CENTRAL EXTENSIONS AND PHYSICS

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Abstract. In this paper two themes are considered; first of all we consider the question under what circumstances a central extension of the Lie algebra of a given Lie group determines a central extension of this Lie group (and how many different ones). The answer will be that if we give the algebra extension in the form of a left invariant closed 2-form ω on the Lie group, then there exists an associated group extension if and only if the group of periods of ω is a discrete subgroup of $\mathbf{R}$ and ω admits a momentum map for the left action of the group on itself.

The second theme concerns the process of prequantization; we show that the construction needed to answer the previous question is exactly the same as the construction of prequantum bundles in geometric quantization. Moreover we show that the formalism of prequantization over a symplectic manifold and the formalism of quantum mechanics (where the projective Hilbert spaces replaces the (symplectic) phase space) can be identified (modulo some "details" concerning infinite dimensions).

§1. Introduction and notation

Central extensions play an important role in quantum mechanics: one of the earlier encounters is by means of Wigner's theorem which states that a symmetry of a quantum mechanical system determines a (anti-) unitary transformation of the Hilbert space, which is unique up to a phase factor $e^{i\vartheta}$. As an immediate consequence of this phase factor, one deduces that given a quantum mechanical symmetry group G there exists an extension G' of G by $U(1)$ (the phase factors) which acts as a group of unitary transformations on the Hilbert space. In most cases physicists have been succesful in hiding these central extensions by using larger symmetry groups: $SO(3)$ gives rise to a nasty sign, so one uses its universal covering $SU(2)$ instead for which this problem with signs disappears (the extension is trivial); the connected component of the Lorentz group has the same problem, so one uses its universal covering $SL(2,\mathbf{C})$ for which this problem disappears. However, there are two important exceptions for which this trick does not work: for the translation group $\mathbf{R}^{2n}$ of translations in both position and momentum it is not possible to hide the phase factors: one obtains the Heisenberg group. The same problem occurs for the Galilei group: it is not a symmetry group of the (non relativistic) Schrödinger equation, but its central extension, the Bargmann group, is. Another area of physics where one encounters central extensions is the quantum theory of conserved currents of a Lagrangian. These currents span an algebra which is closely related to so called affine Kac-Moody algebras, which are the universal central extension of loop algebras. The central terms in this case are known as Schwinger

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terms. More recently, these affine Kac-Moody algebras and their associated groups also appear in the theory of strings.

The central extensions of Lie groups which occur in physics are often defined at the algebra level (e.g. by means of commutation relations). This raises the question whether such an algebra extension determines an extension of the Lie group. If we only have Lie algebras, then the question is easy to answer: according to Lie's third fundamental theorem to each Lie algebra there exists a global Lie group associated to this Lie algebra. However, if the group of the original algebra is fixed, then the answer becomes more complicated. The construction needed to answer this question is already described by É. Cartan in [Ca] where he gave a geometrical proof of Lie's third fundamental theorem. More recently W.T. van Est has memorated in [vE,1] this proof and its applications to the case of infinite dimensional algebras and Banach-Lie groups.

After the complete answer to the above question (existence of a Lie group extension of a given Lie group associated to an algebra extension) we will show that exactly the same construction is used in the process of prequantization (which is the first step in the procedure of geometric quantization [Ko], [Si&Wo], [So,1]), although it does not directly give the well known quantization condition: the quantization condition enters by means of conditions imposed by physics. Finally we discuss the relation between prequantization and central extensions of Lie groups (see also [Gu&St], [Si,1] and [Si,2]) and we show that the relation between prequantization and quantum mechanics is more than a formality: the Schrödinger equation on a Hilbert space can be interpreted as the canonical flow of a Hamiltonian function on a symplectic manifold (infinite dimensional); the Schrödinger equation itself is in this formalism the (unique) lift of the Hamiltonian vector field to the prequantization bundle.

We conclude this introduction with the remark that we do not claim to present essentially new results (all separate facts will be known by various specialists). The aim of this paper is to show a unified approach to various (seemingly unrelated) problems such as (1) does an algebra extension determine a group extension and (2) the quantization problem in physics. It is our hope that such a unified approach will lead to a better understanding of both classical mechanics and quantum mechanics and the connections between them.

We finish this section with some conventions about notation and language.

1. Except when stated otherwise explicitly we will always assume that (i) all manifolds (and especially all Lie groups) are connected, (ii) all manifolds, functions, vector fields and k-forms are smooth (i.e., C^∞).
2. Substitution of a vector field X in a k-form (giving a $(k-1)$-form) is denoted by $\iota(X)$ and the Lie derivative in the direction of X is denoted by $\mathcal{L}(X)$.
3. Except when it denotes the real number $3.1415\ldots$, the letter π will always denote a canonical projection between two spaces; it will be clear from the context which projection is meant.
4. Abelian groups will be denoted additively except the group $U(1) \subset \mathbf{C}$, which is denoted multiplicatively.
5. We assume summation convention: when an index is given twice, summation over this index is understood implicitly.

§2 An origin in physics: Wigner's theorem

In ordinary quantum mechanics a physical system is described by a Hilbert space $\mathcal{H}$ equipped with an inner product denoted by $\langle\ ,\ \rangle$. However, two non zero vectors of $\mathcal{H}$ which differ by a complex constant represent the same state of the system, so the set of states of the system can be described by the complex lines in $\mathcal{H}$, i.e. by the projective Hilbert space $\mathbf{P}\mathcal{H}$. In the projective Hilbert space one can define a structure $P(\ ,\ )$ (with values in $\mathbf{R}_{\geq 0}$) induced by the inner product on $\mathcal{H}$ as follows:

$$P(\pi x, \pi y) = \frac{\langle x, y\rangle \cdot \langle y, x\rangle}{\langle x, x\rangle \cdot \langle y, y\rangle} \qquad \text{for } x, y \in \mathcal{H} \setminus \{0\}\ ,$$

where π denotes the canonical projection $\pi : \mathcal{H} \setminus \{0\} \to \mathbf{P}\mathcal{H}$; $P(\ ,\ )$ is often interpreted as the transition probability between two states.

With these definitions one then defines a symmetry transformation g as a bijection of $\mathbf{P}\mathcal{H}$ to itself which leaves the structure P invariant: $\forall x, y \in \mathbf{P}\mathcal{H}$: $P(gx, gy) = P(x, y)$. According to a theorem due to Wigner such a symmetry transformation can be lifted to a transformation of the Hilbert space $\mathcal{H}$ itself.

Theorem 2.1 (Wigner). *Suppose g is a symmetry of $\mathbf{P}\mathcal{H}$, then there exists either a unitary or an anti-unitary operator $U(g)$ on $\mathcal{H}$ such that the action of g is induced by the action of $U(g)$:*

$$\forall x \in \mathbf{P}\mathcal{H} : gx = \pi\, U(g)\, \pi^{-1} x\ .$$

Moreover, the operator $U(g)$ is unique up to a phase factor $\mathrm{e}^{i\vartheta}$.

For a proof we refer to [Bau] and [Barg,1]. It follows from this theorem that if we have a connected Lie group G of symmetry transformations then each element $g \in G$ will be represented by a *unitary* operator on $\mathcal{H}$ (in a neighbourhood of the origin each element g can be written as the square of another element $g = h^2$, $U(h)^2$ is an (anti-) unitary lift of g ($\Rightarrow U(g) = U(h)^2\,\mathrm{e}^{i\vartheta}$), the square of an anti unitary transformation is unitary and each neighbourhood of the origin generates the whole group if it is connected).

Corollary 2.2. *Suppose G is a connected Lie group of symmetry transformations on $\mathbf{P}\mathcal{H}$, choose for every $g \in G$ a unitary transformation $U(g)$ on $\mathcal{H}$, then there exists a function $\varphi : G \times G \to U(1)$ such that:*

$$U(gh) = U(g)\, U(h)\ \varphi(g, h)\ , \tag{2.1}$$

and by associativity of group multiplication this φ satisfies

$$\varphi(g, h)\, \varphi(gh, k) = \varphi(g, hk)\, \varphi(h, k)\ . \tag{2.2}$$

If the $U(g)$ can be chosen such that the function φ is constant 1 then it follows from formula (2.1) that U is a unitary representation of G on $\mathcal{H}$; in the general case (φ not constant 1) one calls the representation U of G as unitary transformations on $\mathcal{H}$ a "projective representation," "ray representation" or "representation up to a phase factor" (because of the phase factor φ appearing in (2.1)). In the next section a detailed discussion will be given concerning the freedom in φ and its implications; at the moment we will be content with some ad hoc constructions.

With the aid of the function φ one can construct a unitary representation of a central extension G' of G as follows. Define $G' = G \times U(1)$ as *set* with multiplication:

$$(g, \zeta) \cdot (h, \eta) = (gh, \zeta \cdot \eta \cdot \varphi(g, h)^{-1}) \ .$$

It follows from relation (2.2) that this is a well defined group multiplication, and we now can define a *unitary* representation of G' on $\mathcal{H}$ by

$$(g, \zeta)x = \zeta \cdot U(g)x \qquad \text{for } x \in \mathcal{H} \ ,$$

where $U(g)$ is the specific choice of the unitary lift of g to $\mathcal{H}$ used in the construction of φ. The group G' is a central extension of G in the sense that the kernel of the canonical projection $\pi : G' \to G$ is a central subgroup of G' (isomorphic to $U(1)$).

Since G' is represented unitary on $\mathcal{H}$, it leaves the innerproduct $\langle \ , \ \rangle$ and the unit sphere $\mathbf{S}\mathcal{H} = \{x \in \mathcal{H} \mid \langle x, x\rangle = 1\} \subset \mathcal{H}$ invariant, i.e., G' acts on $\mathbf{S}\mathcal{H}$ leaving $\langle \ , \ \rangle$ invariant. In the sequel we will often encounter such a situation in which a group acts on a set, leaving some structure invariant; we will summarise such a statement by saying that the group is a *symmetry group* of the pair (set, structure). With this convention we can summarise the above result: if a connected Lie group G is a symmetry group of $(\mathbf{P}\mathcal{H}, P)$, then there exists a central extension G' of G by $U(1)$ which is a symmetry group of $(\mathbf{S}\mathcal{H}, \langle \ , \ \rangle)$. If we furthermore depict the action of a group on a set by a wiggling arrow $\rightsquigarrow$ then this result can be summarised in the following commutative diagram:

$$\begin{array}{ccc} G' & \rightsquigarrow & (\mathbf{S}\mathcal{H}, \langle \ , \ \rangle) \\ \big\downarrow & & \big\downarrow \\ G & \rightsquigarrow & (\mathbf{P}\mathcal{H}, P) \ . \end{array} \tag{2.3}$$

Remark 2.3. The fibres of the projection $\pi : \mathbf{S}\mathcal{H} \to \mathbf{P}\mathcal{H}$ are all circles and the action of $U(1)$ on $\mathbf{S}\mathcal{H}$ leaves the fibres invariant. Moreover, $U(1)$ acts transitively without fixed points on these fibres, hence $\mathbf{S}\mathcal{H}$ is a pricipal fibre bundle over $\mathbf{P}\mathcal{H}$ with structure group $U(1)$; the central subgroup $U(1)$ of G' behaves as if it were the structure group.

In physics one often asks the question whether the function φ in (2.1) is constant 1, i.e., whether G can be represented as symmetry group of $(\mathbf{S}\mathcal{H}, \langle \ , \ \rangle)$ rather than G' (N.B. an equivalent formulation of the above question is: when is G' isomorphic to $G \times U(1)$ as a group, not only as a set). One can show that if G is a semi simple and simply connected, connected Lie group, then the operators $U(g)$ can be chosen such that $\varphi \equiv 1$, i.e., in such a case G can be represented as a symmetry group of $(\mathbf{S}\mathcal{H}, \langle \ , \ \rangle)$. If one of these two conditions is not fulfilled then in general there is no choice for the $U(g)$ such that $\varphi \equiv 1$. The various cases do occur in physics: (a) the group $SO(3)$ of rotations is semi simple but not simply connected: the operators $U(g)$ can be chosen such that $\varphi(g, h) = \pm 1$. (b) The universal covering group $SU(2)$ of $SO(3)$ is semi simple and simply connected: it is a symmety group of $(\mathbf{S}\mathcal{H}, \langle \ , \ \rangle)$. (c) The restricted Lorentz group L is semi simple but not simply connected: also in this case the $U(g)$ can be chosen such that $\varphi(g, h) = \pm 1$. (d) The double covering $SL(2, \mathbf{C})$ of L is semi simple and simply connected: $SL(2, \mathbf{C})$ is a symmetry group of $(\mathbf{S}\mathcal{H}, \langle \ , \ \rangle)$. (e) The group $\mathbf{R}^{2n}$

acting as translations in position and momentum of a physical system is simply connected but not semi simple: the central extension G' is the Heisenberg group. (f) The Galilei group occurring in non-relativistic mechanics is neither semi simple nor simply connected: the central extension G' is the Bargmann group. A more detailed account of some of these examples can be found in [Barg,2], where the emphasis is on simply connected Lie groups.

Apart from the question whether the $U(g)$ can be chosen such that $\varphi \equiv 1$, another question is even more important: is the extension G' of G a Lie group if G is? The problem is whether the multiplication in G' is smooth; if φ is smooth then the multiplication is smooth, but is this a necessary condition? In physics the groups G' turn out to be Lie groups; the underlying idea is that in physics one constructs a central extension of the corresponding Lie algebra rather than of the group; G' then is a Lie group associated with this Lie algebra extension. This raises immediately the question under what circumstances can an extension of a Lie algebra be "extended" to an extension of the corresponding group? In the next section this question will be discussed in more detail.

§3 The mathematical problem: Central extensions of (Lie) groups and Lie algebras

In this section we will discuss the relation between Lie algebra extensions and Lie group extensions. Therefore we start with the definition of a central extension of an abstract group, together with a classification of the possible different extensions. Then we will do the same for Lie algebras. To obtain the classification of the inequivalent central extensions we need the basic definitions of group and Lie algebra cohomology, so we supply them. Finally we will specialise central extensions of groups to central extensions of Lie groups and we will discuss the connection between group and algebra extension. We will finish with the main question whether an algebra extension determines a group extension. The answer to this question will be given in §5. More information on the use of group cohomology can be found in [St]. For simply connected Lie groups the connection between Lie algebra central extensions and Lie group central extension can be found in [Barg,2], [Si,1] and [Si,2].

§3.1 Central extensions of abstract groups

Definition 3.1. Let G be a group and A an abelian group. A group G' is called a *central extension* of G by A if

(i) A is (isomorphic to) a subgroup of the center of G' and
(ii) G is isomorphic to G'/A.

Two central extensions G'_1 and G'_2 of G by A will be called *equivalent* if there exists an isomorphism $\Phi : G'_1 \to G'_2$ such that

(i) $\pi_2 \circ \Phi = \pi_1$ where π_i is the canonical projection of G'_i to $G \cong G'_i/A$, and
(ii) Φ is the identity on the subgroup A : $\Phi(g'\,a) = \Phi(g')\,a$ for any g' in G'_1.

Remark 3.2. When we use the language of exact sequences (the image of one map is the kernel of the next), a central extension G' of G by A is given by a short exact sequence

$$\{0\} \to A \to G' \to G \to \{1\}$$

of group homomorphisms with the restriction that $\mathrm{im}(A) \subset \mathrm{center}(G')$. An equivalence of central extensions then is a commutative diagram

$$\begin{array}{ccccccccc} & & & & G'_1 & & & & \\ & & & \nearrow & & \searrow & & & \\ \{0\} & \to & A & & \updownarrow & & G & \to & \{1\} \\ & & & \searrow & & \nearrow & & & \\ & & & & G'_2 & & & & \end{array}$$

in which both upper and lower exact sequences represent the central extensions.

Definition 3.3 (group cohomology). Let G be a group, then for any $k \in \mathbf{N}$ denote by $C^k(G, A) = \{\varphi : G^k \to A\}$ the set of k-cochains on G with values in the abelian group A, which forms an abelian group under pointwise addition of functions. One defines the coboundary operator $\delta_k : C^k \to C^{k+1}$ by

$$(3.1)\quad (\delta_k\varphi)(g_1, \ldots, g_{k+1}) = \varphi(g_2, \ldots, g_{k+1}) + \sum_{j=1}^{k} (-1)^j \varphi(g_1, \ldots, g_{j-1}, g_j \cdot g_{j+1}, g_{j+2}, \ldots, g_{k+1}) + (-1)^{k+1}\varphi(g_1, \ldots, g_k)$$

and one calls $Z^k(G, A) = \ker(\delta_k)$ the set of k-cocycles and $B^k(G, A) = \mathrm{im}(\delta_{k-1})$ the set of k-coboudaries. It is easy to verify that $\delta_k \circ \delta_{k-1} = 0$, hence $H^k_{gr}(G, A) = \ker(\delta_k)/\mathrm{im}(\delta_{k-1})$ is a well defined abelian group called the k-th cohomology group of G with values in A (the subscript "gr" denotes "group" to distinguish these cohomology groups from others).

Proposition 3.4. *The inequivalent central extensions of a group G by A are classified by $H^2_{gr}(G, A)$.*

Proof. Let G' be a central extension of G by A, denote by π the canonical projection from G' to G and let $s : G \to G'$ be a section of G' i.e., a map satisfying $\pi \circ s = id(G)$. It follows that for any $g, h \in G$ there exists an element $\varphi(g, h) \in A$ such that:

$$(3.2)\qquad s(gh) = s(g)s(h)\varphi(g, h)$$

(φ measures the extent in which the section s fails to be a homomorphism). Since multiplication in G' is associative it follows that the map $\varphi : G \times G \to A$ is a 2-cocycle on G with values in A. If s' is another section then there exists a function $\chi : G \to A$ such that $s'(g) = \chi(g)s(g)$; the associated 2-cocycle φ' is related to φ by $\varphi' = \varphi - \delta_1\chi$. We conclude that the extension G' determines a unique cohomology class in $H^2_{gr}(G, A)$.

Conversely let φ be a 2-cocycle on G with values in A representing a particular cohomology class, then we define an extension G' by:

$$(3.3)\qquad G' = G \times A \quad \text{with multiplication} \quad (g, a)(h, b) = (gh, a + b - \varphi(g, h)) .$$

One verifies easily the following facts:

(i) the condition φ a cocycle implies that (a) $\varphi(e, e) = \varphi(e, g) = \varphi(g, e)$ for all g in G where e denotes the identity in G, (b) $(e, \varphi(e, e))$ is the identity of G' and (c) this multiplication is associative;

(ii) $\pi : (g, a) \mapsto g$ defines G' as a central extension of G by A; the canonical injection $A \to G'$ is given by $a \mapsto (e, a + \varphi(e, e))$.

By choosing the section $s(g) = (g, 0)$ one shows that this extension determines a cohomology class represented by φ, showing that this construction is a left inverse to the map "extensions $\to$ cohomology classes" constructed above.

Finally let φ_1 and φ_2 be two 2-cocycles with associated extensions G'_1 and G'_2 and suppose Φ is an equivalence between them. By definition of equivalence Φ must be of the form $\Phi(g, a) = (g, a + \chi(g))$ and one verifies easily that "Φ a group homomorphism" implies that $\varphi_2 = \varphi_1 + \delta_1\chi$, proving the proposition. $\boxed{QED}$

Corollary 3.5. *If G is a connected Lie group of symmetries of $\mathbf{P}\mathcal{H}$ then the map φ of corollary 1.2 is a 2-cocycle and the associated cohomology class $[\varphi] \in H^2_{gr}(G, U(1))$ depends only on the action of G on $\mathbf{P}\mathcal{H}$, not on the specific choice of the unitary maps $U(g)$ on the Hilbert space $\mathcal{H}$.*

§3.2 Central extensions of Lie algebras

Definition 3.6. Let $\mathfrak{g}$ be a Lie algebra and $\mathbf{R}^p$ an abelian Lie algebra. A Lie algebra $\mathfrak{g}'$ is called a *central extension* of $\mathfrak{g}$ by $\mathbf{R}^p$ if

(i) $\mathbf{R}^p$ is (isomorphic to) an ideal contained in the center of $\mathfrak{g}'$ and
(ii) $\mathfrak{g}$ is isomorphic to $\mathfrak{g}'/\mathbf{R}^p$.

Two extensions will be called *equivalent* if there exists an isomorphism (of Lie algebras) which is the identity on the ideal $\mathbf{R}^p$ and which is compatible with the "projections" to $\mathfrak{g}$.

Remark 3.7. In the language of exact sequences a central extension $\mathfrak{g}'$ of $\mathfrak{g}$ by $\mathbf{R}^p$ is given by a short exact sequence

$$\{0\} \to \mathbf{R}^p \to \mathfrak{g}' \to \mathfrak{g} \to \{0\}$$

of Lie algebra homomorphisms with the restriction that $\mathrm{im}(\mathbf{R}^p) \subset \mathrm{center}(\mathfrak{g}')$. An equivalence of central extensions then is a commutative diagram

$$\begin{array}{ccccccc} & & & \mathfrak{g}'_1 & & & \\ & & \nearrow & \updownarrow & \searrow & & \\ \{0\} & \to & \mathbf{R}^p & & G & \to & \{0\} \\ & & \searrow & & \nearrow & & \\ & & & \mathfrak{g}'_2 & & & \end{array}$$

in which both upper and lower exact sequences represent the central extensions.

Definition 3.8 (Lie algebra cohomology). Let $\mathfrak{g}$ be a Lie algebra, then for any $k \in \mathbf{N}$ denote by $C^k(\mathfrak{g}, \mathbf{R}^p) = \{\omega : \mathfrak{g}^k \to \mathbf{R}^p \mid \omega \text{ is } k\text{-linear and antisymmetric}\}$ the set of k-cochains on $\mathfrak{g}$ with values in the abelian Lie algebra $\mathbf{R}^p$, which forms a vector space in the obvious way. One defines the coboundary operator $\delta_k : C^k \to C^{k+1}$ by:

$$(3.4) \quad (\delta_k\omega)(X_1, \ldots, X_{k+1}) = \sum_{1 \le i < j \le k+1} (-1)^{i+j}\omega([X_i, X_j], X1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{k+1})$$

and one calls $Z^k(\mathfrak{g}, \mathbf{R}^p) = \ker(\delta_k)$ the set of k-cocycles and $B^k(\mathfrak{g}, \mathbf{R}^p) = \mathrm{im}(\delta_{k-1})$ the set of k-coboudaries. One can verify that $\delta_k \circ \delta_{k-1} = 0$, hence $H^k_{al}(\mathfrak{g}, \mathbf{R}^p) = \ker(\delta_k)/\mathrm{im}(\delta_{k-1})$ is a well defined vector space called the k-th cohomology group of $\mathfrak{g}$ with values in $\mathbf{R}^p$ (the subscript "al" denotes "algebra" to stress that it concerns Lie algebra cohomology).

Remark 3.9. In formula (3.4) one recognises (a part of) the formula for the exterior derivative of a k-form and indeed if G is a Lie group with Lie algebra $\mathfrak{g}$, if ω is a left invariant k-form with values in $\mathbf{R}^p$ (i.e., p separate left invariant k-forms) and if $X_1, \ldots, X_{k+1}$ are $k+1$ left invariant vector fields on G (where we identify left invariant vector fields with elements of the Lie algebra $\mathfrak{g}$) then δ_k is just the exterior derivative on these p k-forms. It follows that the Lie algebra cohomology groups are isomorphic to the de Rham cohomology of left invariant forms with values in $\mathbf{R}^p$: $H^k_{al}(\mathfrak{g}, \mathbf{R}^p) \cong H^k_{li,dR}(G, \mathbf{R}^p) \cong [H^k_{li,dR}(G, \mathbf{R})]^p$.

Proposition 3.10. *The inequivalent central extensions of a Lie algebra* $\mathfrak{g}$ *by* $\mathbf{R}^p$ *are classified by* $H^2_{al}(\mathfrak{g}, \mathbf{R}^p)$.

Proof. Let $\mathfrak{g}'$ be such an extension, $\pi : \mathfrak{g}' \to \mathfrak{g}$ the canonical projection and $\sigma : \mathfrak{g} \to \mathfrak{g}'$ a section of $\mathfrak{g}'$, i.e., a linear map satisfying $\pi \circ \sigma = id(\mathfrak{g})$. It follows that there exists a map ω from $\mathfrak{g} \times \mathfrak{g}$ to $\mathbf{R}^p \subset \mathfrak{g}'$ such that:

$$\sigma([X,Y]) = [\sigma(X), \sigma(Y)] + \omega(X,Y) \ , \tag{3.5}$$

and from the Jacobi identity in $\mathfrak{g}'$ one deduces that ω is a 2-cocycle. The rest of the proof follows the same pattern as the proof of proposition 3.4, so we leave it to the reader. QED

§3.3 Central extensions of Lie groups and the associated extensions of their Lie algebras

In this subsection we will always work with the following situation: a central extension of groups $\{0\} \to A \to G' \to G \to \{1\}$, in which A, G' and G are connected Lie groups and the arrows Lie group homomorphisms (i.e. C^∞ group homomorphisms); in particular $\pi : G' \to G$ denotes the projection. In such a case we will say that G' is a Lie group (central) extension of G by A. Since A is a connected abelian Lie group, its universal covering group is (isomorphic to) $\mathbf{R}^p$ with addition of vectors. It follows that a neigbourhood of the "identity" in A is isomorphic to an open subset of $\mathbf{R}^p$ containing 0, so we have a canonical way to identify the Lie algebra of A (i.e., the tangent space at the identity) with $\mathbf{R}^p$. The Lie algebras of G' and G will be denoted by $\mathfrak{g}'$ and $\mathfrak{g}$. The map $\pi : G' \to G$ induces a map at the algebra level $\pi_* : \mathfrak{g}' \to \mathfrak{g}$ which is readily seen to be a central extension of $\mathfrak{g}$ by $\mathbf{R}^p$ (which is the algebra of A). To facilitate the discussions we introduce the following notations: $\mathrm{Ext}(G,A)$ will denote the set of all inequivalent Lie group central extensions of G by A and $\mathrm{ext}(\mathfrak{g}, \mathbf{R}^p)$ will denote the set of all inequivalent Lie algebra central extensions of $\mathfrak{g}$ by $\mathbf{R}^p$. In the previous section we have seen that $\mathrm{ext}(\mathfrak{g}, \mathbf{R}^p)$ is isomorphic to $H^2_{al}(\mathfrak{g}, \mathbf{R}^p)$. More facts about extensions of Lie groups (not only central ones) can be found in [Sh] and [Ho].

What we now *want* to do is to establish a map Δ between $H^2_{gr}(G,A)$ and $H^2_{al}(\mathfrak{g}, \mathbf{R}^p)$ which associates to the cohomology class representing the extension G' of G the cohomology class representing the extension $\mathfrak{g}'$ of $\mathfrak{g}$. However, there are some difficulties related to the fact that we have to do with Lie groups which are in particular smooth manifolds. If G is a Lie group and φ a 2-cocycle with values in A then we can not guarantee that the group G' defined in formula (3.3) is a Lie group. If this cocycle φ is a smooth function then we are sure that G' is indeed a Lie group (and moreover a Lie group extension by A), which suggests that for Lie groups we have to restrict ourselves in the definition of group cohomology to smooth functions. The cohomology groups derived in this way (using functions $\varphi : G^k \to A$ that are smooth) will be denoted by $H^k_{s,gr}(G, A)$, the subscript "s" denoting "smooth." However, in the general case $H^2_{s,gr}(G, A)$ does not classify all central extensions of G by A, but only those which admit a global smooth section (see §5). The appropriate cohomology theory in which $H^2_{?,gr}(G, A)$ does classify the Lie group central extensions, is the one based on e-smooth cochains, i.e., cochains $\varphi : G^k \to A$ which are smooth in a neighbourhood of the identity $e^k \in G^k$; the resulting cohomology groups will be denoted by $H^2_{es,gr}(G, A)$.

Proposition 3.11. $\mathrm{Ext}(G, A)$ *is "isomorphic" to* $H^2_{es,gr}(G, A)$.

Proof (sketch). If $G' \to G$ is a central extension, then G' is in particular a locally trivial principal fibre bundle over G with structure group A, hence there exists a section $s : G \to G'$ which is smooth in a neighbourhood of e. Consequently the associated 2-cocycle is smooth around $e^2 \in G^2$, hence defines an element of $H^2_{es,gr}(G, A)$.

Conversely if φ is a representative of an element of $H^2_{es,gr}(G, A)$, then φ is smooth around the identity. Using the facts that G and A are topological groups and that φ is continuous around the identity, it is easy to construct a fundamental system of neighbourhoods of the identity $e' \in G' = G \times A$ (the group constructed in formula (3.3)) which turns G' into a topological group. Then using the facts that G and A are Lie groups and that φ is smooth around the identity one shows that G' admits the structure of a Lie group (lemma 2.6.1 of [Va]) for which $\pi : G' \to G$ is a Lie group morphism. *QED*

Remark 3.12. There exist canonically defined maps

$$H^k_{s,gr}(G, A) \to H^k_{es,gr}(G, A) \to H^k_{gr}(G, A) \ ,$$

which associate to a cocycle representing a cohomology class the cohomology class of the same cocycle in the cohomology group with less structure; what we will show in this section is that there exists a map $\Delta : H^2_{es,gr}(G, A) \to H^2_{al}(\mathfrak{g}, \mathbf{R}^p)$ with the property that if $[\varphi]$ represents the extension $\pi : G' \to G$ then $\Delta[\varphi]$ represents the extension $\pi_* : \mathfrak{g}' \to \mathfrak{g}$. We will see in §5 that the map Δ is in general neither surjective nor injective.

Construction 3.13. To define this map we proceed as follows: choose p left invariant vector fields $Y_1, \ldots, Y_p$ on G' which form a basis for the Lie algebra of A; then choose n left invariant vector fields $\Xi_1, \ldots, \Xi_n$ on G' such that $Y_1, \ldots, Y_p, \Xi_1, \ldots, \Xi_n$ is a basis of the Lie algebra $\mathfrak{g}'$ of G'. It follows that the left invariant vectors $X_i = \pi_* \Xi_i$ on G form a basis of the Lie algebra $\mathfrak{g}$ of G. Finally denote by $\beta^1, \ldots, \beta^n$ the left invariant 1-forms on G which are dual to the basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ and denote by $\alpha^1, \ldots, \alpha^p$ the left invariant 1-forms on G' such that $\alpha^1, \ldots, \alpha^p, \pi^*\beta^1, \ldots, \pi^*\beta^n$ are dual to the basis $Y_1, \ldots, Y_p, \Xi_1, \ldots, \Xi_n$, especially $\alpha^i(Y_j) = \delta^i_j$. Since A is contained in the center of G' we have $[Y_i, Y_j] = 0 = [Y_i, \Xi_j]$ and there exist constants c^k_{ij} and d^r_{ij} such that:

$$[\Xi_i, \Xi_j] = c^k_{ij}\, \Xi_k + d^r_{ij}\, Y_r \ . \tag{3.6}$$

It follows immediately that $[X_i, X_j] = c^k_{ij}\, X_k$. By duality one derives the following identities:

$$\mathrm{d}\alpha^r = -\tfrac{1}{2}\, d^r_{ij}\, \pi^*\beta^i \wedge \pi^*\beta^j \qquad \text{(full summation)} \ , \tag{3.7}$$

from which one deduces that $d\alpha^r$ is the pull back of the closed 2-form $\omega^r = -\frac{1}{2}\, d^r_{ij}\, \beta^i \wedge \beta^j$ on G. The map $\sigma(X_i) = \Xi_i$ from $\mathfrak{g}$ to $\mathfrak{g}'$ is a section of the Lie algebra extension $\pi_* : \mathfrak{g}' \to \mathfrak{g}$, and it follows easily from (3.6) and (3.7) that the 2-cocycle ω with values in $\mathbf{R}^p$ associated to this section according to proposition 3.10 is given by $\omega = (\omega^r) \cong \omega_r \otimes Y_r$.

What follows at this point is not directly relevant for the present discussion, but it is appropriate to mention it here. Suppose G is a Lie group, M a manifold, $\Phi : G \times M \to M$ a C^∞ map and denote $\Phi(g,m)$ by $\Phi(g)(m)$, i.e., $\Phi(g) : M \to M$ is a C^∞ map. The map Φ is called a left action of G on M if it satisfies $\Phi(gh) = \Phi(g) \circ \Phi(h)$; it is called a right action if it satisfies $\Phi(gh) = \Phi(h) \circ \Phi(g)$.

Definition 3.14. If Φ is an action (left or right) of G on M and if $X \in \mathfrak{g}$, we define the *fundamental vector field* X_M on M as the vector field associated to the flow $\Phi(\exp(Xt))$ on M.

In the case of a left action we have $\Phi(g)_* X_M = \big(\mathrm{Ad}(g)X\big)_M$; for a right action this formula becomes $\Phi(g)_* X_M = \big(\mathrm{Ad}(g-1)X\big)_M$. We now assume that the reader is familiar with the definition of a principal fibre bundle over a manifold; we recall the definition of a connection.

Definition 3.15. A connection 1-form α on a principal fibre bundle P over M with structure group G (in particular $\Phi : P \times G \to P$ denotes the (right) action of G on P) is a 1-form with values in the Lie algebra $\mathfrak{g}$ of G satisfying:

(i) $\forall X \in \mathfrak{g} : \alpha(X_P) = X$ and
(ii) $\forall g \in G : \Phi(g)^* \alpha = \mathrm{Ad}(g^{-1}) \circ \alpha$.

Proposition 3.16. *The* 1*-form* α *on* G' *with values in the Lie algebra of* $A \cong \mathbf{R}^p$ *defined by* $\alpha = (\alpha^r) \cong \alpha^r \otimes Y_r$ *is a connection* 1*-form on the principal fibre bundle* G' *over* G *with structure group* A.

Construction 3.17. We now proceed with the main discussion of this subsection. Let s be an arbitrary section of the extension $\pi : G' \to G$ by A which is smooth around the identity and let φ be the associated e-smooth 2-cocycle, then formula (3.3) defines an isomorphism between G' and $G \times A$ which is smooth around the identity. Hence this isomorphism defines an identification between $\mathfrak{g}'$ and $\mathfrak{g} \times \mathbf{R}^p$ identified with the tangent spaces in the identities. Using this identification, an elementary calculation shows that for $(X,v),(Y,w) \in \mathfrak{g}'$ with $X, Y? \in \mathfrak{g}$ and $v, w \in \mathbf{R}^p$ the commutator is given by:

$$[\,(X,v),(Y,w)\,] = ([X,Y], \Omega(X,Y))$$

with

$$\Omega(X,Y) = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \Big(\varphi\big(\exp(Xt), \exp(Ys)\big) - \varphi\big(\exp(Ys), \exp(Xt)\big)\Big) \tag{3.8}$$

(N.B. Here we treat the function φ as if it were a function to $\mathbf{R}^p$ instead of to A; this is permitted because we are only interested in the values around the identity of A) Since φ is a 2-cocycle on G with values in A it follows that Ω is a 2-cocycle with values in $\mathbf{R}^p$ on the Lie algebra $\mathfrak{g}$ (it also follows from the fact that $\mathfrak{g}'$ is a Lie algebra in which the commutator satisfies the Jacobi identity). A different section

s' of G', which is equivalent to adding the boundary $\delta\chi$ of a 1-cochain χ to φ, would have changed the 2-cocycle Ω with the boundary of the 1-cochain ψ defined by:

$$\psi(X) = \frac{d}{dt}\bigg|_{t=0} \chi(\exp(Xt)) \ .$$

From these observations we deduce that the map defined by (3.8) from e-smooth group 2-cocycles φ to algebra 2-cocycles Ω induces a map $\Delta : H^2_{es,gr}(G, A) \to H^2_{al}(\mathfrak{g}, \mathbf{R}^p)$.

Remark 3.18. We know in an abstract way that the algebra 2-cocycles ω (defined in construction 3.13) and Ω (defined in construction 3.17) on $\mathfrak{g}$ with values in $\mathbf{R}^p$ have to differ by a 2-coboundary. However, we can indicate explicitly a section s of the bundle G' which yields according to formula (3.8) the algebra 2-cocycle $\omega = (\omega^r)$ of the initial construction ($\pi^*\omega^r = \mathrm{d}\alpha^r$, α^r dual to Y_j). The exponential map from a Lie algebra to an associated Lie group is a diffeomorphism in a neighbourhood of the identity; the 2-cocycle ω is constructed by means of the section σ of $\mathfrak{g}' \to \mathfrak{g}$ defined by $\sigma(X_i) = \Xi_i$, so we can define a section s of the extension $G' \to G$ by: $s\big(\exp_G(X)\big) = \exp_{G'}\big(\sigma(X)\big)$ in a neighbourhood of the identity and arbitrary elsewhere. This section then is e-smooth and calculation of (3.8) for the 2-cocycle associated to this section yields the algebra 2-cocycle ω.

3.18 Summary. We can summarise our results in the following commutative diagram

$$\begin{array}{ccccc}
H^2_{s,gr}(G,A) & \xrightarrow{1} & H^2_{es,gr}(G,A) & \xrightarrow{2} & H^2_{al}(\mathfrak{g},\mathbf{R}^p) \\
& {}_3\searrow & \big\downarrow{\scriptstyle 4} & & \big\downarrow{\scriptstyle 5} \\
& & \mathrm{Ext}(G,A) & \xrightarrow{6} & \mathrm{ext}(\mathfrak{g},\mathbf{R}^p)
\end{array}$$

where the map (1) is the natural inclusion, (2) is the map Δ defined in construction 3.17, (3) is the map defined in proposition 3.4 when restricted to smooth 2-cocycles, (4) is the isomorphism given in proposition 3.11, (5) is the isomorphism given in proposition 3.10, and (6) is the map which associates to the Lie group central extension $\pi : G' \to G$ the extension $\pi_* : T_{e'}G' \to T_eG$ of Lie algebras. Now the question arises naturally whether one of the maps (1), (2), (3) or (6) is invertible or not. As already said in remark 3.12 in general neither of the two maps (2) or (6) is invertible: there exist different group extensions with the same algebra extension and there exist algebra extensions which do not derive from a group extension. Moreover, there exist group extensions which do not admit a global smooth section, i.e., which do not determine an element of $H^2_{s,gr}(G, A)$, hence in general the maps (1) and (3) are not invertible. On the other hand, if G is a simply connected Lie group all the arrows are invertible. In §5 we will derive necessary and sufficient conditions under which an algebra extension is associated to a group extension, from which we can deduce the claims made above.

§4 "Central extensions" of manifolds

As a particular case of the previous section we have seen that if G' is an extension of G by a 1-dimensional abelian Lie group A (i.e., $A \cong \mathbf{R}/D$ with D a discrete subgroup of $\mathbf{R}$), then G' is a principal fibre bundle with connection 1-form α over G with structure group A (proposition 3.16) and $\mathrm{d}\alpha = \pi^*\omega$ where ω is a closed (left invariant) 2-form on G. Moreover we have seen that also in (quantum) mechanics a principal fibre bundle (with fibre $U(1)$) occurs (remark 2.3). This should motivate the questions studied in this section: suppose M is a connected manifold and ω a closed 2-form on M. Under what conditions does there exist a principal fibre bundle $\pi : Y \to M$ with structure group $\mathbf{R}/D$, D a discrete subgroup of $\mathbf{R}$, together with a (connection) 1-form α on Y such that $\mathrm{d}\alpha = \pi^*\omega$? If it exists, what are the different possibilities for (Y, α) ?

In the next sections we will show that the answers to these questions tell us how to construct central extensions of Lie groups, how to construct principal fibre bundles over symplectic manifolds (prequantization) and how to duplicate the situation of diagram (2.3) for classical mechanics.

§4.1 Definition of the group of periods $\mathrm{Per}(\omega)$

We start with a closed 2-form ω on a connected manifold M and we choose a cover $\{U_i \mid i \in I\}$ of M such that all finite intersections $U_{i_1} \cap \cdots \cap U_{i_k}$ are either contractible or empty (e.g., one can choose the U_i geodesically convex sets for some Riemannian metric on M). By contractibility of these finite intersections there exist 1-forms θ_i, functions f_{ij} and constants a_{ijk} such that:

$$(4.1) \qquad \begin{cases} & \mathrm{d}\theta_i = \omega & \text{on } U_i \ (\omega \text{ is closed}) \\ \Rightarrow & \theta_i - \theta_j = \mathrm{d}f_{ij} & \text{on } U_i \cap U_j \ (\text{with } f_{ji} = -f_{ij}) \\ \Rightarrow & f_{ij} + f_{jk} + f_{ki} = a_{ijk} & \text{on } U_i \cap U_j \cap U_k. \end{cases}$$

In order to define the group of periods $\mathrm{Per}(\omega)$ we need a few words on Čech cohomology. The set $\mathrm{Nerve}(\{U_i\})$ is defined by:

$$\mathrm{Nerve} = \{(i_0, \dots i_k) \in I^{k+1} \mid U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset, k = 0, 1, 2, \dots\}$$

and an element $(i_0, \dots i_k)$ is called a k-simplex. With these k-simplices one defines the abelian group $C_k(\{U_i\})$ as the free $\mathbf{Z}$-module with basis the k-simplices, i.e., $C_k(\{U_i\})$ consists of all finite formal sums $\sum c_{i_0,\dots,i_k} \cdot (i_0, \dots i_k)$ with $c_{i_0,\dots,i_k} \in \mathbf{Z}$; elements of $C_k(\{U_i\})$ are called k-chains. Between $C_k(\{U_i\})$ and $C_{k-1}(\{U_i\})$ we can define a homomorphism ∂_k called the boundary operator by:

$$\partial_k(i_0, \dots, i_k) = \sum_{j=0}^{k} (-1)^j \cdot (i_0, \dots, i_{j-1}, i_{j+1}, \dots, i_k)$$

on the basis of k-simplices and extended to $C_k(\{U_i\})$ by "linearity."

A homomorphism $h : C_k(\{U_i\}) \to A$ from the k-chains to an abelian group A is completely determined by its values on the basis of k-simplices $h(i_0, \dots, i_k)$; it is called a k-cochain if and only if it is totally antisymmetric in the sense that if two entries i_p and i_q in $(i_0, \dots, i_k)$ are interchanged then h changes sign. In this sense the constants a_{ijk} defined in (4.1) form a 2-cochain with values in $A = \mathbf{R}$. The set of all k-cochains with values in the abelian group A is denoted by $C^k(\{U_i\}, A)$; equipped with pointwise addition of functions this is an abelian group. By duality we can define coboundary operators $\delta_k : C^k(\{U_i\}, A) \to C^{k+1}(\{U_i\}, A)$ by:

$$(\delta_k h)(c) = h(\partial_{k+1} c) \qquad \text{for } c \in C_{k+1}(\{U_i\}) \ .$$

One can verify that $\partial_k \, \partial_{k+1} = 0$, and hence $\delta_k \, \delta_{k-1} = 0$. It follows that the set $B^k(\{U_i\}, A) = \mathrm{im}(\delta_{k-1})$ (whose elements are called k-coboundaries) is contained in the set $Z^k(\{U_i\}, A) = \ker(\delta_k)$ (whose elements are called k-cocycles), so their quotient $H^k_{\check{C}}(\{U_i\}, A) = Z^k(\{U_i\}, A)/B^k(\{U_i\}, A)$ is a well defined abelian group called the k-th Čech cohomology group associated to the cover $\{U_i\}$ with values in A. It follows directly from (4.1) that the 2-cochain a is a 2-cocycle.

Remark 4.1. One can show that the Čech cohomology groups $H^k_{\check{C}}(\{U_i\}, A)$ do not depend upon the chosen cover $\{U_i\}$ (with the restrictions imposed on such a cover as above), so one usually denotes it by $H^k_{\check{C}}(M, A)$. Moreover one can show that for $A = \mathbf{R}$ these Čech cohomology groups $H^k_{\check{C}}(M, \mathbf{R})$ are isomorphic to the de Rham cohomology groups $H^k_{dR}(M, \mathbf{R})$ (e.g., see [Wa]). In particular the map $[\omega] \to [a]$ which associates to the cohomology class of ω in $H^2_{dR}(M, \mathbf{R})$ the cohomology class of the 2-cocycle a in $H^2_{\check{C}}(M, \mathbf{R})$ is the well defined isomorphism between $H^2_{dR}(M, \mathbf{R})$ and $H^2_{\check{C}}(M, \mathbf{R})$.

Definition 4.2. $\mathrm{Per}(\omega) = \mathrm{im}(a : \ker \partial_2 \to \mathbf{R})$ where a is the 2-cocycle defined in (4.1).

Proposition 4.3. $\mathrm{Per}(\omega)$ *is independent of the various choices which can be made in the construction of the 2-cocycle* a.

Proof. If θ_i is replaced by $\theta_i + \mathrm{d}\phi_i$ (with ϕ_i a function on U_i) then a_{ijk} is not changed; if f_{ij} is replaced by $f_{ij} + c_{ij}$ (with c_{ij} a constant) then a_{ijk} is changed to $(a + \delta c)_{ijk}$ but by definition of $\mathrm{Per}(\omega)$ this does not change $\mathrm{Per}(\omega)$ because δc is zero on $\ker \partial$. Since these changes exhaust the possible choices in the construction of a the proposition is proved. QED

Remark 4.4. If ω is replaced by $\omega + \mathrm{d}\theta$ then θ_i is replaced by $\theta_i + \theta$ and f_{ij} is not changed; it follows that $\mathrm{Per}(\omega)$ depends only upon the cohomology class of ω in the de Rham cohomology group $H^2_{dR}(M, \mathbf{R})$.

Proposition 4.5. *Let D be a subgroup of* $\mathbf{R}$ *containing* $\mathrm{Per}(\omega)$*, then the cocycle a can be chosen such that $a_{ijk} \in D$.*

Proof. N.B. This is a purely algebraic statement; no topological arguments are involved and D might be dense in $\mathbf{R}$. We define a homomorphism $b : C_1(\{U_i\}) \to \mathbf{R}/D$ as follows: on the subspace $\mathrm{im}(\partial_2)$ b is defined by $b = \pi \, a \, \partial_2^{-1}$ where π is the canonical projection $\mathbf{R} \to \mathbf{R}/D$; this is independent of the choice in ∂_2^{-1} by definition of D and $\mathrm{Per}(\omega)$. Now $\mathbf{R}/D$ is a divisible $\mathbf{Z}$-module so there exists an extension b to the whole of $C_1(\{U_i\})$ (see [Hi&St], §1.7). Since $C_1(\{U_i\})$ is a free $\mathbf{Z}$-module, there exists a homomorphism $b' : C_1(\{U_i\}) \to \mathbf{R}$ satisfying $\pi \, b' = b$. Finally we can replace the functions f_{ij} by $f_{ij} - b'_{ij}$ which changes the cocycle a into $a - \delta b'$ and by construction of b' it follows that $\pi(a_{ijk} - (\delta b')_{ijk}) = 0$, showing that this modified cocycle has values in D as claimed. QED

Nota Bene 4.6. The simplex (ijk) is clearly not in $\ker \partial$, nevertheless we have shown that the cocycle a can be chosen such that it takes everywhere values in D containing $\mathrm{Per}(\omega)$!

§4.2 Construction of the $\mathbf{R}/D$ principal fibre bundle (Y, α)

If D is a dense subgroup of $\mathbf{R}$ then $\mathbf{R}/D$ is not a manifold in the usual sense (although it is a diffeological manifold [So,2], [Do], an S-manifold [vE,2] or a Q-

manifold [Barr]), so from now on we will always assume that D is a discrete subgroup of $\mathbf{R}$. Since we also assume that it contains the group of periods $\mathrm{Per}(\omega)$, this condition excludes some pairs (M,ω) (the easiest situation we know of where $\mathrm{Per}(\omega)$ is dense is on $\mathbf{S}^1\times\mathbf{S}^1\times\mathbf{S}^1\times\mathbf{S}^1$ with $\omega = dx_1\wedge dx_2 + t\,dx_3\wedge dx_4$ where t is irrational). On the manifold $\mathbf{R}/D$ we will use the local coordinate x inherited from the standard coordinate on $\mathbf{R}$. Finally we assume that the functions f_{ij} have been chosen such that the cocycle a takes its values in D everywhere. With these assumptions it follows that the functions $g_{ij} : U_i\cap U_j \to \mathbf{R}/D$ defined by $g_{ij}(m) = \pi(f_{ij}(m))$ satisfy the cocycle condition

$$g_{ij} + g_{jk} + g_{ki} = 0 \qquad \text{on } U_i\cap U_j\cap U_k,$$

so they define a principal fibre bundle $\pi : Y \to M$ with structure group $\mathbf{R}/D$, local charts $U_i\times\mathbf{R}/D$ with projection $\pi(m,x_i) = m$, and transitions between charts

$$U_i\times\mathbf{R}/D \ni (m,x_i)\mapsto (m, x_i + g_{ij}(m)) = (m,x_j)\in U_j\times\mathbf{R}/D\ .$$

The action of $y\in\mathbf{R}/D$ on Y is given on a local chart $U_i\times\mathbf{R}/D$ by $(m,x_i)\cdot y = (m,x_i+y)$. On Y we can define a 1-form α as follows: on the local chart $U_i\times\mathbf{R}/D$ it is given by

$$\alpha = \pi^*\theta_i + \mathrm{d}x_i\ , \tag{4.2}$$

which is correctly defined globally because of the construction of the f_{ij} (N.B. although x_i is in general not a global coordinate on $\mathbf{R}/D$, $\mathrm{d}x_i$ is a globally defined 1-form). In fact α is a connection 1-form on the principal fibre bundle Y if we identify the Lie algebra of $\mathbf{R}/D$ with $\mathbf{R}$ itself as tangent space of $0\in\mathbf{R}$. Since (4.2) implies $\mathrm{d}\alpha = \pi^*\omega$, we can now answer the question posed at the beginning of this section.

Proposition 4.7. *A principal $\mathbf{R}/D$ fibre bundle Y over M with a compatible connection form α exists if and only if $\mathrm{Per}(\omega)$ is contained in the discrete subgroup D.*

Proof. The if part is shown in the construction above. For the converse we reason as follows: if (Y,α) exists, then there exist transition functions $g_{ij} : U_i\cap U_j \to \mathbf{R}/D$. For a general connection α the local expression is given by $\alpha = \mathrm{Ad}(g^{-1})\pi^*\theta + g^{-1}\,\mathrm{d}g$; in this case the structure group A is abelian, hence α is expressed locally by $\alpha = \pi^*\theta_i + \mathrm{d}x_i$ and the condition $\mathrm{d}\alpha = \pi^*\omega$ implies $\mathrm{d}\theta_i = \omega$. By contractibility of $U_i\cap U_j$ the existence of functions f_{ij} satisfying (4.1) follows and the original bundle is shown to be obtained by the process described above. In particular $\pi(a_{ijk}) = 0$, i.e., $a_{ijk}\in D$, showing that $\mathrm{Per}(\omega)\subset D$. *QED*

§4.3 Classification of the different possibilities (Y,α)

Two "bundles with connection" (Y,α) and (Y',α') will be called equivalent if there exists a bundle diffeomorphism $\varphi : Y\to Y'$ commuting with the group action (i.e., if $\varphi(m,x) = (m,x')$ then $\varphi(m,x+y) = (m,x'+y)$) such that $\varphi^*\alpha' = \alpha$. With future applications in mind we will relax the notion of equivalence slightly, using the following definition of equivalence.

Definition 4.8. Suppose Θ is a subgroup of the vector space of all 1-forms on M, then we will call an extension (Y,α) over (M,ω) Θ-*equivalent* to an extension (Y',α') over (M,ω') if there exists a 1-form $\theta \in \Theta$ and a principal fibre bundle equivalence $\varphi : Y \to Y'$ (i.e., φ commutes with the projection to M and φ is equivariant with respect to the action of the structure group $\mathbf{R}/D$), such that $\varphi^*\alpha' = \alpha + \pi^*\theta$; it follows that $\omega' = \omega + \mathrm{d}\theta$, i.e., the cohomology classes of ω and of ω' are the same, hence their groups of periods are the same.

Remark 4.9. One sometimes borrows the language of exact sequences to denote principal fibre bundles (e.g. [Br&Di]): the principal fibre bundle $\pi : Y \to M$ with (structure) group A then is denoted by the sequence

$$\{0\} \to A \rightsquigarrow Y \to M \ ,$$

where the first arrow is added to show that the action of A on Y is without fixed points. Using these sequences, two bundles with connection (Y,α) and (Y',α') are Θ-equivalent if there exists a commutative diagram of the following form:

$$\begin{array}{ccccccc} & & & (Y,\alpha+\pi^*\Theta) & & & \\ & & \nearrow & & \searrow & & \\ \{0\} & \to & A & \updownarrow & & (M,\omega+\mathrm{d}\Theta) \\ & & \searrow & & \nearrow & & \\ & & & (Y',\alpha'+\pi^*\Theta) & & & \end{array}$$

in which both the upper and the lower sequences denote the principal fibre bundles, and in which the arrows "preserve" the (collections of) forms.

Before we can state the classification of the Θ-inequivalent extensions over M, we need a few comments. Let θ be any closed 1-form on M and let $\{U_i\}$ be a cover as in §4.1, then on U_i there exists a function Γ_i such that $\mathrm{d}\Gamma_i = \theta$. It follows that $\Gamma_i - \Gamma_j = \gamma_{ij}$ is constant on $U_i \cap U_j$ hence is a 1-cocycle with values in $\mathbf{R}$. It is easy to show that the cohomology class $[\gamma]$ of the 1-cocycle γ depends only upon the cohomology class $[\theta]$ of θ in $H^1_{dR}(M,\mathbf{R})$ (the map $[\theta] \mapsto [\gamma]$ is the equivalence $H^1_{dR}(M,\mathbf{R}) \to H^1_{\check{C}}(M,\mathbf{R})$, see also remark 4.1). When we project the cocycle γ to $\mathbf{R}/D$, we get a cocycle $\pi\gamma$, hence an induced map $\pi : H^1_{\check{C}}(M,\mathbf{R}) \to H^1_{\check{C}}(M,\mathbf{R}/D)$. Finally we denote by Θ_o the set of closed 1-forms in Θ, by $[\Theta_o]$ the image of Θ_o in $H^1_{\check{C}}(M,\mathbf{R})$ and by $[\Theta_o]_D$ the image of $[\Theta_o]$ under π in $H^1_{\check{C}}(M,\mathbf{R}/D)$.

Theorem 4.10. *The Θ-inequivalent possibilities for the principal $\mathbf{R}/D$ bundle with connection $(Y,\alpha+\pi^*\Theta)$ over $(M,\omega+\mathrm{d}\Theta)$ are classified by $H^1_{\check{C}}(M,\mathbf{R}/D)$ mod $[\Theta_o]_D$.*

Proof. Suppose (Y,α) over (M,ω) is constructed with solutions θ_i and f_{ij} to (4.1) and (Y',α') over (M,ω') with θ'_i and f'_{ij} (with $\omega' = \omega + \mathrm{d}\theta$ for some θ in Θ and with the constraint that a_{ijk} and a'_{ijk} should be in $D \supset \mathrm{Per}(\omega) = Per(\omega')$), then the transition functions between two local charts are given by $g_{ij} = \pi f_{ij}$ for (Y,α) and by $g'_{ij} = \pi f'_{ij}$ for (Y',α').

Since U_i is contractible, all closed 1-forms are exact, hence there exist functions F_i on U_i such that $\theta'_i = \theta_i + \theta + dF_i$. Since $U_i \cap U_j$ is contractible there exist constants c_{ij} such that $f'_{ij} = f_{ij} + F_i - F_j + c_{ij}$, and the constraint $a_{ijk}, a'_{ijk} \in D$ translates as $(\delta_1 c)_{ijk} \in D$, which is equivalent to πc being a 1-co*cycle* with values in $\mathbf{R}/D$. In the construction of the cohomology class $[\pi c]$ we have two degrees of freedom: we can modify the functions F_i with a constant b_i, which changes the 1-cochain c with the coboundary δb, hence the cohomology class of the 1-cocycle πc is not changed. In the second place, we can modify the choice of $\theta \in \Theta$ by an element $\theta_o \in \Theta_o$ which modifies the functions F_i with the functions Γ_i (associated to the closed 1-form θ_o on M (see above)), hence πc is modified by $\pi\gamma$ ($\gamma_{ij} = \Gamma_i - \Gamma_j$, hence $\delta\gamma = 0$), so the image of $[\pi c]$ in $H^1_{\check{C}}(M, \mathbf{R}/D) \mod [\Theta_o]_D$ is independent of the possible choices in the construction of the cochain c. It follows that if we classify all bundles (Y', α') relative to (Y, α) then we have constructed a map from the Θ-inequivalent bundles to $H^1_{\check{C}}(M, \mathbf{R}/D) \mod [\Theta_o]_D$, which is surjective as can be verified easily.

Now suppose φ is an equivalence between (Y, α) and (Y', α'); it follows that on U_i φ is given by $\varphi(m, x) = (m, x + \varphi_i(m))$ for some function $\varphi_i : Ui \to \mathbf{R}/D$. Since U_i is contractible (hence simply connected) there exist functions $\phi_i : U_i \to \mathbf{R}$ such that $\pi\phi_i = \varphi_i$. From the equation $\varphi^*\alpha' = \alpha + \theta''$ for some $\theta'' \in \Theta$ one deduces in the first place that $\theta'' = \theta + \theta_o$ with $\theta_o \in \Theta_o$ and $\theta \in \Theta$ as used in the construction of the classifying cocycle πc. In the second place one deduces that $\theta'_i = \theta_i + \theta + \theta_o - \mathrm{d}\phi_i$, from which one deduces that $F_i + \phi_i - \Gamma_i$ is constant ($\mathrm{d}\Gamma_i = \theta_o$). When we now apply the transition functions between two local trivialisations $U_i \times \mathbf{R}/D$ and $U_j \times \mathbf{R}/D$ we find that $g'_{ij} = g_{ij} + \varphi_j - \varphi_i$ which implies that $f'_{ij} = f_{ij} + \phi_j - \phi_i + d_{ij}$ for some constant $d_{ij} \in D$. Combining all these facts we find that the cohomology class $[\pi c]$ in $H^1_{\check{C}}(M, \mathbf{R}/D)$ is the same as the cohomology class of $[\pi\gamma]$ determined by $\theta_o \in \Theta_o$, showing that two equivalent bundles determine the same element in $H^1_{\check{C}}(M, \mathbf{R}/D) \mod [\Theta_o]_D$. Reversing this argument one sees that two bundles which determine the same element in $H^1_{\check{C}}(M, \mathbf{R}/D) \mod [\Theta_o]_D$ are equivalent, which proves the proposition. $\boxed{QED}$

Remark 4.11. As is known the Čech cohomology group $H^1_{\check{C}}(M, A)$ is isomorphic to the group $\mathrm{Hom}(\pi_1(M) \to A)$ of homomorphisms from the first homotopy group of M into A. From this it follows that if M is simply connected then all possible bundles (Y, α) are equivalent.

Remark 4.12. In the classification of Θ-inequivalent bundles we can consider the two extremes: $\Theta = \{0\}$ and $\Theta =$ all 1-forms. When we consider $\Theta = \{0\}$, we consider the classification of the principal A bundles (Y, α) over M such that $\mathrm{d}\alpha = \pi^*\omega$; these are classified by $H^1_{\check{C}}(M, \mathbf{R}/D)$. On the other extreme, when we consider $\Theta =$ all 1-forms, we are obviously not interested in the specific choice of ω as long as the cohomology class $[\omega]$ remains constant. Moreover, in choosing all 1-forms, we also disregard the connection α on Y: we are only interested in the different principal $\mathbf{R}/D$ bundles Y over M which can be constructed by the method of §4.2. To prove this statement, suppose that Y and Y' are two principal $\mathbf{R}/D$ bundles over M constructed by the method of §4.2 (for two closed 2-forms in the cohomology class $[\omega]$) and suppose φ is an equivalence of principal fibre bundles over M between Y and Y'. If α is the connection on Y and α' is the connection on Y' as constructed in §4.2, then $\varphi^*\alpha'$ is a connection on Y which satisfies $d\varphi^*\alpha' \in [\omega]$. It is easy to show that the 1-form $\varphi^*\alpha' - \alpha$ on Y is the pull back of a 1-form on M (it is zero on tangent vectors on Y in the direction of the fibre and its exterior derivative is the pull back of a 2-form on M), hence $\varphi^*\alpha' = \alpha + \pi^*\theta$, which shows that (Y, α) and (Y', α') are Θ-equivalent for $\Theta =$ all 1-forms.

We can summarise the above discussion by the statement that the inequivalent principal $\mathbf{R}/D$ bundles Y over M, which can be constructed by the method of §4.2, regardless of the connection form and regardless of the specific choice of ω in $[\omega]$, are classified by $H^1_{\check{C}}(M,\mathbf{R}/D) \mod \pi H^1_{\check{C}}(M,\mathbf{R})$. If $D=\{0\}$, then this statement reduces to the fact that there is up to equivalence only one such bundle. If $D\cong\mathbf{Z}$, then we can use the long exact sequence in cohomology associated to the short exact sequence of abelian groups $0\to D\to\mathbf{R}\to\mathbf{R}/D\to 0$ to show that this quotient is equivalent to the image of $H^1_{\check{C}}(M,\mathbf{R}/D)$ in $H^2_{\check{C}}(M,D)$, which in turn is equal to the kernel of the map $H^2_{\check{C}}(M,D)\to H^2_{\check{C}}(M,\mathbf{R})$ (specialists in algebraic topology will recognise this as the kernel of the map $H^2_{\check{C}}(M,\mathbf{Z})\to \mathrm{Hom}(H_2(M,\mathbf{Z}),\mathbf{Z})$ where H_2 is the homology group). This result connects cohomology with values in a constant sheaf (the Čech cohomology $H_{\check{C}}(M,A)$) and cohomology with values in a sheaf of germs of smooth functions (which we will denote by $H(M,A_s)$). The long exact sequence in $H(M,A_s)$ associated to the above mentioned short exact sequence, together with the fact that $H^1(M,\mathbf{R}_s)$ and $H^2(M,\mathbf{R}_s)$ are zero (a partition of unity argument, see [Hi]) shows that $H^1(M,(\mathbf{R}/D)_s)$ and $H^2(M,D_s)$ are isomorphic. Now $H^1(M,(\mathbf{R}/D)_s)$ classifies all inequivalent principal $\mathbf{R}/D$ bundles Y over M (the transition functions g_{ij} of §4.2 determine an element of $H^1(M,(\mathbf{R}/D)_s)$) and $H^2(M,D_s)=H^2_{\check{C}}(M,D)$ (because smooth functions to the discrete set D are necessarily (locally) constant), so indeed the image of $H^1_{\check{C}}(M,\mathbf{R}/D)$ in $H^2_{\check{C}}(M,D)$ classifies inequivalent principal $\mathbf{R}/D$ fibre bundles over M.

Remark 4.13. It is possible that Y is topologically trivial but not trivial as "bundle with connection" (in the strict sense with $\Theta=\{0\}$) as can be seen in the example $M=\mathbf{S}^1$ (all 2-forms are zero hence exact hence closed) where the above construction always gives a topologically trivial bundle (see the next proposition) although the $\{0\}$-inequivalent "bundles with connection" are classified by $H^1(\mathbf{S}^1,\mathbf{R}/D)\cong\mathbf{R}/D$.

Proposition 4.14.

(a) (Y,α) *is topologically trivial* $\Longrightarrow$ ω *is exact* $\Longleftrightarrow$ $\mathrm{Per}(\omega)=\{0\}$;
(b) *if* $D=\{0\}$ *then* (Y,α) *is topologically trivial.*

Proof. (i) If Y is trivial then there exists a global (smooth!) section $s: M\to Y$. Since α is a global 1-form on Y, $s^*\alpha$ is a global 1-form on M and $\mathrm{d}\, s^*\alpha = s^*\mathrm{d}\alpha = s^*\pi^*\omega=\omega$, proving that ω is exact.

(ii) If ω is exact it follows directly from its definition that $\mathrm{Per}(\omega)$ equals $\{0\}$.

(iii) Suppose $\mathrm{Per}(\omega)=\{0\}$, and let θ_i and f_{ij} be as in (4.1). Let ρ_i be a partition of unity subordinate to the cover $\{U_i\}$, then the local 1-forms $\theta'_i=\theta_i+\sum_k \mathrm{d}(\rho_k f_{ki})$ are well defined on U_i and they coincide on the intersections $U_i\cap U_j$, thus defining a global 1-form θ' which obviously satisfies $\mathrm{d}\theta'=\omega$.

(iv) Suppose $D=\{0\}$, we want to show that Y is topologically trivial. Let ρ_i be a partition of unity subordinate to the cover $\{U_i\}$, then applying proposition 4.5 it follows that the set of local sections $s_i(m)=\sum_k \rho_k(m)f_{ki}(m)$, given as functions from U_i to $\mathbf{R}$, defines a global smooth section, which proves that Y is topologically trivial. $\boxed{QED}$

Remark 4.15. Part (b) of the above proposition is a partial converse to the implication Y trivial $\Longrightarrow$ ω exact (if $D=\{0\}$ then, since it contains $\mathrm{Per}(\omega)$, ω is exact by part (a)). That in general the implication ω exact $\Longrightarrow$ Y trivial is not true can be seen "easily" by the following example. The real projective plane $\mathbf{P}^2(\mathbf{R})$ is the quotient of the 2-sphere $\mathbf{S}^2$ with respect to the action of the group $\{\pm id\} \cong \mathbf{Z}/2\mathbf{Z}$ seen as transformations of $\mathbf{R}^3$ which leave $\mathbf{S}^2$ invariant. $Y = \mathbf{S}^2 \times \mathbf{S}^1$ with the canonical projection on the first coordinate and the 1-form $\alpha = \mathrm{d}x$ (x the cyclic coordinate on $\mathbf{S}^1$) is a principal $\mathbf{S}^1$ bundle with connection over $\mathbf{S}^2$ and curvature form $\omega = \mathrm{d}\alpha = 0$. On Y acts the group $\{id, r\}$ with $r(u, x) = (-u, x + \pi)$, i.e., r is the reflection (with respect to the origin) in both coordinates. One easily verifies that the quotient Y' of Y with respect to this group is a principal $\mathbf{S}^1$ bundle over $\mathbf{P}^2(\mathbf{R})$ and that the connection form α descends to a connection form α' on Y', transforming (Y', α') into a principal $U(1)$ bundle with connection over $\mathbf{P}^2(\mathbf{R})$ with $\mathrm{d}\alpha' = 0$. We claim that this bundle is not trivial, so suppose it is trivial.

$\mathbf{P}^2(\mathbf{R})$ can be thought of as the upper hemisphere of $\mathbf{S}^2$ with its boundary where opposite points on the boundary have to be identified. If Y' is trivial, it has a global smooth section and such a global section can be identified with a function f from the upper hemisphere to the circle $\mathbf{S}^1$ with the condition that the values of opposite boundary points are opposite on $\mathbf{S}^1$ (a direct consequence of the construction of Y'). Now the upper hemisphere is the full disc with boundary $\mathbf{S}^1$ and the condition on f can be translated as "the restriction of f to the boundary $\mathbf{S}^1$ of the disc is a function with odd winding number." This is a clear contradiction with the fact that f is defined on the whole disc which implies that the restriction to the boundary has winding number zero. Hence the initial assumption has to be false, implying that Y' is not a trivial principal fibre bundle (see also example 5.12).

Remark 4.16. One can show that if the first homology group with values in $\mathbf{Z}$, $H_1(M, \mathbf{Z})$, is without torsion, then the quotient $H^1_{\check{C}}(M, \mathbf{R}/D) \mod \pi H^1_{\check{C}}(M, \mathbf{R})$ classifying principal $\mathbf{R}/D$ bundles equals $\{0\}$. In such a case we have the equivalence (Y, α) trivial $\Longleftrightarrow$ ω exact, because among the possibilities for the extensions is the trivial bundle and the classification tells us that there is only one equivalence class of extensions.

§4.4 Lifting infinitesimal symmetries

Related to the question of the existence of the bundle Y over M is the question about infinitesimal symmetries: suppose the vector field X on M is an infinitesimal symmetry of the pair (M, ω), i.e., the Lie derivative of ω with respect to X is zero: $\mathcal{L}(X)\omega = 0$, or in other words: the flow φ_t of X leaves ω invariant, i.e., $\varphi_t^*\omega = \omega$. Question: does there exist a vector field X' on Y such that $\pi_* X' = X$ and X' is an infinitesimal symmetry of (Y, α), i.e., $\mathcal{L}(X')\alpha = 0$?

Proposition 4.17. *Such X' exists if and only if there exists a function f on M satisfying $\iota(X)\omega + \mathrm{d}f = 0$ (i.e., $\iota(X)\omega$ is exact); if X' exists, it is uniquely determined by this function f, a function which also satisfies $\pi^* f = \alpha(X')$.*

Proof. Suppose X' exists, then $\mathcal{L}(X')\alpha = 0$ and $\pi_* X' = X$ which imply:

$$0 = \iota(X')\mathrm{d}\alpha + \mathrm{d}(\alpha(X')) = \pi^*(\iota(X)\omega) + \mathrm{d}(\alpha(X')) \ .$$

From this equation one immediately deduces that $\alpha(X')$ is the pull back of a function f on M and hence $\pi^*(i(X)\omega + \mathrm{d}f) = 0$, implying the only if part of the proposition because π^* is injective. On the other hand, if $\iota(X)\omega$ is exact, say $\iota(X)\omega + \mathrm{d}f = 0$, then we can define a vector field X' on the local chart $U_i \times \mathbf{R}/D$ by

$$X' = X + \big(f - \theta_i(X)\big) \cdot \frac{\partial}{\partial x_i} \ .$$

Using the local expression for $\alpha : \alpha = \pi^*\theta_i + \mathrm{d}x_i$ one shows that X' is on $U_i \times \mathbf{R}/D$ uniquely determined by the equations $\pi_* X' = X$ and $\alpha(X') = \pi^* f$. Since these equations are global equations it follows that X' is a globally defined vector field satisfying these equations. Finally: $\mathcal{L}(X')\alpha = i(X')\,\mathrm{d}\alpha + \mathrm{d}(\alpha(X')) = \pi^*(\iota(X)\omega + \mathrm{d}f) = 0$. QED

Now suppose G is a Lie group which is a symmetry group (on the left) of (M, ω), i.e., the left action $\Phi : G \times M \to M$ (definition 3.14) satisfies $\Phi(g)^*\omega = \omega$ for all g. Since all $\Phi(g)$ leave ω invariant, it follows that each fundamental vector field X_M (whose flow is given by $\Phi(\exp(Xt))$) is an infinitesimal symmetry of (M, ω).

Definition 4.18. A *momentum map* J for the action of a symmetry group G on (M, ω) is a linear map from the Lie algebra of G to smooth functions on M (i.e., if X is a left invariant vector field on G then J_X is a function on M) such that

$$\iota(X_M)\omega + \mathrm{d}J_X = 0 \ . \tag{4.3}$$

Proposition 4.19. *A momentum map exists if and only if each fundamental vector field X_M can be lifted to an infinitesimal symmetry of (Y, α).*

Proof. The only if part follows from the definition of momentum map and proposition 4.17. Conversely, if each fundamental vector field can be lifted to an infinitesimal symmetry of (Y, α), then by proposition 4.17 for a basis of the Lie algebra there exist functions satisfying (4.3); extending this map to the whole of the Lie algebra by linearity we obtain a momentum map. QED

Remark 4.20. The action of $\mathbf{R}/D$ on Y is a symmetry group of (Y, α); the fundamental vector field associated to the left invariant vector field ∂_x on $\mathbf{R}/D$ is the vector field which is expressed on each local chart $U_i \times \mathbf{R}/D$ as ∂_x. One easily verifies that the freedom in the lift of an infinitesimal symmetry X on (M, ω) to an infinitesimal symmetry X' on (Y, α) is just a multiple of the vector field ∂_x; this freedom corresponds exactly to the freedom in the function f satisfying $\iota(X)\omega + \mathrm{d}f = 0$: f is determined up to an additive constant (we always assume that M is connected!). It follows that the freedom in a momentum map (if it exists!) is an element μ of the dual Lie algebra, i.e., one may add to each function J_X the constant $\mu(X)$.

Remark 4.21. Let us denote by $\mathrm{symm}(Y, \alpha)$ the Lie algebra of infinitesimal symmetries of α (a Lie algebra because $\mathcal{L}([X, Y]) = [\mathcal{L}(X), \mathcal{L}(Y)]$) and denote by $\mathrm{symm}'(M, \omega)$ the Lie algebra of infinitesimal symmetries of ω which satisfy the lifting condition of proposition 4.17 (a Lie algebra because $\iota(X)\omega + \mathrm{d}f = 0$ and $\iota(Y)\omega + \mathrm{d}g = 0$ imply $\iota([X, Y])\omega + \mathrm{d}\omega(X, Y) = 0$). From $\mathrm{d}\mathcal{L}(X)\alpha = \mathcal{L}(X)\pi^*\omega = 0$ for $X \in \mathrm{symm}(Y, \alpha)$ we deduce that $\pi_* X$ is a well defined element of $\mathrm{symm}'(M, \omega)$; it follows from 4.17 and 4.20 that π_* is a surjective Lie algebra morphism with a 1-dimensional kernel consisting of multiples of the fundamental vector field ∂_x associated to the structure group of Y. Moreover, the vector field ∂_x is an element of the center of $\mathrm{symm}(Y, \alpha)$ hence $\mathrm{symm}(Y, \alpha)$ is a (1-dimensional) central extension of $\mathrm{symm}'(M, \omega)$. This is one of the reasons why we call (Y, α) a central extension of (M, ω).

§5 Applications

In this section we will discuss two different applications of central extensions of manifolds, the first one to answer the question posed at the end of §3 concerning central extensions of a Lie group associated to central extensions of the corresponding Lie algebra. The second application is to classical mechanics where it yields (up to a minor difference) the well known prequantization bundle over a symplectic manifold. The difference is that no quantization condition is necessary, only the condition $\mathrm{Per}(\omega)$ discrete; the connection with the usual prequantization construction is discussed. We finish this section with an application which is a combination of the previous two: we duplicate diagram (2.3) for classical mechanics and we give an explicit (easy) way to define the central extension G' occurring in the classical mechanics diagram.

§5.1 Lie group extensions associated to Lie algebra extensions

In this section we will concentrate on the case $p = 1$, i.e., we will consider central extensions of a Lie algebra $\mathfrak{g}$ by $\mathbf{R}$ and we will consider central extensions of a corresponding Lie group G by $\mathbf{R}/D$ with D a discrete subgroup of $\mathbf{R}$. The general case is a relatively straightforward generalization of the 1-dimensional case so we leave it to the reader.

Suppose G is a Lie group with associated Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}'$ be a 1-dimensional central extension of $\mathfrak{g}$ determined by the algebra 2-cocycle ω seen as left invariant closed 2-form on G (remark 3.9). As we have seen in proposition 3.16 if G' is a central extension of G by $\mathbf{R}/D$ for which $\mathfrak{g}'$ is the associated extension of $\mathfrak{g}$, then G' is a principal fibre bundle with connection α over G such that $\mathrm{d}\alpha = \omega$ (at least such an ω can be found in the cohomology class determined by the extension $\mathfrak{g}'$). Proposition 4.7 then tells us that a necessary condition for the existence of G' is that D contains $\mathrm{Per}(\omega)$; since all groups $\mathbf{R}/D$ are equivalent for D infinite discrete, it follows that a necessary condition for the existence of G' is that $\mathrm{Per}(\omega)$ is discrete. However, for the moment this is not a sufficient condition because if $\mathrm{Per}(\omega)$ is contained in D then we have a principal fibre bundle with connection (Y, α) over G, but a principal fibre bundle is not yet a Lie group. What we need is a way to see whether Y can be equipped with the structure of a Lie group such that the projection $\pi : Y \to G$ is a Lie group homomorphism. To tackle this problem we start with some general remarks and terminology on Lie groups.

5.1 Terminology. The graded algebra of left invariant forms on a Lie group is called the *Maurer Cartan algebra* on G, its elements are called *Maurer Cartan forms.* If $\beta^1, \ldots, \beta^n$ are n left invariant 1-forms on G which form a basis of the dual of the Lie algebra $\mathfrak{g}$ of G, then the Maurer Cartan algebra is generated by these 1-forms, i.e., each Maurer Cartan k-form can be written as a linear combination of the forms $\beta^{t_1} \wedge \cdots \wedge \beta^{t_k}$, $(1 \le t_1 < t_2 < \cdots < t_k \le n)$.

Now let $X_1, \ldots, X_n$ be n left invariant vectorfields on a Lie G forming a basis of its Lie algebra $\mathfrak{g}$ and c_{ij}^k constants such that $[X_i, X_j] = c_{ij}^k X_k$. If we denote by $\beta^1, \ldots, \beta^n$ the n left invariant 1-forms on G dual to the basis $X_1, \ldots, X_n$, then these 1-forms satisfy the equations:

$$\mathrm{d}\beta^k = -\tfrac{1}{2}\, c_{ij}^k\, \beta^i \wedge \beta^j \qquad \text{(full summation).} \tag{5.1}$$

From these equations it follows that the Lie algebra $\mathfrak{g}$ and the graded differential algebra of Maurer Cartan forms together with the exterior derivative are completely determined by equations (5.1). One now can ask the converse: does a set of 1-forms on a manifold satisfying equations (5.1) determine a Lie group with Lie algebra $\mathfrak{g}$?

Definition 5.2. Let $\mathfrak{g}$ be a Lie algebra, $X_1, \ldots, X_n$ a basis of $\mathfrak{g}$ and c_{ij}^k constants such that $[X_i, X_j] = c_{ij}^k X_k$. Let M be a manifold and $\tau^1, \ldots, \tau^n$ 1-forms on M, then M is called a $\mathfrak{g}$-*manifold* if the following two conditions are satisfied:

(i) at each point m of M : $\tau^1, \ldots, \tau^n$ is a basis of $T_m^* M$,
(ii) $\mathrm{d}\tau^k = -\frac{1}{2}\, c_{ij}^k\, \tau^i \wedge \tau^j$.

A diffeomorphism of M which leaves the τ^i invariant is called a *Maurer Cartan automorphism* of the $\mathfrak{g}$-manifold M (abbreviated as MC automorphism). The MC automorphisms of a $\mathfrak{g}$-manifold obviously constitute a group called $\mathrm{Aut}_{MC}(M)$ and the $\mathfrak{g}$-manifold M is called a complete $\mathfrak{g}$-manifold if this group acts transitively on M.

Proposition 5.3. *A $\mathfrak{g}$-manifold M can be given the structure of a Lie group with Lie algebra $\mathfrak{g}$ for which the τ^i are left invariant* 1*-forms if and only if it is a complete $\mathfrak{g}$-manifold.*

Proof (sketch). The only if part is obvious since if M is a Lie group, then all left translations are MC automorphisms hence $\mathrm{Aut}_{MC}(M)$ acts transitively.

To prove the if part we make the following comments: denote by τ^i $(i = 1, \ldots, n)$ the 1-forms on M which establish M as a $\mathfrak{g}$-manifold and denote by $\mathrm{pr}_j : M \times M \to M$ the projection on the j-th component $(j = 1, 2)$. The equations $\mathrm{pr}_1^* \tau^i = \mathrm{pr}_2^* \tau^i$ $(i = 1, \ldots, n)$ define a foliation on $M \to M$ of dimension $n = \dim M$ (it defines an involutive distribution on $M \times M$ hence by Frobenius' theorem it is integrable) and we have two obvious types of integral manifolds : (i) the diagonal $\{(m, m) \mid m \in M\}$ and (ii) the graph of an MC automorphism g $\{(m, gm) \mid m \in M\}$. From the uniqueness of integral manifolds we conclude that if the MC automorphism g has a fixed point, then it must be the identity, i.e., $\mathrm{Aut}_{MC}(M)$ acts without fixed points. This shows that if $\mathrm{Aut}_{MC}(M)$ acts transitively as well, we can establish a bijection between $\mathrm{Aut}_{MC}(M)$ and M as follows : choose a point m_o in M then $\phi \mapsto \phi(m_o)$ is a bijection. This defines a group structure on M by $\phi_1(mo) \cdot \phi_2(mo) = \phi_1(\phi_2(m_o))$, $(\phi(m_o))^{-1} = \phi^{-1}(m_o)$ and identity element m_o. Now let $\phi(m_o)$ be any point of M, then left translation by $\phi(m_o)$ is just the MC automorphism ϕ hence the τ^i are left invariant 1-forms. It remains to show that this group action is smooth; as a technical "detail" this is left to the reader. $\boxed{QED}$

We now go back to our original problem: is it possible to equip the bundle with connection (Y, α) with the structure of a Lie group such that $\pi : Y \to G$ is a Lie group homomorphism? Therefore let $\beta^1, \ldots, \beta^n$ be a basis of the left invariant 1-forms on G, then $\alpha, \pi^*\beta^1, \ldots, \pi^*\beta^n$ are 1-forms on Y and moreover, since $\mathrm{d}\alpha = \pi^*\omega$ is a left invariant 2-form on G, it follows (using formulas (3.5) and (3.7)) that Y is a $\mathfrak{g}'$-manifold. Hence Y is a Lie group if and only if it is a complete $\mathfrak{g}'$-manifold, i.e., if $\mathrm{Aut}_{MC}(Y)$ acts transitively.

As a first step in proving the transitive action we note that the action of $\mathbf{R}/D$ on Y leaves the 1-form α invariant and, since its orbits are the fibres, it also leaves the $\pi^*\beta^i$ invariant, so $\mathbf{R}/D$ is contained in $\mathrm{Aut}_{MC}(Y)$ and any two points in one fibre of Y can be joined by an MC automorphism.

If we could lift each left translation L_g on G $(g \in G)$, i.e., an MC automorphism of the $\mathfrak{g}$-manifold G, to an MC automorphism Λ_g of Y (a $\mathfrak{g}'$-manifold) satisfying $\pi\Lambda_g = L_g\pi$, then we would know that Y is complete, because if y and y' are two arbitrary elements of Y then $\pi(y) = h$ and $\pi(y') = h'$ are two elements of G which can be joined by a left translation $h' = L_g h$, hence y' and $\Lambda_g y$ are two elements of the same fibre which can be joined by an MC automorphism, hence y and y' can be joined by an MC automorphism. Finally we note that the condition $\Lambda_g \in \mathrm{Aut}_{MC}(Y)$ reduces to the equation $\Lambda_g^* \alpha = \alpha$ because $\Lambda_g * \pi^* \beta^i = \pi^* L_g^* \beta^i = \pi^* \beta^i$. With these preliminaries we can now state and prove the necessary and sufficient condition for the existence of a Lie group extension associated to a Lie algebra extension.

Theorem 5.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and suppose ω is a Lie algebra 2-cocycle with values in $\mathbf{R}$, i.e., $[\omega] \in H^2_{li,dR}(M, \mathbf{R}) \cong H^2_{al}(\mathfrak{g}, \mathbf{R})$. Then there exists a Lie group central extension G' of G by $\mathbf{R}/D$ associated to the Lie algebra extension $\mathfrak{g}'$ of $\mathfrak{g}$ by $\mathbf{R}$ (defined by ω) if and only if the following two conditions are satisfied:*

(i) $\mathrm{Per}(\omega) \subset D$ *discrete in* $\mathbf{R}$
(ii) *there exists a momentum map for the left action of G on (G, ω).*

If these conditions are satisfied then the inequivalent group extensions (definition 3.1) are classified by $H^1_C(G, \mathbf{R}/D)/[\Theta_o]_D$ with Θ = all left invariant 1-forms (definition 4.8) (see also [Sh] *and* [Ho]*).*

Proof (only if part). Let us first assume that G' exists and let α be the connection form on G' defined in §3.3 (α is a left invariant 1-form on G' dual to the center $\mathbf{R}/D$). According to proposition 4.7 $\mathrm{Per}(\omega)$ is contained in D. Now for any $X \in \mathfrak{g}$ the fundamental vector field X_G associated to the left action of G on itself is the right invariant vector field X^r determined by the vector X_e at the identity (where we identified $\mathfrak{g}$ with T_eG). Since ω is left invariant this vector field X^r is an infinitesimal symmetry of (G, ω). Let X'^r be any right invariant vector field on G' satisfying $\pi_* X'^r = X^r$ (such X'^r exist: choose a lift above the identity e' of G' and extend to a right invariant vectorfield on G') then X'^r is an infinitesimal symmetry of (G', α) because α is left invariant. According to proposition 4.19 we may conclude that there exists a momentum map as specified in the theorem. □

(if part). According to the discussion preceding this theorem we only have to show the existence (for each $g \in G$) of a diffeomorphism Λ_g on G' satisfying $\pi \Lambda_g = L_g \pi$ and $\Lambda_g^* * \alpha = \alpha$ when (G', α) is a principal fibre bundle with connection associated to (G, ω). To do this we proceed as follows. Let $X \in \mathfrak{g}$ and J_X the function on G determined by the momentum map for the left action of G on itself. It follows from proposition 4.17 that J_X determines a unique vector field X' on G' such that $\mathcal{L}(X')\alpha = 0$ (i.e., an infinitesimal symmetry of (G', α)), $\pi_* X' = X_G = X^r$ and $\alpha(X') = \pi^* J_X$. From the fact that the vector field X^r is complete on G one can deduce that the vector field X' is complete on G', hence its flow φ_t is determined for all times t and $\varphi_t^* \alpha = \alpha$. Because X' projects on X, its flow projects on the flow of X which is left multiplication by $\exp(Xt)$. From this we deduce that we have found the maps $\Lambda_{\exp(Xt)} = \varphi_t$ we were looking for. Since exp is a diffeomorphism from a neighbourhood of 0 to a neighbourhood of the identity and since every neighbourhood of the identity generates the whole group (G is connected) we can find lifts Λ_g for all g in G, proving the if part. □

(Classification part). If we can show that two bundles with connection (G', α) and (G'', α') over (G, ω) are Θ-equivalent in the sense of definition 4.8 with Θ the collection of all left invariant 1-forms on G if and only if the Lie group extensions G' and G'' are equivalent extensions, then we are finished because by theorem 4.10

we know the classification of inequivalent bundles with connection to be given by $H^1(G, \mathbf{R}/D)/[\Theta_o]_D$.

Therefore suppose $\Phi : G' \to G''$ is an equivalence of Lie group extensions, then α and $\Phi^*\alpha'$ differ by the pull back of some left invariant 1-form on G because they are both left invariant 1-forms on G' which are dual to the algebra of $\mathbf{R}/D$. Since Φ is also an equivalence of bundles, it follows that G' and G'' are Θ-equivalent bundles. On the other hand, suppose $\Phi : G' \to G''$ is an equivalence of bundles with connection and both G' and G'' are Lie group extensions of G defined by the cohomology class of ω in $H^2_{li,dR}(G, \mathbf{R})$. Since the action of $\mathbf{R}/D$ on G'' is an equivalence of G'' to itself, we can assume that $\Phi(e') = e''$, i.e., Φ maps the identity of G' on the identity of G''. Now Φ^* maps the Maurer Cartan algebra of left invariant forms on G'' onto the Maurer Cartan algebra of left invariant forms on G' because $\pi''\,\Phi = \pi'$ (where π' is the canonical projection from G' on G and π'' from G'' on G) and because α and $\Phi^*\alpha'$ differ by the pull back of a left invariant form on G (we have chosen the collection Θ as all left invariant 1-forms on G). One now verifies easily that these two conditions (MC algebra $\mapsto$ MC algebra and $\Phi(e') = e''$) imply that Φ is a group homomorphism, hence the extensions G' and G'' are equivalent Lie group extensions, proving that inequivalence of Lie group extensions is the same as Θ-inequivalence of bundles with connection. $\boxed{QED}$

Remark 5.5 ([vE,3]**).** Theorem 5.4 remains valid if one replaces "(finite dimensional) Lie group" by "Banach-Lie group." On the other hand, a thorough analysis of finite dimensional Lie groups (which will not be given here) shows that there exist no central extensions of finite dimensional Lie groups for which the group of periods is infinite ($\mathrm{Per}(\omega) \neq \{0\}$), hence for finite dimensional Lie groups a slightly stronger theorem holds: condition (i) can be replaced by (i') $\mathrm{Per}(\omega) = \{0\}$ (or by ω is exact as 2-form on G).

Remark 5.6. The diffeomorphism $\Lambda_{\exp(Xt)}$ of G' occurring in the proof above can be computed explicitly in local coordinates for small t: let $U \times \mathbf{R}/D$ be a local chart of G' on which α is expressed as $\pi^*\theta + \mathrm{d}x$ for some potential θ of ω (θ not necessarily left-invariant). On this chart the lift X' is given by $X' = X^r + (J_X - \theta(X^r))\partial_x$ (see proposition 4.17) hence for small t the flow $\Lambda_{\exp(Xt)}$ of X' is given by

$$\Lambda_{\exp(Xt)}(g, x) = \Big(\exp(Xt)g\ ,\ x + \int_0^t \big(J_X - \theta(X^r)\big)(\exp(Xs)g)\, ds\Big)\ ,$$

from which one can easily see that X' is indeed a complete vector field (N.B. $\pi_* X' = X^r$ is the *right* invariant vector field associated to $X \in \mathfrak{g}$!). In general the function $\varphi(\exp(Xt), g) = \int \big(J_X - \theta(X^r)\big)(\exp(Xs)g)\, \mathrm{d}s$ can not be extended to the whole of $G \times G$; however, if G is simply connected, it can be extended to $G \times G$ as we will show.

If G is a simply connected Lie group then a theorem of Hopf tells us that not only the first de Rham cohomology group $H^1_{dR}(G, \mathbf{R})$ is zero, but also $H^2_{dR}(G, \mathbf{R}) = \{0\}$ (the ingredients of the proof are the Künneth formula and the group structure). It follows in the first place that for any algebra extension defined by ω the conditions of theorem 5.4 are satisfied: $H^2 = \{0\}$ implies that ω is exact hence $\mathrm{Per}(\omega) = \{0\}$ and $H^1 = \{0\}$ implies that all 1-forms $\iota(X^r)\omega$ are exact, implying the existence of a momentum map. In the second place it follows that if ω is a closed (left invariant) 2-form on G then there exists a 1-form θ (not necessarily left-invariant) such that

$\mathrm{d}\theta = \omega$. From the construction of the principal fibre bundle with connection (G', α) in §4.3 we know that G' is the trivial bundle $G' = G \times \mathbf{R}/D$ and that α is given by $\alpha = \theta + dx$ (up to equivalence). Now let $g \in G$ then there exists a function f_g on G such that $\theta - L_g^*\theta = \mathrm{d}f_g$ ($H^1_{dR}(G, \mathbf{R}) = \{0\}$); it follows that the map Λ_g on $G \times \mathbf{R}/D$ defined by

$$\Lambda_g(h, x) = \big(gh, x + f_g(h)\big) \tag{5.2}$$

leaves the 1-form α invariant, hence is the lift one looked for in the proof of theorem 5.4. Moreover, it is easy to show that the function J'_X defined by $J'_X - \theta(X^r) = d_t|_{t=0} f_{\exp(Xt)}$ satisfies the equation $\iota(X^r)\omega + \mathrm{d}J'_X = 0$, hence differs only a constant from the momentum function J_X. This shows the relation between the two methods to lift left translations L_g on G to Λ_g on G' (i.e., the method in the proof of theorem 5.4 and formula (5.2)).

Corollary 5.7. *If G is a simply connected Lie group, then the map Δ defined in construction 3.17, $\Delta : H^2_{s,gr}(G, \mathbf{R}/D) \to H^2_{al}(\mathfrak{g}, \mathbf{R})$ is an isomorphism, and the Lie group central extensions of G by $\mathbf{R}/D$ are classified by either $H^2_{al}(\mathfrak{g}, \mathbf{R})$ or $H^2_{s,gr}(G, \mathbf{R}/D)$.*

Proof. As we have seen above, if G is simply connected, then each closed left invariant 2-form satisfies the conditions of theorem 5.4, hence for every algebra extension there exists a Lie group extension. Moreover, the classification part shows that there exists up to equivalence only one extension, showing that the Lie group central extensions of G by $\mathbf{R}/D$ are classified by $H^2_{al}(\mathfrak{g}, \mathbf{R})$. On the other hand, if G' is such a central extension of G then by proposition 4.14 it is topologically a trivial bundle, hence there exists a global smooth section, showing that G' determines a cohomology class in $H^2_{s,gr}(G, \mathbf{R}/D)$, which in turn shows that the map Δ is surjective. Since each algebra extension determines up to equivalence only one group extension, Δ is also injective. $\boxed{QED}$

Corollary 5.8. *If G is semi simple, then the central extensions of G by $\mathbf{R}/D$ are classified by $H^1_{\check{C}}(G, \mathbf{R}/D)$.*

Proof. G semi simple implies that its Lie algebra $\mathfrak{g}$ is semi simple. For semi simple Lie algebra's one can show that $H^1_{al}(\mathfrak{g}, \mathbf{R}) = H^2_{al}(\mathfrak{g}, \mathbf{R}) = \{0\}$ (see e.g. [Gu&St,§52]). From $H^2_{al}(\mathfrak{g}, \mathbf{R}) = \{0\}$ one deduces that there exists (up to equivalence) only one algebra extension. From $H^1_{al}(\mathfrak{g}, \mathbf{R}) = \{0\}$ one deduces that each closed left invariant 1-form on G is the exterior derivative of a left invariant 0-form, hence each closed left invariant 1-form is exact, hence the set $[\Theta_o]_D$ occurring in theorem 5.4 is zero. According to theorem 5.4 this shows that the inequivalent group extensions associated to the unique algebra extension are classified by $H^1_{\check{C}}(G, \mathbf{R}/D)$. $\boxed{QED}$

Corollary 5.9. *If G is a simply connected, semi simple Lie group, then the only Lie group central extension of G by $\mathbf{R}/D$ is the trivial extension.*

Example 5.10. One of the easiest examples is the group $\mathbf{R}^2$ with addition: it is an abelian simply connected Lie group. However, its Lie algebra cohomology (left invariant de Rham cohomology) is not trivial: left invariant functions are constants, left invariant 1-forms are of the form $\xi\, dx + \eta\, dy$ (where ξ and η are constants and x and y denote the canonical coordinates on $\mathbf{R}^2$) and left invariant 2-forms are of the form $\delta\, dx \wedge dy$ (δ a constant). It follows that the exterior derivative of all left invariant forms is zero, hence $H^2_{li,dR}(G, \mathbf{R}) \cong \mathbf{R}$ and $dx \wedge dy$ is a generator of this cohomology group. If we choose $D = \{0\}$, then the extension G' (associated to the Lie algebra extension defined by the Lie algebra 2-cocycle $dx \wedge dy$) is topologically $\mathbf{R}^3$ with connection $\alpha = \frac{1}{2}(x\, dy - y\, dx) + dt$, hence with group structure

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + \tfrac{1}{2}(x'y2 - 2xy'))\ ,$$

where we have chosen $(0, 0, 0)$ as the identity (the point m_o of proposition 5.3). Using the section $s(x, y) = (x, y, 0)$ we get a 2-cocycle φ which yields after application of formula 3.8 exactly the defining 2-cocycle $dx \wedge dy$ (not only one in the same cohomology class).

When we choose $D = d\mathbf{Z}$ for some element $d \in \mathbf{R}$, then the Lie group extension of $\mathbf{R}^2$ by $\mathbf{R}/D$ can be obtained from the above one by taking the quotient with respect to the normal subgroup $\{0\} \times \{0\} \times D$. Although there should be only one Heisenberg group, all these extensions are called the Heisenberg group (mathematicians usually call the extension with $d = 0$ the Heisenberg group, physicists usually call the extension with $d = 2\pi$ the Heisenberg group).

Example 5.11. The abelian Lie group $G = (\mathbf{R}/\mathbf{Z}) \times \mathbf{R}$ together with the left invariant exact 2-form $dx \wedge dy$ is a slight modification of the previous example. However, G is not simply connected and moreover, there exists no momentum map for the left action of G on $(G, dx \wedge dy)$: the fundamental vector fields associated to the left action are linear combinations of the vector fields ∂_x and ∂_y and then: $\iota(\partial_y)dx \wedge dy = -dx$ which is closed but not exact (x is a cyclic coordinate). Hence there does not exist a Lie group extension G' associated to the Lie algebra extension defined by $dx \wedge dy$ (one can verify by hand that a bundle with connection (Y, α) as constructed in §4 can not be given a group structure such that the projection is a homomorphism).

Example 5.12. $SU(2)$ is a semi simple, simply connected Lie group, hence by corollary 5.9 the only Lie group central extension of $SU(2)$ by $\mathbf{R}/D$ is the trivial extension $SU(2) \times \mathbf{R}/D$.

The group $SO(3)$ is the quotient of $SU(2)$ by the normal subgroup $\{id, -id\}$; the Lie algebra of $SO(3)$ is the same as for $SU(2)$ so there exists only one Lie algebra extension (up to equivalence): the trivial extension. However, $\pi_1(SO(3)) = \mathbf{Z}/2\mathbf{Z} = \{id, -id\}$ and hence $H^1_{\check{C}}(SO(3), \mathbf{R}/D) \cong \mathrm{Hom}(\mathbf{Z}/2\mathbf{Z} \to \mathbf{R}/D)$, which classifies, according to corollary 5.8, the inequivalent group extensions of $SO(3)$ by $\mathbf{R}/D$. Consequently, if $D = \{0\}$ there is only one group extension: the trivial one; if $D = d\mathbf{Z}$ (with $d \in \mathbf{R} \setminus \{0\}$) there are two inequivalent Lie group extensions of $SO(3)$ by $\mathbf{R}/D$. These two extensions can be obtained from the trivial extension $SU(2) \times \mathbf{R}/D$ of $SU(2)$ by taking the quotient with respect to the normal subgroups $\{(id, 0), (-id, 0)\}$ (giving the trivial extension) and $\{(id, 0), (-id, \frac{1}{2}d)\}$ (giving a non trivial extension of $SO(3)$ which will be discussed again in example 5.29).

Example 5.13. The group $SL(2, \mathbf{R})$ is semi simple but not simply connected: topologically $SL(2, R) \cong \mathbf{R}^2 \times \mathbf{S}^1$, hence $\pi_1(SL(2, \mathbf{R})) = \mathbf{Z}$. It follows from corollary 5.8 that the inequivalent Lie group central extensions of $SL(2, \mathbf{R})$ by $\mathbf{R}/D$ are classified by $H^1_{\check{C}}(SL(2, \mathbf{R}), \mathbf{R}/D) \cong \mathrm{Hom}(\mathbf{Z} \to \mathbf{R}/D) \cong \mathbf{R}/D$ (for all discrete $D \subset \mathbf{R}$).

§5.2 Pre-quantization

Prequantization is a construction which can be carried out within the framework of classical mechanics (or in symplectic geometry). The purpose of this construction is to obtain a faithful representation of the Poisson algebra (as vectorfields on a manifold). Such a representation of the Poisson algebra realises a part of the canonical quantization program of Dirac, i.e. constructing a faithful representation of the Poisson algebra as (essentially selfadjoint) operators on a Hilbert space $\mathcal{H}$. In this section we will elucidate these statements.

5.14 The Poisson algebra and the prequantization bundle. The usual (modern) formulation of classical mechanics is by means of a symplectic manifold (M, ω) called the phase space and a Hamilton function $H : M \to \mathbf{R}$ generating the time flow by means of the associated Hamiltonian vectorfield ξ_H defined by the equation $\iota(\xi_H)\omega + dH = 0$. The map $f \mapsto \xi_f$ from functions on M to vector fields on M is clearly a linear map. The Poisson bracket of two functions $\{f, g\} = \xi_f g$ defines the structure of a Lie algebra (infinite dimensional) on the set of all smooth functions on M which is called the Poisson algebra. The commutator of vector fields defines a Lie algebra structure on the set of all smooth vector fields on M and equipped with these Lie algebra structures, the map $f \mapsto \xi_f$ is a homomorphism of Lie algebras. However, this map (representation) is not injective: its kernel consists of all constant functions on M (which is supposed to be connected).

The most natural way to obtain an injective representation of the Poisson algebra (functions on M) out of the Hamiltonian vectorfields is to apply the construction of §4. The idea is roughly as follows: the map $f \mapsto \xi_f$ is a faithful representation of the Poisson algebra modulo the kernel of this map and the Poisson algebra is a 1-dimensional Lie algebra extension of this quotient, so one only needs a way to represent this kernel injectively. This can be done by adding an extra coordinate to the manifold M and representing the constant functions as constant vector fields in the new direction. The actual construction is a bit more delicate: since a symplectic form ω is in particular a closed 2-form one can apply the construction of §4 to obtain a principal fibre bundle with connection (Y, α) over M with $\mathrm{d}\alpha = \omega$ provided $\mathrm{Per}(\omega)$ is discrete. This puts a constraint on the class of classical systems which can be treated in this way (although it is not a quantization condition, see the discussion below). If this condition is satisfied, then an injective representation of the Poisson algebra of functions on M as vector fields of Y is an easy application of proposition 4.17: each Hamiltonian vector field ξ_f satisfies the equation $\iota(\xi_f)\omega + df = 0$, so it defines a unique infinitesimal symmetry η_f of (Y, α) satisfying $\pi_*\eta_f = \xi_f$ and $\alpha(\eta_f) = \pi^* f$. Using the local expression for η_f as given in the proof of proposition 4.17 it is an easy exercise to prove that the map $f \mapsto \eta_f$ is an injective Lie algebra homomorphism of the Poisson algebra to the infinitesimal symmetries of (Y, α). The bundle (Y, α) over M is usually called the *prequantization bundle over* M of (M, ω). The results are summarised in the following theorem.

Theorem 5.15. *Suppose (M, ω) is a symplectic manifold with Poisson algebra $\mathcal{P}$ and suppose that there exists a bundle with connection (Y, α) over M such that $\mathrm{d}\alpha = \omega$, i.e., there exists a prequantization bundle Y over M. Then we can construct a faithful representation η of $\mathcal{P}$ on Y with the following properties:*

$$\pi_*\eta_f = \xi_f \quad , \quad \mathcal{L}(\eta_f)\alpha = 0 \quad , \quad \alpha(\eta_f) = \pi^* f$$

(and remember: $\mathcal{L}(\xi_f)\omega = 0$ *and* $\iota(\xi_f)\omega + \mathrm{d}f = 0$*).*

5.16 Quantization. When one speaks in physics about quantization of a classical system, one is looking for a Hilbert space $\mathcal{H}$ and for (selfadjoint) operators $\mathcal{O}(f)$ on $\mathcal{H}$ representing classical observables f (i.e., $f : M \to \mathbf{R}$). According to a remark of Dirac ([Di],p87) the ideal situation would be the situation in which the map $f \mapsto \mathcal{O}(f)$ is linear and maps the Poisson bracket to i times the commutator of operators; more precisely, a map which satisfies the following conditions [vH], [Go]:

(i) it is an $\mathbf{R}$-linear map from the Poisson algebra to the operators on $\mathcal{H}$ which satisfies the condition $h = [f, g]_P \Rightarrow \mathcal{O}(h) = -i\hbar\,[\mathcal{O}(f), \mathcal{O}(g)]$, where $[\ ,\ ]_P$ is the Poisson bracket of functions, where $[\ ,\ ]$ is the usual commutator of operators and where $\hbar$ is Planck's constant h divided by 2π (in short, it should be a Lie algebra homomorphism for a certain algebra structure on the set of operators on $\mathcal{H}$),

(ii) the function constant 1 is mapped to the identity operator on $\mathcal{H}$ and

(iii) $\mathcal{H}$ is irreducible under the action of the images of a set of canonical coordinates.

Apart from the fact that there does not always exist a set of global canonical coordinates on an arbitrary symplectic manifold, we have the famous no-go theorem of Van Hove [vH] which tells us that even in the case of $\mathbf{R}^{2n}$ with its canonical symplectic structure, such a map can not exist: one has to drop one of the conditions. In quantization procedures one usually relaxes condition (i) to: it should be a Lie algebra homomorphism for a restricted class of observables. We will not discuss this here; what we will do is construct a Hilbert space $\mathcal{H}$ and a map from observables to operators which satisfy (i) and (ii) by means of the injective representation of the Poisson algebra on vector fields on the prequantization bundle (Y, α). This approach is the starting point of the geometric quantization procedure; geometric quantization proceeds by constructing "irreducible" parts out of the Hilbert space $\mathcal{H}$ and defining the operators on such an irreducible part; it follows from Van Hove's theorem that in general condition (i) is no longer satisfied for the class of all observables.

5.17 The Hilbert space and the quantization condition. The symplectic manifold (M, ω) carries a natural volume form: the Liouville volume form ε which is (apart from some numerical constant) given by ω^n where $2n$ denotes the dimension of the symplectic manifold M. With this volume form one can define in a canonical way the Hilbert space $\mathcal{H} = L^2_{\mathbf{C}}(M, \varepsilon)$ of square integrable complex valued functions on M with respect to the measure defined by the volume form ε. Moreover, observables $f : M \to \mathbf{R}$ can be represented in an obvious way as operators on $\mathcal{H}$. If we define the operator $\mathcal{O}(f)$ associated to an observable f by $\mathcal{O}(f) = -i\hbar\,\xi_f$ then this map satisfies condition (i) above, but it does not satisfy condition (ii): the Hamiltonian vector fields of constant functions are zero, so the operator $\mathcal{O}(1)$ associated to the constant function 1 is zero.

The obvious way out is to construct the Hilbert space $\mathcal{H}$ out of functions on Y: on Y we also have a canonical volume element ε (abuse of notation: ε also denotes the Liouville volume element on M) which is (again apart from some numerical constants) given by $\alpha \wedge (\mathrm{d}\alpha)^n$ so we can define the Hilbert space $L^2_{\mathbf{C}}(Y, \varepsilon)$ on which act the operators $\mathcal{O}(f) = -i\hbar\eta_f$ associated to an observable f. The operator $\mathcal{O}(1)$ associated to the constant function 1 is the operator $-i\hbar\,\partial_x$ where x is the coordinate on the fibre $\mathbf{R}/D$ of Y. We see that this operator is not the identity on the whole of $L^2_{\mathbf{C}}(Y,\alpha)$, so we define $\mathcal{H}$ to be the subspace of $L^2_{\mathbf{C}}(Y,\alpha)$ on which it is. On a local chart $U_i \times \mathbf{R}/D$ the equation $\mathcal{O}(1)\phi = \phi$ for $\phi \in L^2_{\mathbf{C}}(Y,\alpha)$ reduces to the equation

$$(5.3) \qquad \frac{\partial\phi}{\partial x}(m,x) = -i\hbar\,\phi(m,x) \quad \Longleftrightarrow \quad \phi(m,x) = \phi(m)\cdot\exp(ix/\hbar)\ .$$

This equation immediately gives rise to two problems: if $D = \{0\}$, then integration of $|\phi|^2$ over the fibre $\mathbf{R}$ diverges (except when $\phi(m) = 0$), so in that case the subspace $\mathcal{H}$ of $L^2_{\mathbf{C}}(Y,\alpha)$ is $\{0\}$; on the other hand, if $D = d\mathbf{Z}$ (with $d \in \mathbf{R}\setminus\{0\}$ a generator of D), then $\exp(ix/\hbar)$ should be well defined on $\mathbf{R}/D$ which implies that $\exp(id/\hbar) = 1$ or in other words: $d \in h\mathbf{Z}$ (where h is Planck's constant).

The first problem is easy to solve: if $D = \{0\}$, it implies (proposition 4.14) that the group of periods of ω is zero: $\mathrm{Per}(\omega) = \{0\}$, so there is no problem when we choose $D = d\mathbf{Z}$: $\mathrm{Per}(\omega)$ remains contained in D. The second problem is more serious: it poses a constraint on the symplectic form ω because (Y,α) exists only if $\mathrm{Per}(\omega)$ is contained in $D = d\mathbf{Z}$, hence the condition $d \in h\mathbf{Z}$ implies that $\mathrm{Per}(\omega)$ should be contained in $h\mathbf{Z}$. If this condition is satisfied then the subspace $\mathcal{H}$ of $L^2_{\mathbf{C}}(Y,\alpha)$ on which the operator $\mathcal{O}(1)$ associated to the constant function 1 acts as the identity is not trivial. An easy way to describe this Hilbert space $\mathcal{H}$ is in terms of the action of the structure group $\mathbf{R}/D$ on Y:

$$(5.4) \qquad \mathcal{H} = \{\phi \in L^2_{\mathbf{C}}(Y,\alpha) \mid \forall a \in \mathbf{R}/D : \phi(y+a) = \phi(y)\cdot\exp(ia/\hbar)\}\ .$$

In this way we have constructed a Hilbert space and operators on it satisfying the conditions (i) and (ii) mentioned above. We have seen that the existence of the prequantization bundle Y is subject to the constraint $\mathrm{Per}(\omega)$ discrete and that the supplementary condition (ii) yields the quantization condition $\mathrm{Per}(\omega) \subset h\mathbf{Z}$ (it is called a quantization condition because it restricts the generator of $\mathrm{Per}(\omega)$ to a discrete subset of $\mathbf{R}$).

5.18 Relations with other prequantization constructions. We have called the principal fibre bundle (Y,α) the prequantization bundle over the symplectic manifold (M,ω) with a hyphen between pre and quantization to distinguish it from the bundles constructed in the usual prequantization procedures. In literature one encounters two versions of prequantization due to J.-M. Souriau [So,1] and B. Kostant [Ko]; we will discuss the relations between these two versions and the version described above. The relation with Souriau's version is very simple: he only considers the case D equal to $h\mathbf{Z}$, so the fibre of Y is fixed and one obtains the quantization condition that a generator of $\mathrm{Per}(\omega)$ should be a multiple of h. In our case with $D \subset h\mathbf{Z}$ one can obtain a bundle with fibre $\mathbf{R}/h\mathbf{Z}$ from the bundle with fibre $\mathbf{R}/D$ by taking the quotient of Y with respect to the action of the subgroup $h\mathbf{Z}/D$ of $\mathbf{R}/D$ on Y; this is equivalent to saying that one takes the "transition functions" f_{ij} of §4.2 not modulo D but modulo $h\mathbf{Z}$. It will be clear from formula (5.4) that there is a canonical unitary equivalence between the Hilbert space of formula (5.4) with the "arbitrary" subgroup D and the Hilbert space constructed with $D = h\mathbf{Z}$, so with respect to the Hilbert spaces there is no difference. It should be noted that the process of taking the quotient of Y with respect to the action of the subgroup $h\mathbf{Z}/D$ of $\mathbf{R}/D$ on Y is called by Souriau in [So,1] "quantization by fusion."

To discuss the relation with Kostant's version we need a small digression on associated vector bundles. Let ρ_k (for $k \in \mathbf{N}$) be the unitary representation of $\mathbf{R}/d\mathbf{Z}$ on $\mathbf{C}$ defined by $\rho_k(x) = \exp(-2k\pi ix/d)$ and denote by L_k the complex line bundle over M associated with the principal fibre bundle (Y, α) and the representation ρ_k (see e.g. [Pij]). The connection α on Y defines a connection ∇ on L_k and one can calculate (an easy exercise) that the curvature 2-form of the connection ∇ is given by:

$$\text{curvature}(\nabla) = \frac{-2k\pi i}{d} \cdot id(\mathbf{C}) \cdot \omega \ .$$

Now one can show that there exists a $1-1$ correspondence between the sections of L_k and the functions ϕ on Y satisfying the condition $\phi(y+a) = \phi(y)\cdot\exp(2k\pi ia/d)$. Comparing this with formula 5.4 we see that the Hilbert space $\mathcal{H}$ defined in formula 5.4 consists of sections of the bundle L_k where the integer k is defined by the equation $d = kh$. It follows that the curvature of ∇ is given by $-(i/\hbar)\omega$, independent of the choice of d (which has of course to satisfy the "equation" $\text{Per}(\omega) \subset d\mathbf{Z} \subset h\mathbf{Z}$). In Kostant's approach one focuses right away on a complex line bundle L over M by asking: given a symplectic manifold (M, ω) does there exist a complex line bundle L over M with connection ∇ such that its curvature is given by $\text{curvature}(\nabla) = -(i/\hbar)\omega$? The answer is given by A. Weil: a necessary and sufficient condition is that $\omega/\hbar$ determines an integer cohomology class, a condition which can also be stated as $\text{Per}(\omega)$ contained in $h\mathbf{Z}$. We see that Kostant's approach is more direct to the goal of determining the Hilbert space, whereas our approach is in two steps: first an injective representation of the Poisson algebra and then the construction of the Hilbert space. Apart from these minor differences in approach, all three methods to obtain a Hilbert space $\mathcal{H}$ are equivalent as we have seen above.

Remark 5.19. So far we have said nothing about the (essential) self adjointness of the operators $\mathcal{O}(f) = -i\hbar\eta_f$ associated to a classical observable $f : M \to \mathbf{R}$. Since η_f is an infinitesimal symmetry of (Y, α), it follows that the Lie derivative of the volume form ε on Y in the direction of η_f is zero which shows that $-i\hbar\eta_f$ is formally a symmetric operator on $L^2_{\mathbf{C}}(Y, \alpha)$. Furthermore, if the vector field η_f is a complete vector field, then its flow is a 1-parameter group of diffeomorphisms of Y which leave the volume form invariant, hence they define a 1-parameter unitary group of transformations of $L^2_{\mathbf{C}}(Y, \alpha)$. By Stone's theorem the infinitesimal generator of this unitary group is a self adjoint operator on $L^2_{\mathbf{C}}(Y, \alpha)$, showing that if η_f is complete, then $\mathcal{O}(f)$ is (essentially) self adjoint.

Example 5.20. Let $M = \mathbf{R}^{2n}$ and let $\omega = \mathrm{d}(p_i\,\mathrm{d}q^i)$ be its standard symplectic form. M is simply connected and ω is exact, so (up to equivalence) there exists for each discrete D only one bundle $Y :\ M\times\mathbf{R}/D$, with connection form $\alpha = p_i\,\mathrm{d}q^i + \mathrm{d}x$. Calculating for $f : M \to \mathbf{R}$ the associated vector field η_f we find:

$$\eta_f = \frac{\partial f}{\partial p_i} \cdot \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \cdot \frac{\partial}{\partial p_i} + (f - p_i \cdot \frac{\partial f}{\partial p_i}) \cdot \frac{\partial}{\partial x} \ .$$

Functions ϕ on Y which satisfy formula 5.4 can be *globally* identified with functions on M by $\phi(q, p, x) = \phi(q, p)\cdot\exp(ix/\hbar)$, where we now assume that $D = h\mathbf{Z}$ (which is allowed because $\text{Per}(\omega) = \{0\}$). With this identification the Hilbert space $\mathcal{H}$ is isomorphic to $L^2_{\mathbf{C}}(\mathbf{R}^{2n}, \mathrm{Leb})$ and the operators $\mathcal{O}(f)$ are given by:

$$\mathcal{O}(f)\phi = -i\hbar \cdot \frac{\partial f}{\partial p_j} \cdot \frac{\partial \phi}{\partial q^j} + i\hbar \cdot \frac{\partial f}{\partial q^j} \cdot \frac{\partial \phi}{\partial p_j} + (f - p_j \frac{\partial f}{\partial p_j}) \cdot \phi$$

or

$$\mathcal{O}(f) = -i\hbar \cdot \frac{\partial f}{\partial p_j} \cdot \frac{\partial}{\partial q^j} + i\hbar \cdot \frac{\partial f}{\partial q^j} \cdot \frac{\partial}{\partial p_j} + (f - p_j \frac{\partial f}{\partial p_j}) \ .$$

Substituting for f the functions p_j and q^j we find $\mathcal{O}(p_j) = -i\hbar\partial_{q^j}$ and $\mathcal{O}(q^j) = q^j + i\hbar\partial_{p_j}$ as operators on $L^2_{\mathbf{C}}(\mathbf{R}^{2n}, \mathrm{Leb})$. When we compare this to the usual Schrödinger quantization with $\mathcal{O}(p_j) = -i\hbar\partial_{q^j}$ and $\mathcal{O}(q^j) = q^j$ as operators on $L^2_{\mathbf{C}}(\mathbf{R}^n, \mathrm{Leb})$, we see that these two quantizations do not resemble each other. In fact, they are not equivalent since it follows from Van Hove's remark [vH] that our quantization on $L^2_{\mathbf{C}}(\mathbf{R}^{2n}, \mathrm{Leb})$ is reducible under the action of the operators $\mathcal{O}(p_j)$ and $\mathcal{O}(q^j)$, while the Schrödinger quantization is irreducible under the action of these operators. However, although $L^2_{\mathbf{C}}(\mathbf{R}^{2n}, \mathrm{Leb})$ is reducible under $\mathcal{O}(p_j)$ and $\mathcal{O}(q^j)$, Van Hove has shown in [vH] that $L^2_{\mathbf{C}}(\mathbf{R}^{2n}, \mathrm{Leb})$ is irreducible under the action of the complete Poisson algebra.

Example 5.21. One of the easiest non-trivial examples of a symplectic manifold is the sphere $\mathbf{S}^2$ with its canonical symplectic form ω given in polar coordinates ϑ, φ by

$$\omega = -\lambda \sin\vartheta \, \mathrm{d}\vartheta \wedge \mathrm{d}\varphi \ ,$$

where λ is a (positive) real parameter. Apart from being non-trivial, this symplectic manifold also plays an important role in the relation between classical and quantum mechanics (see for instance [So,1] and [Du]): it is the phase space of classical spin. Since ω is not exact, the group of periods of ω is infinite and one can show that it is given by $\mathrm{Per}(\omega) = 4\pi\lambda\mathbf{Z}$; it follows that the quantization condition $\mathrm{Per}(\omega) \subset h\mathbf{Z}$ is equivalent to $\lambda = \frac{1}{2}n\hbar$ for some positive integer n, giving the well known quantization of spin if we interpret λ as the classical spin.

The associated principal fibre bundle with connection (Y, α) for $D = \mathrm{Per}(\omega)$ is isomorphic to the Hopf fibration $\mathbf{S}^3 \to \mathbf{S}^2$ which can be obtained in the following way: the sphere $\mathbf{S}^2$ is (isomorphic to) the complex projective line $\mathbf{P}^1(\mathbf{C})$ (the Riemann sphere); when one restricts the canonical projection $\mathbf{C}^2 \to \mathbf{P}^1(\mathbf{C})$ to the sphere $\mathbf{S}^3 \subset \mathbf{C}^2$ one obtains the Hopf fibration. The connection form α is (equivalent to) the restriction to $\mathbf{S}^3$ of the 1-form α' on $\mathbf{C}^2$ given by $\alpha' = i\sum_k (z_k \, \mathrm{d}\overline{z}_k - \overline{z}_k \, \mathrm{d}z_k)$ where (z_1, z_2) are the complex coordinates on $\mathbf{C}^2$ and $\overline{z}$ denotes the complex conjugate of z (see also §6). The bundles with $D = h\mathbf{Z}$ for $\lambda = \frac{1}{2}n\hbar$ can be obtained from the Hopf fibration by taking the n-fold tensor product of the Hopf fibration with itself.

§5.3 Duplicating diagram (2.3) in classical mechanics

The contents represented by diagram (2.3) can be stated as follows:

(i) in quantum mechanics the elements of $\mathbf{P}\mathcal{H}$ represent bijectively the states of a system (where we ignore super-selectionrules, non-physical states etc.).

(ii) There exists a principal fibre bundle $\mathbf{S}\mathcal{H}$ over $\mathbf{P}\mathcal{H}$ with structure group $U(1)$.

(iii) If G is a symmetry group of $(\mathbf{P}\mathcal{H}, P(\ ,\ ))$ then there exists a central $U(1)$ extension G' of G which is a symmetry group of $(\mathbf{S}\mathcal{H}, \langle\ ,\ \rangle)$.

(iv) The action of G' on $\mathbf{S}\mathcal{H}$ projects onto the action of G on $\mathbf{P}\mathcal{H}$ and the action of the kernel of $G' \to G \cong U(1)$ is the action of the structure group.
(v) With these properties G' is unique (up to equivalence of central extensions).

When we combine the previous two applications of the "central extensions of manifolds," we can formulate "exactly" the same results for classical mechanics, where the points of a symplectic manifold (M, ω) represent bijectively the states of a system.

Theorem 5.22. *Suppose (M, ω) is a symplectic manifold and $\mathrm{Per}(\omega)$ is discrete; suppose G is a connected Lie group which is a symmetry group of (M, ω) and admits a momentum map J, then for any discrete $D \supset \mathrm{Per}(\omega)$:*

(i) *There exists a principal fibre bundle (Y, α) with connection over (M, ω) (the prequantization bundle) with structure group $\mathbf{R}/D \cong U(1)$.*
(ii) *There exists a Lie group central $\mathbf{R}/D$ extension $G' \to G$ which is a symmetry group of (Y, α).*
(iii) *The action of G' on Y projects onto the action of G on M and the action of $\ker(G' \to G) \cong \mathbf{R}/D$ is the action of the structure group.*
(iv) *With these properties G' is unique up to equivalence.*

When given in a diagram this copies diagram (2.3) in the context of classical mechanics:

$$\begin{array}{ccc} G' & \rightsquigarrow & (Y, \alpha) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ G & \rightsquigarrow & (M, \omega)\ . \end{array} \tag{5.5}$$

Proof. According to proposition 4.7 (Y, α) exists, so we choose one (if there is any choice). Now choose any point $m \in M$ and define the evaluation map $E_m : G \to M$ by $E_m(g) = \Phi(g)(m)$, where $\Phi(g)$ is the action of the element $g \in G$ on M. Since each $\Phi(g)$ conserves the symplectic form ω, the pull back $\omega' = E_m^*\omega$ is a left invariant closed 2-form on G, hence it defines a Lie algebra extension of the Lie algebra $\mathfrak{g}$ of G. We now define G' as the pull back of the bundle Y by means of E_m, i.e., $G' = \{(g, y) \in G \times Y \mid E_m(g) = \pi(y)\}$, we denote by pr_i the projection of G' on the i-th coordinate ($i = 1, 2$), and we define the form $\alpha' = pr_2^*\alpha$. Tracing the construction of G', ω' and α' one can show that $pr_1 : G' \to G$ is a principal fibre bundle over G with structure group $\mathbf{R}/D$ for which α' is a connection with $\mathrm{d}\alpha' = \pi^*\omega'$ (and in particular one can show that $\mathrm{Per}(\omega') \subset \mathrm{Per}(\omega)$). Since the action of G on (M, ω) admits a momentum map J, it follows easily that the left action of G on (G, ω') admits a momentum map J' (just the composite of E_m with J), hence according to (the proof of) theorem 5.4 G' is a Lie group central extension of G associated to the Lie algebra extension defined by ω'.

To show that G' is a symmetry group of (Y, α) we use the ingredients of the proof of theorem 5.4. The element $\Lambda_{\exp(Xt)}(e')$ of G' is obtained by the flow through the identity e' of G' of the lifted vector field X' (the lift to an infinitesimal symmetry of (G', α') of the right invariant vector field X^r on G, uniquely defined by means of the momentum map J'; see §5.2); the action of this element on Y now is the flow (during a time t) of the unique vector field η_f defined by the function $f = J_X$ which is the image under the momentum map of $X \in \mathfrak{g}$, or in other words, η_f is the unique lift of the fundamental vector field X_M on M defined by the momentum map. If we define the action of the central extension subgroup $\mathbf{R}/D$ to be the action of the structure group $\mathbf{R}/D$ on the principal fibre bundle Y, then it is an easy excercise to show that we have indeed defined an action Φ' of G' (which is the pull back of Y!) on Y; moreover, this action leaves the 1-form α invariant because the action of the structure group and the flows of the vector fields η_f leave α invariant.

We finally outline that G' is unique, so suppose G'' is a second Lie group with the properties (ii), (iii) and (iv). We start with the choice of a fixed element $y \in Y$. Let $g' \in G'$, denote by $g = \pi(g') \in G$ then by property (iii) there is a unique $g'' \in G''$ such that $\pi(g'') = g$ and such that the images $g'(y)$ and $g''(y)$ are equal. Using the connectedness of M and the fact that G' and G'' leave the form α on Y invariant one can show that the action of g' is equal to the action of g'' on the whole of Y (not only on the fixed element y). It follows easily that the correspondence $g' \mapsto g''$ is an isomorphism of group extensions. To show that it is also a Lie group morphism one uses a local description of $G'^{/\prime\prime}$ in a neighbourhood of the identity as $G'^{/\prime\prime} \cong G \times \mathbf{R}/D$ and the properties (ii), (iii) and (iv) of $G'^{/\prime\prime}$. QED

Remark 5.23. The above construction of G' is carried out after the choice of a special point $m \in M$. However, one can show, independently of the proof of uniqueness of G', that the cohomology class of $\omega' = E_m^*\omega$ is locally constant as function of m ([So,3]), hence (since we assume that M is connected) the associated Lie algebra extensions are equivalent.

Remark 5.24. In the language of exact sequences as introduced for extensions and principal fibre bundles (remarks 3.2 and 4.9) diagram (5.5) can be enlarged to a "commutative" diagram in which the upper sequence denotes the Lie group extension and the lower sequence the principal fibre bundle:

$$
\begin{array}{ccccccccc}
 & & & & G' & \longrightarrow & G & \longrightarrow & \{1\} \\
 & & & \nearrow & & & & & \\
\{0\} & \longrightarrow & \mathbf{R}/D & & \wr\downarrow & & \wr\downarrow & & \\
 & & & \searrow & & & & & \\
 & & & & (Y,\alpha) & \longrightarrow & (M,\omega) & &
\end{array}
$$

Remark 5.25. One can formulate conditions on M or on G which guarantee the existence of a momentum map independent of the action of G on M and independent of the symplectic form ω ([So,1]). (i) If $H^1_{dR}(M,\mathbf{R}) = \{0\}$ then all closed 1-forms are exact, especially the 1-forms $\iota(X_M)\omega$, hence there exists a momentum map. (ii) If $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ (where $\mathfrak{g}$ is the Lie algebra of G), then there exists a momentum map (because of the identity $\iota([X_M,Y_M])\omega + \mathrm{d}\omega(X_M,Y_M) = 0$). In the general case the existence of the momentum map depends upon the triple (G,M,ω), G acting on M. If one of the above two conditions is satisfied and if $\mathrm{Per}(\omega)$ is discrete, then the conditions of theorem 5.22 are satisfied and we obtain diagram (5.5) as classical analogue of diagram (2.3).

Remark 5.26. When we compare theorem 5.22 with Wigner's theorem (theorem 2.1), one difference is obvious: in 2.1 a single symmetry can be lifted to the principal bundle (obtaining a unitary or anti-unitary transformation) and in 5.22 a whole group is lifted simultaneously provided some conditions are satisfied. Even in classical mechanics one can sometimes lift a single symmetry. If $g : M \to M$

is a diffeomorphism of M which leaves ω invariant and if (Y, α) is prequantization bundle over M with structure group $\mathbf{R}/D \subset \mathrm{Per}(\omega)$ then the pull-back g^*Y of Y by means of g is a prequantization bundle over M. Hence if $H^1_C(M, \mathbf{R}/D) = \{0\}$ then g^*Y is equivalent to Y and this equivalence defines a lift of g (see also [So,1,§18] and [Wo,§5.6]).

Example 5.27. Let M be $\mathbf{R}^{2n}$ with its standard symplectic form (example 5.20) and let G be the abelian group $\mathbf{R}^{2n}$ which acts as translations on M. Denoting the coordinates on M by (q, p) and on G by (a, b) we have $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i$ and $\Phi(a, b)(q, p) = (q + a, p + b)$, hence the pull back ω' of ω to G is the same for all $m \in M$: $\omega' = \mathrm{d}b_i \wedge \mathrm{d}a^i$. Combining examples 5.10 and 5.20 we see that the Lie group extension G' (which is the Heisenberg group) is the same as the prequantization bundle Y and that the action of G' on Y is the left action of G' on itself. Note that the extension G' in example 5.10 is defined by means of another connection form α than the prequantization bundle Y of example 5.20 (i.e., with $\frac{1}{2}(p\,\mathrm{d}q - q\,\mathrm{d}p) + \mathrm{d}x$ instead of $p\,\mathrm{d}q + \mathrm{d}x$), so one has to adapt one set of coordinates to get the exact correspondence.

Example 5.28. In this example we modify the previous example slightly (see also example 5.11): as symplectic manifold we use $M = T^*\mathbf{S}^1 = \mathbf{R}/\mathbf{Z} \times \mathbf{R}$ with the canonical symplectic form $\omega = \mathrm{d}p \wedge \mathrm{d}q$ and as symmetry group we take the abelian group $G = \mathbf{R} \times \mathbf{R}$ with the action Φ of G on M defined by $\Phi(a, b)(q \mod \mathbf{Z}, p) = (q + a \mod \mathbf{Z}, p + b)$. The pull back of the symplectic form ω to G is the standard left invariant form $\mathrm{d}b \wedge \mathrm{d}a$ on G which defines the Heisenberg extension. However, the action of G on M does not allow a momentum map so we can not apply theorem 5.22. What we can do is the following: since ω is exact there exists a prequantization bundle $Y = T^*\mathbf{S}^1 \times \mathbf{R}/\mathbf{Z}$ with connection 1-form $\alpha = p\,\mathrm{d}q + \mathrm{d}x$. When we ask which elements of G can be lifted to a diffeomorphism of Y which leaves the form α invariant and which commutes with the projection onto M (see remark 5.26), an easy calculation shows that only those elements $(a, b) \in G$ can be lifted for which $b \in \mathbf{Z}$. This fact should be compared with the fact that in quantum mechanics the linear momentum of a particle on a circle is quantised.

Example 5.29. The group $SO(3)$ acts as symmetries on the symplectic manifold $(\mathbf{S}^2, \omega)$ described in example 5.21, it admits a momentum map and $\mathrm{Per}(\omega)$ is discrete, so we can apply theorem 5.22. Although ω is not exact, the pull back to $SO(3)$ is (see example 5.12); however, the extension G' constructed as the pull back of the Hopf fibration over $\mathbf{S}^2$ is not the trivial extension (in the sense of either principal fibre bundles or groups), but the one obtained from $SU(2) \times U(1)$ by taking the quotient with respect to the normal subgroup $\{(id, 1), (-id, -1)\}$ (see also remark 4.15). This phenomenon is known in quantum mechanics where one says that for $SO(3)$ one can get rid of the phase factor (corollary 2.2) up to a sign.

If we use $SU(2)$ as symmetry group of $(\mathbf{S}^2, \omega)$ instead of $SO(3)$ (where we can visualise the action of $SU(2)$ on $\mathbf{S}^2$ either via the projection on $SO(3)$ or as the action of $SU(2)$ on $\mathbf{P}^1(\mathbf{C}) = \mathbf{S}^2$ induced by the action of $SU(2)$ on $\mathbf{C}^2$), then the extension G' defined by theorem 5.22 is necessarily the trivial extension $G' = SU(2) \times U(1)$ and the action of G' on the Hopf fibration $\mathbf{S}^3$ then is given as follows: $(A, \exp(i\vartheta)) \in SU(2) \times U(1)$ acts on $\mathbf{C}^2$ as first A and then $\exp(i\vartheta) \cdot id(\mathbf{C}^2)$; this is a unitary action on $\mathbf{C}^2$ so it induces an action on $\mathbf{S}^3$, which is the action of G' on Y as defined by theorem 5.22.

It is now easy to see how the non trivial extension of $SO(3)$ by $U(1)$ above arises: the element $(-id, 1) \in SU(2) \times U(1)$ does not act on $\mathbf{C}^2$ as the identity whereas $(-id, -1)$ does!

§6 More analogies

In this section we will show that the resemblance between diagram 2.3 and diagram 5.5 is not as superficial as one might think: quantum mechanics can be cast in the symplectic formalism in such a way that (i) $\mathbf{P}\mathcal{H}$ is a symplectic manifold (with a canonically defined symplectic form), (ii) $\mathbf{S}\mathcal{H}$ is the associated principal fibre bundle with connection over $\mathbf{P}\mathcal{H}$ and (iii) the Schrödinger equation is equivalent to the flow of a vector field η_f on the prequantization bundle $\mathbf{S}\mathcal{H}$ which is the lift of a Hamiltonian vector field on the phase space $\mathbf{P}\mathcal{H}$ (see also §5.2). To achieve these goals we will work in the category of smooth manifolds modeled on a Hilbert space; our basic references are [La] and [Ch&Ma].

6.1 The manifolds. Suppose $\mathcal{H}$ is a separable complex Hilbert space with inner product $\langle\ ,\ \rangle$ (which we assume to be linear in the second variable and anti-linear in the first), then it is clearly a manifold modeled on the underlying real Hilbert space and $\mathbf{S}\mathcal{H}$ is a submanifold of $\mathcal{H}$; moreover, the complex projective space $\mathbf{P}\mathcal{H}$ (the set of complex lines in $\mathcal{H}$) is also a manifold modeled on a Hilbert space for which the projection $\pi : \mathcal{H} \setminus \{0\} \to \mathbf{P}\mathcal{H}$ is a submersion. N.B. If $\mathcal{H} = \mathbf{C}^n$ then $\mathcal{H}$ is modeled on $\mathbf{R}^{2n}$, $\mathbf{S}\mathcal{H} = \mathbf{S}^{2n-1}$ is modeled on $\mathbf{R}^{2n-1}$ and $\mathbf{P}\mathcal{H} = \mathbf{P}^{n-1}(\mathbf{C})$ is modeled on $\mathbf{R}^{2n-2}$; if $\mathcal{H} = \ell^2(\mathbf{C})$ then $\mathcal{H}$, $\mathbf{S}\mathcal{H}$ and $\mathbf{P}\mathcal{H}$ are all modeled on $\ell^2(\mathbf{R})$.

6.2 Local charts. For any $m \in \mathcal{H}$ with $\langle m, m\rangle = 1$ we can define $U_m \subset \mathbf{P}\mathcal{H}$ as the set of lines $\mathbf{C}x$ in $\mathcal{H}$ with $\langle m, x\rangle \neq 0$, hence $\pi^{-1}(U_m) = \{x \in \mathcal{H} \mid \langle m, x\rangle \neq 0\}$; defining the (complex) Hilbert space $\mathcal{H}'$ by $\mathcal{H}' = \{x \in \mathcal{H} \mid \langle m, x\rangle = 0\}$ (all $\mathcal{H}'$ for different m are isomorphic!) then $\mathbf{P}\mathcal{H}$ is modeled on $\mathcal{H}'$ and the projection $\pi : \pi^{-1}(U_m) \to U_m \cong \mathcal{H}'$ is given by:

$$\pi : x \mapsto z = \langle m, x\rangle^{-1} \cdot x - m \ .$$

When we restrict the projection π to the unit sphere $\mathbf{S}\mathcal{H}$, the local chart $U_m \cong \mathcal{H}'$ defines a local chart $\mathcal{H}' \times U(1)$ for $\mathbf{S}\mathcal{H}$ ($\mathbf{S}\mathcal{H}$ is a $U(1)$ bundle over $\mathbf{P}\mathcal{H}$) by:

$$\mathcal{H}' \times U(1) \ni (z, \mathrm{e}^{i\vartheta}) \leftrightarrow (1 + \langle z, z\rangle)^{-1/2} \cdot \mathrm{e}^{i\vartheta} \cdot (z + m) \in \mathbf{S}\mathcal{H} \subset \mathcal{H} \ , \tag{6.1}$$

where one has to recall that $z \in \mathcal{H}' \Rightarrow \langle m, z\rangle = 0$. It is easy to see that the action of the structure group $U(1)$ of the principal fibre bundle $\mathbf{S}\mathcal{H}$ over $\mathbf{P}\mathcal{H}$ is given on the local chart $\mathcal{H}' \times U(1)$ just by multiplication in the second coordinate, hence these charts are bundle charts.

6.3 Connection and symplectic forms. On the complex Hilbert space $\mathcal{H}$ one has the canonically defined real symplectic form σ defined by $\sigma_x(v, w) = 2\,\mathrm{Im}\langle v, w\rangle$ where we have identified $T_x\mathcal{H}$ with $\mathcal{H}$. However, this form is **not** the pull back of a 2-form on $\mathbf{P}\mathcal{H}$, so if we want to define a symplectic form on $\mathbf{P}\mathcal{H}$ we have to be more careful. When we restrict the 2-form σ to the unit sphere $\mathbf{S}\mathcal{H}$ we obtain a 2-form on $\mathbf{S}\mathcal{H}$ which **is** the pull back of a 2-form ω on $\mathbf{P}\mathcal{H}$ (a pull back to $\mathbf{S}\mathcal{H}$)! On the local chart $\mathcal{H}'$ of $\mathbf{P}\mathcal{H}$ the form ω is given by

$$\omega_z(v, w) = \frac{\langle v, w\rangle - \langle w, v\rangle}{i(1 + \langle z, z\rangle)} - \frac{\langle v, z\rangle\langle z, w\rangle - \langle w, z\rangle\langle z, v\rangle}{i(1 + \langle z, z\rangle)(1 + \langle z, z\rangle)} \ ,$$

or in terms of forms by

$$\omega_z = \frac{\langle \mathrm{d}z \wedge \mathrm{d}z\rangle}{i(1+\langle z,z\rangle)} - \frac{\langle \mathrm{d}z, z\rangle \wedge \langle z, \mathrm{d}z\rangle}{i(1+\langle z,z\rangle)(1+\langle z,z\rangle)} \ .$$

Since σ is closed, it is closed on $\mathbf{S}\mathcal{H}$ and because $\pi^*\omega = \sigma$ (on $\mathbf{S}\mathcal{H}$!) ω is closed. Moreover, it is not hard to show that ω is strongly symplectic in the sense that the map $v \mapsto \iota(v)\omega_z$ from $T_z\mathbf{P}\mathcal{H}$ to $T_z^*\mathbf{P}\mathcal{H}$ is an isomorphism.

The form σ is not only closed, but also exact: it is the exterior derivative of the 1-form α on $\mathcal{H}$ defined by $\alpha_x(v) = \mathrm{Im}\langle x, v\rangle$. When we restrict this form to $\mathbf{S}\mathcal{H}$, we can express it in terms of the coordinates $(z, \mathrm{e}^{i\vartheta})$ defined by formula 6.1:

$$\alpha_{(z,\vartheta)}(v, \tau\partial_\vartheta) = \tau + \frac{\langle z, v\rangle - \langle v, z\rangle}{2i(1+\langle z,z\rangle)}$$

or in terms of forms:

$$\alpha_{(z,\vartheta)} = \mathrm{d}\vartheta + \frac{\langle z, \mathrm{d}z\rangle - \langle \mathrm{d}z, z\rangle}{2i(1+\langle z,z\rangle)} \ .$$

From this formula it is clear that α restricted to $\mathbf{S}\mathcal{H}$ is a connection 1-form on the principal fibre bundle $\mathbf{S}\mathcal{H}$ over $\mathbf{P}\mathcal{H}$ with $\mathrm{d}\alpha = \pi^*\omega$. Moreover, one can show [vE,3] that in all cases (finite or infinite dimensional) $\mathrm{Per}(\omega) = 2\pi\mathbf{Z}$, showing that $(\mathbf{S}\mathcal{H}, \alpha)$ is a prequantization of the (strongly) symplectic manifold $(\mathbf{P}\mathcal{H}, \omega)$ in the sense of §5.2 with $D = \mathrm{Per}(\omega)$. Because the complex projective spaces are simply connected, this prequantization is unique.

Remark 6.4. In the finite dimensional case $\mathcal{H} = \mathbf{C}^n$ the form α is usually given in the complex coordinates $(x_1, \dots, x_n)$ by:

$$\alpha_x = \frac{i}{2} \cdot \sum_j (x_j \mathrm{d}\overline{x}_j - \overline{x}_j \mathrm{d}x_j) \ ,$$

where $\overline{x}$ denotes as usual the complex conjugate of z. In real coordinates (p, q) with $x_j = p_j + iq_j$ the form α is expressed as $\sum_j (p_j \,\mathrm{d}q_j - q_j \,\mathrm{d}p_j)$ showing that (up to a factor 2) the symplectic form σ on $\mathcal{H}$ is the canonical symplectic form on $\mathbf{R}^{2n}$.

In the case $n = 2$ the associated projective space $\mathbf{PC}^2 = \mathbf{P}^1(\mathbf{C}) = \mathbf{S}^2$ is the Riemann sphere and the symplectic form ω defined above is the same one as the symplectic form defined in example 5.21 with $\lambda = \frac{1}{2}$.

Remark 6.5. As already said, σ is not the pull back of ω, so one might ask which form is the pull back of ω:

$$(\pi^*\omega)_x = \frac{\langle \mathrm{d}x \wedge \mathrm{d}x\rangle}{i\langle x, x\rangle} - \frac{\langle \mathrm{d}x, x\rangle \wedge \langle x, \mathrm{d}x\rangle}{i\langle x,x\rangle\langle x,x\rangle} = \mathrm{d}\frac{\langle x, \mathrm{d}x\rangle - \langle \mathrm{d}x, x\rangle}{2i\langle x,x\rangle} = \mathrm{d}\beta \ .$$

This pull back is not defined in the origin, which is obvious since the projection to $\mathbf{P}\mathcal{H}$ is only defined on non-zero vectors. We also have a projection from $\mathcal{H} \setminus \{0\}$ to the unit sphere $\mathbf{S}\mathcal{H}$ ($x \mapsto x/\|x\|$) and the 1-form β above is the pull back to $\mathcal{H} \setminus \{0\}$ of the restriction to $\mathbf{S}\mathcal{H}$ of the 1-form α on $\mathcal{H}$.

6.6 The Schrödinger equation. Let $\mathbf{H}$ be a self adjoint operator on $\mathcal{H}$ which defines the time evolution of the quantum system by means of the Schrödinger equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = \frac{i}{\hbar} \cdot \mathbf{H}x(t) \ .$$

We can view this equation as the equation of the flow of a vector field v on $\mathcal{H}$ defined by $v_x = \frac{i}{\hbar}\mathbf{H}x$, which poses only one problem: this vector field is not defined for all x (in general). However, our point of view is that the important feature of a vector field is the flow it determines and a self adjoint operator defines a flow (even a 1-parameter group of unitary transformations of $\mathcal{H}$), hence we will not be bothered by the fact that in the sequel some functions and vector fields are not defined everywhere.

The important property of the vector field v associated to the Schrödinger equation is that it projects down to $\mathbf{P}\mathcal{H}$, more precisely: denote by H the real function on $\mathbf{P}\mathcal{H}$ defined by the equation

$$(\pi^* H)(x) = \frac{\langle x, \mathbf{H}x\rangle}{\langle x, x\rangle} \qquad \text{(the expectation value of H in state x)},$$

then the vector field v obeys the equation

$$\iota(v)(\pi^*\hbar\omega) + \mathrm{d}\pi^* H = 0 \ ,$$

which shows that $\pi_* v$ is the Hamiltonian vector field on $\mathbf{P}\mathcal{H}$ with respect to the symplectic form $\hbar\omega$ associated to the observable H. We can do even more: the restriction of v to the unit sphere $\mathbf{S}\mathcal{H}$ satisfies $\hbar\alpha(v) = \pi^* H$, hence v (restricted to $\mathbf{S}\mathcal{H}$) is the unique lift (proposition 4.17) of the Hamiltonian vector field associated to the observable $H : \mathbf{P}\mathcal{H} \to \mathbf{R}$. The above results are summarised in the following proposition.

Proposition 6.7. *When we apply the process of prequantization of §4 and §5.2 to the symplectic manifold* $(\mathbf{P}\mathcal{H}, \hbar\omega)$ *with discrete* $D = \mathrm{Per}(\hbar\omega) = \hbar\mathbf{Z}$*, then we obtain the unique prequantization bundle* $(\mathbf{S}\mathcal{H}, \hbar\alpha)$ *over* $\mathbf{P}\mathcal{H}$*. Moreover, if* $\mathbf{H}$ *is any self adjoint operator on* $\mathcal{H}$ *and* $H : \mathbf{P}\mathcal{H} \to \mathbf{R}$ *the associated expectation value function on* $\mathbf{P}\mathcal{H}$*, then the unique lift of the Hamiltonian vector field* ξ_H *on* $\mathbf{P}\mathcal{H}$ *to a vector field* η_H *on* $\mathbf{S}\mathcal{H}$ *is the restriction of the Schrödinger equation to the unit sphere* $\mathbf{S}\mathcal{H}$*.*

Remark 6.8. Applying the process of geometric quantization to the symplectic manifold $(\mathbf{P}\mathcal{H}, \hbar\omega)$ we can do even better: using the anti-holomorphic polarization on the Kähler manifold $(\mathbf{P}\mathcal{H}, \hbar\omega)$ the Hilbert space derived by means of geometric quantization is the same as the original Hilbert space $\mathcal{H}$ we started with.

For those who are familiar with geometric quantization we briefly outline the proof of the above statement. The Hilbert space constructed by geometric quantization consists of $\mathbf{C}$-valued smooth functions f on the prequantization bundle satisfying certain conditions, among which is the condition that $f(\mathrm{e}^{i\vartheta}\, x) = \mathrm{e}^{i\vartheta}\, f(x)$. Now for any $y \in \mathcal{H}$ we can define a function f_y on $\mathbf{S}\mathcal{H}$ by $f_y(x) = \langle y, x\rangle$ which has this property. If we can show that these functions are the only ones, then we have proven our claim. Therefore let $f : \mathbf{S}\mathcal{H} \to \mathbf{C}$ be a smooth function satisfying $f(\mathrm{e}^{i\vartheta}\, x) = \mathrm{e}^{i\vartheta}\, f(x)$, then we can define a smooth function $f^\bullet$ on $\mathcal{H} \setminus \{0\}$ by $f^\bullet(x) = \|x\| f(x/\|x\|)$ where $\|x\| = \sqrt{\langle x, x\rangle}$ denotes the norm of x. The function

$f^\bullet$ satisfies the equation $f^\bullet(\lambda x) = \lambda f^\bullet(x)$ for $\lambda \in \mathbf{C}$ and is obviously smooth on $\mathcal{H} \setminus \{0\}$. Having chosen the anti-holomorphic polarization, the condition that f should be covariantly constant in the direction of this polarization together with the homogeneity property translates to the condition that $f^\bullet$ should be holomorphic on $\mathcal{H} \setminus \{0\}$, i.e., the (real) derivative of $f^\bullet$ should be complex linear. Now take any two vectors $x, y \in \mathcal{H} \setminus \{0\}$, then the function $f^\sharp$ defined on $\mathbf{C}^2 \setminus \{0\}$ by $f^\sharp(\lambda, \mu) = f^\bullet(\lambda x + \mu y)$ is holomorphic, hence by Hartog's theorem it is defined on the whole of $\mathbf{C}^2$; since it is homogeneous of degree 1 it is necessarily complex linear, which shows that the original function $f^\bullet$ is complex linear. Finally $f^\bullet$ is linear and smooth on $\mathcal{H} \setminus \{0\}$, hence it is continuous in 0, so it defines an element y of $\mathcal{H}$ such that $f^\bullet$ is given by $f^\bullet(x) = \langle y, x\rangle$.

One final remark: apart from the condition that the function f should be covariantly constant in the direction of the polarization, the process of geometric quantization imposes the condition that f should be square integrable over the symplectic manifold. In the finite dimensional case this poses no extra condition on the functions f, but in the infinite dimensional case one should integrate over the infinite dimensional non-compact manifold $\mathbf{P}\mathcal{H}$, which is (as far as we know) not defined. This missing procedure is the only shadow on the above result that geometric quantization applied to $(\mathbf{P}\mathcal{H}, \hbar\omega)$ recovers the original Hilbert space $\mathcal{H}$.

6.9 On Wigner's theorem. Now that we know that quantum mechanics can be formulated as symplectic mechanics, we have two definitions of symmetry: a quantum mechanical one which tells us that a diffeomorphism $g : \mathbf{P}\mathcal{H} \to \mathbf{P}\mathcal{H}$ is a symmetry if it conserves transition probabilities (the absolute value squared of the inner product, see §2) and a classical one which tells us that g is a symmetry if it leaves the symplectic form $\hbar\omega$ invariant. The question arises naturally whether these two definitions coincide, or more generally, whether there is any connection between these two definitions.

In the finite dimensional case ($\dim \mathcal{H} < \infty$) this question can be answered: a diffeomorphism g conserves transition probabilities if and only if g leaves the symplectic form invariant and conserves the complex structure (i.e., g should be holomorphic). The only if part is an easy consequence of Wigner's theorem (where we forget the possibility that the lift to $\mathcal{H}$ can be anti-unitary because we are always interested in connected Lie groups of symmetries). The if part requires (a consequence of) a theorem of Chow: if g is a holomorphic diffeomorphism of $\mathbf{P}^n(\mathbf{C})$ then it is projective linear, i.e., it is induced by a linear isomorphism of $\mathbf{C}^{n+1}$. Using this theorem it is an elementary calculation to show that if g also conserves the symplectic form, then it is induced by a unitary transformation of $\mathcal{H}$, which implies that g conserves the transition probabilities.

In the infinite dimensional case we do not know of an equivalent of Chow's theorem, so (for us) it is an open question whether there exists such a relation between the two definitions of symmetries.

§7 Summary and discussion

In the previous sections we have studied two questions; the first one a mathematical problem: given a Lie group G with Lie algebra $\mathfrak{g}$, what are necessary and sufficient conditions on a Lie algebra central extension (of dimension 1) of $\mathfrak{g}$ such that there exists a Lie group central extension of G by a 1-dimensional Lie group associated to this algebra extension? The second question was a question in physics: in quantum mechanics a symmetry group G does not a priori act on the Hilbert space, but only upon the projective Hilbert space; using Wigner's theorem one shows that a central extension of G (by $U(1)$) acts as (unitary) transformations on the Hilbert space (diagram (2.3)). We asked if an analogous situation occurs in classical mechanics.

We showed that the answers to both questions involved the same construction: a central extension of a manifold; more precisely: the construction of a principal fibre bundle with connection over the manifold with specified curvature form[1]. Moreover, the answers are "the same:" given a Lie algebra extension defined by a closed left invariant 2-form ω, a Lie group extension exists if and only if the group of periods of ω is discrete and if ω admits a momentum map (for the left action of G on itself); diagram (2.3) in quantum mechanics can be duplicated in classical mechanics provided that the group of periods of the symplectic form ω on the phase space is discrete (to guarantee the existence of a prequantum bundle) and admits a momentum map. It should not come as a surprise that the answers are "the same" because the central extension of a symmetry group G in the classical analogue of diagram (2.3) is constructed with as closed left invariant 2-form the pull back of the symplectic form.

Although we can duplicate diagram (2.3) for classical mechanics, there are some important differences. Wigner's theorem tells us that for every symmetry group there exists a central extension which acts on the Hilbert space; in classical mechanics not every symmetry group admits a central extension which acts on the prequantum bundle. The example to illustrate this difference is a 1-dimensional particle with the circle $\mathbf{S}^1$ as configuration space (hence $T^*\mathbf{S}^1$ as phase space: example 5.28): the classical symmetry group of translations in position and momentum does not admit a momentum map, hence cannot be "lifted" to the prequantum bundle. In quantum mechanics this group is not a symmetry group; the 1-dimensional group of translations in position is a quantum mechanical symmetry group (with the momentum operator as infinitesimal generator), but a position operator does not exist. However, these differences disappear and the analogy between quantum mechanics and the prequantization formulation of classical mechanics becomes striking if we realise that translations in position admits clasically a momentum map and can hence be "lifted," that a discrete subset of the translations in momentum can be lifted to the prequantum bundle (example 5.28), and that there exists in quantum mechanics a discrete symmetry group (of unitary transformations on the Hilbert space) representing shifts in the discrete spectrum of the momentum operator. Hence, although Wigner's theorem and theorem 5.22 differ superficially in the conditions imposed on the symmetry group, nevertheless upon closer scrutiny they offer striking analogies between quantum mechanics and the prequantization formalism (i.e., the construction of the prequantum bundle over the phase space). In the last section we showed that the analogy between quantum mechanics and the prequantization formalism can be extended even further: quantum mechanics itself can be formulated as a classical system. Its phase space is the (infinite dimensional) projective Hilbert space with its canonical (strongly) symplectic form and the Schrödinger equation defines the flow of a Hamiltonian vector field on the pro-

[1]The question about the relation of arbitrary abelian extensions (not only central ones) of Lie groups and Lie algebras can be treated in the same way, requiring a minor modification of the construction of "extensions of manifolds."

jective Hilbert space associated to the (real) function which gives the expectation value of the Hamilton operator. Moreover, the unit sphere in the Hilbert space is the unique prequantum bundle over the quantum mechanical phase space, which shows that the prequantum formulation of classical mechanics fits naturally into the quantum theory. Last but not least: application of the procedure of geometric quantization to the classical system defined by the projective Hilbert space yields the original Hilbert space (modulo some technical "details"), which completes the circle: from quantum mechanics with a Hilbert space to classical mechanics in the projective Hilbert space to quantum mechanics in the original Hilbert space by means of geometric quantization.

Finally we want to point out another use of the general method of central extensions of manifolds in physics, one which is closely related with prequantization (in fact it is prequantization but not on a symplectic manifold). The usual description of classical mechanics is by means of a symplectic manifold (M, ω) (the phase space) and a Hamiltonian function H (the energy, whose Hamiltonian vector field determines the time evolution). An equivalent description is by means of an evolution space $E = M \times \mathbf{R}$ with a closed 2-form $\sigma = \omega - \mathrm{d}H \wedge \mathrm{d}t$ where t is the coordinate on the time axis $\mathbf{R}$ in $E = M \times \mathbf{R}$. The time evolution of the system then is described by the kernel of this 2-form σ: the trajectories (in time) of the system are the integral curves of the 1-dimensional kernel of σ (this description of classical mechnics is strongly favored by Souriau [So,1]; we will show some reasons why). One of the advantages of this description is that the Hamiltonian function H now may depend on the time t: we can describe in a natural way time dependent systems. In this context a symmetry group is a group of diffeomorphisms of E which leave σ invariant (i.e., if they arise from symmetries on the symplectic manifold they have to be diffeomorphisms which leave ω and H invariant). Again the advantage is that time dependent symmetries can be described easily.

Since σ is a closed 2-form we can construct a prequantum bundle (Y, α) over the evolution space (E, σ) according to the construction of §4 (assuming that the group of periods of σ is discrete, which it is if ω has discrete periods). If G is a symmetry group of E which admits a momentum map, then this defines a central extension G' of G which is a symmetry group of Y; the construction is exactly the same as in the symplectic case: G' is the pull-back of Y (see theorem 5.22). Let us explain why this is so interesting: if the Galilei group Gal (which contains time dependent transformations!) is a symmetry group of the classical system described by the evolution space E, then the existence of a momentum map exhibits all 10 conserved Noether quantities (not only the widely known 7 conserved quantities: total energy (1), total linear momentum (3) and total angular momentum (3); see also [So,1]). Moreover, the central extension Gal' which acts upon the prequantization bundle Y is the Bargmann group, which usually occurs in quantum physics as the symmetry group of the non-relativistic Schrödinger equation. Finally, the prequantization bundle Y we obtain in this fashion is exactly the same prequantization bundle one uses in geometric quantization when one applies this quantization method to time dependent systems (see [Sn,§9] and references cited there).

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