).
Proof. Since the right action of $A/D$ on $Y$ is given on a local chart by $(m, x_i) \cdot t = (m, x_i + t)$, it follows immediately that this action preserves the 1-form $\alpha$ and thus $\Omega$. Since the group is abelian, the adjoint action is trivial and thus the second condition of a connection 1-form is satisfied. For the first condition, we note that the fundamental vector fields associated to the Lie algebra elements $c_0 = \partial_x, c_1 = \partial_t$ are $\partial_{x_i}$ and $\partial_{t_i}$. It follows that the contraction of $\Omega$ with a fundamental vector field gives the initial Lie algebra element. This proves that $\Omega$ is a connection 1-form. That $\Omega$ is its curvature is (again) a direct consequence of the fact that the structure group is abelian: for abelian structure groups the curvature 2-form is the exterior derivative of the connection 1-form.

\[ \text{QED} \]

8.8 Theorem. There exists a principal fiber bundle $\pi : Y \to M$ with structure group $A/D$ and connection $\Omega$ whose curvature is $\Omega$ if and only if $\text{Per}(\omega)$ is contained in $D$, in which case $Y$ is obtained by the above construction.

Proof. We only have to prove the only if part, so suppose we have such a bundle $Y$. Without loss of generality we may assume that $Y$ is trivial above each $U_i$. We thus have transition functions $g_{ij} : U_i \cap U_j \to A/D$. Since $A \to A/D$ is a covering and since each $U_i \in U$ is contractible, there exist smooth functions $s_{ij} : U_i \cap U_j \to A$ such that $g_{ij} = s_{ij} \mod D$. On the trivializing chart $U_i \times A/D$ the connection $\Omega$ is necessarily of the form

$$\Omega \equiv \left( (\theta_i)_0 + dx_i \right) \otimes c_0 + \left( (\theta_i)_1 + d\xi_i \right) \otimes c_1 = \theta_i + dx_i$$

for some local 1-form $\theta$. The fact that the curvature of $\Omega$ is $\Omega$ implies that $d\theta_i = \omega$. Comparing the local expressions above $U_i$ and $U_j$ for the global connection $\Omega$ using the transition (8.6) gives us

$$\theta_j + d(x_i + g_{ij}) = \theta_i + dx_i.$$

Hence $\theta_i - \theta_j = dg_{ij} \equiv df_{ij}$. It now suffices to note that the functions $g_{ij}$ satisfy the cocycle condition (8.5) to conclude that $\alpha_{ijk} \in D$. And thus $\text{Per}(\omega) \subset D$. From the above analysis the last statement is also immediate.

\[ \text{QED} \]

Instead of performing our construction of $Y$ with the $A$-valued functions $s_{ij}$, we could have restricted our attention to the even part only. This means that we use the functions $(g_{ij})_0 : U_i \cap U_j \to A_0/D$ to define a principal fiber bundle $Y_{[0]} \to M$ with structure group $A_0/D$. Another way to describe $Y_{[0]}$ is to say that it is the subbundle of $Y$ corresponding to the subgroup $A_0/D \subset A/D$. Said this way, it is clear that the construction of $Y_{[0]}$ is intrinsic. We also only consider the even part of $\alpha : \alpha_0 = (\pi^*\theta)_0 + dx_i$. And as for $\alpha$, $\alpha_0 \otimes c_0$ is a connection 1-form on the principal fiber bundle $Y_{[0]}$. We could and will say that $(Y_{[0]}, \alpha_0)$ is the even part of $(Y, \alpha)$. The importance of the even part of $Y$ lies in the result [8.10], which needs a definition.

8.9 Definitions. A principal fiber bundle $\pi : Y \to M$ with structure group $G$ is called *topologically trivial* if there exists a global smooth section $s : M \to Y$. A $D$-prequantum bundle for the symplectic manifold $(M, \omega)$ is a principal fiber bundle $\pi : Y \to M$ with structure group $A/D$ and connection $\Omega$ whose curvature is $\Omega$. A $D$-prequantum bundle will usually be denoted as a couple $(Y, \alpha)$. Two $D$-prequantum bundles $(Y, \alpha)$ and $(Y', \alpha')$ are called *equivalent* (as $D$-prequantum bundles) if there exists a diffeomorphism $\phi : Y \to Y'$ such that $\pi' \circ \phi = \pi$, commuting with the right-actions of $A/D$ in the sense that $\phi(y \cdot t) = \phi(y) \cdot t$, and such that $\phi^* \alpha' = \alpha$. 
8.10 Theorem. Let \((Y, \alpha)\) be a \(D\)-prequantum bundle.

(i) If \(D = \{0\}\), then \(Y\) is topologically trivial.

(ii) If \(Y\) is a topologically trivial, then \(\text{Per}(\omega) = \{0\}\) and \((Y, \alpha)\) is equivalent to the \(D\)-prequantum bundle \(M \times \mathcal{A}/D\) with connection \(\theta + d\xi\), where \(\theta\) is a global 1-form satisfying \(d\theta = \omega\).

(iii) \(Y\) is equivalent to the \(D\)-prequantum bundle \(Y[0] \times \mathcal{A}_1\) with connection \(\alpha_0 \otimes c_0 + (\theta_1 + d\xi) \otimes c_1\), where \(\theta_1\) is a global odd 1-form on \(M\) satisfying \(d\theta_1 = \omega_1\).

(iv) Inequivalent \(D\)-prequantum bundles are classified by \(H^1_C(M, \mathbb{R}/D)\).

Proof. According to \([8, 8]\) we assume that \(Y\) is constructed with the ingredients \(\theta, f_{ij}\) and \(a_{ijk} \in \text{Per}(\omega)\) associated to the cover \(U\). We let \(\rho_k\) be a partition of unity subordinated to the cover \(U\).

- (i) If \(D = \{0\}\), then \(\text{Per}(\omega) = \{0\}\) because it is supposed to be included in \(D\), but also the functions \(f_{ij}\) act as transition functions for the principal fiber bundle. We define local sections \(s_i : U_i \rightarrow \pi^{-1}(U_i) \cong U_i \times \mathcal{A}\) in terms of the local trivializations by

\[
s_i(m) = (m, \sum_k \rho_k(m)f_{ki}(m)) .
\]

In order to show that these local sections glue together to a global section we first note that since \(\text{Per}(\omega) = \{0\}\), all constants \(a_{ijk}\) are zero. This together with the skew-symmetry of the \(f_{ij}\) implies that \(f_{ij} + f_{jk} = f_{ik}\). Then we recall that the transition functions of \(Y\) are given by \((8.6)\), i.e., \(s_i(m)\) is mapped to \((m, f_{ij}(m) + \sum_k \rho_k(m)f_{ki}(m))\). But

\[
f_{ij}(m) + \sum_k \rho_k(m)f_{ki}(m) = \sum_k \rho_k(m)(f_{ki}(m) + f_{ij}(m)) = \sum_k \rho_k(m)f_{kj}(m) .
\]

In other words, \(s_i(m)\) is mapped by the transition functions to \(s_j(m)\). Hence the local sections \(s_i\) glue together to form a global smooth section.

- (ii) If \(s : M \rightarrow Y\) is a global section, we can define the 1-form \(\theta = s^*\alpha\) on \(M\). Obviously \(d\theta = s^*d\alpha = s^*\pi^*\omega = \omega\). Hence \(\omega\) is exact and thus \(\text{Per}(\omega) = \{0\}\). We now define the map \(\phi : M \times \mathcal{A}/D \rightarrow Y\) by

\[
\phi(m, x) = s(m) \cdot x ,
\]

where on the right hand side we use the action of the structure group \(\mathcal{A}/D\) on \(Y\). If the section \(s\) is represented on a local trivializing chart \(U_l\) by the function \(s_l : U_l \rightarrow \mathcal{A}/D, s(m) = (m, s_i(m))\), then \(\phi\) is given by \(\phi(m, x) = (m, s_i(m) + x)\), from which it follows immediately that \(\phi\) is a diffeomorphism such that \(\pi \circ \phi = \pi_1\) (\(\pi_1 : M \times \mathcal{A}/D \rightarrow M\) the canonical projection) and commuting with the \(\mathcal{A}/D\)-action. In the local trivialization we also have \(\alpha = \theta_l + d\xi_l\), and thus, since \(s^*\alpha = \theta\), we have \(\theta = \theta_l + ds_i\) on \(U_l\). Finally, still in the same trivializing chart we have \(\phi^*\alpha = \theta_l + d(s_i + x) = \theta + dx\).

- (iii) Since the constants \(a_{ijk}\) are real, we have \((f_{ij})_1 = (g_{ij})_1\). We define the isomorphism \(\phi : Y \rightarrow Y[0] \times \mathcal{A}_1\) on local trivializing charts \(U_i \times (\mathcal{A}_0/D) \times \mathcal{A}_1\) by

\[
\phi(m, x_i) = (m, x_i + \sum_k \rho_k(m)f_{ik}(m))_1 .
\]
It is obvious that this is a diffeomorphism between the local trivializing charts, commuting with the \(A/D\) action and compatible with the bundle structure. Remains to be verified that it is globally well defined. For the bundle \(Y\) the transition functions are given by (8.6), for \(Y_{[0]} \times A_1\) they are given by \((m, x_i + \xi) \mapsto (m, x_i + (g_{ij}(m))_0 + \xi)\). We now start with a point \((m, x_i) \in U_i \times A/D \subset Y\). If we first apply \(\phi\) and then change charts to \(U_j \times A/D\) in \(Y_{[0]} \times A_1\), we obtain

\[
(m, x_i + (g_{ij}(m))_0 + \sum_k \rho_k(m)(f_{ik}(m))_1) .
\]

On the other hand, if we first change charts to \(U_j \times A/D\) in \(Y\) and then apply \(\phi\), we obtain

\[
(m, x_i + (g_{ij}(m))_0 + (f_{ij}(m))_1 + \sum_k \rho_k(m)(f_{jk}(m))_1) .
\]

Since \((f_{ij}(m))_1 + (f_{jk}(m))_1 = -(f_{ki}(m))_1 = (f_{ik}(m))_1\) according to (8.5), these two results are the same. Hence \(\phi\) is a well defined global isomorphism of principal fiber bundles.

Under \(\phi^{-1}\) the local odd 1-forms \((\theta_i)_1\) change to \(\hat{\theta}_i = (\theta_i)_1 - d \sum_k \rho_k(f_{ik})_1\). On non-empty intersections \(U_i \cap U_j\) we have \(\hat{\theta}_i - \hat{\theta}_j = f_{ij} - \sum_k \rho_k(f_{ik} - f_{jk})\Lambda_1 = d(f_{ij} - \sum_k \rho_k f_{ij})_1 = 0\). It follows that the local odd 1-forms \(\hat{\theta}_i\) glue together to form a global odd 1-form \(\theta_1\), which obviously satisfies \(d\theta_1 = \omega_1\). By construction we have \((\phi^{-1})^* \alpha = a_0 + \theta_1 + d\xi\), which finishes the proof of (iii).

(iv) Let \((Y', \alpha')\) be a fixed (reference) \(D\)-prequantum bundle constructed with the ingredients \(\theta'\), \(f'\), \(a'_{ijk} \in \text{Per}(\omega)\) and let \((Y, \alpha)\) be an arbitrary \(D\)-prequantum bundle constructed with the ingredients \(\theta\), \(f\), \(a_{ijk} \in \text{Per}(\omega)\). Since \(U_i\) is contractible, there exist \(g_i : U_i \to A\) such that \(\theta_i - \theta'_i = dg_i\). Since \(U_i \cap U_j\) is contractible, there exist constants \(b_{ij} \in \mathbb{R}\) such that \(f_{ij} - f'_{ij} = g_i - g_j + b_{ij}\). Since \(a'_{ijk}, a_{ijk} \in \text{Per}(\omega) \subset D\), it follows that \(\pi b_{ij} + \pi b_{jk} + \pi b_{ki} = 0\), where \(\pi\) denotes the projection \(\pi : \mathbb{R} \to \mathbb{R}/D\) (abuse of notation without confusion). Hence the 1-cocycle \(\pi b_{ij}\) is actually a 1-cocycle and thus determines an element \([\pi b] \in H^1(M, \mathbb{R}/D)\). The only freedom in the construction of this cohomology class is the choice of \(g_i\); if we change \(g_i\) to \(g_i + c_i\) for constants \(c_i \in \mathbb{R}\), the cocycle \(\pi b\) is changed to \(\pi b + \pi \delta c\), and thus the cohomology class \([\pi b]\) is independent of this choice. We thus have constructed a map from the set of \(D\)-prequantum bundles \((Y, \alpha)\) to \(H^1(M, \mathbb{R}/D)\). The next step is to prove that this induces a bijection on equivalence classes of \(D\)-prequantum bundles.

Let us first assume that \((Y, \alpha)\) and \((Y', \alpha')\) determine the same cohomology class, i.e., there exist constants \(c_i \in \mathbb{R}\) such that \(\pi b'_{ij} = \pi b_{ij} + \pi c_i - \pi c_j\). From the definitions it follows that we have

\[
\pi f'_{ij} - \pi f_{ij} = \pi(g'_i - g_i + c_i) - \pi(g'_j - g_j + c_j) .
\]

We now define a map \(\phi : Y \to Y'\) on local trivializing charts by \(\phi(m, x_i) = (m, x_i - \pi(g'_i(m) - g_i(m) + c_i))\). If we first change charts in \(Y\) to \(U_j\) and then apply this \(\phi\) we obtain

\[
(m, x_i + \pi f_{ij}(m) - \pi(g'_j(m) - g_j(m) + c_j)) ,
\]
while if we first apply this $\phi$ and then change charts in $Y'$ we obtain

$$(m, x_i - \pi(g'_i(m) - g_i(m) + c_i) + \pi f'_{ij}(m)).$$

According to (8.11) these two results are the same, showing that $\phi$ is a globally well defined diffeomorphism satisfying $\pi' \circ \phi = \pi$ and compatible with the action of $A/D$.

Since $\theta'_i - \theta_i = dg'_i - dg_i$, it follows that $\phi^*(\theta'_i + dx_i) = \theta_i' + dx_i - (dg'_i - dg_i) = \theta_i + dx_i$.

In other words, $\phi^* \alpha' = \alpha$. This shows that $D$-prequantum bundles mapping to the same cohomology class are equivalent.

Conversely, suppose that $(Y, \alpha)$ and $(Y', \alpha')$ are equivalent via the diffeomorphism $\phi : Y \to Y'$. This implies that there exist smooth functions $\chi_i : U_i \to A/D$ such that on a local trivializing chart we have

$$\phi(m, x_i) = (m, x_i + \chi_i(m)).$$

Since $U_i$ is contractible, there exist smooth functions $h_i : U_i \to A$ such that $\pi h_i = \chi_i$. The fact that $\phi$ is globally defined implies that these functions $h_i$ satisfy the condition $\pi(h_i + f'_{ij}) = \pi(f_{ij} + h_j)$. The condition that $\phi^* \alpha' = \alpha$ translates to the fact that $\phi^* (\theta'_i + dx_i) = \theta_i + dx_i$. This gives us $\theta'_i - \theta_i = -dh_i$. Since this is also equal to $dg'_i - dg_i$, there exist constants $c_i$ such that $g'_i - g_i + h_i = c_i$ ($U_i$ is connected). If we now apply the definitions, we obtain without difficulty that $\pi b'_{ij} - \pi b_{ij} = \pi c_i - \pi c_j$, i.e., $\pi b' - \pi b = \pi b_0 c$ and thus $Y$ and $Y'$ determine the same cohomology class. We conclude that the map from equivalence classes of $D$-prequantum bundles to $H^1_{C}(M, R/D)$ is injective.

To finish the proof, let $b_{ij}$ be constants such that $\pi b$ is a 1-cocycle. We have to construct a $D$-prequantum bundle $(Y', \alpha')$ which determines this cocycle under the map from $D$-prequantum bundles to $H^1_{C}(M, R/D)$. The reference bundle is constructed with the transition functions $\pi f'_{ij}$. Since $\pi b$ is a cocycle, the functions $\pi f'_{ij}$ with $f'_{ij} = f'_{ij} + b_{ij}$ also satisfy the cocycle condition (8.5). Since the $b_{ij}$ are constants, the local 1-forms $\theta'_i + dx_i$ still glue together to form a global 1-form (connection) $\alpha$. We thus obtain a $D$-prequantum bundle $(Y, \alpha)$, and it is immediate that this bundle maps to the cohomology class $[\pi b]$. $\quad \Box$

Remarks. • If $D = \{0\}$, then necessarily $\text{Per}(\omega) = \{0\}$ (because it is contained in $D$) and thus part (ii) of [8.10] is a partial converse to part (i). However, it is possible that $\text{Per}(\omega) = \{0\}$ and that $Y$ is not a topologically trivial bundle, but this can happen only if $D$ is different from $\{0\}$ (see [TW] for an explicit example).

• Part (iii) of [8.10] can be interpreted in different ways. In the first place, $A_1$ is a vector space and thus a simple partition of unity argument shows that any principal fiber bundle with structure group $A_1$ is topologically trivial (see [Hi]). In the second place, any closed odd 2-form is exact, and thus, if we perform our construction with a single $U = M$, we directly obtain the direct product $M \times A_1$ with connection $\theta_1 + d\eta$ for the odd-part of the principal bundle.

• It is a standard result in algebraic topology that $H^1_{C}(M, R/D)$ is isomorphic to $\text{Hom}(\pi_1(M) \to R/D)$. With this result we can give another interpretation of [8.10](iv). If $(Y', \alpha')$ is a reference bundle, we denote by $(\overline{Y}', \overline{\pi})$ the $D$-prequantum bundle over the simply connected covering $\overline{M}$ of $M$ obtained by pull-back. It follows that $\pi_1(M)$ acts on $\overline{Y}'$ commuting with the right action of $A/D$, and that $Y' = \overline{Y}' / \pi_1(M)$. Now the various inequivalent $D$-prequantum
bundles can be obtained by the following procedure. For any homomorphism \( \phi : \pi_1(M) \to \mathbb{R}/D \) we define a modified action \( \Phi_\phi : \pi_1(M) \times \mathbb{Y}' \to \mathbb{Y}' \) of \( \pi_1(M) \) on \( \mathbb{Y}' \) by
\[
\Phi_\phi(g, \overline{y}) = \Phi(g, \overline{y}) \cdot \phi(g^{-1}) ,
\]
where \( \Phi \) denotes the previously mentioned action of \( \pi_1(M) \) on \( \mathbb{Y}' \). This is a (left) action because \( \phi \) is a homomorphism and because \( \Phi \) commutes with the \( \mathcal{A}/D \) action. Taking the quotient of \( (\mathbb{Y}', \overline{\pi}') \) with respect to this modified action gives us a \( D \)-prequantum bundle \((Y_\phi, \alpha_\phi)\). Varying the homomorphism \( \phi \) gives all inequivalent \( D \)-prequantum bundles.

Readers might have wondered why we introduced the subgroup \( D \subset \mathbb{R} \). The reason is simply to have a natural way to state the prequantization construction according to Souriau. According to Souriau, a prequantization of a symplectic manifold \((M, \omega)\) is a \( D \)-prequantum bundle \((Y, \alpha)\) with \( D = 2\pi \hbar \mathbb{Z} \). This is always possible if \( \text{Per}(\omega) = \{0\} \), it gives a quantization condition if \( \text{Per}(\omega) \) is discrete (if \( \text{Per}(\omega) = \lambda \mathbb{Z} \), the condition is \( \lambda \in 2\pi \hbar \mathbb{Z} \)), and it is never possible if \( \text{Per}(\omega) \) is dense in \( \mathbb{R} \).

After this discussion on \( D \)-prequantum bundles leading to prequantization in the sense of Souriau, we now come back to prequantization in the sense of Kostant. This means looking for a complex line bundle \( L \) over \( M \) with connection \( \nabla \) whose curvature is \(-i\omega/\hbar \). Unfortunately, I am unable to define a vector bundle over \( M \) with typical fiber the “complex line” \( \mathcal{A}^C = \mathcal{A} \oplus i\mathcal{A} = \mathcal{A} \otimes \mathbb{C} \) and linear connection \( \nabla \) whose curvature is the (mixed) symplectic form. The reason is that the curvature of a linear connection is necessarily even. However, there is an answer if we are slightly less demanding. A reasonable way to obtain a connection on a vector bundle is to start with a principle fiber bundle with a connection and to construct an associated vector bundle via a representation of the structure group. This is what is done in the ungraded case and which provides there the equivalence between the approaches of Souriau and Kostant. In the graded case a natural representation \( \rho \) of \( \mathcal{A}/D \) on \( \mathcal{A}^C \) is given by \( \rho(t, \tau) = e^{-2\pi it/d} \), where \( e^{i\tau} \) should be interpreted as the (even) automorphism of \( \mathcal{A}^C \) of multiplication by \( e^{i\tau} \). This representation is injective on \( \mathcal{A}/D \) but ignores the odd part of \( \mathcal{A}/D \). When one computes the curvature of the connection \( \nabla \) induced on the associated vector bundle by the connection \( \overline{\pi} \) on \( Y \), one finds
\[
\text{curvature}(\nabla) = \rho, \text{curvature}((\overline{\pi})) = \frac{-2\pi i}{d} \cdot \text{id}(\mathcal{A}^C) \cdot \omega_0 .
\]

As was to be expected, we only recover the even part of \( \omega \) in the curvature of the connection \( \nabla \). Note that the \( \mathcal{A} \)-vector space \( C \) is not involved: the curvature of \( \nabla \) is a 2-form with values in the endomorphisms of \( \mathcal{A}^C \), which is canonically isomorphic to \( \mathcal{A}^C \) (matrices of size \( 1 \times 1 \)). It also follows immediately that if we want this curvature to be (the even part of) \(-i\omega/\hbar \), we have to choose \( d = 2\pi h \), which gives in turn, via \( \text{Per}(\omega) \subset d\mathbb{Z} \), the quantization condition that a generator of \( \text{Per}(\omega) \) should be a multiple of \( 2\pi h \). These heuristic arguments can be made rigorous and give the following proposition.

8.12 Proposition. There exists an \( \mathcal{A}^C \)-line bundle \( L \) over \( M \) with connection \( \nabla \) whose curvature is \(-i\omega_0/\hbar \) if and only if \( \text{Per}(\omega) \subset 2\pi h \mathbb{Z} \). If that is the case, \( L \)
is the line bundle associated to a $D$-prequantum bundle $Y$ with $D \equiv 2\pi \hbar \mathbb{Z}$ by the representation $\rho(x, \xi) = e^{-ix/\hbar}$ of $D$ on $\mathcal{A}^C$.

Proof. • Suppose first that $\text{Per}(\omega) \subset D \equiv 2\pi \hbar \mathbb{Z}$. Then there exists a $D$-prequantum bundle $Y$. Using the representation $\rho : D \to \text{End}(\mathcal{A}^C)$, $\rho(x, \xi) = e^{-ix/\hbar}$, we then form the associated line bundle $L$ with the connection $\nabla$ induced from the connection $\mathcal{F}$ on $Y$. The curvature of this connection is given by the formula

$$\text{curvature}(\nabla) = \rho_* \text{curvature}(\mathcal{F}) = -i\omega_0/\hbar,$$

where we used $\rho_* \partial_x = -i/\hbar$ and $\rho_* \partial_\xi = 0$. This proves the if part and the second statement.

• Next we suppose that $(L, \nabla)$ exists and we use a cover $\mathcal{U}$ as in §8bis such that $L$ is trivial above each $U_i$. The standard (partition of unity) argument that the structure group of a vector bundle can be reduced to the orthogonal group applies here as well and shows that we may assume that the (even) transition functions $g_{ij} : U_i \cap U_j \to \text{Aut}(\mathcal{A}^C) \cong \{ x \in \mathcal{A}^C | \text{B} x \neq 0 \}$ are of the form $g_{ij}(m) = e^{-i\varphi_{ij}(m)}$ for some function $\varphi_{ij} : U_i \cap U_j \to \mathcal{A}^C_0$. Since we assume that $U_i \cap U_j$ is contractible, we also may assume that $\varphi_{ij}$ is smooth.

If $s$ is a (global) section of $L$, it is locally above $U_i$ represented by a function $s_i : U_i \to \mathcal{A}^C$. If $X$ is a global vector field, the covariant derivative $\nabla_X s$ is locally represented by the function $(\nabla_X s)_i$ given by the expression

$$(\nabla_X s)_i = Xs_i + \iota(X)\Gamma_is_i$$

for some even $\mathcal{A}^C$-valued 1-form $\Gamma_i$ on $U_i$. The curvature of the connection $\nabla$ is locally given by the $\mathcal{A}^C$-valued 2-form $d\Gamma_i$ (the group $\text{Aut}(\mathcal{A}^C)$ is abelian, so the term $\frac{1}{2} [\Gamma_i, \Gamma_i]$ in $d\Gamma_i = d\Gamma_i + \frac{1}{2} [\Gamma_i, \Gamma_i]$ vanishes). On the overlap of two trivializing charts $U_i$ and $U_j$ the local functions $s_i$ and $s_j$ are related by $s_j = s_i \cdot g_{ij}$ and similarly $(\nabla_X s)_j = (\nabla_X s)_i \cdot g_{ij}$. This gives us the relation

$$\Gamma_i = \Gamma_j - id\varphi_{ij},$$

because $g_{ij} = e^{-i\varphi_{ij}}$ is even and thus commutes with everything. Choosing a global potential $\theta_1$ for $\omega_1$, i.e., $d\theta_1 = \omega_1$, we introduce the 1-forms $\theta_i = \theta_1 + \theta_1$. We also introduce the functions $f_{ij} = \hbar \varphi_{ij}$. With these definitions we have on the one hand $d\theta_i = \omega$ (the curvature of $\nabla$ is $-i\omega_0/\hbar$) and on the other hand $\theta_i - \theta_j = df_{ij}$. Since the transition functions $g_{ij}$ satisfy the cocycle condition (8.5) (in multiplicative notation), it follows that $\varphi_{ij} + \varphi_{jk} + \varphi_{ki} \in 2\pi \mathbb{Z}$, and thus the constants $a_{ijk}$ constructed in §8bis are in $2\pi \hbar \mathbb{Z} \equiv D$. This implies directly that $\text{Per}(\omega) \subset D$ (we have seen that the odd part of $\omega$ does not contribute to the group of periods). And thus we have proved the only if part. Moreover, it is immediate from the above construction that we have a $D$-prequantum bundle whose transition functions $f_{ij}$ mod $D$ map under the representation $\rho$ exactly to the transition functions $g_{ij}$ of $L$. Hence $L$ can be seen as the $\mathcal{A}^C$-line bundle associated to this prequantum bundle. \[QED\]

We see that the weakened prequantization in the sense of Kostant exists under exactly the same conditions as prequantization in the sense of Souriau and that both are directly related. Some of the motivations that led to the introduction
of prequantization are given in the next section, as well as an argument that it is reasonable to weaken prequantization in the sense of Kostant for mixed symplectic forms.

Remark. Most texts on prequantization in terms of line bundles (e.g., [Ko1], [Sn], [Wo]) also talk about compatible inner products, something which has been ignored in this discussion. This additional structure has two purposes. It is used in the definition of equivalent line bundle prequantizations, and it is used in the definition of a scalar product on a space of functions. We skipped the first item because it will give us exactly the same classification as for $D$-prequantum bundles [8.10]. And the second item is beyond the scope of this paper.

9. Representations and invariant subspaces

Quantization and representation theory both are interested in (a kind of) irreducible representations. We start this section by investigating in more detail the representation aspect of prequantization in the sense of Souriau.

9.1 Definition. An infinitesimal symmetry of a $D$-prequantum bundle $(Y, \alpha)$ is a (smooth) vector field $Z$ on $Y$ preserving the connection $\pi$, i.e., the Lie derivative of $\pi$ in the direction of $Z$ is zero: $L(Z)\pi = 0$. This is equivalent to the condition that $Z$ preserves the homogeneous parts of $\alpha$: $L(Z)\alpha_0 = 0 = L(Z)\alpha_1$. The set of all infinitesimal symmetries of $(Y, \alpha)$ is denoted by $\text{Symm}(Y, \alpha)$; it is a subset of the set of all (smooth) vector fields on $Y$.

9.2 Proposition. Infinitesimal symmetries enjoy the following properties.

(i) $\text{Symm}(Y, \alpha)$ is a Lie algebra when equipped with the commutator of vector fields.

(ii) For each $f \in P$ there exists a unique $\eta_f \in \text{Symm}(Y, \alpha)$ such that $\iota(\eta_f)\pi = \pi^*f$.

(iii) The map $f \mapsto \eta_f$, $P \to \text{Symm}(Y, \alpha)$ is an isomorphism of $\mathbb{R}$-Lie algebras with the additional property that $\pi_*\eta_f = X_f$.

(iv) In a local chart $U_i \times A/D$ the vector field $\eta_f$ takes the form

\begin{equation}
\eta_f = X_f + \left(f^0 - \iota(X_f)(\theta_i)\right)\frac{\partial}{\partial x_i} + \left(f^1 - \iota(X_f)(\theta_i)\right)\frac{\partial}{\partial \xi_i}.
\end{equation}

Proof. • (i): Since $\pi$ is even, if $Z$ preserves $\pi$, its homogeneous parts $Z_0, Z_1$ do too. In order to show that the commutator of two infinitesimal symmetries is again an infinitesimal symmetry, we thus may assume that they are homogeneous. But then the result is obvious because for homogeneous vector fields $Z$ and $Z'$ we have $[L(Z), L(Z')] = L(Z) \circ L(Z') \pm L(Z') \circ L(Z)$ with the sign determined by the parities.

• (ii)–(iv): Let $Z \in \text{Symm}(Y, \alpha)$ be arbitrary. From the equation $L(Z)\pi = 0$ and the fact that $\pi^*\pi = \pi^*\pi$ we deduce that at each point $y \in Y$ we have

\begin{equation}
(d\iota(Z)\pi)|_y + \pi^*\iota(\pi_*Z|_y)\pi|_{\pi(y)} = 0.
\end{equation}

This implies that the derivatives of the function $\iota(Z)\pi$ in the direction of the fiber coordinates must be zero. Since the fibers are connected this implies that this function is independent of the fiber coordinates, and thus that there exists a (unique)
function \( f \in C^\infty(M, C) \) such that \( \iota(Z_{\overline{\alpha}}) = \pi^* f \). But then, using that \( \pi^* \) is injective, we have the equation
\[
\iota(\pi_* Z|_y)_{\overline{\alpha}(y)} + df|_{\pi(y)} = 0 .
\]
This implies that \( \pi_* Z|_y \) depends only upon \( \pi(y) \) (\( \omega \) is homogeneously non-degenerate), i.e., \( \pi_* Z \) is a well defined vector field on \( M \). But then we have the global equation \( \iota(\pi_* Z)|_\overline{\alpha} = -df \), which shows that \( f \) belongs to \( \mathcal{P} \) and that \( \pi_* Z = X_f \) is the associated hamiltonian vector field.

Now let \( Z' \in \text{Symm}(Y, \alpha) \) be another infinitesimal symmetry, with \( \iota(Z'|_{\overline{\alpha}}) = \pi^* f' \), \( f' \in \mathcal{P} \). We then compute:
\[
\iota([Z, Z'])_{\overline{\alpha}} = [\mathcal{L}(Z), \iota(Z')_{\overline{\alpha}}]_{\overline{\alpha}} = \mathcal{L}(Z)\iota(Z')_{\overline{\alpha}} - \sum_{\beta, \gamma=0}^1 (-1)^{\beta} \iota(Z'_\beta)\mathcal{L}(Z_\gamma)|_{\overline{\alpha}}
= \pi^*\mathcal{L}(\pi_* Z) f' = \pi^* X_f f' = \pi^* \{f, f'\} .
\]
This shows that the map \( Z \mapsto f \) from \( \text{Symm}(Y, \alpha) \) to \( \mathcal{P} \) is a morphism of \( \mathcal{A} \)-Lie algebras. Since \( \overline{\alpha} \) is even, this map is also even. It thus remains to prove that it is bijective. For \( f \in \mathcal{P} \) we thus have to find \( Z \in \text{Symm}(Y, \alpha) \) such that \( \iota(Z)|_{\overline{\alpha}} = \pi^* f \), which implies that \( \pi_* Z = X_f \). In a local trivializing chart \( U_i \times \mathcal{A}/D \) the looked for vector field \( Z \) thus must be of the form
\[
Z|_{(m, x_i)} = X_f|_m + g(m, x_i) \frac{\partial}{\partial x_i}|_{(m, x_i)} + \chi(m, x_i) \frac{\partial}{\partial \xi_i}|_{(m, x_i)} ,
\]
for some local functions \( g \) and \( \chi \). Since \( \overline{\alpha} \) has the local form \( \overline{\alpha} = \theta_i + d \xi_i \) (8.7), the condition \( \iota(Z)|_{\overline{\alpha}} = \pi^* f \) gives us
\[
g = f^0 - \iota(X_f)(\theta_i)_0 , \quad \chi = f^1 - \iota(X_f)(\theta_i)_1 .
\]
It follows that \( Z \) is uniquely determined on the local trivializing chart \( U_i \times \mathcal{A}/D \) by the equations \( \pi_* Z = X_f \) and \( \iota(Z)|_{\overline{\alpha}} = \pi^* f \). Since these equations are global, it follows that \( Z \) exists globally and is unique. These two equations also guarantee that \( Z \) belongs to \( \text{Symm}(Y, \alpha) \).

**Remark.** Since the fundamental vector fields associated to the right action of \( \mathcal{A}/D \) on \( Y \) reproduce the Lie algebra elements when contracted with the connection, it follows that they are infinitesimal symmetries. Moreover, one can deduce from the local expression [9.3] that they correspond to the constant functions in \( \mathcal{P} \). In other words, the vector fields \( \eta_f \), with \( f \) running through the constant functions in \( \mathcal{P} \), generate the action of the structure group \( \mathcal{A}/D \) on the principal fiber bundle \( Y \). If \( M \) is connected we thus can say that the kernel of the map \( \mathcal{P} \to \text{HSymm}(M, \omega) \) corresponds to the (infinitesimal) action of the structure group on \( Y \).

**9.4 Proposition.** Let \( G \) be a symmetry group of a connected symplectic \( \mathcal{A} \)-manifold \( (M, \omega) \), let \( \pi : \mathfrak{h} \to \mathfrak{g} \) be the central extension of the \( \mathcal{A} \)-Lie algebra \( \mathfrak{g} \) of \( G \) determined by \( \omega \) [5.1], and let \((Y, \alpha) \) be a \( D \)-prequantum bundle over \( M \) with projection \( \pi^Y : Y \to M \). Then there exists a momentum map for the \( G \)-action if and only if there exists a Lie algebra morphism \( H : \mathfrak{B} \mathfrak{h} \to \text{Symm}(Y, \alpha) \) compatible with
the $G$-action, i.e., each $H(V), V \in \mathfrak{b}\mathfrak{h}$ projects to the fundamental vector field of $\pi(V) \in \mathfrak{g} : \pi_Y^*(H(V)) = (\pi(V))^M$.

**Proof.** • Suppose first that the map $H$ exists. From [9.2](iii) we deduce that for each $V \in \mathfrak{b}\mathfrak{h}$ there exists $f \in \mathcal{P}$ such that $H(V) = \eta_f$ and thus $(\pi(V))^M = X_f$, i.e., $(\pi(V))^M$ is globally hamiltonian. Since the $\pi(V), V \in \mathfrak{b}\mathfrak{h}$ generate $\mathfrak{g}$, we have proven the existence of a momentum map.

• Now suppose that there exists a momentum map $J$. Combining [4.8] and [5.3] we see that the bracket on $\mathfrak{h} = \mathfrak{g} \times \mathbb{C}$ is given by

$$[[v, e], \{w, f\}] = \{[v, w], \iota(v, w)\Omega J\}.$$ 

We now define the map $\hat{H} : \mathfrak{b}\mathfrak{h} \to \mathcal{P}$ by $\hat{H}(v, e) = \langle v | \mathcal{J} \rangle + e$. Computing $\hat{H}([[v, e], \{w, f\}])$ we find:

$$\hat{H}([[v, e], \{w, f\}]) = \{\langle v, w \rangle | \mathcal{J} \rangle + \iota(v, w)\Omega J\}$$

$$= \{\langle v | \mathcal{J} \rangle, \langle w | \mathcal{J} \rangle\} = \{\hat{H}(v), \hat{H}(w)\},$$

where the last equality follows from the fact that constant functions have zero Poisson bracket. It follows that $\hat{H}$ is a Lie algebra morphism. Combining it with the isomorphism $\mathcal{P} \cong \text{Symm}(Y, \alpha)$ we obtain the desired result. QED

The results [9.2] and [9.4] give us a motivation for the introduction of a $D$-prequantum bundle for a symplectic manifold. It provides us with an injective representation of the Poisson algebra as vector fields on $Y$, contrary to the representation by hamiltonian vector fields on $M$ in which the (locally) constant functions disappear. Even better, it gives us an isomorphism between the Poisson algebra and the infinitesimal symmetries of the $D$-prequantum bundle. It also gives a nice interpretation of the existence of a momentum map: it exists if and only if the infinitesimal action of the central extension of $\mathfrak{g}$ determined by the symplectic form can be lifted to an infinitesimal action on the prequantum bundle $Y$.

In order to provide some motivation for prequantization in the sense of Kostant and our weakened version in case the symplectic form is not even, we will use the idea of quantization. Quantization is usually formulated as finding a representation of the Poisson algebra on some space of functions with additional requirements. In this paper we will forego the conditions concerning a Hilbert space structure and irreducibility, nor will we discuss the incompatibility between the various conditions. For this the interested reader is referred to the existing literature (e.g., [TW], [GGT] and references cited therein). We will focus on a representation $Q$ of the Poisson algebra $\mathcal{P}$ on a space of functions $\mathcal{E}$ with the additional condition that constant functions $r \cdot c_0 \in \mathcal{P}$ ($r \in \mathbb{R}$) are represented by $r \cdot id(\mathcal{E})$. To be more precise, we will look at even linear maps $Q : \mathcal{P} \to \text{End}(\mathcal{E})$ satisfying the representation condition

$$[Q(f), Q(g)] = -i\hbar Q(\{f, g\}).$$

For a symplectic manifold $(M, \omega)$ we have the obvious candidate $\mathcal{E} = C^\infty(M, \mathcal{A}_C)$ of $\mathcal{A}_C$-valued smooth functions on $M$ with $Q(f) = -i\hbar X_f$. But this does not fulfill the addition condition because the hamiltonian vector fields of constant functions are zero.
Given a $D$-prequantum bundle $Y$, we can improve upon this situation by taking $\mathcal{E} = C^\infty(Y,A^C)$ and $Q(f) = -i\hbar \eta_f$. Since $f \mapsto \eta_f$ is an injective representation, $Q$ is injective and no longer sends constant functions to the zero operator. In order to see whether $c_0 \in \mathcal{P}$ ($r \in \mathbb{R}$) is represented by $id(\mathcal{E})$, we recall that the vector field $\eta_{c_0}$ is the same as the fundamental vector field $\partial_x$ associated to the action of $\mathcal{A}_0/D$ on $Y$. For $Q(c_0)$ to act as the identity operator on a function $\phi \in C^\infty(Y)$, the function $\phi$ should be of the form

\begin{equation}
\phi(m,x+\xi) = e^{ix/h} \hat{\phi}(m,\xi)
\end{equation}

on a local trivializing chart for the bundle $Y$. Since $x$ is a coordinate modulo $d$, this implies that if we want the function $\hat{\phi}$ to be non-zero, then necessarily $d/h \in 2\pi \mathbb{Z}$. If this is the case, the condition that $\phi$ is of the form (9.5) can be stated as the condition

$$
\phi(y \cdot t) = e^{it/h} \phi(y) \quad y \in Y, \ t \in \mathcal{A}_0/D,
$$

where $y \cdot t$ denotes the right action of $t \in \mathcal{A}_0/D$ on $Y$. This description has the advantage that it is globally valid. Combining the local expression (9.5) with the local expression (9.3) we see that the subspace on which $Q(c_0)$ acts as the identity is invariant under the action of the vector fields $\eta_f$. We thus obtain the following result.

**9.6 Proposition.** There exists a non-zero subspace of $C^\infty(Y,A^C)$ on which $Q(c_0)$ acts as the identity if and only if $D \subset 2\pi \hbar \mathbb{Z}$ (which implies but is not equivalent to $\text{Per}(\omega) \subset 2\pi \hbar \mathbb{Z}$). If this condition is satisfied, the subspace in question is invariant under the representation $Q$ and can be described as

$$
\{ \phi \in C^\infty(Y,A^C) \mid \forall a, y : \phi(y \cdot a) = e^{ia/h} \phi(y) \}.
$$

If $\mathcal{A}_0/D$ were the whole structure group of the principal fiber bundle, this would describe exactly the sections of an associated vector bundle (associated to the representation $\rho(\hbar \mod d) = e^{-i\hbar/d}$ of $\mathcal{A}_0/D$ on $A^C$). However, our structure group has the additional factor $A_1$. Moreover, we have focused attention on the functions $r \cdot c_0 \in \mathcal{P}$, which are constant functions in the Poisson algebra. But they do not constitute all constant functions: a general constant function in $\mathcal{P}$ is of the form $r \cdot c_0 + s \cdot c_1$, $r, s \in \mathbb{R}$. Since $c_1$ is odd and $Q$ a representation and thus even, the result $Q(c_1)$, which is odd, can never be a non-zero multiple of the identity operator, which is even. I now arbitrarily decide that we want to represent $Q(c_1)$ as the zero operator. In order to motivate this decision, note that [8.10](iii) implies that

$$
C^\infty(Y,A^C) \cong C^\infty(Y_{[0]},A^C) \oplus C^\infty(Y_{[0]},A^C) = C^\infty(Y_{[0]},A^C)^2,
$$

in which the pair of functions $\phi^0, \phi^1 \in C^\infty(Y_{[0]},A^C)$ corresponds to the function $\phi \in C^\infty(Y,A^C)$ given by

$$
\phi(y,\xi) = \phi^0(y) + \xi \cdot \phi^1(y) \quad y \in Y_{[0]}.
$$

Moreover, it is immediate from (9.3) that the first factor $C^\infty(Y_{[0]},A^C) \cong \{ (\phi,0) \mid \phi \in C^\infty(Y_{[0]},A^C) \} \subset C^\infty(Y,A^C)$ of functions independent of $\xi$ is invariant under all operators $Q(f)$, $f \in \mathcal{P}$. This is also the largest subspace on which $Q(c_1)$ acts
as the zero operator because $Q(c_1) = i\hbar \partial_{\xi}$ maps $(\phi^0, \phi^1)$ to $(i\hbar \phi^1, 0)$. We thus see that $C^\infty(Y, A^C)$ naturally splits as a direct sum of two copies of $C^\infty(Y[0], A^C)$, one of which is the null space of $Q(c_1)$ which is invariant under the action of $\mathcal{P}$. If we think of quantization as a quest for an irreducible representation, the choice to look at this null space of $Q(c_1)$ becomes natural.

Since this null space is the space of functions independent of the global coordinate $\xi$, we can also describe it as

$$\{ \phi \in C^\infty(Y, A^C) \mid \forall \tau, y : \phi(y \cdot \tau) = \phi(y) \},$$

where $y \cdot \tau$ denotes the right action of $A_1 \subset A/D$ on $Y$. If we are interested in the intersection of this null space with the space on which $Q(c_0)$ acts as the identity, we can combine the two descriptions to obtain the function space $E = \{ Q(c_0) = id \} \cap \{ Q(c_1) = 0 \}:

$$E = \{ \phi \in C^\infty(Y, A^C) \mid \forall t, y : \phi(y \cdot t) = e^{it/\hbar} \phi(y) \}.$$

From this we see that $E$ is the set of smooth sections of the vector bundle $L$ with typical fiber $A^C$ associated to the principal fiber bundle $Y$ by the representation $\rho : A/D \to \mathrm{End}(A^C)$, $\rho(t) = e^{it/\hbar}$. And as said before, this representation makes sense only if $D \subset 2\pi \hbar \mathbb{Z}$. Since $Y$ has a connection $\nabla$, we have an induced connection $\nabla$ on $L$. Computing the curvature of $\nabla$, we find

$$\text{curvature}(\nabla) = \rho_* \text{curvature}(\nabla) = -i\omega_0/\hbar.$$

We see that this curvature is independent of the choice of $D$. But of course we have to choose $D$ such that $\text{Per}(\omega) \subset D \subset 2\pi \hbar \mathbb{Z}$. Choosing $D = 2\pi \hbar \mathbb{Z}$ imposes the least number of restrictions on $\text{Per}(\omega)$. We can summarize this as follows.

9.7 Proposition. There exists a non-zero subspace $E$ of $C^\infty(Y, A^C)$ on which $Q(c_0)$ acts as the identity and on which $Q(c_1)$ acts as the zero operator if and only if $\text{Per}(\omega) \subset 2\pi \hbar \mathbb{Z}$.

If this condition is satisfied, $E$ is invariant under the representation $Q$ and can be described as

$$E = \{ \phi \in C^\infty(Y[0], A^C) \mid \forall \ t \in A_0/D, y \in Y[0] : \phi(y \cdot t) = e^{it/\hbar} \phi(y) \},$$

i.e., $E$ is the set of smooth sections of the complex line bundle $L$ over $M$ associated to $Y$ by the representation $\rho(t) = e^{-it/\hbar}$. It can also be described as

$$E = \{ \phi \in C^\infty(Y[0], A^C) \mid \forall \ t \in A_0/D, y \in Y[0] : \phi(y \cdot t) = e^{it/\hbar} \phi(y) \}.$$

Said differently, $L$ is the complex line bundle over $M$ associated to $Y[0]$ by the representation $\rho(t) = e^{-it/\hbar}$ of $A_0/D$ on $A^C$. The induced linear connection $\nabla$ on $L$ has curvature $-i\omega_0/\hbar$. In other words, $L$ is the prequantization line bundle in the sense of Kostant. In terms of this line bundle the representation $Q$ takes the form

$$Q(f)s = -i\hbar \nabla_X \phi s + f^0 s,$$

for $f \in \mathcal{P}$ and $s$ a smooth section of $L$. 
Proof. The only thing that remains to be proven is the expression for \( Q(f) \) in terms of the line bundle. In the equivalence between sections of a line bundle and functions on the principal fiber bundle, the action of \( \nabla_X \) on a section corresponds to the action of \( \tilde{X} \) on the corresponding function with \( \tilde{X} \) being the horizontal lift of \( X \). It is immediate from (8.7) that \( \tilde{X}_f \) is (locally) given by

\[
\tilde{X}_f = X_f - \iota(X_f)(\theta_i)\partial_i - \iota(X_f)(\theta_i)\partial_i \xi_i,
\]

and thus

\[
\eta_f = \tilde{X}_f + f^0 \frac{\partial}{\partial x_i} + f^1 \frac{\partial}{\partial \xi_i}.
\]

Applying \(-ih\eta_f\) to an element of \( \mathcal{E} \) in the local form (9.5) and independent of \( \xi \) is the same as applying \(-ih\tilde{X}_f\) plus multiplication by \( f^0 \).

To summarize: we started with a \( D \)-prequantum bundle and we looked for a representation of the Poisson algebra satisfying certain conditions. Such a quest is quite natural in quantization. We ended up with the line bundle of prequantization in the sense of Kostant in its weaker form. The only unjustified point in our argument was the condition to look at the subspace on which \( Q(c_1) \) acts as the zero operator. Future research in unitary representations of \( \mathcal{A} \)-Lie groups and in quantization of symplectic \( \mathcal{A} \)-manifolds should determine whether this ad hoc assumption is the right thing to do. Note however that in the ungraded case (no \( \mathcal{A}_1 \), no \( c_1 \)) it is the subspace \( \mathcal{E} \) which is the starting point of the geometric quantization procedure/orbit method.

To end, let us look at the special cases in which the symplectic form is homogeneous (and the manifold connected). If \( \omega \) is even, we know that the Poisson algebra is (isomorphic to) \( C_\infty(M) \otimes \mathbb{R} \); more precisely, \( f = f^0 c_0 + f^1 c_1 : M \to C \), \( f^\alpha \in C_\infty(M) \) belongs to \( \mathcal{P} \) if and only if \( f^1 \) is constant (and thus real). Moreover, the subset \( C_\infty(M) \cong C_\infty(M) \oplus \{0\} \subset \mathcal{P} \) is a subalgebra. For this subalgebra the definition of hamiltonian vector field and Poisson bracket is exactly the classical ungraded definition: \( \omega \) is non-degenerate, \( \iota(X_f)\omega = -df \) and \( \{f,g\} = X_f g \). Any prequantum bundle \( Y \) is of the form \( Y_{[0]} \times \mathcal{A}_1 \) with a trivial connection \( d\xi \otimes c_1 \) on the \( \mathcal{A}_1 \) part [8,10]. And the representation \( Q \) on \( \mathcal{E} \) is an injective representation of \( C_\infty(M) \subset \mathcal{P} \) (it kills the additional part of functions \( r \cdot c_1, r \in \mathbb{R} \)). We see that in the even case we can completely forget about the \( c_1 \) part, it does not contain any relevant information. We also have equivalence between prequantization according to Souriau and Kostant. And when we forget about the \( c_1 \) part, the Poisson algebra becomes just \( C_\infty(M) \), of which \( Q \) is an injective representation.

If \( \omega \) is odd, the Poisson algebra is again (isomorphic to) \( \mathbb{R} \oplus C_\infty(M) \), but this time we have \( f = f^0 c_0 + f^1 c_1 \in C_\infty(M, C) \) belongs to \( \mathcal{P} \) if and only if \( f^0 \) is constant (and thus real). As in the even case, the subset \( C_\infty(M) \cong \{0\} \oplus C_\infty(M) \subset \mathcal{P} \) is a subalgebra. For this subalgebra the definition of hamiltonian vector field and Poisson bracket is again exactly the classical ungraded definition: \( \omega \) is non-degenerate, \( \iota(X_f)\omega = -df \) and \( \{f,g\} = X_f g \). The difference with the even case lies in parity. The even part \( \mathcal{P}_0 \) of the Poisson algebra is a subalgebra. Hence the intersection of \( \mathcal{P}_0 \) with the previously defined subalgebra \( C_\infty(M) \) is also a subalgebra. But since \( c_1 \) is odd we have \( \mathcal{P}_0 = \mathbb{R} \oplus C_\infty(M)_1 \) and \( \mathcal{P}_0 \cap C_\infty(M) = \)
Any prequantum bundle $Y$ is of the form $Y[0] \times A_1$, but now with $(Y[0], \alpha_0)$ a principal $A_0/D$-fiber bundle with connection whose curvature is zero (one could choose the trivial bundle $Y[0] = M \times A_0/D$ and $\alpha_0 = dx \otimes c_0$), and a non trivial connection $(\theta_1 + d\xi) \otimes c_1$ on the $A_1$ part \[8.10\]. The representation $Q$ kills the constant functions in $C^\infty(M)$, but it represents the constant functions $r \cdot c_0$, $r \in R$ as $r \cdot id_E$. The restriction of $Q$ to $C^\infty(M)$ is not injective, but its restriction to $\mathcal{P}_0$ (and thus a fortiori $C^\infty(M)_1$) is.

References