The Homotopy of $E_n$-operads

Lecture 2: How to Recognize $E_n$-operads?

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Introduction

▶ **Reminder:** A weak-equivalence of operads (in topological spaces) is an operad morphism $\phi : P \to Q$ of which components $\phi(r) : P(r) \to Q(r)$ are weak-equivalences (of topological spaces). An $E_n$-operad (in topological spaces) is an operad $E_n$ connected to the operad of little $n$-discs $D_n$ (or little $n$-cubes) by a chain of weak-equivalences:

$$E_n \xleftarrow{\sim} \cdots \xrightarrow{\sim} \cdots \xrightarrow{\sim} D_n.$$ 

▶ **Remark:** There is such a chain of operad weak-equivalences between the little $n$-discs and the little $n$-cubes operads. The little discs and the little cubes models accordingly define the same notion of $E_n$-operad.

▶ **Problem:** How to recognize $E_n$-operads?
§0. Basic Cases

- **Reminder:**
  - For each $r$, we have $D_1(r) \sim \Sigma_r$, and the little 1-discs operad $D_1$ is weakly-equivalent to the set operad of associative monoids $As$, regarded as a discrete operad in the category of topological spaces.
  - For each $r$, we have $D_\infty(r) \sim pt$, and the operad $D_\infty$ is weakly-equivalent to the set operad of associative and commutative monoids $Com$, regarded as a discrete operad in the category of topological spaces.

- **Corollaries:**
  - An operad $P$ is an $E_1$-operad if and only if each space $P(r)$ has contractible components indexed by permutations, and $\pi_0 P$ is equal to $As$ as an operad in sets.
  - An operad $P$ is an $E_\infty$-operad if and only if each space $P(r)$ is contractible so that $\pi_0 P$ is equal to $Com$ as an operad in sets.
§1. The fundamental groupoid operad of little 2-discs

Reminder: The disc center mapping

\[ \in D_2(4) \in F(\mathbb{D}^2, 4) \]

defines a homotopy equivalence between the little \( n \)-discs spaces \( D_n(r) \) and the configuration space of \( r \) points in the open disc \( F(\mathbb{D}^n, r) \).
Observations: In the case $n = 2$, we have $\pi_i(F(\hat{D}^2, r)) = \ast$ for $i \neq 1$, and $P_r = \pi_1 F(\hat{D}^2, r)$ defines the pure braid group on $r$ strands.

The homotopy equivalence $D_2(r) \sim F(\hat{D}^2, r)$ therefore implies that the space $D_2(r)$ is an Eilenberg-MacLane space $K(P_r, 1)$.

Problem: Base points are not preserved by symmetric group actions and composition operations.

The idea is to use the fundamental groupoids $\pi D_2(r)$ in order to get a combinatorial model of the operad $D_2$. 
Construction:

- The fundamental groupoid $\pi D_2(r)$ has $0b \pi D_2(r) = D_2(r)$ as object set.
- The morphisms of $\text{Mor}_{\pi D_2(r)}(a, b)$ are homotopy classes of paths $\gamma : [0, 1] \to D_2(r)$ going from $\gamma(0) = a$ to $\gamma(1) = b$.
- The map

$$\text{Mor}_{\pi D_2(r)}(a, b) \xrightarrow{\text{disc centers}} \sim \text{Mor}_{\pi F(\mathbb{D}^2, r)}(a, b)$$

identifies the morphism sets of this groupoid with cosets of the pure braid group $P_r$ inside the braid group $B_r$. Thus, a morphism in this groupoid can be represented by a picture of the form:

\[
\gamma = \includegraphics[width=0.5\textwidth]{diagram.png}
\]

\[
\in \text{Mor}_{\pi D_2(2)}
\]

\[
= a
\]

\[
= b
\]
The structure of the fundamental groupoid operad:

- The groupoids $\pi D_2(r)$ inherit a symmetric structure, as well as operadic composition products

$$\circ_i : \pi D_2(m) \times \pi D_2(n) \to \pi D_2(m + n - 1),$$

and hence form an operad in the category of groupoids.

- In the braid picture, the operadic composition products can be depicted as cabling operations:

\[
\begin{align*}
\circ_1 : \pi D_2(2) \times \pi D_2(2) & \to \pi D_2(3) \\
\in \text{Mor } \pi D_2(2) & \in \text{Mor } \pi D_2(2) \\
\end{align*}
\]
§2. The operad of parenthesized braids

Ideas:

- There is no need to consider the whole $D_2(r)$ as object set.
- The groupoids of parenthesized braids $PaB(r)$ are full subgroupoids of the fundamental groupoid $\pi D_2(r)$ defined by appropriate subsets of little 2-discs configurations $\Omega(r) \subset D_2(r)$ as object sets.
- These object sets $\Omega(r)$ are preserved by the operadic composition structure of little 2-discs so that the collection of groupoids $PaB(r)$, $r \in \mathbb{N}$, forms a suboperad of $\pi D_2$. 
The sets $\Omega(r), r = 2, 3, 4, \ldots,$ prescribing the origin and end-points of paths in $\text{PaB}(r)$, consist of little 2-disc configurations of the following form:

(\text{the indices } i, j, \ldots \text{ run over all permutations of } 1, 2, \ldots ).

These configurations represent the iterated operadic composites of the following element

\[
\mu = \begin{array}{c}
\includegraphics{12.png}
\end{array} \in \mathbb{D}_2(2)
\]

and $\Omega$ is formally defined as the suboperad of $\mathbb{D}_2$ generated by this element.

**Proposition:** This operad $\Omega$ is identified with the Magma operad (the free operad on one generator).
Fundamental morphisms of $\text{PaB}$ include:

- the associator

\[
\alpha = \in \text{Mor } \text{PaB}(3)
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \quad 2 \quad 3
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\phi \circ_1 \phi
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\phi \circ_2 \phi
\end{array}
\end{array}
\]

- and the braiding

\[
\tau = \in \text{Mor } \text{PaB}(2)
\]

\[
\begin{array}{c}
\begin{array}{c}
1 \quad 2
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\mu
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
t \mu
\end{array}
\end{array}
\]

where $t = (12) \in \Sigma_2$. 

In the morphism set of the operad PaB:

- the associator satisfy the pentagon equation

and we have two hexagon equations combining associators and braidings.
The first one reads:
and the second one reads:
Theorem (Mac Lane + Joyal-Street): An operad morphism
\( \phi : \text{PaB} \to P \), where \( P \) is any operad in the category of categories, is uniquely determined by:

- an object \( m \in \text{Ob} \ P(2) \), which represents the image of the little 2-disc configuration \( \mu \in D_2(2) \) under the map \( \phi : \text{Ob} \ \text{PaB}(2) \to \text{Ob} \ P(2) \),
- an isomorphism \( a \in \text{Mor} \ P(3)(m \circ_1 m, m \circ_2 m) \), which represents the image of the associator \( \alpha \in \text{Mor} \ \text{PaB}(3)(\mu \circ_1 \mu, \mu \circ_2 \mu) \),
- and an isomorphism \( c \in \text{Mor} \ P(2)(m, tm) \), which represents the image of the braiding \( \tau \in \text{Mor} \ \text{PaB}(2)(\mu, t\mu) \),
- so that \( a \) and \( c \) satisfy the analogue of the usual pentagon and hexagon relations of braided monoidal categories in \( \text{Mor} \ P \).

This result implies that \( \text{PaB} \) is generated by operations defining the structure of a braided monoidal category (without unit).
Observation: The operad \( \mathbf{PaB} \) has a unitary extension \( \mathbf{PaB}_+ \) with \( \mathbf{PaB}_+(0) = \ast \), and where the operadic composition with the additional arity zero element \( \ast \in \mathbf{PaB}_+(0) \) is given by a rectification process in the little 2-discs space \( D_2 \). This operads govern braided monoidal category structures with strict units.

Theorem (variant of a result of Fiedorowicz): The operad \( \mathcal{B}(\mathbf{PaB})_+ \) formed by the classifying spaces of the groupoids \( \mathbf{PaB}_+(r) \) is an \( E_2 \)-operad.

Corollary: The classifying space \( \mathcal{B}(\mathcal{M}) \) of a braided monoidal category \( \mathcal{M} \) is weakly-equivalent to a double loop space \( \Omega^2 X \).
§3. Generalizations

▷ Batanin’s recognition theorem: operads arising from a symmetrization of a contractible $n$-operad are $E_n$-operads. The $n$th Fulton-MacPherson operad is an operad equipped with a structure of this form.

▷ Berger’s recognition theorem: operads equipped a cell structure shaped on complete graphs give rise to nested sequences of operads which are weakly-equivalent to the sequence of the little discs operads. The Barratt-Eccles operad, consisting of the contractible bar construction of the symmetric groups $E(r) = ESigma_r$, is an instance of an operad equipped with a structure of this form.