The Homotopy of E_n-operads
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Lecture 4: Outlook

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§1. The Deligne conjecture on Hochschild cochains

- Reminder: The Hochschild cochain complex $\mathtt{C}^*_{HH}(A,A)$ of a (plain) associative algebra A is defined by the collection of modules
$$\mathtt{C}^n_{HH}(A,A) = \mathtt{Hom}(A^{\otimes n}, A)$$
together with the differential $\partial : \mathtt{C}^*_{HH}(A,A) \to \mathtt{C}^*_{HH}(A,A)$ such that:
$$\begin{aligned}\partial\alpha(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 \cdot \alpha(a_2 \otimes \cdots \otimes a_{n+1}) \\ + \sum_{i=1}^{n} \pm\alpha(a_1 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ + \pm\alpha(a_1 \otimes \cdots \otimes a_n) \cdot a_{n+1}\end{aligned}$$
for every $\alpha \in \mathtt{C}^n_{HH}(A,A)$.

- The Hochschild cohomology of an associative algebra $\mathtt{HH} = \mathtt{HH}^*(A, A)$ is defined as the cohomology of the Hochschild cochain complex $\mathtt{C}^*_{HH}(A, A)$.
- The Hochschild cohomology inherits a commutative (and associative) product

$$\smile : \mathtt{HH} \otimes \mathtt{HH} \to \mathtt{HH},$$

referred to as the cup-product, and an even Lie bracket

$$[-, -] : \mathtt{HH} \otimes \mathtt{HH} \to \mathtt{HH},$$

of (cohomological) degree -1, usually called the Gerstenhaver bracket.
- These operations satisfy the following distribution relation

$$[a \smile b, c] = \pm[a, c] \smile b + \pm a \smile [b, c],$$

and, all together, define a Gerstenhaber algebra structure on $\mathtt{HH} = \mathtt{HH}^*(A, A)$.

- The cup-product $\smile$ is defined, at the cochain level, by the formula:

$$(\alpha \smile \beta)(a_1 \otimes \cdots \otimes a_{m+n}) \\ = \alpha(a_1 \otimes \cdots \otimes a_m) \cdot \beta(a_{m+1} \otimes \cdots \otimes a_{m+n}),$$

for every $\alpha \in C^m_{HH}(A, A)$ and every $\beta \in C^n_{HH}(A, A)$.
- The Gerstenhaber bracket is given, at the cochain level, by the anti-commutator $[\alpha, \beta] = \alpha \smile_1 \beta + \pm\beta \smile_1 \alpha$ of the operation such that:

$$(\alpha \smile_1 \beta)(a_1 \otimes \cdots \otimes a_{m+n-1}) \\ = \sum_{i=1}^{m} \pm\alpha(a_1 \otimes \cdots \otimes \beta(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes \cdots \otimes a_{m+n-1}).$$

This operation represents an obstruction to the commutativity of the cup-product at the cochain level.

Theorem (F. Cohen):

- *We have* $\mathtt{H}_*(\mathsf{D}_2) = \mathsf{Gerst}_2$, *where* Gerst_2 *is the operad governing Gerstenhaber algebras, abstractly defined as the operad generated by*
  - *a cup-product operation* $\mu = \mu(x_1, x_2) \in \mathsf{Gerst}_2(2)$, *of degree* 0, *and such that* $\mu(x_2, x_1) = \mu(x_1, x_2)$,
  - *a Lie bracket operation* $\lambda = \lambda(x_1, x_2) \in \mathsf{Gerst}_2(2)$, *of (homological) degree* 1, *and such that* $\lambda(x_2, x_1) = \lambda(x_1, x_2)$,

  modulo an ideal of relations corresponding to the relations of cup-products and Lie brackets in Gerstenhaber algebras.
- *The operad* Gerst_2 *is also equipped with a unitary operation* $e \in \mathsf{Gerst}_2(0)$, *such that* $\mu(e, x_1) = \mu(x_1, e) = x_1$ *and* $\lambda(e, x_1) = \lambda(x_1, e) = 0$. *The identity* $\mathtt{H}_*(\mathsf{D}_2) = \mathsf{Gerst}_2$ *holds in the category of unitary operads.*

- Theorem (Deligne conjecture, proved by Kontsevich-Soibelman, McClure-Smith, ...): *The cup-product and the Gerstenhaber bracket can be embodied in an action of an E_2-operad on Hochschild cochains $\mathtt{C}_{HH} = \mathtt{C}^*_{HH}(A, A)$ so that this action gives a realization, at the cochain level, of the action of the Gerstenhaber operad $\mathsf{Gerst}_2 = \mathtt{H}_*(\mathsf{D}_2)$ on the cohomology $\mathtt{HH} = \mathtt{HH}^*(A, A)$.*
- Remarks:
  - We have $\mathtt{HH}^0(A, A) = \mathtt{Z}(A)$ (the center of the associative algebra A), and the Hochschild cochain complex represents a derived version of this center construction.
  - The Deligne conjecture therefore asserts that the center is not fully commutative.

§2. Applications to deformation-quantization

- **Theorem (Hochschild-Kostant-Rosenberg):** *In the case* $A = \mathbb{R}[[x_1, \ldots, x_n]]$, *we have:*

$$\mathrm{HH}^*(\mathbb{R}[[x_1, \ldots, x_n]], \mathbb{R}[[x_1, \ldots, x_n]]) = \mathrm{T}^*_{\mathbb{R}^n},$$

where $\mathrm{T}^*_{\mathbb{R}^n}$ *denotes the graded algebra of polyvector fields on* $\mathbb{R}^n$, *the graded commutative algebra over* $\mathbb{R}[[x_1, \ldots, x_n]]$ *generated by the derivations* $\xi_i = \partial/\partial x_i$ *in degree* 1. *We explicitly have:*

$$\mathrm{T}^*_{\mathbb{R}^n} = \mathbb{R}[[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]].$$

- The cup-product corresponds to the wedge product, and the Gerstenhaber product is identified with the Schouten-Nijenhuis bracket, which can also be defined by the following explicit formulas:

$$[\xi_i, x_j] = \delta_{ij},\ [\xi_i, \xi_j] = 0,\ [x_i, x_j] = 0.$$

- Definition: Let $\mathfrak{g}$ be any differential graded Lie algebra. The set of Maurer-Cartan elements in $\mathfrak{g}$ is defined by:

$$\mathcal{MC}(\mathfrak{g}) = \{\gamma \in \mathfrak{g}^1 \,|\delta(\gamma) + \tfrac{1}{2} \cdot [\gamma, \gamma] = 0\}.$$

  This set inherits an action of the group $G = \exp(\mathfrak{g}^0)$, and we also set:

$$\mathtt{MC}(\mathfrak{g}) = \mathcal{MC}(\mathfrak{g}) / \exp(\mathfrak{g}^0).$$

- Examples: Let $\mathtt{M}[[\hbar]] = \mathtt{M} \otimes \mathbb{R}[[\hbar]]$. Let $\mathtt{M}[[\hbar]]_+ = \mathtt{M} \otimes \hbar\,\mathbb{R}[[\hbar]]$.
  - In the case $\mathfrak{g} = \mathtt{C}^*_{HH}(A, A)[[\hbar]]_+$, the Maurer-Cartan elements are identified with power series

$$a \star_\hbar b = a \cdot b + \hbar \cdot \mu_1(a, b) + \cdots + \hbar^n \cdot \mu_n(a, b) + \cdots$$

    defining an associative product on $A[[\hbar]]$, and so that we have

$$(a \star_\hbar b)|_{\hbar = 0} = a \cdot b,$$

    for any $a, b \in A$.
  - In the case $\mathfrak{g} = \mathtt{T}^*_{\mathbb{R}^n}[[\hbar]]_+$, the Maurer-Cartan elements are identified with power series

$$\pi_\hbar(a, b) = \hbar \cdot \pi_1(a, b) + \cdots + \hbar^n \cdot \pi_n(a, b) + \cdots$$

    defining a Poisson structure on $A[(\hbar]] = \mathbb{R}[[x_1, \ldots, x_n]][[\hbar]]$.

- Let $\mathtt{D}^*_{\mathbb{R}^n} = \mathtt{C}^*_{HH}(\mathbb{R}[[x_1, \ldots, x_n]], \mathbb{R}[[x_1, \ldots, x_n]])$.
- Theorem (Kontsevich's formality theorem): *The quasi-isomorphism* $q : \mathtt{T}^*_{\mathbb{R}^n} \xrightarrow{\sim} \mathtt{D}^*_{\mathbb{R}^n}$ *giving the identity of the Hochschild-Kostant-Rosenberg can be upgraded to an isomorphism in the homotopy category of Lie algebras:*

$$\mathtt{T}^*_{\mathbb{R}^n} \sim \mathtt{D}^*_{\mathbb{R}^n}.$$

- Corollary: *We have a bijection:*

$$\mathtt{MC}(\mathtt{T}^*_{\mathbb{R}^n}[[\hbar]]_+) \simeq \mathtt{MC}(\mathtt{D}^*_{\mathbb{R}^n}[[\hbar]]_+).$$

- Thus, we have a one-to-one correspondence between isomorphism classes of Poisson deformations and $\star_\hbar$-products. This statement notably implies that every Poisson structure on $\mathbb{R}^n$ has a deformation-quantization.

- Proof ideas (Step 1): The formality of the little 2-discs operad implies that any E_2-operad in chain complexes E_2 is connected to the Gerstenhaber operad Gerst_2 by a chain of operad quasi-isomorphisms:
$$\mathsf{E}_2 \xleftarrow{\sim} \mathsf{M}_2 \xrightarrow{\sim} \mathsf{Gerst}_2.$$
- Then:
  - The Gerstenhaber algebra structure on $\mathtt{T}^*_{\mathbb{R}^n}$ restricts to an action of the operad M_2.
  - The E_2-operad action on $\mathtt{D}^*_{\mathbb{R}^n}$ (given by the Deligne conjecture) restricts to an action of the operad M_2 as well.
- Thus the Lie algebra structures of the cochain complexes $\mathfrak{g} = \mathtt{T}^*_{\mathbb{R}^n}$ and $\mathfrak{g} = \mathtt{D}^*_{\mathbb{R}^n}$ can be embodied in an action of the same operad M_2 such that $\mathsf{M}_2 \xrightarrow{\sim} \mathsf{Gerst}_2$.

- Proof ideas (Step 2): We have:

$$\tilde{\mathrm{H}}^*_{\mathsf{Gerst}_2}(\mathtt{T}^*_{\mathbb{R}^n}, \mathtt{T}^*_{\mathbb{R}^n})^{\mathtt{Aff}(\mathbb{R}^n)} = 0,$$

  where $\tilde{\mathrm{H}}^*_{\mathsf{Gerst}_2}(-,-)$ denotes a truncated version of the natural cohomology theory associated to the Gerstenhaber operad.
- This statement implies that the algebra of polyvector fields $\mathtt{T}^*_{\mathbb{R}^n}$ is intrinsically formal:
  - *any* M_2*-algebra* $\mathtt{R}^*$ *equipped with an action of the affine group* $\mathtt{Aff}(\mathbb{R}^n)$
  - *and satisfying* $\mathrm{H}^*(\mathtt{R}^*) = \mathtt{T}^*_{\mathbb{R}^n}$
  - *is quasi-isomorphic to* $\mathtt{T}^*_{\mathbb{R}^n}$ *as an* M_2*-algebra.* □

Outlook

- The set of Drinfeld associators inherits an action of a pro-unipotent version of the Grothendieck-Teichmüller group $\mathtt{GT}(\mathbb{Q})$, a device introduced by Grothendieck in Galois theory in order to encode the information that can be captured from actions of the absolute Galois group $\Gamma = \mathtt{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on curves.
- Kontsevich's proposal: This group $\mathtt{GT}(\mathbb{Q})$ acts on deformation-quantization functors through associators.

- Theorem (BF): *We have:*

$$\mathtt{GT}(\mathbb{Q}) = \mathtt{Aut}_{\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)}(\mathrm{D}_2)^{\wedge}_{\mathbb{Q}},$$

where:

  - $(\mathrm{D}_2)^{\wedge}_{\mathbb{Q}}$ *denotes a rationalization of the little 2-discs operad* D_2*,*
  - $\mathtt{Ho}(\mathcal{T}op\mathcal{O}p)$ *denotes the homotopy category of operads in topological spaces (where weak-equivalences become isomorphisms).*

  Remark: an analogous result in chain complexes has been obtained by Willwacher.

- Question: higher dimensional analogues of the Grothendieck-Teichmüller group?
- Embedding spaces $\mathtt{Emb}(\mathbb{R}^m, \mathbb{R}^n)$ have rational models involving E_n-operads (works of Budney, Sinha, Salvatore, Lambrechts-Turchin-Volić, Arone-Turchin, Dwyer-Hess, Turchin, ...).

  Proposal: Study actions of (higher) Grothendieck-Teichmüller groups on these models through homotopy automorphisms of E_n-operads.

Thank you for your attention!

References:

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  - Master degree course associated with the book (preprint and reference material): http://math.univ-lille1.fr/~fresse/operads2012.html
  - Book project web-page (draft documents): http://math.univ-lille1.fr/~fresse/OperadHomotopyBook