The Homotopy of E_n-operads
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Lecture 3: Formality and Associators

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Introduction

- Observations: The homology of the little n-discs spaces $\mathrm{H}_*(\mathsf{D}_n(r))$, $r \in \mathbb{N}$, forms an operad in the category of graded modules. The collection of cohomology algebras $\mathrm{H}^*(\mathsf{D}_n(r))$, $r \in \mathbb{N}$, inherits the dual structure of a Hopf cooperad (a cooperad in the category of graded commutative algebras).
- Goal: Explain that a rationalization (respectively, realification) of the topological operad D_n can be retrieved from this cohomology Hopf cooperad

  $$\mathrm{H}^*(\mathsf{D}_n) = \{\mathrm{H}^*(\mathsf{D}_n(r)), r \in \mathbb{N}\},$$

  where we take $\Bbbk = \mathbb{Q}$ (respectively, $\Bbbk = \mathbb{R}$) as field of coefficients.

§1. The Sullivan model of the little 2-discs spaces

- Reminder: We have $\mathsf{D}_2(r) \sim \mathsf{F}(\mathring{\mathbb{D}}^2, r)$, for each $r \in \mathbb{N}$, where $\mathsf{F}(\mathring{\mathbb{D}}^2, r)$ denotes the configuration space of r points in the open disc $\mathring{\mathbb{D}}^2$.
- Theorem (Arnold): *We have:*

$$\mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^2, r)) = \mathtt{S}(\omega_{ij}, 1 \leq i \neq j \leq r)/(\textit{Arnold's relation}),$$

where we take the (graded) symmetric algebra $\mathtt{S}(-)$ *on generating elements* ω_{ij} *in (cohomological) degree* 1 *modulo the ideal generated by the Arnold relation*

$$\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} \equiv 0,$$

for $i \neq j \neq k$.

- Definition: The rth Drinfeld-Kohno Lie algebra (the Lie algebra of infinitesimal braids on r strands) is a Lie algebra defined by a presentation

$$\mathfrak{p}(r) = \mathbb{L}(t_{ij}, 1 \leq i \neq j \leq r)/(\text{commutation}, \text{Yang-Baxter})$$

  where we take the free Lie algebra $\mathbb{L}(-)$ on generating elements t_{ij} associated to each pair $1 \leq i \neq j \leq r$ modulo the ideal generated by:
  - the commutation relations
  $$[t_{ij}, t_{kl}] = 0 \quad (\forall i \neq j \neq k \neq l),$$
  - and the Yang-Baxter relations
  $$[t_{ij}, t_{ik} + t_{kj}] = 0 \quad (\forall i \neq j \neq k).$$
- The enveloping algebra of this Lie algebra $\mathbb{U}\,\mathfrak{p}(r)$ can be defined by the same presentation, where we just set $[u, v] = uv - vu$.

- Theorem (Kohno): *Let* $\mathtt{C}^*_{CE}(\mathfrak{p}(r))$ *denotes the Chevalley-Eilenberg cochain complex of the Lie algebra* $\mathfrak{p}(r)$*. For each* r*, we have a dg-algebra quasi-isomorphism*

$$\mathtt{C}^*_{CE}(\mathfrak{p}(r)) \xrightarrow[\sim]{\kappa} \mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^2, r)).$$

- Reminder: The Chevalley-Eilenberg cochain complex of a Lie algebra $\mathfrak{p}(r) = \mathfrak{g}$ is a quasi-free commutative dg-algebra

$$\mathtt{C}^*_{CE}(\mathfrak{g}) = (\mathtt{S}(\Sigma^{-1}\mathfrak{g}^\vee), \partial^\vee_{CE})$$

where: the desuspension Σ^{-1} puts the module $\mathfrak{g}^\vee$ in (cohomological) degree 1, and the differential

$$\partial^\vee_{CE} : \mathtt{S}(\Sigma^{-1}\mathfrak{g}^\vee) \to \mathtt{S}(\Sigma^{-1}\mathfrak{g}^\vee)$$

is given by the dual of the Lie bracket map $[-,-] : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ on this generating module $\Sigma^{-1}\mathfrak{g}^\vee \subset \mathtt{S}(\Sigma^{-1}\mathfrak{g}^\vee)$.

Explanations:

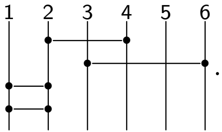
- The Lie algebra $\mathfrak{p}(r)$ is equipped with a weight grading such that each generating element t_{ij} is homogeneous of weight 1.
- The morphism $\kappa : \mathtt{C}^*_{CE}(\mathfrak{p}(r)) \to \mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^2, r))$ is defined on the dual elements on the monomial basis of $\mathfrak{p}(r)$ by the following assignment:
  - $\kappa(t_{ij}^\vee) = \omega_{ij}$, for each generator t_{ij},
  - $\kappa(\alpha^\vee) = 0$, when $\alpha^\vee$ has weight > 1.
- The theorem implies that the quasi-free commutative dg-algebra $\mathtt{C}^*_{CE}(\mathfrak{p}(r))$ is the minimal model of the cohomology algebra $\mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^2, r))$.
- For a quasi-free dg-algebra of this form $\mathtt{C}^*_{CE}(\mathfrak{p}(r)) = \mathtt{C}^*_{CE}(\mathfrak{g})$, we have an identity in the homotopy category of simplicial sets:

$$\mathtt{G}_\bullet(\mathtt{C}^*_{CE}(\mathfrak{g})) = \mathtt{Mor}_{dg\,\mathcal{C}om}(\mathtt{C}^*_{CE}(\mathfrak{g}), \Omega^*(\Delta^\bullet)) \sim \mathtt{B}(\mathbb{G}\,\hat{\mathbb{U}}(\mathfrak{g})),$$

  where $\mathbb{G}(-)$ denotes the group-like element functor from (complete) Hopf algebras to groups.

§2. The Drinfeld-Kohno Lie algebra operad

- The monomials $t_{i_1 j_1} \cdots t_{i_r j_r} \in \mathbb{U}(\mathfrak{p}(r))$ are usually represented by chord diagrams on r strands. For instance:

$t_{12} t_{12} t_{36} t_{24} =$ 

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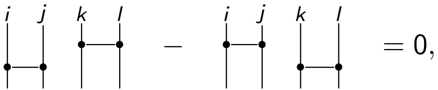
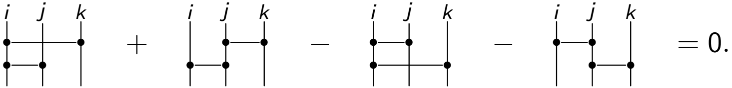
- In this representation, the commutator and Yang-Baxter relations read:

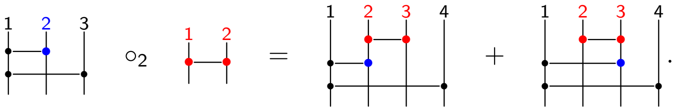




- Observations: The collection $\mathfrak{p} = \{\mathfrak{p}(r), r \in \mathbb{N}\}$ forms an operad in the category of Lie algebras (the Drinfeld-Kohno Lie algebra operad), and $\mathbb{U}\,\mathfrak{p} = \{\mathbb{U}\,\mathfrak{p}(r), r \in \mathbb{N}\}$ forms an operad in the category of Hopf algebras.
- The operad structure of the Drinfeld-Kohno Lie algebra operad is given by:
  - an action of the symmetric group Σ_r on $\mathfrak{p}(r)$, for each $r \in \mathbb{N}$, which is given by the permutation of strand indices,
  - composition operations

$$\mathfrak{p}(m) \oplus \mathfrak{p}(n) \xrightarrow{\circ_k} \mathfrak{p}(m+n-1)$$
$$\Leftrightarrow \mathbb{U}\,\mathfrak{p}(m) \otimes \mathbb{U}\,\mathfrak{p}(n) \xrightarrow{\circ_k} \mathbb{U}\,\mathfrak{p}(m+n-1),$$

defined for all $m, n \in \mathbb{N}$, $k = 1, \ldots, m$, and which, in the chord picture, are given by the following insertion operations:



Construction:

- Take $\hat{\mathbb{U}}\,\mathfrak{p}(r)$ the completion of the enveloping algebras $\mathbb{U}\,\mathfrak{p}(r)$ with respect to their augmentation ideals,
- and the sets of group like elements

$$\mathbb{G}\,\hat{\mathbb{U}}\,\mathfrak{p}(r) = \{u \in \hat{\mathbb{U}}\,\mathfrak{p}(r) |\, \epsilon(u) = 1\,,\ \Delta(u) = u \hat{\otimes} u\}$$

  in these complete Hopf algebras.
- Regard these groups as the morphism sets of groupoids $\mathrm{CD}^{\wedge}_{\mathbb{Q}}(r)$ such that $\mathtt{Ob}\,\mathrm{CD}^{\wedge}_{\mathbb{Q}}(r) = *$.
- These groupoids $\mathrm{CD}^{\wedge}_{\mathbb{Q}}(r)$ form an operad (in the category of groupoids) $\mathrm{CD}^{\wedge}_{\mathbb{Q}}$, with composition products

$$\underbrace{\mathtt{Mor}\,\mathrm{CD}^{\wedge}_{\mathbb{Q}}(m)}_{=\mathbb{G}\,\hat{\mathbb{U}}\,\mathfrak{p}(m)} \times \underbrace{\mathtt{Mor}\,\mathrm{CD}^{\wedge}_{\mathbb{Q}}(n)}_{=\mathbb{G}\,\hat{\mathbb{U}}\,\mathfrak{p}(n)} \xrightarrow{\circ_k} \underbrace{\mathtt{Mor}\,\mathrm{CD}^{\wedge}_{\mathbb{Q}}(m+n-1)}_{=\mathbb{G}\,\hat{\mathbb{U}}\,\mathfrak{p}(m+n-1)}$$

  induced by the operadic composition of chord diagrams.

- Observation (Tamarkin): The collection

$$\mathtt{C}^*_{CE}(\mathfrak{p}) = \{\mathtt{C}^*_{CE}(\mathfrak{p}(r)), r \in \mathbb{N}\}$$

inherits the structure of a Hopf dg-cooperad (a cooperad in commutative dg-algebras) and the collection of quasi-isomorphisms $\kappa : \mathtt{C}^*_{CE}(\mathfrak{p}(r)) \xrightarrow{\sim} \mathtt{H}^*(\mathsf{D}_2(r))$ in §1, where we also use the identity $\mathtt{H}^*(\mathsf{D}_2(r)) = \mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^2, r))$, defines a quasi-isomorphism of Hopf cooperads

$$\kappa : \mathtt{C}^*_{CE}(\mathfrak{p}) \xrightarrow{\sim} \mathtt{H}^*(\mathsf{D}_2).$$

- Proposition: *The collection*

$$\mathtt{G}_\bullet(\mathtt{C}^*_{CE}(\mathfrak{p})) = \{\mathtt{Mor}_{dg\,\mathcal{C}om}(\mathtt{C}^*_{CE}(\mathfrak{p}(r)), \Omega^*(\Delta^\bullet)), r \in \mathbb{N}\}$$

forms an operad in simplicial sets which is isomorphic to the operad of classifying spaces $\mathtt{B}(\mathsf{CD}^\wedge_{\mathbb{Q}}(r)) = \mathtt{B}(\mathbb{G}\,\hat{\mathbb{U}}\,\mathfrak{p}(r))$ *in the homotopy category of operads in simplicial sets.*

§3. Drinfeld's associators

- Reminder: The operad of parenthesized braids is an operad in groupoids PaB such that any morphism $\phi : \mathsf{PaB} \to \mathsf{P}$ is uniquely determined by giving:
  - an object $m \in \mathtt{Ob}\,\mathsf{P}(2)$,
  - and isomorphisms

$$a \in \mathtt{Mor}_{\mathsf{P}(3)}(\underbrace{m(m(x_1, x_2), x_3)}_{=m\circ_1 m}, \underbrace{m(x_1, m(x_2, x_3))}_{=m\circ_2 m}),$$

$$c \in \mathtt{Mor}_{\mathsf{P}(2)}(\underbrace{m(x_1, x_2)}_{=m}, \underbrace{m(x_2, x_1)}_{=(1\ 2)m}),$$

  satisfying coherence constraints, expressed by pentagon and hexagon equations in $\mathtt{Mor}\,\mathsf{P}$.
- This operad has a unitary extension PaB_+ with an additional element $e \in \mathsf{PaB}_+(0)$ representing a strict unit operation for these structures which the parenthesized braid operad models.

- Definition: The set of Drinfeld's associators is the set of operad morphisms $\phi : \mathsf{PaB}_+ \to \mathsf{CD}^{\wedge}_{\mathbb{Q}}$ so that we have:

$$c = \exp(\hbar t_{12}/2)$$

for some parameter $\hbar \in \mathbb{Q}^{\times}$. The associativity isomorphism $a \in \mathtt{Mor}\,\mathsf{CD}^{\wedge}_{\mathbb{Q}}$ associated to such a morphism can be expressed as the exponential of a Lie power series, so that we get:

$$a = \exp\phi(t_{12}, t_{23}) \in \exp\hat{\mathbb{L}}(t_{12}, t_{23}).$$

- Theorem (Drinfeld): *The set of Drinfeld associators is non-empty.*

- Proposition: *The morphism of operads in simplicial sets* $\mathtt{B}(\phi) : \mathtt{B}(\mathsf{PaB}_+) \to \mathtt{B}(\mathsf{CD}^{\wedge}_{\mathbb{Q}})$ *associated to a Drinfeld associator induces an iso at the rational cohomology level.*
- Conclusion: Recall that we also have $\mathsf{D}_2 \sim \mathtt{B}(\mathsf{PaB}_+)$. The operad $\mathtt{B}(\mathsf{CD}^{\wedge}_{\mathbb{Q}}) \sim \mathtt{G}_\bullet(\mathtt{C}^*_{CE}(\mathfrak{p}))$, determined by a model of the cohomology cooperad of little 2-discs, can therefore be identified with a rationalization of the little 2-discs operad in the category of topological spaces.

Generalization to higher dimension cases:

- The cohomology of configuration spaces $\mathrm{F}(\mathring{\mathbb{D}}^n, r)$ have the same description in the case $n > 2$ has in the case $n = 2$ (up to degree shifts and parity changes).
- The definition of the model $\kappa : \mathtt{C}^*_{CE}(\mathfrak{p}) \xrightarrow{\sim} \mathtt{H}^*(\mathrm{D}_n)$ extends to all $n \geq 2$ as well, and the operad in simplicial sets $\mathtt{G}_\bullet(\mathtt{C}^*_{CE}(\mathfrak{p}))$ represents a realification (conjecturally, a rationalization) of the little n-discs operad D_n, for any such $n \geq 2$ (Kontsevich).