The Homotopy of $E_n$-operads

Lecture 1: Introduction to the Theory

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Introduction

- The operad of little $n$-discs was introduced in topology in order to model structures attached to $n$-fold loop spaces (Boardman-Vogt, May).
- The little discs operads are also used to define a hierarchy of homotopy commutative structures, from fully homotopy associative but non-commutative ($n = 1$), up to fully homotopy associative and commutative ($n = \infty$).
- For such applications, one usually deals with operads weakly-equivalent to the reference model of little $n$-discs, and the name of an $E_n$-operad is used to refer to this notion.
Chapter 0. The notion of an operad

**Definition:** An operad $P$ in a symmetric monoidal category $\mathcal{M}$ (e.g. spaces, modules, ... ) consists of a collection of objects $P(r) \in \mathcal{M}$, $r \in \mathbb{N}$, equipped with:

- an action of the symmetric group $\Sigma_r$ on $P(r)$, for each $r \in \mathbb{N}$;
- composition operations $\circ_i : P(m) \otimes P(n) \to P(m + n - 1)$, defined for each $m, n \in \mathbb{N}$, $i = 1, \ldots, m$, and satisfying natural equivariance, unit and associativity relations.

*Intuitively, the elements $p \in P(r)$ (whenever the notion of an element makes sense) model operations on $r$ variables $p = p(x_1, \ldots, x_r)$, and the definition of an operad formalizes the structures naturally associated with operations of this form.*
Fundamental example: Let $M \in \mathcal{M}$. The collection of hom-objects $\text{End}_M(r) = \text{Hom}(M^\otimes r, M)$ forms an operad associated to $M$, the endomorphism operad of $M$. In the point-set (module) context:

- The action of a permutation $\sigma \in \Sigma_r$ on an element $f \in \text{End}_M(r)$ is defined by:
  $$(\sigma f)(x_1, \ldots, x_r) = f(x_{\sigma(1)}, \ldots, x_{\sigma(r)}).$$

- The operadic composite of elements $f \in \text{End}_M(m)$ and $g \in \text{End}_M(n)$ is defined by:
  $$(f \circ_i g)(x_1, \ldots, x_{m+n-1}) = f(x_1, \ldots, g(x_i, \ldots, x_{i+n-1}), \ldots, x_{m+n-1}).$$

The structure of an operad is modeled on this fundamental example.
1. The little discs and little cubes models

The little \( n \)-discs spaces \( D_n(r) \) consist of collections of \( r \) little \( n \)-discs with disjoint interiors inside a fixed unit \( n \)-disc \( \mathbb{D}^n \) (see Figure).

The configuration spaces \( F(\mathbb{D}^n, r) \) consist of collections of \( r \) distinct points in the open disc \( \mathbb{D}^n \) (see Figure).

There is an obvious homotopy equivalence \( D_n(r) \xrightarrow{\sim} F(\mathbb{D}^n, r) \).
The symmetric group $\Sigma_r$ acts on $D_n(r)$ by permutation of the little disc indices (and on the configuration space similarly).

The little $n$-discs spaces (unlike the configuration spaces) inherit operadic composition operations

$$\circ_k : D_n(r) \times D_n(s) \to D_n(r + s - 1)$$

given by the following substitution process

The little $n$-discs operad $D_n$ is the structure defined by the collection of little $n$-discs spaces $D_n = \{D_n(r), r \in \mathbb{N}\}$ equipped with this symmetric operadic composition structure.
The little $n$-cubes operad $C_n$ is a variant of the operad of $r$ little $n$-discs, where we consider configurations of cubes in a fixed unit cube $I^n = [0, 1]^n$ instead of discs (see Figure).

The symmetric and composition structure of this operad is defined as in the little discs setting.

The little cubes spaces $C_n(r)$ also inherit obvious homotopy equivalences $C_n(r) \sim F(I^n, r)$. 
Reminder:

- The little $n$-cubes operad (and the little $n$-discs operad similarly) encodes operations acting on $n$-fold loop spaces $\Omega^n X$.

- **Theorem (May’s approximation theorem):** If $X$ is a connected base space, then we have a weak-equivalence $\iota : S_\ast(C_n, X) \sim \Omega^n \Sigma^n X$, where $S_\ast(C_n, X)$ is a base version of the free $C_n$-algebra generated by $X$.

- **Theorem (Boardman-Vogt’, May’s recognition theorem):** If $Y$ is a connected (or more generally a group-like) base space equipped with an action of the operad $C_n$, then we have $Y \sim \Omega^n X$ for some $n$-fold loop space $\Omega^n X$. 
§2. The notion of an $E_n$-operad

Definition: A weak-equivalence of operads (in topological spaces) is an operad morphism $\phi : P \to Q$ of which components $\phi(r) : P(r) \to Q(r)$ are weak-equivalences (of topological spaces). The notation $\sim \rightarrow$ will be used to mark any distinguished class of weak-equivalences in a category (e.g. in the category of topological spaces, in the category of operads).

Definition: An $E_n$-operad (in topological spaces) is an operad $E_n$ connected to the operad of little $n$-discs $D_n$ (or little $n$-cubes) by a chain of weak-equivalences of operads

$$E_n \leftrightarrow \sim \rightarrow \ldots \sim \rightarrow D_n.$$

Not such an easy exercise: There is such a chain of operad weak-equivalences between the little $n$-discs and the little $n$-cubes operads. The little discs and the little cubes models accordingly define the same notion of $E_n$-operad.

Problem: How to recognize $E_n$-operads?
For each \( r \), we have a topological inclusion \( \iota : D_{n-1}(r) \hookrightarrow D_n(r) \) mapping any collection of little \((n-1)\)-discs

\[
c_i(D^{n-1}) = P_i + R_i \cdot D^{n-1}, \quad i = 1, \ldots, r,
\]

to the collection of little \( n \)-discs with the same radius \( R_i \) and the same center \( P_i \) in the equatorial hyperplane of the unit \( n \)-discs \( D^n \), as in the following picture:

These maps define an inclusion of topological operads \( \iota : D_{n-1} \hookrightarrow D_n \).
Thus the operads of little discs form a nested sequence of operads in topological spaces:

$$D_1 \hookrightarrow D_2 \hookrightarrow \cdots \hookrightarrow D_n \hookrightarrow \cdots .$$

We also set $D_\infty = \text{colim}_n D_n$.

**Proposition:**

- For each $r$, we have $D_1(r) \sim \Sigma_r$, and the little 1-discs operad $D_1$ is weakly-equivalent to the set operad of associative monoids $\text{As}$, regarded as a discrete operad in the category of topological spaces.

- For each $r$, we have $D_\infty(r) \sim \text{pt}$, and the operad $D_\infty$ is weakly-equivalent to the set operad of associative and commutative monoids $\text{Com}$, regarded as a discrete operad in the category of topological spaces.