A spectral sequence for the homology of a finite algebraic delooping

Birgit Richter
joint work in progress with Stephanie Ziegenhagen

Lille, October 2012
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Aim:
1) Approximate Quillen homology for $E_n$-algebras by Quillen homology of Gerstenhaber algebras.
2) Reduce this further to Quillen homology of graded Lie-algebras and of commutative algebras, aka Andr´ e-Quillen homology.
3) Apply this for instance to the Hodge decomposition of higher order Hochschild homology (in the sense of Pirashvili).
$E_n$-homology

A resolution spectral sequence

A Blanc-Stover spectral sequence

Hodge decomposition
Little $n$-cubes

Let $C_n$ denote the operad of little $n$-cubes. Then $(C_* C_n(r))_r, \ r \geq 1$ is an operad in the category of chain complexes. Let $E_n$ be a cofibrant replacement of $C_* C_n$. 

Theorem [Fresse 2011] There is an $n$-fold bar construction for $E_n$-algebras, $B_n$, such that $H_{E_n s}(\bar{A}_*) \cong H_s(\Sigma^{-n} B_n(\bar{A}_*))$. 

I.e., $E_n$-homology is the homology of an $n$-fold algebraic delooping.
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Let $C_n$ denote the operad of little $n$-cubes. Then $(C_\ast C_n(r))_r$, $r \geq 1$ is an operad in the category of chain complexes. Let $E_n$ be a cofibrant replacement of $C_\ast C_n$. For an augmented $E_n$-algebra $A_\ast$ let $\bar{A}_\ast$ denote the augmentation ideal.
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The $s$th $E_n$-homology group of $\bar{A}_*$, $H^E_n(s)(\bar{A}_*)$ is then the $s$th derived functor of indecomposables of $\bar{A}_*$. 

I.e., it is Quillen homology of the $E_n$-algebra $A_*$. 

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The $s$th $E_n$-homology group of $\bar{A}_\ast$, $H^E_{s,n}(\bar{A}_\ast)$ is then the $s$th derived functor of indecomposables of $\bar{A}_\ast$. I.e., it is Quillen homology of the $E_n$-algebra $A_\ast$. 

Theorem [Fresse 2011]

There is an $n$-fold bar construction for $E_n$-algebras, $B_n$, such that $H^E_{s,n}(\bar{A}_\ast) \sim H_s(\Sigma^{-n} B_n(\bar{A}_\ast))$. I.e., $E_n$-homology is the homology of an $n$-fold algebraic delooping.
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Some results

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Fresse (2011): $X$ a nice space: $B^n(C^*(X))$ determines the cohomology of $\Omega^n X$. 

Livernet-Richter (2011): Functor homology interpretation for $H^E_n$ for augmented commutative algebras. $H^E_n(\overline{A}^*) \sim = \text{HH}[^n*](A;k), \text{Hochschild homology of order } n \text{ in the sense of Pirashvili.}$

Can we gain information about $\text{HH}[^n*](A;k)$, at least rationally?

What is $H^E_n(\overline{A}^*)$ in other interesting cases such as Hochschild cochains, $A^* = C^*(B, B)$, or $A^* = C^*(\Omega^n X)$?
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$H_{*}^{E_{n}}(\overline{A}) \cong HH_{*+n}^{[n]}(A; k)$, Hochschild homology of order $n$ in the sense of Pirashvili.
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Cartan (50s): $H^E_{\ast n}$ of polynomial algebras, exterior algebras and some more.
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$H^E_{\ast n}(\overline{A}) \cong HH^{[n]}_{\ast + n}(A; k)$, Hochschild homology of order $n$ in the sense of Pirashvili.
Can we gain information about $HH^{[n]}_{\ast}(A; k)$, at least rationally?
What is $H^E_{\ast n}(\overline{A}_\ast)$ in other interesting cases such as Hochschild cochains, $A_\ast = C^\ast(B, B)$, or $A_\ast = C_\ast(\Omega^n X)$?
In the following $k$ is a field, most of the times $k = \mathbb{F}_2$ or $k = \mathbb{Q}$. The underlying chain complex of $A_*$ is non-negatively graded.
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1-restricted Lie algebras

**Definition**

A *1-restricted Lie algebra over* $\mathbb{F}_2$ *is a non-negatively graded* $\mathbb{F}_2$-*vector space*, $g_*$, together with two operations, a Lie bracket of degree one, $[-,-]$ and a restriction, $\xi$:

\[
[-,-] : \ g_i \times g_j \rightarrow g_{i+j+1}, \quad i,j \geq 0,
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\xi : \ g_i \rightarrow g_{2i+1} \quad i \geq 0.
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These satisfy the relations

1. The bracket is bilinear, symmetric and satisfies the Jacobi relation

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[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \text{ for all homogeneous } a, b, c \in \mathfrak{g}_*.
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1-rL: The category of 1-restricted Lie algebras.
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2. The restriction interacts with the bracket as follows:

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- (Poisson relation)

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\xi(ab) = a^2 \xi(b) + \xi(a)b^2 + ab[a, b] \quad \text{for all homogeneous } a, b \in G_*.
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1-rG: the category of 1-restricted Gerstenhaber algebras.
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1-rG: the category of 1-restricted Gerstenhaber algebras.
In particular, the bracket and the restriction annihilate squares: 
\[ [a, b^2] = 2b[a, b] = 0 \text{ and } \xi(a^2) = 2a^2 \xi(a) + a^2[a, a] = 0. \] Thus if 1 denotes the unit of the algebra structure in $G_*$, then $[a, 1] = 0$ for all $a$ and $\xi(1) = 0$. 
Free objects and indecomposables

For a graded vector space $V_*$ let $1rL(V_*)$ be the free 1-restricted Lie algebra on $V_*$. 

The free graded commutative algebra $S(1rL(V_*))$ has a well-defined $1rG$ structure and is the free $1rG$ Gerstenhaber algebra generated by $V_*$. 

For $G_* \in 1rG$ let $Q_{1rG}(G_*)$ be the graded vector space of indecomposables. 

Note: $Q_{1rG}(G_*) = Q_{1rL}(Q_a(G_*)).$
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Homology of free objects

Lemma

\[ H_*(E_2(\bar{A}_*)) \cong 1rG(H_*(\bar{A}_*)). \]
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Proof: Let \( X \) be a space. We have Cohen’s identification of \( H_*(C_2(X); \mathbb{F}_2) \).
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Observation by Haynes Miller: \( H_*(C_2(X); \mathbb{F}_2) \cong 1rG(\bar{H}_*(X; \mathbb{F}_2)) \).

(Dyer-Lashof operations only give algebraic operations.)
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Lemma

\[ H_* (E_2(\overline{A}_*)) \cong 1rG(H_* (\overline{A}_*)). \]

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(Dyer-Lashof operations only give algebraic operations.)
Take \( X \) with \( \overline{H_* (X; \mathbb{F}_2)} \cong H_* (\overline{A}_*) \), then

\[
H_* (E_2(\overline{A}_*)) \cong \bigoplus_r H_* (E_2(r) \otimes_{\mathbb{F}_2[\Sigma_r]} H_* (\overline{A}_*) \otimes r)
\]

\[
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\]

\[
\cong H_* (C_2(X); \mathbb{F}_2) \cong 1rG(H_* \overline{A}_*). \]
Theorem
There is a spectral sequence

\[ E_{p,q}^2 \cong (\mathbb{L}_p Q_1 r\mathbb{G}(H_* (\bar{A}_*)))_q \Rightarrow H_{p+q}^{E_2} (\bar{A}_*). \]
Resolution spectral sequence

**Theorem**
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\[ E^2_{p,q} \cong (\mathbb{L}_p Q_{1_{rG}}(H_*(\bar{A}_*))))_q \Rightarrow H^{E_2}_{p+q}(\bar{A}_*). \]

Proof: Standard resolution \( E^{\bullet+1}_2(\bar{A}_*) \).
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Proof: Standard resolution \( E_2^{\bullet+1}(\overline{A}_*) \).

\[ E^1_{p,q} : H^E_q(E^p_{2+1}(\overline{A}_*)) \cong H_q(E^p_2(\overline{A}_*)) \]
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\[ H_q(E^p_2(\bar{A}_*)) \cong 1rG^p(H_*(\bar{A}_*))_q \cong Q_{1rG}(1rG^{p+1}(H_*(\bar{A}_*)))_q. \]

\( d^1 \) takes homology wrt resolution degree.
For $X$ connected:

$$(\mathbb{L}_p Q_{1rG}(H_{\ast}(C_{\ast}(\Omega^2 \Sigma^2 X; \mathbb{F}_2))))_{\ast} = (\mathbb{L}_p Q_{1rG}(1rG(\tilde{H}_{\ast}(X; \mathbb{F}_2))))_{\ast}.$$
For $X$ connected:

\[
(\mathbb{L}_p Q_{1rG}(H_*(C_*(\Omega^2\Sigma^2 X; \mathbb{F}_2))))_* = (\mathbb{L}_p Q_{1rG}(1rG(\tilde{H}_*(X; \mathbb{F}_2))))_*.
\]

This reduces to $\tilde{H}_q(X; \mathbb{F}_2)$ in the $(p = 0)$-line and

\[
H^{E_2}_q(\tilde{C}_*(\Omega^2\Sigma^2 X; \mathbb{F}_2)) \cong \tilde{H}_q(X; \mathbb{F}_2).
\]
Rational case

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The rational case is much easier:

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the free $n$-Gerstenhaber algebra generated by the homology of $\tilde{A}_\ast$. We get:

\[ E^2_{p,q} \cong (\mathbb{L}_p Q nG(H_\ast (\tilde{A}_\ast)))_q \Rightarrow H^{E_{n+1}}_{p+q} (\tilde{A}_\ast) \]

for every $E_{n+1}$-algebra $\tilde{A}_\ast$ over the rationals.
Let $C$ and $B$ be some categories of graded algebras (e.g., Lie, Com, $n$-Gerstenhaber etc.) and let $A$ be a concrete category (such as graded vector spaces) and $T : C \to B$, $S : B \to A$.  

▶ Note: $T$, $S$ non-additive.

▶ $\overline{S}t(\pi^*B) = \pi t(SB)$ if $B$ is free simplicial; otherwise it is defined as a coequaliser.

▶ $\overline{S}$ takes the homotopy operations on $\pi^*B$ into account ($B$ a simplicial object in $B$): $\pi^*B$ is a $\Pi$-$B$-algebra.

▶ $B = \text{Com}$: $\pi^*(B)$ has divided power operations.

▶ $B = r\text{Lie}$: $\pi^*(B)$ inherits a Lie bracket and has some extra operations.
General Blanc-Stover setting

Let $\mathcal{C}$ and $\mathcal{B}$ be some categories of graded algebras (e.g., Lie, Com, $n$-Gerstenhaber etc.) and let $\mathcal{A}$ be a concrete category (such as graded vector spaces) and $T: \mathcal{C} \to \mathcal{B}$, $S: \mathcal{B} \to \mathcal{A}$. If $TF$ is $S$-acyclic for every free $F$ in $\mathcal{C}$, then there is a Grothendieck composite functor spectral sequence for all $C$ in $\mathcal{C}$

$$E_{s,t}^2 = (\mathbb{L}_s \bar{S}_t)(\mathbb{L}_s T)C \Rightarrow (\mathbb{L}_{s+t}(S \circ T))C.$$
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\[
E^2_{s,t} = (\mathbb{L}_s \bar{S}_t)(\mathbb{L}_* T)C \Rightarrow (\mathbb{L}_{s+t}(S \circ T))C.
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- Note: $T, S$ non-additive.
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Let $\mathcal{C}$ and $\mathcal{B}$ be some categories of graded algebras (e.g., Lie, Com, $n$-Gerstenhaber etc.) and let $\mathcal{A}$ be a concrete category (such as graded vector spaces) and $T: \mathcal{C} \to \mathcal{B}$, $S: \mathcal{B} \to \mathcal{A}$. If $TF$ is is $S$-acyclic for every free $F$ in $\mathcal{C}$, then there is a Grothendieck composite functor spectral sequence for all $C$ in $\mathcal{C}$

$$E^2_{s,t} = (\mathbb{L}_s \tilde{S}_t)(\mathbb{L}_t S)C \Rightarrow (\mathbb{L}_{s+t}(S \circ T))C.$$ 

▶ Note: $T, S$ non-additive.

▶ $\tilde{S}_t(\pi_* B) = \pi_t(SB)$ if $B$ is free simplicial; otherwise it is defined as a coequaliser.

▶ $\tilde{S}$ takes the homotopy operations on $\pi_* B$ into account ($B$ a simplicial object in $\mathcal{B}$): $\pi_* B$ is a $\Pi_\mathcal{B}$-algebra.

▶ $\mathcal{B} = \text{Com}$: $\pi_*(B)$ has divided power operations.
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- $\tilde{S}_t(\pi_* B) = \pi_t (SB)$ if $B$ is free simplicial; otherwise it is defined as a coequaliser.
- $\tilde{S}$ takes the homotopy operations on $\pi_* B$ into account ($B$ a simplicial object in $\mathcal{B}$): $\pi_* B$ is a $\Pi$-$\mathcal{B}$-algebra.
- $\mathcal{B} = \text{Com}$: $\pi_*(B)$ has divided power operations.
- $\mathcal{B} = r\text{Lie}$: $\pi_* B$ inherits a Lie bracket and has some extra operations.
In our case

**Theorem**

- \( k = \mathbb{F}_2 \): For any \( C \in 1rG \):

\[
E^2_{s,t} = \mathbb{L}_s((\bar{Q}_{1rL})^t)(AQ_* (C|\mathbb{F}_2, \mathbb{F}_2)) \Rightarrow \mathbb{L}_{s+t}(Q_{1rG})(C).
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- For \( k = \mathbb{Q} \) we get for all \( n \)-Gerstenhaber algebras \( C \):
  \[
  \mathbb{L}_s((\bar{Q}_{nL})_t)(AQ_*(C|\mathbb{Q}, \mathbb{Q})) \Rightarrow \mathbb{L}_{s+t}(Q_{nG})(C).
  \]
Hodge decomposition for $k = \mathbb{Q}$

Let $HH_{\ast}^{[n]}(A; \mathbb{Q})$ denote Hochschild homology of order $n$ or $A$ with coefficients in $\mathbb{Q}$. Over $\mathbb{Q}$, $HH_{\ast}^{[n]}(A; \mathbb{Q})$ has a decomposition, the Hodge decomposition:
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Theorem [Pirashvili 2000] For odd $n$ we obtain

$$HH^{[n]}_{\ell+n}(A; \mathbb{Q}) = \bigoplus_{i+nj=\ell+n} HH^{(j)}_{i+j}(A; \mathbb{Q}).$$

Here $HH^{(j)}_*(A; \mathbb{Q})$ is the $j$-th Hodge summand of ordinary Hochschild homology.
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$$HH_*^{[n]}(A; \mathbb{Q}) = \bigoplus_{i+nj=\ell+n} \text{Tor}_i^\Gamma(\theta^j, \mathcal{L}(A, \mathbb{Q})).$$
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Here, $\theta^j[n]$ is the dual of the $\mathbb{Q}$-vector space that is generated by the $S \subset \{1, \ldots, n\}$ with $|S| = j$. 
Relationship to Taylor towers

The groups $\text{Tor}^\Gamma_i(\theta^j, \mathcal{L}(A, \mathbb{Q}))$ are related to a variant of Goodwillie’s calculus of functors for $\Gamma$-modules.
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**Theorem [R,2000]**

$$\text{Tor}^\Gamma_i(\theta^j, \mathcal{L}(A, \mathbb{Q})) \cong H_i(D_j(\mathcal{L}(A, \mathbb{Q}))[1])$$

where $D_j$ is the $j$th homogenous piece in the Taylor tower of $\mathcal{L}(A, \mathbb{Q})$

$$D_j(\mathcal{L}(A, \mathbb{Q}))[\ast] = \text{cone}_{\ast+1}(P_j(\mathcal{L}(A, \mathbb{Q})) \rightarrow P_{j-1}(\mathcal{L}(A, \mathbb{Q})))$$

with

$$\ldots P_n(\mathcal{L}(A, \mathbb{Q})) \rightarrow P_{n-1}(\mathcal{L}(A, \mathbb{Q})) \rightarrow \ldots \rightarrow P_1(\mathcal{L}(A, \mathbb{Q})) \rightarrow \mathbb{Q}.$$
**Theorem** Let $A$ be a commutative augmented $\mathbb{Q}$-algebra. For all $\ell, k \geq 1$ and $m \geq 0$:

$$HH^{(\ell)}_{m+1}(A; \mathbb{Q}) \cong (\mathbb{L}m Q_{2kG} \tilde{A})_{(\ell-1)2k}.$$
Hodge summands as Quillen homology of Gerstenhaber algebras

**Theorem** Let $A$ be a commutative augmented $\mathbb{Q}$-algebra. For all $\ell, k \geq 1$ and $m \geq 0$:

1. $HH_{m+1}^{(\ell)}(A; \mathbb{Q}) \cong (\mathbb{L}_m Q_{2kG \bar{A}})(\ell-1)2k$.

2. $\text{Tor}^\Gamma_{m-\ell+1}(\theta^\ell, \mathcal{L}(A; \mathbb{Q})) \cong (\mathbb{L}_m Q_{(2k-1)G \bar{A}})(\ell-1)(2k-1)$. 


Idea of proof

First we prove a stability result

\[(\mathbb{L}_m \mathcal{Q}_{nG} \bar{A})_{qn} \simeq (\mathbb{L}_m \mathcal{Q}_{(n+2)G} \bar{A})_{q(n+2)}.\]
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We show this by producing an isomorphism of the corresponding Blanc-Stover spectral sequences. The remaining argument is just a matching of the decomposition pieces in the Hodge decomposition and the resolution spectral sequence.