The Homotopy of E_n-operads
–
Lecture 1: Introduction to the Theory

Benoit Fresse

Université Lille 1

YTM 2013, Lausanne
July 8-12, 2013

Introduction

- The operad of little n-discs was introduced in topology in order to model structures attached to n-fold loop spaces (Boardman-Vogt, May).
- The little discs operads are also used to define a hierarchy of homotopy commutative structures, from fully homotopy associative but non-commutative ($n = 1$), up to fully homotopy associative and commutative ($n = \infty$).
- For such applications, one usually deals with operads weakly-equivalent to the reference model of little n-discs, and the name of an E_n-operad is used to refer to this notion.

§0. The notion of an operad

Definition: An operad P in a symmetric monoidal category $\mathcal{M}$ (e.g. spaces, modules, ...) consists of a collection of objects $\mathsf{P}(r) \in \mathcal{M}$, $r \in \mathbb{N}$, equipped with:

- an action of the symmetric group Σ_r on $\mathsf{P}(r)$, for each $r \in \mathbb{N}$;
- composition operations

$$\circ_i : \mathsf{P}(m) \otimes \mathsf{P}(n) \to \mathsf{P}(m+n-1),$$

defined for each $m, n \in \mathbb{N}$, $i = 1, \dots, m$, and satisfying natural equivariance, unit and associativity relations.

Intuitively, the elements $p \in \mathsf{P}(r)$ (whenever the notion of an element makes sense) model operations on r variables $p = p(x_1, \dots, x_r)$, and the definition of an operad formalizes the structures naturally associated with operations of this form.

Fundamental example: Let $M \in \mathcal{M}$. The collection of hom-objects $\mathtt{End}_M(r) = \mathtt{Hom}(M^{\otimes r}, M)$ forms an operad associated to M, the endomorphism operad of M. In the point-set (module) context:

- The action of a permutation $\sigma \in \Sigma_r$ on an element $f \in \mathtt{End}_M(r)$ is defined by:

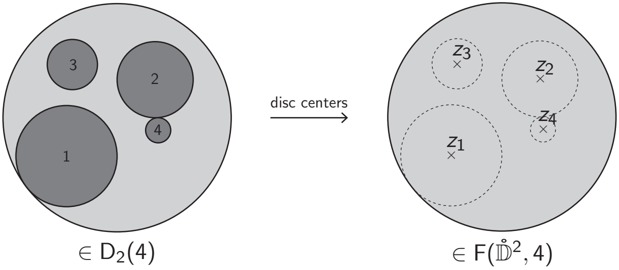
$$(\sigma f)(x_1, \dots, x_r) = f(x_{\sigma(1)}, \dots, x_{\sigma(r)}).$$

- The operadic composite of elements $f \in \mathtt{End}_M(m)$ and $g \in \mathtt{End}_M(n)$ is defined by:

$$(f \circ_i g)(x_1, \dots, x_{m+n-1}) \\ = f(x_1, \dots, g(x_i, \dots, x_{i+n-1}), \dots, x_{m+n-1}).$$

The structure of an operad is modeled on this fundamental example.

§1. The little discs and little cubes models

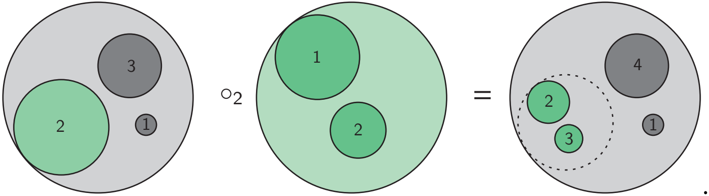


- The little n-discs spaces $D_n(r)$ consist of collections of r little n-discs with disjoint interiors inside a fixed unit n-disc $\mathbb{D}^n$ (see Figure).
- The configuration spaces $F(\mathring{\mathbb{D}}^n, r)$ consist of collections of r distinct points in the open disc $\mathring{\mathbb{D}}^n$ (see Figure).
- There is an obvious homotopy equivalence $D_n(r) \xrightarrow{\sim} F(\mathring{\mathbb{D}}^n, r)$.

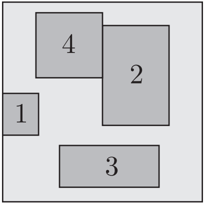
- The symmetric group Σ_r acts on $\mathrm{D}_n(r)$ by permutation of the little disc indices (and on the configuration space similarly).
- The little n-discs spaces (unlike the configuration spaces) inherit operadic composition operations

$$\circ_k : \mathrm{D}_n(r) \times \mathrm{D}_n(s) \to \mathrm{D}_n(r+s-1)$$

given by the following substitution process



- The little n-discs operad D_n is the structure defined by the collection of little n-discs spaces $\mathrm{D}_n = \{\mathrm{D}_n(r), r \in \mathbb{N}\}$ equipped with this symmetric operadic composition structure.



- The little n-cubes operad C_n is a variant of the operad of r little n-discs, where we consider configurations of cubes in a fixed unit cube $\mathbb{I}^n = [0,1]^n$ instead of discs (see Figure).
- The symmetric and composition structure of this operad is defined as in the little discs setting.
- The little cubes spaces $\mathsf{C}_n(r)$ also inherit obvious homotopy equivalences $\mathsf{C}_n(r) \xrightarrow{\sim} \mathsf{F}(\mathring{\mathbb{I}}^n, r)$.

Reminder:

- The little n-cubes operad (and the little n-discs operad similarly) encodes operations acting on n-fold loop spaces $\Omega^n X$.
- Theorem (May's approximation theorem): *If X is a connected base space, then we have a weak-equivalence $\iota : \mathrm{S}_*(\mathrm{C}_n, X) \xrightarrow{\sim} \Omega^n \Sigma^n X$, where $\mathrm{S}_*(\mathrm{C}_n, X)$ is a base version of the free C_n-algebra generated by X.*
- Theorem (Boardman-Vogt', May's recognition theorem): *If Y is a connected (or more generally a group-like) base space equipped with an action of the operad C_n, then we have $Y \sim \Omega^n X$ for some n-fold loop space $\Omega^n X$.*

§2. The notion of an E_n-operad

- Definition: A weak-equivalence of operads (in topological spaces) is an operad morphism $\phi : \mathsf{P} \to \mathsf{Q}$ of which components $\phi(r) : \mathsf{P}(r) \to \mathsf{Q}(r)$ are weak-equivalences (of topological spaces). The notation $\xrightarrow{\sim}$ will be used to mark any distinguished class of weak-equivalences in a category (e.g. in the category of topological spaces, in the category of operads).
- Definition: An E_n-operad (in topological spaces) is an operad E_n connected to the operad of little n-discs D_n (or little n-cubes) by a chain of weak-equivalences of operads

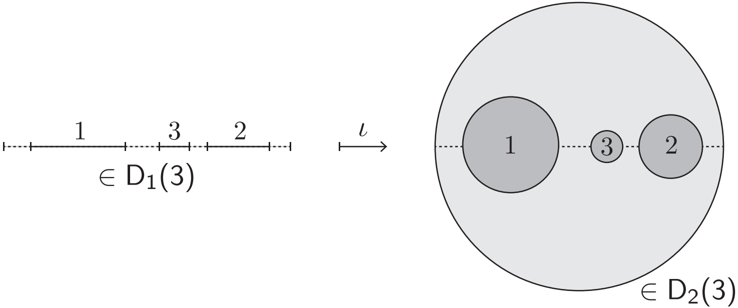
$$\mathsf{E}_n \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdot \ldots \cdot \xrightarrow{\sim} \mathsf{D}_n \, .$$

- Not such an easy exercise: There is such a chain of operad weak-equivalences between the little n-discs and the little n-cubes operads. The little discs and the little cubes models accordingly define the same notion of E_n-operad.
- Problem: How to recognize E_n-operads?

- For each r, we have a topological inclusion $\iota : \mathsf{D}_{n-1}(r) \hookrightarrow \mathsf{D}_n(r)$ mapping any collection of little $(n-1)$-discs

$$c_i(\mathbb{D}^{n-1}) = P_i + R_i \cdot \mathbb{D}^{n-1}, \quad i = 1, \dots, r,$$

  to the collection of little n-discs with the same radius R_i and the same center P_i in the equatorial hyperplane of the unit n-discs $\mathbb{D}^n$, as in the following picture:



- These maps define an inclusion of topological operads $\iota : \mathsf{D}_{n-1} \hookrightarrow \mathsf{D}_n$.

- Thus the operads of little discs form a nested sequence of operads in topological spaces:

$$\mathsf{D}_1 \hookrightarrow \mathsf{D}_2 \hookrightarrow \cdots \hookrightarrow \mathsf{D}_n \hookrightarrow \cdots .$$

  We also set $\mathsf{D}_\infty = \operatorname{colim}_n \mathsf{D}_n$.
- Proposition:
  - *For each r, we have $\mathsf{D}_1(r) \sim \Sigma_r$, and the little 1-discs operad D_1 is weakly-equivalent to the set operad of associative monoids* As*, regarded as a discrete operad in the category of topological spaces.*
  - *For each r, we have $\mathsf{D}_\infty(r) \sim pt$, and the operad D_∞ is weakly-equivalent to the set operad of associative and commutative monoids* Com*, regarded as a discrete operad in the category of topological spaces.*