Rational Normal Form for Dynamical Systems by Carleman Linearization*

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Abstract

We propose in this paper a rational normal form for dynamical systems or vector fields. That is to say if the coefficients of the system are in a field K (which, in practice, is $\mathbf{Q}, \mathbf{R}$), so is the computed rational normal form. We give an algorithm for an effective computation of the normal form up to a finite order. We don't need to compute the Jordan canonical form of the leading matrix neither its eigenvalues. We give also a rational method to test the existence of resonance of a fixed order without computing the eigenvalues of the linear part. Our normal form is a refinement of the classical normal forms. Our method is applicable for both the nilpotent and the non nilpotent cases.

1 Introduction

Let K be a commutative field of characteristic zero. We consider the Poincaré-Dulac normal form problem for an autonomous dynamical system or the associated vector field

$$\dot{x} = \frac{dx}{dt} = F(x) \quad \text{or} \quad D = f_1\partial_{x_1} + \cdots + f_n\partial_{x_n} \tag{1}$$

where $x = (x_1, \cdots, x_n)^t$. $F(x) = (f_1(x), \cdots, f_n(x))^t$ is a vector whose components are formal power series without constant terms, *i.e.* $F(0) = 0$ (0 is a singularity for the dynamical system). One writes $F(x) = \sum_{k\geq 1} F_k(x)$ where $F_k(x)$ is a vector of homogeneous polynomials of degree k. $F_1(x) = Ax$, $A \in \mathcal{M}(n, n)$, is the linear part of the system, where $\mathcal{M}(k, m)$ denotes the vector space of $k \times m$ matrices with entries in K.

Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be the eigenvalues of A and suppose that A is in the Jordan canonical form with $\lambda_1, \cdots, \lambda_n$ in the diagonal. Let $q = (q_1, \cdots, q_n) \in \mathbf{N}^n$ with

$$|q| = q_1 + \cdots + q_n \geq 2.$$

We define $x^q = x_1^{q_1} \cdots x_n^{q_n}$. A monomial $a_q x^q$ in $f_j(x)$ is called resonant of order $|q|$ if $\lambda_j = \langle q, \lambda \rangle = q_1\lambda_1 + \cdots + q_n\lambda_n$.

The Poincaré-Dulac normal form theorem states as follows:

Theorem (Poincaré-Dulac). *Consider a system of the form* (1) *where $F(x) = Ax + \cdots$ is a vector of formal power series, and A is the Jordan canonical form as above. Then* (1) *can be reduced to a "normal form" $\dot{y} = Ay + h(y)$ by means of a formal transformation $x = y + \phi(y)$, where $\phi(y)$ is a vector of formal power series without constant term and linear part, and $h(y)$ contains only resonant monomials.*

In practice one uses the Poincaré-Dulac Theorem to get the normal form up to some degree, say N, to obtain

$$\dot{y} = Ay + z(y) + O(\|y\|^{N+1})$$

where $z(y)$ is a vector of polynomials of degree less than or equal to N without constant and linear terms.

A fundamental improvement of the above theorem is given by Takens in [20]. Systematic procedures for constructing normal forms has been given in many ways. A method of Lie brackets is given in [8], [10], [20] and [22], a method by use of an inner product in the spaces of homogeneous polynomials is given in [12], a method by direct computation is given in [2] and [6], a method by use of Carleman linearization is given in [21]. The nilpotent case (A is a nilpotent matrix) is treated in [11] by use of representation theory of $\mathrm{sl}_2(\mathbf{R})$, and in [7] by use of Carleman linearization. The Carleman linearization, introduced in [3] has been used in the study of the normal form theory of nonlinear vector fields in [17], [21] and [7].

All the above works use the Jordan canonical form of the leading matrix A and the possibility of computing the resonances. As it is well known, the computation of the

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eigenvalues and the Jordan canonical forms are very difficult in Computer Algebra. Due to this fact these methods are not effective for an implementation in a Computer Algebra system.

We propose in this paper a new refinement of the classical normal form for dynamical systems or vector fields. Our approach is based on the computation of a rational normal form for dynamical systems of the form (1). Rational normal form means that if the coefficients of the system are in K (which is $\mathbf{Q}$ in practice), so is the computed normal form and all the computations are done in K. We don't need to compute the Jordan canonical form of A neither its eigenvalues. We also give a rational method to test the existence of resonance of any fixed order without computing the eigenvalues of A. Our method is applicable for both the non nilpotent and the nilpotent cases.

We shall give in the Section 2 the general formalism which fits well for our purpose, that is to say the Carleman linearization process. In the Section 3 we give the classical normal form theorem in our context and a new refinement of the classical normal form. In the Section 4 we give the definition of a rational normal form and a rational algorithm for the computations of both the classical and our refined normal form. We also introduce a method for testing the existence of resonant monomials of any order which is not used in this paper. In the Section 5 we shall consider examples of normal forms computed by our algorithm implemented in Maple V. And finally in the last section we present an algorithm which transforms any dynamical system to a single differential equation if the matrix of the linear part is a companion matrix.

2 Carleman linearization

We use in the following the ring of formal power series, $K[[x]]$, where $x = (x_1, x_2, \cdots, x_n)$ with $n \geq 2$. For all integer $k \geq 1$, $H_k(x)$ will denote the space of homogeneous polynomials of degree k in n variables $x_1, x_2, \cdots, x_n$. In $H_k(x)$ we choose the canonical basis x^q, with an n-tuple index $q = (q_1, \cdots, q_n)$, $q_i \in \mathbf{N}$ and $|q| = k$. We choose an order on the monomials x^q (for instance the lexicographical order induced by the order $x_1 < x_2 < \cdots < x_n$). We denote this basis by

$$e_1^k = x_1^k,\ e_2^k = x_1^{k-1}x_2,\ \cdots,\ e_{d_k}^k = x_n^k$$

and $m_k = (e_1^k, \cdots, e_{d_k}^k)^t$ where $d_k = \binom{n+k-1}{k}$ is the dimension of $H_k(x)$. So any element of $H_k(x)$ can be written as

$$P(x) = \sum_{|q|=k} \alpha_q x^q = (a_1, \cdots, a_{d_k}) m_k.$$

The basic idea of Carleman linearization is to associate to the system (1) a derivation D acting on $K[[x]]$. This derivation is defined as the directional derivation in the direction of the vector field defined by F: $D\phi = \langle F, \bigtriangledown\phi\rangle$ (where $\bigtriangledown\phi$ is the gradient of ϕ). In particular $Dx_i = f_i(x)$ where $f_i(x)$ is the i-th component of $F(x)$. So

$$Dm_1 = \begin{pmatrix} Dx_1 \\ \vdots \\ Dx_n \end{pmatrix} = \sum_{i=1}^{+\infty} D_{1i} m_i \tag{2}$$

where D_{1i} are $n \times d_i$ matrices, $F_k(x) = D_{1k} m_k$ is the homogeneous part of degree k of the system and $D_{11} = A$.

We extend the action of D to the elements of the basis m_i of $H_i(x)$ for $i \geq 2$. For example, the action of D on the component x_1x_2 of m_2 is $D(x_1x_2) = x_2Dx_1 + x_1Dx_2$, where Dx_1 and Dx_2 are known by (2). Similarly, $D(x_1^2) = 2x_1D(x_1)$ and $D(x_2^2) = 2x_2D(x_2)$. For $n = 2$ we have

$$Dm_2 = D\begin{pmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} 2x_1 & 0 \\ x_2 & x_1 \\ 0 & 2x_2 \end{pmatrix} \begin{pmatrix} Dx_1 \\ Dx_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 & 0 \\ x_2 & x_1 \\ 0 & 2x_2 \end{pmatrix} \sum_{j\geq 1} D_{1j} m_j = \sum_{k \geq 2} D_{2k} m_k.$$

By recurrence, for $i \geq 2$, we can calculate:

$$Dm_i = \sum_{j \geq i} D_{ij} m_j.$$

Let $H^\infty = H_1 \oplus H_2 \oplus + \cdots$. D is a linear map from H^∞ to H^∞. It has an infinite matrix representation $T_m(D)$ in the basis

$$m = (m_1^t, m_2^t, m_3^t, \cdots)^t.$$

We can write

$$Dm = T_m(D)m$$

where $T_m(D)$ is an infinite upper block triangular matrix of the form:

$$\begin{pmatrix} D_{11} & D_{12} & D_{13} & \cdots \\ & D_{22} & D_{23} & \cdots \\ & & D_{33} & \cdots \\ & & & \ddots \end{pmatrix}.$$

with $D_{ij} (i \leq j)$ a $d_i \times d_j$ matrix.

Remark 2.1 The matrix $T_m(D)$ is uniquely determined by the matrices $D_{1k} (k \geq 1)$. So we will denote also this matrix by $\mathrm{Der}_m(D_{11}, D_{12}, \cdots)$. And we will denote by

$$\mathrm{Der}_m^k(D_{11}, \cdots, D_{1k}) \quad \text{or} \quad T_m^{(k)}(D),$$

the following truncated matrix:

$$T_m^{(k)}(D) = \begin{pmatrix} D_{11} & D_{12} & D_{13} & \cdots & D_{1k} \\ & D_{22} & D_{23} & \cdots & D_{2k} \\ & & D_{33} & \cdots & D_{3k} \\ & & & \ddots & \vdots \\ & & & & D_{kk} \end{pmatrix} \tag{3}$$

i.e. the truncation of the matrix at the order k.

Let $\mathcal{G}_n$ be the group of formal automorphisms tangent to identity of $K[[x]]^n$ and

$$\varphi = (\varphi_1(x), \cdots, \varphi_n(x))^t \in \mathcal{G}_n.$$

We now introduce the matrix representation of φ. Let us write

$$\varphi(m_1) = m_1 + \sum_{i \geq 2} T_{1i} m_i.$$

We can extend these representations to other basis vectors:

$$\varphi(m_j) = m_j + \sum_{k > j} T_{jk} m_k = \sum_{k \geq j} T_{jk} m_k$$

using the properties of φ. For example, if $n=2$ one has

$$\varphi(m_2)=\varphi\begin{pmatrix}x_1^2\\ x_1x_2\\ x_2^2\end{pmatrix}=\begin{pmatrix}\varphi_1(x)^2\\ \varphi_1(x)\varphi_2(x)\\ \varphi_2(x)^2\end{pmatrix}=m_2+\cdots.$$

Then φ, as a linear map on H^∞ still denoted by φ, can be represented by the following infinite upper block triangular matrix:

$$T_m(\varphi)=\begin{pmatrix}I_1 & T_{12} & T_{13} & \cdots\\ & I_2 & T_{23} & \cdots\\ & & I_3 & \cdots\\ & & & \ddots\end{pmatrix} \tag{4}$$

where in the diagonal I_i is the identity matrix of order d_i and T_{ij} is a $d_i\times d_j$ matrix. We can write

$$\varphi(m)=T_m(\varphi)m.$$

We will use φ as a change of variables for the dynamical system (1). If $m_i'=\varphi(m_i)$, then $m'=({m_1'}^t,{m_2'}^t,\cdots)^t$ is a basis in H^∞ and $m'=T_m(\varphi)m$. One has then

$$Dm'=T_m(\varphi)Dm=T_m(\varphi)T_m(D)m$$

$$=T_m(\varphi)T_m(D)T_m(\varphi)^{-1}m'.$$

Hence the new matrix of D in the basis m' is

$$T_{m'}(D)=T_m(\varphi)T_m(D)T_m(\varphi)^{-1}.$$

We have to take care about the fact that if $\psi,\varphi\in\mathcal{G}_n$ then $\psi\circ\varphi\in\mathcal{G}_n$ and $T_m(\psi\circ\varphi)=T_{m'}(\psi)T_m(\varphi)$. In fact if $T_{m'}(\psi)=(T_{ij})$ and $T_m(\varphi)=(U_{ij})$ then for any $j\geq 1$,

$$\psi\circ\varphi(m_j)=\sum_{i\geq j}T_{ji}\varphi(m_i)=\sum_{i\geq j}\sum_{k\geq i}T_{ji}U_{ik}m_k$$

$$=\sum_{k\geq j}\sum_{j\leq i\leq k}T_{ji}U_{ik}m_k=\sum_{k\geq j}V_{jk}m_k$$

where

$$V_{jk}=\sum_{j\leq i\leq k}T_{ji}U_{ik}.$$

Hence $T_m(\psi\circ\varphi)=T_{m'}(\psi)T_m(\varphi)$.

Remark 2.2 The matrix representation (4) depends only on the matrices: $I_1,T_{12},T_{13},\cdots$. Therefore we can also use the notation $\mathrm{Diff}_m(I_1,T_{12},T_{13},\cdots)$ for this matrix and we denote by $\mathrm{Diff}_m^k(I_1,T_{12},\cdots,T_{1k})$ or $T_m^{(k)}(\varphi)$ the corresponding matrix truncated at the order k as for the derivation above.

3 Fundamental theorems of normal form theory

The aim of normal form theory is to answer the following question: D being a derivation of $K[[x]]$ or a formal vector field of the form (1), can we build φ belonging to $\mathcal{G}_n$, such that it's action on D defined by: $\varphi\circ D\circ\varphi^{-1}$ be as simple as possible? That is to say (in this context) we try to eliminate as many as possible nonlinear components of D.

Our approach of the problem uses intensively the infinite dimensional matrix formalism. We want to build $T_m(\varphi)$ such that $T_m(\varphi)\cdot T_m(D)\cdot T_m(\varphi)^{-1}$ be in a simpler form.

Let N be an integer ≥ 2. Consider the derivation D associated to (1) defined in the above section. As stated in the introduction we want to build a normal form of order N. For this purpose we shall work with the truncated representation of D, *i.e.* $\mathrm{Der}_m^N(D_{11},\cdots,D_{1N})$. The previous normal form problem is reduced to eliminating as many as possible nonzero elements in the matrices $D_{1j}(2\leq j\leq N)$.

The general theory of normal form consists of fixing a normal form of order $\leq k-1$ and looking for the normal form of order k. So we look for a formal diffeomorphism $\varphi(x)=x+\varphi^k(x)$ where φ^k is a vector of homogeneous polynomials of degree k, *i.e.*

$$m_1'=\varphi(m_1)=m_1+E_km_k.$$

Then one can compute

$$T_m(\varphi)=\mathrm{Diff}_m(I,0,\cdots,0,E_k,0,\cdots),$$

its matrix representation as in the Section 2. In the matrix $T_m(\varphi)\cdot T_m(D)\cdot T_m(\varphi)^{-1}$ the first $k-1$ rows and columns are unchanged. Since we are building the normal form of order k the changes for the degrees $>k$ don't concern us at the moment. So for simplicity we work with the following matrix instead of $\mathrm{Der}_m^N(D_{11},\cdots,D_{1N})$:

$$M=\begin{pmatrix}D_{11} & D_{1k}\\ 0 & D_{kk}\end{pmatrix}.$$

Denote by $\mathrm{Elem}(E_k)$ the elementary matrix of the following form

$$\mathrm{Elem}(E_k)=\begin{pmatrix}I_1 & E_k\\ 0 & I_k\end{pmatrix},$$

where I_j is the identity matrix of order d_j for any $j\geq 1$ and E_k is an $n\times d_k$ matrix. It is clear that

$$\mathrm{Elem}(E_k)^{-1}=\mathrm{Elem}(-E_k)$$

and $\mathrm{Elem}(E_k)\cdot\mathrm{Elem}(E_k')=\mathrm{Elem}(E_k+E_k')$. Our problem is to find a matrix E_k such that in the resulting matrix

$$\mathrm{Elem}(E_k)\cdot M\cdot\mathrm{Elem}(-E_k)=\begin{pmatrix}D_{11} & D_{1k}'\\ 0 & D_{kk}\end{pmatrix}$$

the matrix D_{1k}' contains as many as possible zeros. Define a linear map

$$L_k:\mathcal{M}(n,d_k)\longrightarrow\mathcal{M}(n,d_k)$$

by $L_k(E_k)=D_{11}E_k-E_kD_{kk}$. We then have

$$D_{1k}'=D_{1k}-L_k(E_k).$$

Let $\mathcal{R}^k$ be the range of L_k in $\mathcal{M}(n,d_k)$ and $\mathcal{C}^k$ be any of its supplementary subspace in $\mathcal{M}(n,d_k)$. We have the following decomposition:

$$\mathcal{M}(n,d_k)=\mathcal{R}^k\oplus\mathcal{C}^k. \tag{5}$$

In our context the classical normal form theorem (cf. [20]) can be written as follows:

Theorem 3.1 *Let notations be as above. Suppose that we have a decomposition of the form (5) for any $2\leq k\leq N$. Then there exists a formal diffeomorphism $\varphi\in\mathcal{G}_n$ with*

$$T_m^{(N)}(\varphi)=\mathrm{Diff}_m^N(I_1,T_{12},\cdots,T_{1N})$$

such that

$$T_m^{(N)}(\varphi\circ D\circ\varphi^{-1})=\mathrm{Der}_m^N(D_{11},D_{12}',\cdots,D_{1N}')$$

where $D_{1k}'\in\mathcal{C}^k$ for all $2\leq k\leq N$.

We consider now further reduction of the above normal form.

Let ℓ be an integer such that $\dim \mathcal{C}^\ell \geq 1$. We have then $\dim \mathrm{Ker}(L_\ell) = \dim \mathcal{C}^\ell \geq 1$. For any $E'_\ell \in \mathrm{Ker}(L_\ell)$,

$$L_\ell(E_\ell + E'_\ell) = L_\ell(E_\ell).$$

Let $k > \ell$. According to the Theorem 3.1 and the above section there exists a formal diffeomorphism φ with

$$T^{(k)}_m(\varphi) = \mathrm{Diff}^k_m(I, 0, \cdots, 0, E_\ell + E'_\ell, E_{\ell+1}, \cdots, E_k)$$

such that the transformed system

$$\mathrm{Der}^k_m(D_{11}, \cdots, D'_{1\ell}, \cdots, D'_{1k})$$

is in the normal form of the Theorem 3.1, *i.e.* $D'_{1j} \in \mathcal{C}^j$ for $\ell \leq j \leq k$. One can write $D'_{1k} = \tilde{D}_{1k} - \hat{D}_{1k}$ where $\tilde{D}_{1k}$ and $\hat{D}_{1k}$ belong to $\mathcal{C}^k$ and where $\hat{D}_{1k}$ contains all terms depending on E'_ℓ.

Define a (nonlinear) operator

$$N_{\ell,k} : \mathrm{Ker}(L_\ell) \longrightarrow \mathcal{C}^k$$

by $N_{\ell,k}(E'_\ell) = \hat{D}_{1k}$. Let $\mathcal{R}^k_2$ be a subspace contained in the range of $N_{\ell,k}$. Then one can find a supplementary subspace $\mathcal{C}^k_2$ in $\mathcal{C}^k$ such that

$$\mathcal{C}^k = \mathcal{R}^k_2 \oplus \mathcal{C}^k_2. \qquad (6)$$

The following is a new refinement of the classical normal form Theorem 3.1.

Theorem 3.2 *Let notations be as above. Assume that ℓ is an integer such that* $\mathrm{Ker}(L_\ell) \neq \{0\}$. *Suppose that there exist $E_j (\ell \leq j \leq k)$ such that there is a non trivial subspace $\mathcal{R}^k_2$ contained in the range of $N_{\ell,k}$ with the decomposition* (6). *Then there exists $E'_\ell \in \mathrm{Ker}(L_\ell)$, hence a formal diffeomorphism φ with*

$$T^{(k)}_m(\varphi) = \mathrm{Diff}^k_m(I_1, 0, \cdots, 0, E_\ell + E'_\ell, \cdots, E_k),$$

such that the transformed system is in the form

$$\mathrm{Der}^k_m(D_{11}, \cdots, D''_{1\ell}, \cdots, D''_{1,k-1}, D''_{1k})$$

where $D''_{1j} \in \mathcal{C}^j (\ell \leq j \leq k-1)$ and $D''_{1k} \in \mathcal{C}^k_2$.

Note that Ushiki in [22] and Gaeta in [13] used $\mathrm{Ker}(L_\ell)$ to give further reduction of normal forms of higher order in a different way. All the examples given in [13] are semisimples. Comparisons of normal forms with the method of Ushiki is given in [6].

We give in the next section an algorithm for an effective computation of both the classical normal form (Theorem 3.1) and the refined normal form (Theorem 3.2). Examples are given in the Section 5 to show how our algorithm works.

4 Rational normal form and algorithm

The traditional way to handle the problem is to transform the linear part of the derivation to its Jordan canonical form. This way of handling the problem introduces both theoretical and practical difficulties that are unsolved by the traditional algorithms. For instance, the computation of the eigenvalues of the matrices, the well known related problem of recognition of the resonant monomials. We propose in this paper to start with a linear part reduced to its Frobenius canonical form (which can be easily computed by rational algorithms as implemented in several Computer Algebra systems, see also [4], [14], [15] and [18]).

Let N be an integer ≥ 2 and D the derivation defined as above by

$$D(m_1) = \sum_{j=1}^{\infty} D_{1j} m_j \qquad (7)$$

We suppose that the linear part D_{11} of the system is in Frobenius canonical form. One can write

$$D_{11} = \mathrm{diag}(C_1, \cdots, C_r) \qquad (8)$$

where C_i is a companion matrix. For instance

$$C_i = \mathrm{companion}(a_j)_{1 \leq j \leq n_i}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_{n_i} \end{pmatrix}.$$

Remark that we need that, in (8), all the matrices C_i are in the companion form. But we don't need that the characteristic polynomial of C_i divides that of C_{i-1} as is needed in the Frobenius canonical form.

We now introduce some useful definitions to describe the normal form of the derivation (7). We denote by D_0 the linear derivation associated to the linear part of D, *i.e.* the linear operator such that $D_0 m_1 = D_{11} m_1$. It is clear that D_0 induces a linear map on each $H_k(x)$ (for $k \geq 2$).

Definition 4.1 *A sequence $\{e_1, e_2, \cdots, e_j\}$ of elements in $H_k(x)$ is called a Frobenius chain of D_0(or relative to D_0) if they are linearly independent and if*

$$D_0 e_i = e_{i+1}, i = 1, \cdots, j-1;$$

$$D_0 e_j = \textstyle\sum_{i=1}^j c_i e_i \text{ with } c_i \in K.$$

A basis of $H_k(x)$ is called a Frobenius basis if it is composed of Frobenius chains.

As in the general theory of normal forms we fix a normal form of order $k-1$ and then build a normal form of order k. We have the following:

Theorem 4.1 *Let D_{11} be in the Frobenius canonical form* (8). *Let k be any integer such that $k \geq 2$. One can compute a Frobenius basis of $H_k(x)$ relative to D_0 which consists of s Frobenius chains. If we denote by $p_1, \cdots, p_s$ the first elements of each Frobenius chains, then one can compute a formal diffeomorphism in $\mathcal{G}_n$ such that the homogeneous part of order k of the derivation* (7) *is transformed to the following form:*

$$D(x_i) = \cdots + \sum_{j=1}^{s} \alpha_j p_j + \cdots \quad (\alpha_j \in K).$$

where the polynomial part of degree $< k$ is unchanged.

Furthermore if there is a Frobenius chain which consists only of one element then this element appears only in the equations corresponding to the last row of each block of D_{11}.

Remark 4.1 In many cases one may reduce some or all of the α_j to zero by the algorithm of this paper.

We will prove this theorem by producing an algorithm for building the expected normal form.

Description of the algorithm:

(i) First step:

Consider the truncated matrix representation (3), *i.e.* the finite matrix $\mathrm{Der}_m^N(D_{11}, \cdots, D_{1N})$. We compute at first the Frobenius canonical form of the matrices D_{ii} for $i = 1, 2, \cdots, N$. That is equivalent to the computation of a Frobenius basis in each H_i relative to D_0. In fact if P_i is a matrix such that $P_i D_{ii} P_i^{-1}$ is in its Frobenius canonical form, then the components of $\sigma_i = P_i m_i$ form a Frobenius basis of H_i. In the new bases $\sigma = (\sigma_1, \cdots, \sigma_N)$ we have a truncated matrix representation denoted by

$$T_\sigma^{(N)}(D) = \begin{pmatrix} D'_{11} & D'_{12} & D'_{13} & \cdots & D'_{1N} \\ 0 & D'_{22} & D'_{23} & \cdots & D'_{2N} \\ 0 & 0 & D'_{33} & \cdots & D'_{3N} \\ & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & D'_{NN} \end{pmatrix} \tag{9}$$

where $D'_{ii} = P_i D_{ii} P_i^{-1}$ are in Frobenius canonical form and where $D'_{ij} = P_i D_{ij} P_j^{-1}$ for $1 \le i \le j$.

As we have supposed that $D_{11} = A$ is in the Frobenius form (8), $P_1 = I$ and $\sigma_1 = m_1$.

(ii) Second step:

Let $2 \le k \le N$. We suppose that we have obtained a rational normal form for the degrees $\le k-1$. We want to compute a k-th order normal form. We are going to make transformations on the matrix $T_\sigma^{(N)}(D)$ which let the (block) columns 1 to $k-1$ unchanged, and make changes in the k-th column (and also the columns from $k+1$ to N but it does not concern us at the moment since we are looking for a normal form of order k).

So to simplify the notations we work instead of (9) with the following matrix:

$$M' = \begin{pmatrix} D'_{11} & D'_{1k} \\ 0 & D'_{kk} \end{pmatrix}.$$

Remark 4.2 At this stage we can see a useful criterion for answering the existence problem for resonances without computing the eigenvalues of A. It is stated in the following proposition.

Proposition 4.1 *Let notations be as above. There is no resonance of order k if and only if D_{11} (or D'_{11}) and D_{kk} (or D'_{kk}) have no common eigenvalue.*

This fact is not used in this paper due to the fact that we are working with rational normal form. Recall that D'_{11} and D'_{kk} are in Frobenius canonical form. It is easy to test whether they have common eigenvalue by computing the gcd of their characteristic polynomials.

Write $D'_{kk} = \mathrm{diag}(C'_1, \cdots, C'_s)$ where each C'_i is a companion matrix. In fact D'_{kk} is the representation of the linear derivation in a Frobenius basis which is composed of s Frobenius chains, each of which is represented by a C'_i.

Proposition 4.2 *Let $F_1 = \mathrm{companion}(a_i)_{1 \le i \le \nu}$ and $F_2 = \mathrm{companion}(b_i)_{1 \le i \le \mu}$ be two matrices in Frobenius canonical form of dimension ν and μ (with $\nu, \mu \ge 2$). Then for any $\nu \times \mu$ matrix P there exists a $\nu \times \mu$ matrix Q such that*

$$QF_2 - F_1 Q + P = B$$

where B is a $\nu \times \mu$ matrix with nonzero entries only in the first column.

Proof. Denote by P_i, Q_i and B_i the i-th column of the matrices P, Q and B respectively. We look for a matrix Q as claimed by the proposition. Let Q_μ be given arbitrarily. The j-th column of the matrix equation in the proposition can be written as:

$$Q_{j-1} + b_j Q_\mu - F_1 Q_j + P_j = B_j$$

where $j = \mu, \mu - 1, \cdots, 2$. One can determine $Q_{\mu-1}$ such that $B_\mu = 0$ and recursively

$$Q_{j-1} = F_1 Q_j - b_j Q_\mu - P_j$$

such that $B_j = 0$ for $j = \mu, \mu-1, \cdots, 2$. B is in the desired form. □

Remark 4.3 All the Q_j depends linearly on Q_μ. For $j = 1$ one has

$$b_1 Q_\mu - F_1 Q_1 + P_1 = B_1.$$

Since Q_1 depends linearly on Q_μ, the last equation may be used to choose some or all elements of Q_μ such that some or all elements of B_1 equal to zero.

In the particular case where F_1 and F_2 have no common eigenvalue, one can determine Q_μ such that $B_1 = 0$, *i.e.* in this case $B = 0$.

Proposition 4.3 *Let $F_1 = \mathrm{companion}(a_i)_{1 \le i \le \nu}$, $F_2 = (b)$ be two matrices in Frobenius canonical form of dimension ν and 1. Then for any $\nu \times 1$ matrix P there exists a $\nu \times 1$ matrix Q such that*

$$QF_2 - F_1 Q + P = B$$

where B is a $\nu \times 1$ matrix with at most one nonzero element in the last row.

Furthermore if $b = 0$ and $a_1 \ne 0$ then we can choose Q such that $B = 0$.

Proof. Denote P_i, Q_i and B_i the element in the i-th row of the matrices P, Q and B respectively. Let Q_1 be given arbitrarily. We have for $1 \le i \le \nu - 1$,

$$bQ_i - Q_{i+1} + P_i = B_i.$$

One determines Q_{i+1} using Q_i to cancel all the B_i for $1 \le i \le \nu - 1$.

If $b \ne 0$ then all $Q_i (2 \le i \le \nu)$ depend linearly on Q_1 and

$$B_\nu = bQ_\nu - a_1 Q_1 - a_2 Q_2 - \cdots - a_\nu Q_\nu + P_\nu$$

depends also linearly on Q_1. If the coefficient of Q_1 is nonzero then one can find Q_1 such that $B_\nu = 0$ otherwise B_ν may be nonzero.

If $b = 0$ then $Q_i (2 \le i \le \nu)$ don't depend on Q_1 and $B_\nu = -a_1 Q_1 - a_2 Q_2 - \cdots - a_\nu Q_\nu + P_\nu$. Therefore if $a_1 \ne 0$ then one can choose Q_1 such that $B_\nu = 0$, *i.e.* all elements of B are zero. □

The following table gives a clear description of the blocks in our matrix representation M':

$$
\begin{array}{|ccc|ccc|}
\hline
C_1 & & & P_{11} & \cdots & P_{1s} \\
& \ddots & & \vdots & & \vdots \\
& & C_r & P_{r1} & \cdots & P_{rs} \\
\hline
& & & C'_1 & & \\
& & & & \ddots & \\
& & & & & C'_s \\
\hline
\end{array}
$$

We apply the above propositions with $F_1 = C_i$, any block of D_{11}, $F_2 = C'_j$, any block of D'_{kk} and $P = P_{ij}$, the corresponding block of the matrix D'_{1k}. We obtain a matrix which we denote by Q_{ij}. It is clear that what remain in the normal form are the first column of the matrix B_{ij}, *i.e.* the column corresponding to the first element of each Frobenius chain. We form the matrix $\mathrm{Elem}(E_k)$ with

$$E_k = \begin{pmatrix} Q_{11} & \cdots & Q_{1s} \\ \vdots & & \vdots \\ Q_{r1} & \cdots & Q_{rs} \end{pmatrix}.$$

Then

$$\mathrm{Elem}(E_k) M' \mathrm{Elem}(E_k)^{-1} = \begin{pmatrix} D'_{11} & D''_{1k} \\ 0 & D'_{kk} \end{pmatrix}$$

where D''_{1k} is in the desired form of the Theorem 4.1.

(iii) Third step:

We now return to (9). We have computed in the second step a matrix $\mathrm{Elem}(E_k)$ to perform elimination for those elements in D'_{1k} of M' which do not correspond to the first element of a Frobenius chain. According to the Remark 2.2, we can compute a formal diffeomorphism φ such that

$$T_\sigma^{(N)}(\varphi) = \mathrm{Diff}_\sigma^N(I_1, 0, \cdots, 0, E_k, 0, \cdots, 0).$$

We have $\varphi(\sigma_1) = \sigma_1 + E_k \sigma_k$. To get the matrix representation of φ, we compute $\varphi(\sigma_i)$ for $i = 2, \cdots, N$ as functions of $\sigma_j (1 \le j \le N)$. This can be done as in the Section 2 with the bases σ_j in places of m_j. For instance $m_1 = \sigma_1$,

$$\varphi(m_1) = \varphi(\sigma_1) = m_1 + E_k P_k m_k.$$

We compute $\varphi(m_2)$ as in the Section 2. Hence

$$\varphi(\sigma_2) = \varphi(P_2 m_2) = P_2 \varphi(m_2).$$

One uses $m_i = P_i^{-1} \sigma_i$ to find the matrix representation of $\varphi(\sigma_2)$ in the basis σ. According to the second step above,

$$T_\sigma^{(N)}(\varphi) T_\sigma^{(N)}(D) T_\sigma^{(N)}(\varphi)^{-1} =$$
$$\mathrm{Der}_\sigma^N(D'_{11}, \cdots, D''_{1k}, \cdots, D''_{1N})$$

is in rational normal form of degree k described in the Theorem 4.1.

One can repeat the second and third steps on the matrix

$$\mathrm{Der}_\sigma^N(D'_{11}, \cdots, D''_{1k}, \cdots, D''_{1N})$$

to obtain a rational normal form of degree $k+1$, etc. The algorithm can be repeated until the rank N.

(iv) Further reductions:

The algorithm can be used to make further reductions of the normal form.

In the Propositions 4.2 and 4.3 there may exist arbitrary elements of Q. So in the second step, there exist elements in E_k undetermined if $\mathcal{C}_k \neq \{0\}$. These undetermined elements may be used for finding higher order normal form. In fact we continue to compute the normal form of order $k+1$. If in the homogeneous part of degree $k+1$ of the normal form, there is a term α depending (linearly) on an undetermined element of E_k, then one can solve the equation $\alpha = 0$ for this element of E_k. This reduces one more parameter to zero in the normal form of order $k+1$. The same procedure applies to higher order normal forms.

This leads to the refined normal form as stated in the Theorem 3.2.

We have implemented this algorithm in Maple V. Examples of computations are given in the next section for both the classical normal forms and our refined normal forms.

5 Examples of normal forms

The previous theorem shows us that for computing a rational normal form of any dynamical system with a certain linear part A, one only needs to perform the computation of a Frobenius basis in $H_k(x)$ for any k.

Example 5.1 Consider dynamical systems of dimension 2, *i.e.* $n = 2$ with

$$A = \begin{pmatrix} 0 & 1 \\ c_0 & c_1 \end{pmatrix}$$

and let D_0 be the linear derivation associated to A, $D_0 x = Ax$. One has

$$D_0 x_1 = x_2, \quad D_0 x_2 = c_0 x_1 + c_1 x_2,$$

i.e. $\{x_1, x_2\}$ is a Frobenius basis in H_1.

We now compute a Frobenius basis in $H_k(x)$. Let $y_0 = x_1^k$. One computes $y_1 = D_0 y_0 = k x_1^{k-1} x_2$. One proves by recurrences that if $y_j = D_0 y_{j-1}$ for $j = 1, \cdots, k$ then $\{y_0, \cdots, y_k\}$ are linearly independent. They form a Frobenius basis in $H_k(x)$ with only one Frobenius chain. According to the Theorem 4.1, the general rational normal forms of any dynamical systems are in the form $\begin{pmatrix} \alpha_1 x_1^k \\ \alpha_2 x_1^k \end{pmatrix}$ (see also [20] and [6]). Further reductions are possible depending on the values of c_0 and c_1.

Consider now dynamical systems of dimension 2 with a nilpotent leading matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(see also [6], [11], [12], [20] and [22]). Denote

$$f_1 = x_2 + \sum_{|q| \ge 2} \alpha_q x^q, \quad f_2 = \sum_{|q| \ge 2} \beta_q x^q.$$

The coefficients α_q and β_q of the system are arbitrary. To construct a second order normal form we obtain at the end of the second step of our algorithm the following matrix

$$E = \begin{bmatrix} -\frac{1}{2}\beta_{0,2} - \frac{1}{2}\alpha_{1,1} & Z_2 - \frac{1}{2}\alpha_{0,2} & Z_1 \\ -\frac{1}{2}\beta_{1,1} & -\frac{1}{2}\beta_{0,2} & Z_2 \end{bmatrix}$$

where we have denoted by Z_1 and Z_2 the undetermined elements in the last column of the matrix Q. This leads to a second order normal form with

$$D''_{12} = \begin{pmatrix} \frac{1}{2}\beta_{1,1} + \alpha_{2,0} & 0 & 0 \\ \beta_{2,0} & 0 & 0 \end{pmatrix}$$

and the other matrices in $T_\sigma(D)$ may depend on Z_1 and Z_2. We compute then a third order normal form. We obtain a classical normal form as in the Theorem 3.1 whose homogeneous part of degree 3 is

$$D''_{13} = \begin{pmatrix} u & 0 & 0 & 0 \\ v & 0 & 0 & 0 \end{pmatrix}$$

where $v = \beta_{2,0}\,\alpha_{1,1} + \beta_{3,0} - \beta_{1,1}\,\alpha_{2,0}$ and $u = -\frac{1}{3}\beta_{2,0}\,\alpha_{0,2} + \frac{1}{3}\beta_{2,1} + \frac{1}{6}\beta_{1,1}\,\alpha_{1,1} - \frac{1}{3}\alpha_{2,0}\,\beta_{0,2} + \frac{1}{6}\beta_{1,1}\,\beta_{0,2} + \alpha_{3,0} + 2\,Z_2\,\beta_{2,0}$.

If $\beta_{2,0} \neq 0$, one determines Z_2 such that $u = 0$ and the third order normal form is

$$D''_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ v & 0 & 0 & 0 \end{pmatrix}.$$

The algorithm can be used to compute higher order normal form.

We obtain a refined normal form of order 7 (if $\beta_{2,0} \neq 0$ and $\gamma_1 = \frac{1}{2}\beta_{1,1} + \alpha_{2,0} \neq 0$):

$$\begin{aligned} \dot{x}_1 &= x_2 + \gamma_1 x_1^2 + \gamma_3 x_1^4 + \gamma_4 x_1^5 + \gamma_5 x_1^7 + O(\|x\|^8), \\ \dot{x}_2 &= \beta_{2,0} x_1^2 + \gamma_2 x_1^3 + O(\|x\|^8). \end{aligned}$$

One can also compute normal forms in the degenerate cases (for example, $\beta_{2,0} = 0$) by our algorithm.

For comparison, Ushiki (see [22] and [10]) obtained a normal form of order 4 and we have obtained in [6] a normal form of order 9 by studying directly the homological equations.

Example 5.2 Consider dynamical systems of dimension 3 with a nilpotent matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

as its linear part and

$$f_1 = x_2 + \sum_{|q|\geq 2} a_q x^q,\ f_2 = x_3 + \sum_{|q|\geq 2} b_q x^q,\ f_3 = \sum_{|q|\geq 2} c_q x^q.$$

For $k = 2$, the Frobenius canonical form of D_{22} has two blocks of order 5 and 1 respectively. We obtain a Frobenius basis with two chains: $\{x_1^2 + \frac{1}{3}x_2^2 - \frac{2}{3}x_1 x_3,\ 2x_1x_2,\ 2x_2^2 + 2x_1x_3,\ 6x_2x_3,\ 6x_3^2\}$ and $\frac{1}{3}x_2^2 - \frac{2}{3}x_1x_3$. We then obtain a normal form of order 2

$$D'_{12} = \begin{pmatrix} \gamma_1 & 0 & 0 & 0 & 0 & 0 \\ \gamma_2 & 0 & 0 & 0 & 0 & 0 \\ \gamma_3 & 0 & 0 & 0 & 0 & \gamma_4 \end{pmatrix}$$

where $\gamma_3 = c_{2,0,0}$ and $\gamma_4 = -c_{2,0,0} - c_{0,2,0} + c_{1,0,1}$, *etc.*

For $k = 3$, D_{33} has a Frobenius canonical form with two blocks of order 7 and 3 respectively. If $c_{2,0,0} \neq 0$ we obtain a normal form of order 3

$$D'_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_5 & 0 & 0 \\ \gamma_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which may have 3 nonzero parameters.

For $k = 4$, the Frobenius canonical form of the matrix D_{44} has three blocks of order 9, 5 and 1 respectively. We obtain, in case that $c_{2,0,0} \neq 0$, a 3×15 matrix D'_{13} which may have at most 2 nonzero elements: $D'_{13}[3,1]$ and $D'_{13}[3,10]$.

Note that, for comparison, Ushiki obtained a normal form of order 3 in this case (see [22] and [10]).

Example 5.3 Consider dynamical systems of dimension 4 with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{pmatrix}$$

as the linear part.

For $k = 2$, under the canonical basis of $H_2(x)$, the matrix D_{22} has a Frobenius canonical form with two blocks C'_1 and C'_2, where

$$C'_1 = \text{companion}(0, 0, 0, -64, 0, -84, 0, -12, 0)$$

and $C'_2 = (0)$. A Frobenius basis in $H_2(x)$ is composed of two chains $(f_1, \cdots, f_9)$ and f'_1. We apply the Proposition 4.2 to $F_1 = A, F_2 = C'_1$ and we obtain a matrix Q such that $B = 0$. And by applying the Proposition 4.3 to $F_1 = A, F_2 = C'_2$, one gets $B = 0$. Therefore in the normal form of a dynamical system with linear part A, the homogeneous part of order 2 is zero.

For $k = 3$ the matrix D_{33} is a 20×20 matrix which has the Frobenius canonical form $\text{diag}(C_1, C_2)$, where

$$C_1 = \text{companion}(-6561, 0, -29160, 0, -51516, 0,$$
$$-45720, 0, -21286, 0, -5080, 0, -636, 0, -40, 0)$$

and $C_2 = \text{companion}(-1, 0, -2, 0)$.

Then one obtains a Frobenius basis in H_3 which is composed by 2 Frobenius chains $(g_1, \cdots, g_{16})$ and $(h_1, \cdots, h_4)$. The homogeneous part of degree 3 in the normal form is in the form:

$$(\alpha_1 g_1 + \beta_1 h_1, \alpha_2 g_1 + \beta_2 h_1, \alpha_3 g_1 + \beta_3 h_1, \alpha_4 g_1 + \beta_4 h_1)^t$$

that has 8 parameters which may be nonzero.

For $k = 4$, we obtain a 25×25 matrix D_{44} whose Frobenius matrix has three blocks of order 25, 9 and 1 the last block is a zero matrix. The homogeneous part of degree 4 is reduced to zero by the algorithm in this paper.

For $k = 5$, we obtain a 56×56 matrix whose Frobenius matrix has three blocks of order 36, 16 and 4. Therefore the homogeneous part of degree 5 in the normal form has 12 parameters which may be nonzero, some of which may be reduced to zero by the above algorithm.

Remark 5.1 Since the matrix A is in its Frobenius canonical form the matrices D_{kk} contains only few nonzero entries.

6 Reduction to a single differential equation

We suppose, in this section, that the matrix A is a companion matrix, *i.e.*

$$D_{11} = A = \text{companion}(c_1, \cdots, c_n) \tag{10}$$

We now describe an algorithm that transforms the dynamical system $\dot{x} = Ax + \cdots$ to a single nonlinear differential equation.

We use the same notations as in the above sections.

Theorem 6.1 *Consider a dynamical system of the form* (1) *with* A *in the companion matrix form* (10). *Let* k *be any integer such that* $k \geq 2$. *One can compute a* $n \times d_k$ *matrix* Q_{1k} *such that*

$$Q_{1k}D_{kk} - D_{11}Q_{1k} + D_{1k} = D'_{1k} \tag{11}$$

where only the last row of D'_{1k} *may have nonzero elements.*
Let $\phi \in \mathcal{G}_n$ *be an automorphism such that*

$$\phi(m_1) = m_1 + Q_{1k}m_k$$

and $T_m(\phi)$ *be its matrix representation by the Carleman linearization. Then*

$$T_m(\phi)T_m(D)T_m(\phi)^{-1} = \mathrm{Der}_m(D_{11}, \cdots, D'_{1k}, \cdots).$$

Therefore by the change of variables $y = \phi(x)$ *the dynamical system* (1) *is converted to a new system of the form*

$$\dot{y} = Ay + \cdots + G^k(y) + \cdots$$

where $G^k(y) = (0, \cdots, 0, g(y))^t$ *is the homogeneous part of degree* k, $g(y)$ *is a homogeneous polynomial of degree* k.

Proof. For $j = 1, \cdots, n$ we denote by p_j, q_j and b_j the j-th row of the matrices D_{1k}, Q_{1k} and D'_{1k} respectively. Let q_1 be given arbitrarily.

For $j = 1, \cdots, n-1$, the j-th row of the matrix equation (11) is

$$q_jD_{kk} - q_{j+1} + p_j = b_j$$

One can determine q_{j+1} such that $b_j = 0$ for $1 \leq j \leq n-1$. In fact for $1 \leq j \leq n-1$,

$$q_{j+1} = q_jD_{kk} + p_j.$$

The matrix D'_{1k} is in the desired form, the first part of the theorem is proved.

One can compute $\phi(x)$ by the same procedure as in the third step of our algorithm given in the Section 4. □

We have implemented this algorithm in Maple V. Examples of normal forms are given in [5].

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