HOMOTOpy OF OPERADS
&
GROTHENDIECK-TEICHMÜLLER GROUPS

by

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The Computation of Homotopy Automorphism Spaces of Operads
Homotopy of Operads & Grothendieck-Teichmüller Groups
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HOMOTOPY OF OPERADS AND
GROTHENDIECK-TEICHMÜLLER GROUPS

The Computation of Homotopy Automorphism Spaces of Operads

This document is the concluding part of a research monograph in preparation on the homotopy of operads and Grothendieck-Teichmüller groups. The ultimate objective of this work is to prove that the Grothendieck-Teichmüller group is the group of homotopy automorphisms of a rational completion of the little 2-discs operad.

The full monograph has two volumes. This first volume provides a comprehensive introduction to the fundamental concepts of operad theory, a survey chapter on little discs operads and $E_n$-operads, a detailed study of the connections between little 2-discs and braids, an introduction to the theory of Hopf algebras and the Malcev completion of groups, and a report on the definition of the Grothendieck-Teichmüller group from the viewpoint of the theory of algebraic operads. We conclude this volume with the definition of a map from the pro-unipotent Grothendieck-Teichmüller group towards the group of homotopy automorphism classes of the rationalization of the little 2-disc operad. Most concepts are carefully reviewed in order to make this account accessible to a broad readership, which should include graduate students as well as researchers coming from various fields of mathematics related to our main topics.

The second volume provides a comprehensive introduction to the methods of homotopical algebra, a survey on the definition of the Sullivan model for the rational homotopy of spaces, and the definition of an extension of this model to study the rational homotopy of operads in spaces. We then give an account of the applications of Bousfield-Kan’ spectral sequences to the computation of the homotopy of function spaces of operads. We use these methods to prove the main result of this book: the pro-unipotent Grothendieck-Teichmüller group is isomorphic to the group of homotopy automorphism classes of the rationalization of the little 2-disc operad.

We give this concluding part of the book, from the definition of Bousfield-Kan’ spectral sequences up to our computation of homotopy automorphism spaces of operads, in this excerpt. This part is widely self-contained. We mainly assume that the reader is familiar with the basic concepts of the theory of operads and with the classical methods of homotopy theory which we review in other parts of the book.

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Bibliography
CHAPTER 1

Results on the Homotopy Automorphism Space of $E_2$-operads and Plans

The purpose of this part is to determine the space of homotopy automorphisms of $E_2$-operads. To be more precise, we establish the following statement, which is the ultimate goal of this book:

**Theorem A.** Let $E^+_2$ denote a cofibrant-fibrant model of the rationalization of the (unitary) operad of little 2-discs $D^+_2$. We have an identity:

$$h\text{Aut}_{\text{Top} \mathcal{O}_p}(E^+_2) \sim GT(\mathbb{Q}) \ltimes SO(2)^\wedge$$

in the homotopy category of monoids in topological space, where we consider:

- the (geometric realization of the) homotopy automorphism space of the operad $E^+_2$ on the left-hand side;
- the cartesian product of the pro-unipotent Grothendieck–Teichmüller group $GT(\mathbb{Q})$ with the rationalization of the topological group $SO(2)$ on the right-hand side.

We consider the category of unitary operads in topological spaces $\mathcal{O}_p = \text{Top} \mathcal{O}_p$ in order to simplify the formulation of this statement. We continue to deal with topological objects for a brief moment in order to explain our result. We go back to the framework of simplicial sets afterwards.

Recall that we use a $+$ lower-script to distinguish the objects of the category of unitary operad $P_+ \in \mathcal{O}_p$, which we characterize by $P_+(0) = *$. The notation $P \in \mathcal{O}_p$ (without this mark $+$) then refers to the non-unitary operad, satisfying $P(0) = \emptyset$, and which we defined by dropping the arity zero term $P_+(0) = *$ from any such unitary operad $P_+ \in \mathcal{O}_p$. In our proofs, we also use that we can retrieve the composition structure of the unitary operad $P_+$ from an extra structure, which we encode in the notion of a $\Lambda$-operad, and which we define at the level of this non-unitary operad $P$. We go back to this technical point later on in this introduction. We only aim to give a basic account of our main results at this stage. We therefore focus on the applications of our constructions to unitary operads. We just drop the $+$ lower-script in the expression of intermediate objects which we form in the category of non-unitary operads.

We first outline the definition of the mapping which relates the homotopy automorphism space $h\text{Aut}_{\mathcal{O}_p}(E^+_2)$ to the group $GT(\mathbb{Q}) \ltimes SO(2)^\wedge$. We give more details on this construction in the concluding chapter of this part, where we recap our results before completing the proof of our statement. To begin with, we have an obvious group action $\rho : SO(2) \times D_2 \to D_2$, defined at the level of the topological little 2-discs operad $D_2$, and given by the rotation action of the ambient unit disc $\mathbb{D}^2$ on the collections of little 2-discs $c = (c_i(\mathbb{D}^2) \subset \mathbb{D}^2, i = 1, \ldots, r)$ that define
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the points of the spaces $D_2(r), r > 0$. This action gives, after rationalization, the factor $SO(2)^\wedge$ of the space $\mathcal{h}\text{Aut}_{\mathcal{O}_p}(E_{2+}^\wedge)$ in the result of our theorem.

We pass to simplicial sets in order to define this rationalization $SO(2)^\wedge$ of the topological group $SO(2)$, and the weak-equivalence of our theorem $\mathcal{h}\text{Aut}_{\mathcal{O}_p}(E_{2+}^\wedge) \sim GT(Q) \ltimes SO(2)^\wedge$. We therefore take the category of simplicial sets $s\text{Set}$ (instead of $\mathcal{M} = \mathcal{T}\text{op}$) as our fundamental instance of a base category from now on, and we use our short notation $\mathcal{O}_p = s\text{Set}\mathcal{O}_p$ to refer to the category of unitary operads in this base category $\mathcal{M} = s\text{Set}$. We take the classifying space of the parenthesized braid operad $\mathcal{B}(\mathcal{P}\mathcal{A}\mathcal{B})$ as our working model of an $E_2$-operad in $s\text{Set}$. We also take the classifying space of the Malcev completion of this operad $\mathcal{B}(\mathcal{P}\mathcal{A}\mathcal{B}^\wedge)$ as our working model of a rational $E_2$ operad.

We have $SO(2) = S^1 = \mathcal{B}(\mathbb{Z}) \Rightarrow SO(2)^\wedge = \mathcal{B}(\mathbb{Q})$, where we consider the additive groups $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$. We actually take this classifying space $\mathcal{B}(\mathbb{Q})$ as a model for the rationalization of the topological group $SO(2)$. We use that the addition $+: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ induces a multiplication map $\mu: \mathcal{B}(\mathbb{Z}) \times \mathcal{B}(\mathbb{Z}) \to \mathcal{B}(\mathbb{Z})$ which provides the classifying space $\mathcal{B}(\mathbb{Z})$ with a group structure equivalent to the composition of rotations. We have a similarly defined multiplication map $\mu: \mathcal{B}(\mathbb{Q}) \times \mathcal{B}(\mathbb{Q}) \to \mathcal{B}(\mathbb{Q})$ which gives the group structure of our model $\mathcal{B}(\mathbb{Q})$ of the rationalization of the space $SO(2) = \mathcal{B}(\mathbb{Z})$. We define a simplicial group action $\rho: \mathcal{B}(\mathbb{Z}) \times \mathcal{B}(\mathcal{P}\mathcal{A}\mathcal{B}) \to \mathcal{B}(\mathcal{P}\mathcal{A}\mathcal{B})$ to get a model, within the category of simplicial sets, of the action of the rotation group $SO(2)$ on the operad in topological spaces $D_2$. We just pass to the Malcev completion in order to get an action of the simplicial group $\mathcal{B}(\mathbb{Q})$ on our model of a rational $E_2$-operad $\mathcal{B}(\mathcal{P}\mathcal{A}\mathcal{B}^\wedge)$. We provide a reminder on the definition of the parenthesized braid operad in the concluding chapter of this part, and we explain the definition of these actions with full details at this moment.

The factor $SO(2)^\wedge = \mathcal{B}(\mathbb{Q})$ in the weak-equivalence of our theorem $\mathcal{h}\text{Aut}_{\mathcal{O}_p}(E_{2+}^\wedge) \sim GT(Q) \ltimes SO(2)^\wedge$ can be obtained by using this action of the simplicial group $\mathcal{B}(\mathbb{Q})$ on the operad $\mathcal{B}(\mathcal{P}\mathcal{A}\mathcal{B}^\wedge)$. The other factor $GT(Q)$ comes from the definition of the Grothendieck–Teichmüller group as a group of automorphisms of the operad $\mathcal{P}\mathcal{A}\mathcal{B}^\wedge$. In this case, we just use the functoriality of the classifying space construction to assign an automorphism $\phi_\gamma: \mathcal{B}(\mathcal{P}\mathcal{A}\mathcal{B}^\wedge) \to \mathcal{B}(\mathcal{P}\mathcal{A}\mathcal{B}^\wedge)$ to any element $\gamma \in GT(Q)$.

Recall that the group of connected components of the homotopy automorphism space of an object $X$ in a simplicial model category $\mathcal{C}$ represents the group of automorphisms of this object in the homotopy category $\text{Aut}_{\text{h}\text{Aut}_{\mathcal{C}}}(X)$. The claim of Theorem A implies that the discrete group $GT(Q)$ is identified with the group of connected components of the space $\mathcal{h}\text{Aut}_{\mathcal{O}_p}(E_{2+}^\wedge)$. Thus, from our result, we also obtain the following statement:

**Theorem B.** Let $E_{2+}^\wedge$ denote any model (not necessarily cofibrant-fibrant) of the rationalization of the (unitary) operad of little 2-discs $D_{2+}$. We have an identity of groups:

$$\text{Aut}_{\text{h}\text{Aut}_{\mathcal{O}_p}}(E_{2+}^\wedge) = GT(Q),$$

where we consider:

- the group of automorphisms of the object $E_{2+}^\wedge$ in the homotopy category of (unitary) operads $\text{h}\text{Aut}_{\mathcal{O}_p}$ on the left-hand side;
- the pro-unipotent Grothendieck–Teichmüller group $GT(Q)$ on the right-hand side.
The factor $SO(2)^\sim = B(Q)$, in the formula of Theorem A, represents the connected component of the identity map in the space $\text{hAut}_{\mathcal{O}_p}(E_{2^+})$. The semi-direct product $GT(Q) \ltimes SO(2)^\sim$, which occurs in our expression, reflects the action of the group $\pi_0 \text{hAut}_{\mathcal{O}_p}(E_{2^+}) = GT(Q)$ on the set of connected components. In the definition of the Grothendieck–Teichmüller group $GT(Q)$ this action is yielded by the map $\lambda : GT(Q) \to \text{Aut}_{\mathcal{G}p}(P_2)$, which assigns an automorphism of the Malcev completion of the pure braid group on two strands $P_2^\sim$ to any element $\gamma \in GT(Q)$. Recall that we have $P_2^\sim = Q \Rightarrow \text{Aut}_{\mathcal{G}p}(P_2) = Q^\sim$. We simply use the functoriality of the classifying space construction to determine the action of the Grothendieck–Teichmüller group $GT(Q)$ on the simplicial group $SO(2)^\sim = B(Q)$ from this action on the group $P_2^\sim = Q$.

Recall that we just consider topological operads for the statement of our results. In practice, in our constructions, we still deal with a simplicial set of homotopy automorphism space $h\text{Aut}_{\mathcal{O}_p}(E_{2^+})$ of the Teichmüller group of the classifying space construction to determine the action of the Grothendieck–Teichmüller group $GT(Q)$ on the homotopy automorphism space of topological operads $h\text{Aut}_{\mathcal{O}_p}(E_{2^+})$ rather than the homotopy automorphism space of topological operads $h\text{Aut}_{\mathcal{O}_p}(E_{2^+})$ considered in our statement.

For the moment, we have only considered an action of the group $GT(Q) \ltimes B(Q)$ on the operad $B(PaB)$, which is possibly not cofibrant-fibrant. In principle, we have to take a cofibrant-fibrant replacement of this operad $E_{2}^\sim \Rightarrow B(PaB)$ in order to get the right homotopy automorphism space $h\text{Aut}_{\mathcal{O}_p}(E_{2^+})$, and then we have to perform lifting constructions in order to map our group $GT(Q) \ltimes B(Q)$, acting on $B(PaB)$, into this space $h\text{Aut}_{\mathcal{O}_p}(E_{2^+})$. To ease our construction, we can use that the model category $\mathcal{O}_p$ admits functorial cofibrant (and fibrant) resolutions. We then have a map of simplicial monoids $\rho : GT(Q) \ltimes B(Q) \to h\text{Aut}_{\mathcal{O}_p}(E_{2^+})$ and our main purpose consists in proving that this map defines a weak-equivalence in the category of spaces.

For this aim, we rather use a function space $\text{Map}_{\mathcal{O}_p}((\text{Res} B(PaB)_+^\sim, E_{2^+}),)$ on which the monoid $h\text{Aut}_{\mathcal{O}_p}(E_{2^+})$ acts, and where we just have to take a cofibrant resolution $\text{Res} B(PaB)$ of the operad $E_{2}^\sim = B(PaB)$ together with a fibrant model of a rational $E_2$-operad $E_2^{\sim \sim}$ on the target. We moreover have a weak-equivalence $\text{Map}_{\mathcal{O}_p}((\text{Res} B(PaB)_+^\sim, E_{2^+})) \Rightarrow \text{Map}_{\mathcal{O}_p}((\text{Res} B(PaB)_+^\sim, E_{2^+}))$ by definition of the rationalization functor. We can therefore drop the rationalization process on the source of our function spaces.

We now explain our computation method. We can temporarily forget about our parenthesized braid operad model of $E_2$-operads and we write $E_2$ for any choice of such a model of an $E_2$-operad. We also write $R_2 = \text{Res}(E_2)$ for the cofibrant resolution of this operad $E_2$ which we consider in our construction. We only go back to the parenthesized braid operad model $E_2 = B(PaB)$ at the moment where we have to interpret the result of our computations.

We have not been precise about our model structure on operads so far. We have a standard model structure on the category of unitary operad $P_+ \in \mathcal{O}_p$, which we obtain by adjunction from the classical projective model structure of symmetric sequences (see for instance [22]). We use the expression of the projective model structure to refer to this classical model structure on operads. We do not use this model category actually. We form our models in the category of non-unitary $\Lambda$-operads. We give a short reminder on this notion before continuing. Briefly recall that the category of non-unitary $\Lambda$-operads, denoted by $\Lambda \mathcal{O}_{\varnothing} = s \mathcal{S}et \Lambda \mathcal{O}_{\varnothing}$, basically consists of non-unitary operads equipped with an extra diagram structure
modeling the composition with an arity zero operation. The mapping \( P_+ \to P \), which carries a unitary operad \( P_+ \in O_P \) to the underlying non-unitary object \( P \in O_{\mathcal{P}} \), accordingly induces an isomorphism from the category of unitary operads \( O_P \) to the category of non-unitary \( \Lambda \)-operads \( \Lambda O_{\mathcal{P}} \) (see §1.3.2). In §8.4, we prove that the category of non-unitary \( \Lambda \)-operads in simplicial sets inherits a good model structure, which differs from the classical projective model structure of unitary operads, but where we have the same weak-equivalences. We deduce the definition of this model category from the general notion of the Reedy model structure on a category of diagrams. We therefore use name of the Reedy model structure to refer to this new model structure on the category of non-unitary \( \Lambda \)-operads.

The model category of non-unitary \( \Lambda \)-operads is simplicial, and is therefore equipped with a natural function space bifunctor. We actually form our function to this new model structure on the category of non-unitary \( \Lambda \)-operads, category of diagrams. We therefore use name of the Reedy model structure to refer to the underlying non-unitary \( \Lambda \)-operads \( \Lambda O_{\mathcal{P}} \) formed by the (abelian) Eilenberg-MacLane spaces \( \mathbb{R} \) whose fiber is identified with the operad: \( \mathbb{R} \) that this operad \( \mathbb{R} \) the rationalized operad \( \mathbb{R} \) the classifying space \( \mathbb{R} \) geometric-realization \( \mathbb{R} \) space. Simply recall for the moment that this cotriple resolution is given by the cotriple \( \mathbb{R} \) in simplicial sets, whose fiber is identified with the operad:

\[
\mathbb{R} \simeq \mathbb{R} \Rightarrow \mathbb{R} \Rightarrow \mathbb{R} \Rightarrow \cdots
\]

where each map \( \mathbb{R} : \mathbb{R} \Rightarrow \mathbb{R} \Rightarrow \mathbb{R} \Rightarrow \cdots \) is a fibration in the category of \( \Lambda \)-operads in simplicial sets, whose fiber is identified with the operad:

\[
\mathbb{K}(\mathbb{E}_m^m p, 1, r > 0) = \{ \mathbb{K}(p)_m r, 1, r > 0 \}
\]

formed by the (abelian) Eilenberg-MacLane spaces \( \mathbb{K}(p)_m r, 1, r > 0 \) on the weight \( m \) homogeneous components \( \mathbb{K}(\mathbb{E}_m^m p, r) = \mathbb{K}(p)_m r, 1 \), of the Drinfeld-Kohno Lie algebras \( \mathbb{K}(p), r > 0 \) (see §12.1). In what follows, we just lowers the indexing of this tower by one. Thus, we get \( \mathbb{K}(\mathbb{E}_m^m p, 1) \) as fiber of level \( m \) in our tower, and we have a first non-trivial term at level \( m = 0 \).

The function space \( \mathbb{M} \Lambda O_P (R_2, E_2) \), now inherits a double decomposition, which arises from the simplicial structure of our resolution \( R_2 = [\mathbb{R} \mathbb{E}(E_2)] \) in
one direction, and from the decomposition of the operad $\tilde{E}_2 = B(\mathcal{CD})$ into the limit of a tower of fibrations in the other direction. We apply general constructions of Bousfield-Kan [37, §§IX-X] to obtain a double spectral sequence:

\[
\pi_* \text{Map}_{\Lambda \mathcal{O}_p}(R_2, K(\mathcal{E}_m^{m+1}, p, 1)) \Rightarrow \pi_* \text{Map}_{\Lambda \mathcal{O}_p}(R_2, \tilde{E}_2)
\]

\[
\pi^* \text{Map}_{\Lambda \mathcal{O}_p}(\text{Res}_\star(\mathcal{E}_2), K(\mathcal{E}_m^{m+1}, p, 1)) \Rightarrow \pi^* \text{Map}_{\Lambda \mathcal{O}_p}(\text{Res}_\star(\mathcal{E}_2), \tilde{E}_2)
\]

from this double decomposition.

We compute the terms of this double spectral sequence in order to determine the homotopy of our function space. We have to choose a computation order. We actually compute the vertical spectral sequence (I) first, and the horizontal spectral sequence (II) afterwards. We formally have a whole tower of spectral sequences (I), whose levels are indexed by the values of the parameter $m \geq 0$, and of which abutment determines the input of this upper horizontal spectral sequence (II) which we compute in second. We obtain the following result:

**Theorem C.**

(a) The vertical spectral sequences $\text{I}^r \Rightarrow \pi_* \text{Map}_{\Lambda \mathcal{O}_p}(R_2, K(\mathcal{E}_m^{m+1}, p, 1))$ degenerate at their second page, for all $m \geq 0$. Furthermore, we have $\text{I}^2 = 0$ in total degree $* > 1$, for every value of the level parameter $m \geq 0$, while in degree $* = 1$ we get:

\[
\text{I}^2_* = \begin{cases} 
0, & \text{in the case } m > 0, \\
\mathbb{Q}, & \text{in the case } m = 0,
\end{cases}
\]

and in degree $* = 0$ we get:

\[
\text{I}^2_0 = \text{grt}_m, \quad \text{for every } m \geq 0,
\]

where $\text{grt}_m$ denotes the weight $m$ homogeneous component of the graded Grothendieck-Teichmüller Lie algebra $\text{grt}$.

(b) The horizontal spectral sequence $\text{II}^r \Rightarrow \pi_* \text{Map}_{\Lambda \mathcal{O}_p}(R_2, B(\mathcal{CD}))$ degenerates at its first page and gives rise to a filtration on its abutment so that we have an identity of weight graded objects:

\[
\text{E}^0 \pi_* \text{Map}_{\Lambda \mathcal{O}_p}(R_2, B(\mathcal{CD})) = \begin{cases} 
\mathbb{Q}, & \text{in degree } * = 1 \text{ (with our module concentrated in weight } m = 0), \\
\text{grt}, & \text{in degree } * = 0,
\end{cases}
\]

while we trivially have $\pi_* \text{Map}_{\Lambda \mathcal{O}_p}(R_2, B(\mathcal{CD})) = 0$ in degree $* > 1$.

We deduce the result of our main theorem, Theorem A, from this computation.
on the source. In fact, we only examine the case of $E_2$-operads, for which we get the result of Theorem C, in the final step of our verifications.

We devote the core chapters of this part ($\S\S$3-5) to the problem of computing the vertical spectral sequence of our operadic function spaces. To be more specific, we explain a general computation method in chapters $\S\S$3-4, while we devote chapter $\S$5 to the applications of this method to general $E_n$-operads, $n \geq 2$, and then to the particular case of $E_2$-operads, for which we get the result of Theorem C.

Let us explain our procedure. (We refer to the introduction of the cited chapters for further details on these plans.) We focus on the case of $E_2$-operads for simplicity. We divide our method into two main steps. We first establish that the second page of our vertical spectral sequence in Theorem C is identified with a cotriple cohomology of the Gerstenhaber operad $\text{Gerst}$ with coefficients in the Drinfeld-Kohno Lie algebra operad $\mathfrak{p}$. The Gerstenhaber operad $\text{Gerst}$ occurs in this picture as the homology of the little 2-discs operad $\mathbb{H}_*(D_2)$. We then use the operadic bar duality, of which we recall the definition in our appendix on cofree cooperads $\S$C, and the fact that the Gerstenhaber operad is Koszul (as an operad) to reduce our cotriple cohomology complex into a small complex of derivations on an operadic Koszul construction. We eventually compute the cohomology of this complex to get the result of Theorem C(a).

We devote the concluding chapter of the part ($\S$6) to the interpretation of our spectral sequence computation in the $E_2$-operad case and to the proof of Theorem A. We still give a short reminder on the definition of the Grothendieck–Teichmüller groups $\text{GT}(\mathbb{Q})$, and we explain the definition of our mapping from the Grothendieck–Teichmüller group to the homotopy automorphism space of rational $E_2$-operads, before tackling these ultimate verifications.

We already alluded to that our method applies to all $E_n$-operads, for any $n \geq 2$, and not only to $E_2$-operads. Therefore, in $\S$5, we still explain our reduction to Koszul complexes in the case of $E_n$-operads, before focusing to $E_2$-operads. We get complexes of the same form each time. We just get another spreading of the homogeneous components of our dg-modules (and more terms in positive degrees) in dimension $n > 2$. We still expect, however, that the degeneration phenomena of Theorem C have a suitable generalization in the higher dimensional cases. We actually design our spectral sequence method to compute homotopy automorphism spaces for the whole hierarchy of $E_n$-operads, $n \geq 2$, while the result for $E_2$-operads (Theorem B) could be obtained by other approaches. We give an overview of these generalizations (in progress) and of further applications of our method to formality problems for $E_n$-operads in the epilogue.

For simplicity, we take the field of rational numbers as ground ring $\mathbb{k} = \mathbb{Q}$ all through this part. We can entirely work in this setting for the computation of the homotopy automorphism space of rational $E_2$-operads. But our methods have a straightforward extension in the case where we take a characteristic zero field as ground ring. We just devote a few side remarks to this generalization of our constructions.
CHAPTER 2

Homotopy Spectral Sequences and Function Spaces of Operads

The purpose of this chapter is to explain the definition of the double spectral sequence of Theorem C. We make the construction of this double spectral sequence explicit in the general case of an operadic function space \( \text{Map}_{\Lambda \text{Op}}(R, Q) \) in the category of non-unitary connected \( \Lambda \)-operads in simplicial sets \( \Lambda \text{Op}_{\Lambda} \) where \( R \) is the resolution \( R = |\text{Res}_*(P)| \) of an operad \( P \) in this model category, and \( Q = Q^\wedge \) is a rational operad equipped with a tower decomposition \( Q^\wedge = \lim_{m} Q_{(m)} \) of the same form as the classifying space of the operad of chord diagrams. Recall that we just need to assume that \( P \) is cofibrant as a symmetric sequence (\( \Sigma \)-cofibrant) in order to ensure that \( R = |\text{Res}_*(P)| \) is cofibrant as a non-unitary connected \( \Lambda \)-operad (see §2.2.1 and §8.5).

We deduce our double spectral sequence from a double decomposition of the operadic function space \( \text{Map}_{\Lambda \text{Op}}(R, Q) \): as the totalization of a cosimplicial space in one direction; as the limit of a tower of fibrations in the other direction. To get our result, we essentially rely on general constructions of Bousfield-Kan [37, §§IX-X], which return a homotopy spectral sequence for any space equipped with such decompositions. We therefore provide comprehensive recollections on the general definition of Bousfield-Kan' homotopy spectral sequences before tackling the applications to operadic function spaces.

We devote a preliminary section (§2.0) to a brief account of general conventions on bi-graded structures which we use in the spectral sequence context. We explain the definition of Bousfield-Kan’ homotopy spectral sequences afterwards (§2.1), and we devote a concluding section to the applications to operads (§2.2). We will review the definition of our function spaces of operads at this moment. By the way, we also make precise the general definition of the tower decompositions of operads which we use all through this part.

2.0. Conventions on bi-graded structures

The pages of the spectral sequences which we define in this chapter are double-indexed collections \( E^r = \{ E^r_{st}, t \geq s \geq 0 \} \) such that \( E^r_{st} \) is a based set in the case \( t - s = 0 \), a general (possibly non-abelian) group in the case \( t - s = 1 \), and an abelian group in the case \( t - s > 1 \). We refer to the difference \( n = t - s \) as the total grading associated to the terms of our spectral sequence, and to the index \( s \geq 0 \) as the horizontal grading, while we say that \( t \geq 0 \) defines the vertical grading. We generically use the expressions of “total degree”, “horizontal degree”, and “vertical degree” to refer to these gradings. The name “degree”, without any further precision, always refers to the total degree when we deal with an object equipped with multiple graded structures.
We implicitly assume in the above definitions that the horizontal grading of our objects is equivalent to an upper (cochain) grading, while the vertical grading is supposed to form a lower grading, and similarly as regards the total grading. We therefore relate these gradings by the equation \( n = t - s \), with a minus sign in the expression of the horizontal degree reflecting the equivalence between lower and upper gradings. In other contexts, where we assume that the horizontal grading \( s \) is given by a lower (chain) grading, we set \( n = t + s \) to define the total grading of our object.

The objects \( E^s_t, t \geq s \geq 0 \), are basically defined by homotopy class sets, and the grading reflects (at least in one direction) the natural grading of the homotopy of a space. In this situation, we also use the expression of “dimension” to refer to our grading. We may still adopt the dimension terminology in the context of the homology of a simplicial module, when we want to stress that this homology actually represents the homotopy of our object in the category of simplicial sets (see §5.0.5). We also employ the terminology of dimension in the cosimplicial module context.

In certain specific situations, we use the term “weight” to refer to an extra grading on a given object. We mainly use this expression when we can determine our grading from the decomposition of a tensor structure (like symmetric algebras, tensor algebras, free operads) into a sum of homogeneous components, and when we want to stress this relationship. We resume this survey in the next chapter, where we use dg-modules equipped with extra gradings in order to compute the terms of our spectral sequences.

2.1. Homotopy spectral sequences

The purpose of this section is to review the general definition of the Bousfield-Kan spectral sequence in homotopy theory. We basically consider homotopy spectral sequences for spaces equipped with a decomposition into a limit of a tower of fibrations on one hand, and for the totalization of cosimplicial spaces on the other hand. Recall that the totalization \( \text{Tot}(X^\bullet) \) of a cosimplicial object \( X = X^\bullet \) in a model category \( C \) admits a canonical decomposition \( \text{Tot}(X^\bullet) = \lim_s \text{Tot}_s(X) \) which is shaped on a tower of fibrations \( \cdots \to \text{Tot}_s(X) \to \text{Tot}_{s-1}(X) \to \cdots \to \text{Tot}_{-1}(X) = \text{pt} \) when \( X \) is Reedy fibrant (see §3.3). We actually use the homotopy spectral sequence associated to this tower when we define the Bousfield-Kan homotopy spectral sequence of a cosimplicial space. We therefore explain the general definition of the homotopy spectral sequence of a tower of fibrations in a first step, and we address the applications of this construction to the case of cosimplicial spaces afterwards.

We skip some verifications in the definition of these homotopy spectral sequences since the main purpose of this reminder is to provide a reference for our subsequent computations. We refer to Bousfield-Kan’ monograph [37] and to the account of the textbook [97] for further details.

2.1.1. The homotopy exact sequences associated with a tower of fibrations. Let \( X \) be a simplicial set. We assume that \( X \) arises as the limit of a tower:

\[
X = \lim_n X_n \longrightarrow \cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow \text{pt} ,
\]

\[
\cdots \quad F_n \quad F_{n-1} \quad \cdots \quad F_0
\]
where each map $p_n : X_n \to X_{n-1}$ is a fibration. We assume that $X$ is equipped with a base point, denoted by $\phi \in X$. We take the image of this base point at each level of our tower in order to provide the space $X_n$ with a base point $\phi \in X_n$, for any $n \in \mathbb{N}$. We then consider the fiber $F_n$ of the map $p_n : X_n \to X_{n-1}$ over this base point (at level $n - 1$).

The definition of the Bousfield-Kan spectral sequence arises from a suitable generalization of the notion of a derived exact couple. We start with the homotopy exact sequences

\[
\cdots \longrightarrow \pi_2(X_{n-1}) \xrightarrow{\partial} \pi_1(F_n) \longrightarrow \pi_1(X_n) \longrightarrow \pi_1(X_{n-1}) \longrightarrow \pi_0(F_n) \longrightarrow \pi_0(X_n) \longrightarrow \pi_0(X_{n-1}) \xrightarrow{\partial} \pi_0(F_{n-1}) \longrightarrow \cdots
\]

associated to the fibration sequences $F_n \to X_n \to X_{n-1}$, $n \geq 0$, where all homotopy groups $\pi_*(\phi) = \pi_*(-, \phi)$ are taken at our chosen base point $\phi$ in the tower (1).

These homotopy exact sequences (2) a priori consist of abelian groups in dimension $* \geq 2$, of general groups in dimension $* = 1$, and of pointed sets in dimension $* = 0$ (with the class of our element $\phi$ as base point). The abelian group $\pi_2(X_{n-1})$ is mapped to the center of the group $\pi_1(F_n)$, and we also have an action of the group $\pi_1(X_{n-1})$ on the set $\pi_0(F_n)$ which extends the natural action of this group (by left translation) on the image of the connecting map $\partial : \pi_1(X_{n-1}) \to \pi_0(F_n)$. The homotopy exact sequence (2) is exact in the usual sense in the range $* \geq 1$, where all our objects are equipped with a group structure. In the degree 0 extension, we still get that the kernel of any map, which we define as the pre-image of the base point when we have no group structure, is equal to the image of the previous map occurring in the sequence. In the case of the homotopy class set $\pi_0(F_n)$, we moreover get that elements $[\alpha], [\beta] \in \pi_0(X_n)$ have the same image in $\pi_0(F_n)$ if and only if these elements lie on the same orbit when we consider the action of the group $\pi_1(X_{n-1})$ on this set $\pi_0(F_n)$. We refer to the textbooks [97, Lemma II.7.3] and [167, §7] for a detailed account on the definition of the homotopy exact sequence of a fibration in simplicial sets and for the proof of these assertions.

2.1.2. The derived homotopy tower associated to a tower of fibrations. The idea of the spectral sequence is to take the homotopy class sets $\pi_*(X_n)$, $n \in \mathbb{N}$, which we associate to the levels of our tower §2.1.1(1), as a first approximation for a tower of homotopy class sets $q_n \pi_*(X) = \text{im}(\pi_*(X) \to \pi_*(X_n))$, for $n \in \mathbb{N}$, which we regard as the quotients of a filtration on the homotopy of the total space of our tower $X$. We also take the homotopy groups of the fibers in our tower $\pi_*(F_n)$ as a first approximation of the sub-quotients $E^0 \pi_*(X) = \text{ker}(q_n \pi_*(X) \to q_{n-1} \pi_*(X))$ associated to this filtration on $\pi_*(X)$. We interpret the homotopy exact sequence §2.1.1(2) as the expression of the relationship between this approximation $E^1 = \pi_*(F_n)$ of the sub-quotients $E^0 \pi_*(X)$ and our given approximation $\pi_*(X_n(0)) = \pi_*(X_n)$ of the tower of filtration quotients $q_n \pi_*(X)$, $n \in \mathbb{N}$.

Then we simply set:

\[
\pi_*(X_n^{(r-1)}) = \text{im}(\pi_*(X_{n+r-1}) \to \pi_*(X_n)),
\]

for any $r \geq 1$, in order to refine the approximation defined by our tower of homotopy class sets $\pi_*(X_n)$, $n \in \mathbb{N}$, which basically correspond to the case $r = 1$ of this construction. We see that this object $\pi_*(X_n^{(r-1)})$ forms a subgroup of the abelian group $\pi_*(X_n)$ in dimension $* \geq 2$, a (possibly not normal) subgroup of the (possibly
non-abelian) group \( \pi_1(X_n) \) in dimension \( * = 1 \), and a subset of the set \( \pi_0(X_n) \) in dimension \( * = 0 \).

We aim to define objects \( E^r = \pi_*(F_n^{(r-1)}) \), fitting the same relationship with respect to these refined approximations \( \pi_*(X_n^{(r-1)}) \) as the homotopy of the fibers of our fibrations \( \pi_*(F_n) \) with respect to the homotopy class sets \( \pi_n(X_n) \), \( n \in \mathbb{N} \), and which we may take as a suitable refinement of our approximation for the homotopy sub-quotients \( E^r \pi_*(X) \) associated to the total space of our tower \( X \). We explain the definition of these objects \( E^r = \pi_*(F_n^{(r-1)}) \) in the next paragraph, as sub-quotients of the homotopy of the fibers of our fibrations. In turn, we make explicit our generalization of the homotopy exact sequence which gives the connection between these subquotients and our tower of homotopy class sets \( \pi_n(X_n^{(r-1)}) \), \( n \in \mathbb{N} \). We go back to the definition of the tower associated to the homotopy of our total space afterwards. We will also examine the relationship between our approximations and the sub-quotients of this tower \( E^r \pi_*(X) = \ker(q_n \pi_*(X) \to q_{n-1}(X)) \) at this moment, when we address the tower decomposition of the homotopy of our total space.

2.1.3. The derived homotopy subquotients associated to a tower of fibrations.

We first consider an \( r \)th derived cycle set \( Z^r \subset \pi_*(F_n) \) such that:

\[
(1) \quad Z^r = \ker(\pi_*(F_n) \to \pi_*(X_n)/\pi_*(X_n^{(r-1)})).
\]

In dimension \( * \geq 2 \), we can perform the standard quotient and kernel constructions (in the category of abelian groups) to define this object \( Z^r \). We may also see that \( Z^r \) represents the pre-image of the subset \( \pi_* (X_n^{(r-1)}) = \text{im}(\pi_*(X_n^{(r-1)}) \to \pi_*(X_n)) \subset \pi_*(X_n) \) under the map \( \pi_*(F_n) \to \pi_*(X_n) \). We just take this interpretation of the object \( Z^r \) to give a sense to our definition in dimension \( * = 0, 1 \). The cycle set \( Z^r \) accordingly consists of homotopy classes \( [\alpha] \in \pi_*(F_n) \) which admit a lifting up to the \( n + r - 1 \)th level of our tower §2.1.1(1).

We define \( E^r = \pi_*(F_n^{(r-1)}) \) as a quotient of this cycle set \( Z^r \) under a set of boundary relations, which we deduce from an action of the group

\[
(2) \quad B^r = \ker(\pi_{*+1}(X_{n-1}) \to \pi_{*+1}(X_{n-r}))
\]

on this object. We formally set:

\[
(3) \quad \pi_*(F_n^{(r-1)}) = \ker(\pi_*(F_n) \to \pi_*(X_n)/\pi_*(X_n^{(r-1)})) \quad = \frac{Z^r}{\ker(\pi_{*+1}(X_{n-1}) \to \pi_{*+1}(X_{n-r}))}. \quad = B^r
\]

We proceed as follows to define this quotient. In the case \( * \geq 2 \), we may just observe that the connecting morphism \( \partial : \pi_{*+1}(X_{n-1}) \to \pi_*(F_n) \) maps \( B^r \subset \pi_{*+1}(X_{n-1}) \) into \( Z^r \subset \pi_*(F_n) \) and we define \( \pi_*(F_n^{(r-1)}) \) as the quotient of these abelian groups. In the case \( * = 1 \), we use that the connecting morphism \( \partial : \pi_2(X_{n-1}) \to \pi_1(F_n) \) maps \( B^r \) into the center of the group \( Z^r \) inside \( \pi_1(F_n) \) in order to extend our quotient construction (3) and to get a (possibly non-abelian) group \( \pi_1(F_n^{(r-1)}) \) from our objects. In the case \( * = 0 \), we use that the action of the group \( \pi_1(X_{n-1}) \) on \( \pi_0(F_n) \) restricts to an action of the subgroup \( B^r \subset \pi_1(X_{n-1}) \) on the cycle set \( Z^r \subset \pi_0(F_n) \), and we define \( \pi_0(F_n^{(r-1)}) \) as the quotient of the set \( Z^r \) under this action.

We have the following statement:
Proposition 2.1.4 (Bousfield-Kan [37, §IX.4.1]). The objects \( \pi_*(X^{(r-1)}_n) \) and \( \pi_*(F^{(r-1)}_n) \), such as defined in §2.1.2 and in §2.1.3, fit in derived homotopy exact sequences

\[
\cdots \longrightarrow \pi_2(X^{(r-1)}_{n-2r+1}) \longrightarrow \pi_1(F^{(r-1)}_{n-r+1}) \longrightarrow \pi_1(X^{(r-1)}_{n-r}) \longrightarrow \pi_1(X^{(r-1)}_r) \longrightarrow \pi_0(F^{(r-1)}_n) \longrightarrow \pi_0(X^{(r-1)}_r) \longrightarrow \pi_0(X^{(r-1)}_1),
\]

which are exact in the same sense as the homotopy exact sequences of a tower of fibrations in §2.1.1, for any \( r \geq 1 \).

Explanations. The maps of these derived homotopy exact sequences are defined by a straightforward restriction process from the maps of the homotopy exact sequences §2.1.1(2) associated to our tower of fibrations §2.1.1(1). We also have an action of the group \( \pi_1(X^{(r-1)}_n) \) on the sub-quotient set \( \pi_0(F^{(r-1)}_n) \) which we obtain by a restriction of the action of the group \( \pi_1(X_n) \) on the set \( \pi_0(F_n) \), for each \( n \geq 0 \). The derived homotopy exact sequences, which we define by this restriction process, are exact by construction. We just have to check that we can form these restrictions coherently with respect to the kernel and quotient processes of §§2.1.2-2.1.3. We leave the details of these verifications as an understanding exercise for the readers.

2.1.5. The homotopy spectral sequence of a tower of fibrations. The Bousfield-Kan spectral sequence of a tower of fibrations §2.1.1(1) is a sequence \( E^r, r = 0, 1, \ldots, \) of which terms are double collections \( E^r = \{E^r_{st}, t \geq s \geq 0\} \), referred to as the pages of the spectral sequence, such that:

\[
E^r_{st} = \pi_{t-s}(F^{(r-1)}_s),
\]

for all \( t \geq s \geq 0 \), where we consider the sub-quotient objects, such as defined in §2.1.3, of the derived homotopy exact sequence of Proposition 2.1.4. In the case \( r = 1 \), we simply have \( E^1_{st} = \pi_{t-s}(F_s) \), where we consider the homotopy class sets of the fibers of our tower which form the initial term of our derived homotopy exact sequence construction. Recall that this construction depends on the choice of a base point \( \phi \in X \) which we transport to the whole tower §2.1.1(1) and which determines a base point in the homotopy class sets of our exact sequences.

We should specify that the pages of the spectral sequence are equipped with the bi-grading \((s, t)\) such that \( s = n\) corresponds to the level of our tower of fibrations, and \( t - s\) corresponds to the dimension of homotopy class sets. We still use the expression of level to refer to the first component of this bi-grading \( s \geq 0 \) and we also use the notation \( E^r_s \) for the collection formed by the terms of the spectral sequence of a given level \( s \geq 0 \):

\[
E^r_s = \{E^r_{st}, t \geq s\}.
\]

We go back to the definition of the objects \( E^r_{st} \) in the next paragraph.

2.1.6. The derived homotopy complexes of a tower of fibrations. The objects \( E^r_{st} \), defining the terms of the \( r \)th page of our spectral sequence, can actually be determined from the terms of the previous page \( E^r_{st} \) by an extension of a standard chain complex construction \( E^r_{st}, d^r \). To be explicit, we consider the maps

\[
d^r : E^r_{st} \to E^r_{s+r+1},
\]
defined by the composites

\[ (2) \quad \pi_{t-s}(F^{(r-1)}_s) \to \pi_{t-s}(X^{(r-1)}_s) \xrightarrow{\partial} \pi_{t-s-1}(F^{(r-1)}_{s+r}) \]

which we obtain by fitting together the derived homotopy exact sequences associated to different levels of our tower of fibrations. In what follows, we mainly use the case \( r = 1 \) of this definition. We then deal with the morphism induced by the canonical embeddings \( F_s \to X_s \) in homotopy, followed by the connecting map \( \partial \) of the homotopy exact sequence \( \S2.1.1(2) \).

The definition of the maps \( d^r \) makes sense for \( t-s \geq 1 \) in our setting. In the case \( t-s > 2 \), our construction returns a morphism in the category of abelian groups. In the case \( t-s = 2 \), we get a morphism from the abelian group \( E^r_{s+2} \) towards the center of the (possibly non-abelian) group \( E^s_{s+1} \). In the case \( t-s = 1 \), we also have an action of the group \( E^r_{s+1} \) on the set \( E^r_{s+r_s+r_t-1} \) which extends the natural action of this group (by left translation) on the image of our map \( d^r: E^r_{s+1} \to E^r_{s+r_s+r_t-1} \).

The terms of the \( r + 1 \)th page of our spectral sequence can be determined as quotients:

\[ (3) \quad E^{r+1}_{s+1} = \ker(d^r: E^r_{s+t} \to E^r_{s+r_s+r_t-1})/\operatorname{im}(d^r: E^r_{s-r_s-r_t-1} \to E^r_{s+t}), \]

in the range \( t-s \geq 1 \). In the case \( t-s = 0 \), we still get that the set \( E^{r+1}_{s+1} \) is identified with a subset of the quotient of the object \( E^r_s \) under the action of the group \( E^r_{s-r_s-r_t-1} \), but we would need some negative dimensional information (which we do not have) in order to determine this subset from the \( rt \)th page of our spectral sequence.

The above formula is valid for all \( r \geq 1 \). In the case \( r > s \), we assume that \( E^r_{s-r_s-r_t-1} \) is the trivial group, and our claim then amounts to the assertion that \( E^{r+1}_{s+1} \) is identified with a certain sub-object of the term \( E^r_{s+t} \) in the \( rt \)th page of our spectral sequence. We use this inclusion relation \( E^{r+1}_{s+t} \subset E^r_{s+t} \) when \( r > s \) to define the abutment \( E^{\infty} \) of the spectral sequence of a tower of fibrations \( E^1, E^2, \ldots, E^r, \ldots \).

We explicitly set:

\[ (4) \quad E^{\infty}_{s+t} = \bigcap_{r>s} E^r_{s+t}, \]

for all \( t \geq s \geq 0 \). We expect that, in good cases, this abutment gives information on the homotopy of our space \( X \). We examine this question in the next paragraph.

2.1.7. The homotopy filtration quotients associated to a tower of fibrations. In order to formalize the correspondence between the abutment of our spectral sequence and the homotopy of the space \( X = \lim_n X_n \), we consider the homotopy class sets:

\[ (1) \quad q_n \pi_*(X) = \operatorname{im}(\pi_*(X) \to \pi_*(X_n)), \]

which we identify with the quotients of a filtration on this homotopy \( \pi_*(X) \), for \( n \in \mathbb{N} \), and we set:

\[ (2) \quad E^0_{s+t} \pi_*(X) = \ker(q_s \pi_{t-s}(X) \to q_{s-1} \pi_{t-s}(X)), \]

for all \( t \geq s \geq 0 \). We adopt the same conventions for the collection of these objects \( E^0 \pi_*(X) = \{ E^0_{s+t} \pi_*(X), t \geq s \geq 0 \} \) as in the case of the collections \( E^r = \{ E^r_{s+t}, t \geq s \geq 0 \} \) that define our spectral sequence. In particular, we write \( E^0_{s+t} \pi_*(X) \) for the collection formed by the terms \( E^0_{s+t} \pi_*(X), t \geq 0 \), of given level \( s \geq 0 \), and which represent the \( st \)th sub-quotient of the filtration associated to our tower (1)
on the homotopy of the space \( X \). In dimension \( t - s = 0 \), we still define the object \( E^0_{s,t} \pi_s(X) \) as the pre-image of the base point \( \phi \in q_{s-1} \pi_0(X) \) in \( q_s \pi_0(X) \).

Each collection \( E^0_{s,t} \pi_s(X) \) explicitly consists of homotopy classes \([\alpha] \in \pi_s(X)\) which lie at level \( s \) of our tower, but admit a lifting in the homotopy of the limit space \( X = \lim_{n} X_n \), and become trivial one level lower, in the homotopy of the space \( X_{s-1} \). The homotopy exact sequence \( \pi \lim \) sequence associated to this tower in \( X \) for the readers.

We have (a) Theorem 2.1.9 (Bousfield-Kan [37, IX.5.3]). We have a natural injective map \( E^0_{s,t} \pi_s(X) \hookrightarrow E^\infty_{s,t} \pi_s(X) \) from the sub-kernels \( E^0_{s,t} \pi_s(X) \) of the filtration quotients \( q_s \pi_s(X) \) in \( \pi \lim \) into the abutment terms \( E^\infty_{s,t} \) of our homotopy spectral sequence in \( \pi \lim \), for all \( t \geq s \geq 0 \).

PROOF. The verification of this proposition follows from a straightforward unraveling of definitions, and we leave this proof as another understanding exercise for the readers.

Bousfield-Kan provide conditions, in terms of a derived limit functor \( \pi \lim \), to ensure that the limit space of a tower \( X = \lim_{n} X_n \) satisfies \( \pi_s(X) = \lim_{n} \pi_s(X_n) \) and to establish that the inclusion relation of the above proposition is an identity (at least when \( t - s \geq 1 \)). We just record the following statement which summarizes some results obtained by these authors and which we use in this book:

**Theorem 2.1.9** (Bousfield-Kan [37, IX.5.4]). We assume that \( X \) is the limit space \( X = \lim_{n} X_n \) of a tower of fibrations as in \( \pi \lim \). We consider the spectral sequence associated to this tower in \( \pi \lim \). If the terms of this spectral sequence satisfy \( \pi \lim_{t-s} E^r_{s,t} = 0 \) for all \( t - s \geq 1 \) and each \( s \geq 0 \), then:

(a) We have \( \lim_{n} \pi_{s+1}(X_n) = 0 \), \( \pi_s(X) = \lim_{n} \pi_s(X_n) \) for all \( s \geq 0 \), and the filtration quotients of \( \pi \lim \) provide a decomposition of the homotopy class sets of our space

\[
\begin{array}{cccccc}
\pi_s(X) & \to & \cdots & \to & q_s \pi_s(X) & \to & q_{s-1} \pi_s(X) & \to & \cdots & \to & q_0 \pi_s(X) & \to & * \\
\uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
E^0_{s} \pi_s(X) & & E^0_{s-1} \pi_s(X) & & \cdots & & E^0_{0} \pi_s(X)
\end{array}
\]

such that \( \pi_s(X) = \lim_{s} q_s \pi_s(X) \) for all \( s \geq 0 \).

(b) We moreover have an identity \( E^0_{s,t} \pi_s(X) = E^\infty_{s,t} \pi_s(X) \) between the sub-kernels of this filtration \( E^0_{s,t} \pi_s(X) \) and the abutment of our spectral sequence \( E^\infty_{s,t} \), for any \( s \geq 0 \) when \( t - s \geq 1 \). We still have an inclusion relation \( E^0_{s,t} \pi_s(X) \subset E^\infty_{s,t} \) for any \( s \geq 0 \) in dimension \( t - s = 0 \).}

**Explanations and references.** The derived limit functor \( \pi \lim \) is classically defined in the setting of abelian groups, by using standard methods of homological algebra (see for instance [220, §3.5]). The groups considered in our statement are possibly not abelian in dimension \( * = 1 \), and we refer to [37, IX.2] for a general
definition of the functor $\lim^1$ which makes sense in this (non-abelian) setting. The homotopy class sets of the limit of a tower of fibrations $X = \lim_n X_n$ fit in a short exact sequence

$$\ast \to \lim^1_n \pi_{s+1}(X_n) \to \pi_*(X) \to \lim_n \pi_*(X_n) \to \ast,$$

for every $s \geq 0$ (see [37, Theorem IX.3.1]). The proof of Theorem 2.1.9 relies on the identity $\lim^1_n \pi_*(X_n) = \lim^1_n \pi_*(X_n^{\pi(n)})$ and on applications of the derived homotopy exact sequences in the range $r > s$ (we refer to loc. cit. for the detailed argument line). We actually have an equivalence

$$\lim_r^1 E^s_r = 0 \ (\forall s) \iff \lim^1_n \pi_*(X_n) = 0 \quad \text{and} \quad E^0_\ast \pi_*(X) = E^\infty_\ast \ (\forall s)$$

but we only use the direct implication $\Rightarrow$ (expressed in our theorem) of this statement.

In our applications, we do not use the explicit definition of the derived limit functor $\lim^1$ either. Let us simply mention that the vanishing condition of the theorem $\lim^1_{\ast} E^s_t = 0$ is fulfilled if, for each pair such that $t - s \geq 1$, we have the identity $E^0_\ast = E^1_\ast = \cdots = E^\infty_\ast$ from some rank $r > s$ (depending on $s$ and $t$) on. We then say that the spectral sequence converges in the Mittag-Leffler sense (see [37, §IX.5.5-5.6]).

In applications to the function spaces of $E_2$-operads, we basically check that the classes at some page of our spectral sequence are all hit by actual elements in the homotopy of our limit space, and we therefore get the identity of assertion (b) by a direct comparison argument. By the way, we also get an identity $\pi_\ast \lim^1_\ast X = \pi_\ast E^\infty_\ast \ast$ in dimension $s = 0$, while the spectral sequence only ensures us that $E^1_\ast \pi_*(X)$ forms a subset of the abutment term $E^\infty_\ast$ in this case.

The constructions of the previous paragraphs are clearly functorial with respect to the space $X$ equipped with a decomposition $X = \lim_n X_n$ of the form considered in §2.1.1(1). We therefore set $E^\ast = E^\ast(X)$ when we need to specify the space of which we consider the spectral sequence. We similarly write $Z^\ast = Z^\ast(X)$ for the cycle set which we associate to our space $X$ in §2.1.3 and we write $B^\ast = B^\ast(X)$ for the group of boundary relations. We assume in these expressions that the space $X$ is given together with a canonical decomposition §2.1.1(1) which we regard as a part of the structure associated to this space $X$.

2.1.10. The spectral sequence of a cosimplicial space. We now consider the case of the total space $\text{Tot}(X)$ of a cosimplicial object in the category of simplicial sets $X = X^\bullet \in c \text{Set}$ (a cosimplicial space for short). We already recalled that this space $\text{Tot}(X)$ admit a decomposition $\text{Tot}(X) = \lim_n \text{Tot}_n(X)$ such that each map $\text{Tot}_n(X) \to \text{Tot}_{n-1}(X)$ is a fibration when $X$ is fibrant in the Reedy sense (see §3.1.9). We therefore have a homotopy spectral sequence

$$E^\ast = E^\ast(\text{Tot}(X))$$

(1)

canonically associated to this total space, and which we deduce from the construction of §2.1.5. To be precise, when we form this spectral sequence, we also assume that $X$ is equipped with a coaugmentation over the one-point set $\eta : pt \to X$, which is equivalent to a collection of base points $\phi \in X^n$, given in each cosimplicial dimension $n \in \mathbb{N}$, and preserved by the structure maps $u_\ast : X^m \to X^n$ of our cosimplicial object $X$. This coaugmentation also determine a natural base point in the total space of our object $\phi \in \text{Tot}(X)$. 
2.1. HOMOTOPY SPECTRAL SEQUENCES

We already observed that the spectral sequence associated to a tower of fibrations is functorial with respect to the tower datum. We established in §3.3 that the total space $\text{Tot}(X)$ associated to a cosimplicial object $X$ does not depend on choices in general (see Theorem 3.3.14 and Theorem 3.3.15). We deduce from this coherence result that the mapping $X \mapsto \text{Tot}(X)$ defines a functor in $X \in c\, s\, \text{Set}$ (at least up to homotopy). We can also easily check that the decomposition $\text{Tot}(X) = \lim_s \text{Tot}_s(X)$ associated with our total space is natural (up to homotopy) too. We deduce from these observations that the homotopy spectral sequence $E^r = E^r(\text{Tot}(X))$, associated to the totalization of a cosimplicial space $X$, defines a functor in $X$. We use the notation

$$E^r(X) = E^r(\text{Tot}(X))$$

when we wish to regard our spectral sequence as a functor of this form, which we define on the fibrant objects of the category of cosimplicial spaces $X$ when we wish to regard our spectral sequence as a functor of this form, which we define on the fibrant objects of the category of cosimplicial spaces $X \in c\, s\, \text{Set}$. We just use the notation $E^r = E^r(\text{Tot}(X))$ when we go back to the definition of this spectral sequence in terms of the tower decomposition of the total space $\text{Tot}(X)$ associated to $X$.

2.1.11. The conormalized homotopy complex associated to a cosimplicial space.

We aim to give a description of the first and of the second page of the spectral sequence

\[ \pi_n(X) \cong \bigcap_{j=0}^{n-1} \ker(s^j : \pi_\ast(X^n) \to \pi_\ast(X^{n-1})) \]

in cosimplicial dimension $n \in \mathbb{N}$, where we consider the homotopy of the space $X^n$, and the kernels of the maps $s^j : \pi_\ast(X^n) \to \pi_\ast(X^{n-1})$ induced by the codegeneracy operators $s^j : X^n \to X^{n-1}$ of our cosimplicial space $X$.

In homotopical dimension $* \geq 2$, where the homotopy of a space forms an abelian group, we just use the conormalized complex functor on a module category (see §5.2.1) when we define this object. In dimension $* = 1$, we deal with morphisms of possibly non-abelian groups, and our construction returns an object in the category of all groups $\mathcal{G}_{\text{rp}}$. In dimension $* = 0$, we deal with maps of base sets. In this case, we again take the convention that our kernels represent the pre-image of the base point $\phi \in \pi_0(X^{n-1})$ to extend our definition of our normalized complex. Our construction accordingly returns a base set in dimension 0.

In dimension $* \geq 2$, we may still use the construction of §5.2.1 to equip the conormalized complex (1) with a differential so that our objects (1) form a cochain graded dg-module in the category of abelian groups. The differential $\partial : \mathbb{N}^n \pi_\ast(X^\bullet) \to \mathbb{N}^{n+1} \pi_\ast(X^\bullet)$ is then defined as the alternate sum $\partial = \sum (-1)^i d^i$, for any cosimplicial dimension $n \in \mathbb{N}$, where we consider the maps $d^i : \pi_\ast(X^n) \to \pi_\ast(X^{n+1})$ induced by the coface operators $d^i : X^n \to X^{n+1}$ of our cosimplicial space $X$. In what follows, we use the notation

$$\pi^n \pi_\ast(X) = H^n(\mathbb{N}^n \pi_\ast(X), \partial)$$

for the cohomology of this particular conormalized complex $\mathbb{N}^n \pi_\ast(X)$, which gives an abelian group $\pi^n \pi_\ast(X)$ naturally associated to $\pi_\ast(X)$, for any $n \in \mathbb{N}$. We also say that the collection of these abelian groups $\pi^n \pi_\ast(X)$, $s \in \mathbb{N}$, defines the cohomotopy of the cosimplicial object $\pi_\ast(X^\bullet)$. 
In homotopical dimension \( * = 1 \), where our conormalized complex construction returns a collection of possibly non-abelian groups, we may still define a cohomotopy class group \( \pi^0 \pi_1(X) \) and a pointed cohomotopy class set \( \pi^1 \pi_1(X) \). The cohomotopy class group in cosimplicial dimension \( s = 0 \) is defined by the formula:

\[
\pi^0 \pi_1(X) = \{ c \in \pi_1(X^0) \mid d^0(c) = d^1(c) \},
\]

where we just consider the equalizer of the maps \( d^0, d^1 : \pi_1(X^0) \to \pi_1(X^1) \) induced by the coface operators \( d^0, d^1 : X^0 \to X^1 \) of our cosimplicial space \( X \). The cohomotopy class set in cosimplicial dimension \( s = 1 \) is defined by a quotient:

\[
\pi^1 \pi_1(X) = \mathbb{Z}^1 \pi_1(X)/\pi_1(X),
\]

where we set

\[
\mathbb{Z}^1 \pi_1(X) = \{ z \in \mathbb{N}^1 \pi_1(X) \mid d^0(z) \cdot d^1(z)^{-1} \cdot d^2(z) = 1 \},
\]

and we take the orbit set under the \( \pi_1(X^0) \)-action such that \( g \cdot z = d^1(g) \cdot z \cdot d^0(g)^{-1} \), for any \( g \in \pi_1(X^0) \), \( z \in \mathbb{Z}^1 \pi_1(X) \).

In dimension \( * = 0 \), we may still consider the equalizer of the maps \( d^0, d^1 : \pi_0(X^0) \to \pi_0(X^1) \) to get a homotopy class set \( \pi^0 \pi_0(X) \) associated to our space \( X \). Note that we have \( \pi_* (X^0) = \mathbb{N}^0 \pi_* (X) \) and the formula

\[
\pi^0 \pi_* (X) = \ker (d^0, d^1 : \pi_* (X^0) \to \pi_* (X^1))
\]

actually holds in every homotopical dimension \( * \geq 0 \).

We have the following observation:

**Proposition 2.1.12** (Bousfield-Kan [37, §X.6-7]). *The homotopy spectral sequence* \( E^r(X) = E^r(\text{Tot}(X)) \) *associated to the totalization of a (Reedy fibrant) cosimplicial space* \( X \in c \mathbf{sSet} \) *in §2.1.10 satisfies:*

\[
E^1_{st}(X) = \mathbb{N}^s \pi_t(X^\bullet) \quad \text{and} \quad E^2_{st}(X) = \pi^s \pi_t(X^\bullet),
\]

*for each pair* \( t \geq s \geq 0 \), *where we consider the conormalized homotopy complex* \( \mathbb{N}^s \pi_t(X^\bullet) \) *and the cohomotopy class sets* \( \pi^s \pi_t(X^\bullet) \) *defined in the previous paragraph (see §2.1.11).*

**Explanations and References.** We give brief explanations on the result of this proposition. We refer to [37, §X.6-7] and [97, VIII.1] for further details.

By definition, the first page of our spectral sequence \( E^1_{st}(X) = E^1_{st}(\text{Tot}(X)) \) is given by the homotopy of the fiber of the maps \( \text{Tot}_s(X) \to \text{Tot}_{s-1}(X) \) associated to the decomposition of the total space \( \text{Tot}(X) \). The result of Proposition 3.3.17, where we prove that this map \( \text{Tot}_s(X) \to \text{Tot}_{s-1}(X) \) fits in a natural pullback, implies that we have an identity:

\[
\text{fib} (\text{Tot}_s(X) \to \text{Tot}_{s-1}(X)) = \text{fib}((X^s)^{\Delta^s} \to M^s(X)^{\Delta^s} \times_{M^s(X)^{\partial \Delta^s}} (X^s)^{\partial \Delta^s}),
\]

where we consider the matching space \( M^s(X) \) of our cosimplicial space \( X \in c \mathbf{sSet} \) and the map of our pullback on the right-hand side. We can easily determine the fiber of this map. We use the definition of the matching space as an equalizer, and basic limit interchange relations. We eventually get:

\[
\text{fib} (\text{Tot}_s(X) \to \text{Tot}_{s-1}(X)) = \Omega^s \mathbb{N}^s(X),
\]

where \( \mathbb{N}^s(X) \) denotes the simplicial set defined by the same intersection \( \mathbb{N}^s(X) = \bigcap_{j=0}^{s-1} \ker (s^j : X^s \to X^{s-1}) \) as our conormalized homotopy complex in §2.1.11, and
2.2. Applications to operads

The general purpose of our constructions is to compute operadic function spaces $\text{Map}_{\Lambda\mathcal{O}_p}(R, Q)$ in the category of non-unitary $\Lambda$-operads in simplicial sets $\Lambda\mathcal{O}_p = \text{sSet}\Lambda\mathcal{O}_p$, where the target object is rational as an operad $Q = Q^\sim$ (see §§9-10).

Recall simply that the category $\Lambda\mathcal{O}_p = \text{sSet}\Lambda\mathcal{O}_p$ inherits a nice model structure, where the fibrations are created in a Reedy model category of $\Lambda$-diagrams, and the weak-equivalences are created arity-wise in the model category of simplicial sets (see §8.4).

In principle, we have to assume that the object $R$ is cofibrant in this model category $\Lambda\mathcal{O}_p$, which we take as a target object in our function space $\text{Map}_{\Lambda\mathcal{O}_p}(R, Q)$. We make explicit the definition of our cofibrant operads, and we explain the definition of our double spectral sequence afterwards.
2.2.1. The setting of non-unitary connected \( \Lambda \)-operads. Recall that the category of connected non-unitary \( \Lambda \)-operads in simplicial sets \( \Lambda \mathcal{O}_\varnothing \) is the full subcategory of the category of non-unitary \( \Lambda \)-operads \( \Lambda \mathcal{O}_\varnothing \) consisting of the objects \( \mathcal{P} \) which are reduced to the one-point set in arity one \( \mathcal{P}(1) = pt \) with the operadic unit 1 in \( \mathcal{P}(1) \) as single element. In §8.5, we observe that \( \Lambda \mathcal{O}_\varnothing = s\text{Set} \Lambda \mathcal{O}_\varnothing \) inherits a model structure of the same form as the Reedy model structure of the category of all non-unitary \( \Lambda \)-operads. Recall that we use this model category in our study of the rational homotopy of unitary operads in simplicial sets.

We explained the definition of these cocartesian squares in the next paragraphs.

We proved in §8.5 that the category embedding \( \iota : s\text{Set} \Lambda \mathcal{O}_\varnothing \hookrightarrow s\text{Set} \Lambda \mathcal{O}_\varnothing \) fits in a Quillen adjunction \( \iota : s\text{Set} \Lambda \mathcal{O}_\varnothing \rightleftarrows s\text{Set} \Lambda \mathcal{O}_\varnothing : \tau \) such that we also have \( \tau_l = id \). We observed in §8.5 that the functor \( \iota : s\text{Set} \Lambda \mathcal{O}_\varnothing \hookrightarrow s\text{Set} \Lambda \mathcal{O}_\varnothing \) preserves fibrations too, and we can compute the function space of our main theorem in the model category of connected non-unitary \( \Lambda \)-operads rather than in the model category of all non-unitary \( \Lambda \)-operads with no difference on the result.

We actually work in the model category of connected non-unitary \( \Lambda \)-operads \( \Lambda \mathcal{O}_\varnothing = \text{Simp} \Lambda \mathcal{O}_\varnothing \) all through this chapter.

We also consider a reduced version of the cotriple resolution construction, which we form in the category of connected non-unitary \( \Lambda \)-operads, rather than in the model category embedding \( ? \hookrightarrow \text{Simp} \Lambda \mathcal{O}_\varnothing \). We observed in §8.5 that the category embedding \( \iota : s\text{Set} \Lambda \mathcal{O}_\varnothing \hookrightarrow s\text{Set} \Lambda \mathcal{O}_\varnothing \) fits in a Quillen adjunction, reviewed in §3.3, in the applications of this chapter.

2.2.2. Postnikov decomposition of operads. We rely on spectral sequence methods to compute function spaces \( \text{Map}_{\Lambda \mathcal{O}_\varnothing}(\mathcal{R}, \mathcal{Q}) \) in the category of non-unitary \( \Lambda \)-operads. We then assume that our target operad \( \mathcal{Q} = \mathcal{Q}' \) arises as the limit of a tower of connected non-unitary \( \Lambda \)-operads:

\[
\begin{align*}
\mathcal{Q} &= \lim_m \mathcal{Q}_{(m)} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_{(m-1)} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_{(-1)} = pt,
\end{align*}
\]

and where each morphism \( p_m : \mathcal{Q}_{(m)} \twoheadrightarrow \mathcal{Q}_{(m-1)} \), \( m \geq 0 \), fits in a cocartesian square of the form:

\[
\begin{array}{ccc}
\mathcal{Q}_{(m)} & \xrightarrow{p_m} & \Gamma_* (\mathcal{Q} \mathcal{P} \mathcal{N}_{(m)}) \\
\downarrow & & \downarrow \\
\mathcal{Q}_{(m-1)} & \xrightarrow{k_m} & \Gamma_* (\mathcal{Q} \mathcal{P} \mathcal{N}_{(m)})
\end{array}
\]

We explain the definition of these cocartesian squares in the next paragraphs.

We first assume that the object \( \mathcal{N}_{(m)} \), for any fixed \( m \geq 1 \), is a connected non-unitary \( \Lambda \)-operad in the category of chain graded dg-modules, where we consider the additive symmetric monoidal structure, given by the direct sum operation, instead of the usual multiplicative symmetric monoidal structure of dg-modules. The operadic composition structure of our object \( \mathcal{N}_{(m)} \) is therefore defined by additive operations of the form \( \circ_i : \mathcal{N}_{(m)}(k) \oplus \mathcal{N}_{(m)}(l) \to \mathcal{N}_{(m)}(k+l-1), \) for \( k, l > 0 \) and
2.2. APPLICATIONS TO OPERADS

We consider the arity-wise suspension of this additive operad $\Sigma N_{(m)}$ and the arity-wise cylinder object $\text{Cyl} N_{(m)}$.

Recall that we have $\Sigma C = Q e_1 \otimes C$, for any $C \in dg_*\text{Mod}$, where $e_1$ is a homogeneous element of degree 1, and $\text{Cyl} C = (Q e_0 \oplus Q e_1, \delta) \otimes C$, where we consider the dg-module $E = (Q e_0 \oplus Q e_1, \delta)$ spanned by a homogeneous element $e_0$ in degree 0, a homogeneous element $e_1$ in degree 1, together with the differential such that $\delta(e_1) = e_0$. We consider the morphism $q : \text{Cyl} N_{(m)} \to \Sigma N_{(m)}$ induced by the canonical projection $Q e_0 \oplus Q e_1 \to Q e_1$ which still forms a morphism of additive non-unitary $\Lambda$-operads in the category of chain graded dg-modules. We then apply the Dold-Kan functor arity-wise to get the morphism $q : \Gamma_* (\text{Cyl} N_{(m)}) \to \Gamma_* (\Sigma N_{(m)})$ of our pullback diagram (2). We actually get a morphism of (additive) connected non-unitary $\Lambda$-operads in simplicial modules in this construction, and not only a morphism of connected non-unitary $\Lambda$-operads in simplicial sets. We will exploit this feature in the definition of our spectral sequence in the next paragraph.

We generally assume that the matching morphisms $\mu : N_{(m)}(r) \to M(N_{(m)})(r)$ associated to the underlying $\Lambda$-sequence of the additive operad $N_{(m)}$ are surjective for all $r > 0$, or equivalently, that the object $N_{(m)}$ is fibrant as a $\Lambda$-sequence in dg-modules, for each $m \geq 0$ (see §8.3). We easily check that the relative matching morphisms $(\mu, q)$ associated to $q : \Gamma_* (\text{Cyl} N_{(m)}) \to \Gamma_* (\Sigma N_{(m)})$ are surjections of simplicial modules, and hence, define fibrations in the category of simplicial sets, when our object $N_{(m)}$ fulfills this fibration requirement. We obtain, therefore, that our morphism $q : \Gamma_* (\text{Cyl} N_{(m)}) \to \Gamma_* (\Sigma N_{(m)})$ defines a fibration in the category of connected non-unitary $\Lambda$-operads in simplicial sets (as marked in our diagram). We automatically obtain that the morphisms $p_m : Q_{(m)} \to Q_{(m-1)}$ in our tower decompositions are fibrations too.

In our study, we will consider the case where the objects $N_{(m)}$ occurring in this tower decomposition form cofree $\Lambda$-sequences over symmetric sequences $S N_{(m)}$, which we determine from the structure of these objects. We already used this notion in §7??, where we study the rational model of $E_n$-operads. We go back to this subject in the course of our constructions. For the moment, simply mention that any $\Lambda$-sequence satisfies our fibration requirement as soon as it has such a cofree structure (see Proposition 3.3.8).

2.2.3. The spectral sequence of a tower of fibrations of operadic function spaces.

We now examine the applications of the homotopy spectral sequences of the previous section to operadic function spaces $\text{Map}_{\Lambda \odot p}(R, Q)$, where $Q = Q^\sim$ is a connected non-unitary $\Lambda$-operad equipped with a tower decomposition of the form specified in the previous paragraph.

In a first step, we assume that $R$ is any cofibrant object of the model category of non-unitary (connected) $\Lambda$-operads. We also assume that $R$ is equipped with a morphism $\phi : R \to Q^\sim$ which we take as a base point for the function space $\text{Map}_{\Lambda \odot p}(R, Q)$. We then have a tower of fibrations of function spaces

\[
\begin{align*}
(1) \quad \text{Map}_{\Lambda \odot p}(R, Q) &= \lim_m \text{Map}_{\Lambda \odot p}(R, Q_{(m)}) \to \cdots \\
&\quad \cdots \to \text{Map}_{\Lambda \odot p}(R, Q_{(m)}) \to \text{Map}_{\Lambda \odot p}(R, Q_{(m-1)}) \to \cdots \\
&\quad \cdots \to \text{Map}_{\Lambda \odot p}(R, Q_{(-1)}) = pt
\end{align*}
\]
which we deduce from the tower decomposition of our operad $Q' = \lim_n Q_{(n)}$ and from the observation that the functor $\text{Map}_{\Lambda \text{ Op}_P}(R, -)$ carries our operadic cartesian squares in §2.2.2 to cartesian squares of the form

\[
\begin{array}{c}
\text{Map}_{\Lambda \text{ Op}_P}(R, Q_{(n)}) \\
\downarrow_{(p_n)_*} \\
\text{Map}_{\Lambda \text{ Op}_P}(R, Q_{(n-1)}) \longrightarrow \text{Map}_{\Lambda \text{ Op}_P}(R, \Gamma_\bullet(C\text{yl }N_{(n)}))
\end{array}
\]

in the category of simplicial sets.

Recall that the objects $N_{(n)}$ which we consider in our tower decomposition are non-unitary $\Lambda$-operads in the category of chain graded dg-modules in the additive sense (see §2.2.2), and the morphisms of generalized Eilenberg-MacLane space operads $q : \Gamma_\bullet(C\text{yl }N_{(n)}) \to \Gamma_\bullet(\Sigma N_{(n)})$, occurring in our cartesian squares, still form morphisms of (additive) non-unitary $\Lambda$-operads in simplicial modules. The function spaces $\text{Map}_{\Lambda \text{ Op}_P}(R, E)$ associated to these objects $E = \Gamma_\bullet(C\text{yl }N_{(n)}), \Gamma_\bullet(\Sigma N_{(n)})$ inherit a natural simplicial module structure, and the map

\[
q_* : \text{Map}_{\Lambda \text{ Op}_P}(R, \Gamma_\bullet(C\text{yl }N_{(n)})) \rightarrow \text{Map}_{\Lambda \text{ Op}_P}(R, \Gamma_\bullet(\Sigma N_{(n)}))
\]

in our diagram of function spaces (3) defines a morphism of simplicial modules. In the sequel, we use that this map inherits a principal fibration structure.

To be more explicit, recall that $C\text{yl }N_{(n)}$ is given by the arity-wise tensor product of the object $N_{(n)}$ with the dg-module $E = (Q \otimes Q_0 \oplus Q_0, \delta)$ spanned by a homogeneous element $\delta_0$ in degree 0, a homogeneous element $\delta_1$ in degree 1, together with the differential such that $\delta(\delta_1) = \delta_0$. We use the identity $N_{(n)} = Q \otimes Q_0 \otimes N_{(n)} \subset C\text{yl }N_{(n)}$ to identify the additive operad in simplicial modules $\Gamma_\bullet(N_{(n)})$ with a subobject of the additive operad $\Gamma_\bullet(C\text{yl }N_{(n)})$. The function space $\text{Map}_{\Lambda \text{ Op}_P}(R, \Gamma_\bullet(N_{(n)}))$ still inherits an obvious simplicial module structure from the object $\Gamma_\bullet(N_{(n)})$. We essentially consider the translation action $\Gamma_\bullet(N_{(n)}) \simeq \Gamma_\bullet(C\text{yl }N_{(n)})$, which we determine from our inclusion relation $\Gamma_\bullet(N_{(n)}) \subset \Gamma_\bullet(C\text{yl }N_{(n)})$, to get an action of this function space $\text{Map}_{\Lambda \text{ Op}_P}(R, \Gamma_\bullet(N_{(n)}))$ on $\text{Map}_{\Lambda \text{ Op}_P}(R, \Gamma_\bullet(C\text{yl }N_{(n)}))$.

In order to identify the map of our pullback diagram (3) with a principal fibration, we still have to check that the function space $\text{Map}_{\Lambda \text{ Op}_P}(R, \Gamma_\bullet(\Sigma N_{(n)}))$ represents the quotient of the space $\text{Map}_{\Lambda \text{ Op}_P}(R, \Gamma_\bullet(C\text{yl }N_{(n)}))$ under this translation action. We use an indirect argument to establish this result. We first use our fibration requirement on the object $N_{(n)}$ in the category of $\Lambda$-sequences (see §2.2.2) to establish that $q : \Gamma_\bullet(C\text{yl }N_{(n)}) \to \Gamma_\bullet(\Sigma N_{(n)})$, defines a fibration of non-unitary $\Lambda$-operads in simplicial sets, and that our map of simplicial modules $q_* : \text{Map}_{\Lambda \text{ Op}_P}(R, \Gamma_\bullet(C\text{yl }N_{(n)})) \to \text{Map}_{\Lambda \text{ Op}_P}(R, \Gamma_\bullet(\Sigma N_{(n)}))$ defines a fibration in the category of simplicial sets. We readily check, in turn, that the space $\text{Map}_{\Lambda \text{ Op}_P}(R, \Gamma_\bullet(N_{(n)}))$ represents the fiber of this map, and the identity of this fibration with our principal fibration follows. By the way, we also obtain the expression:

\[
E^1_n = \pi_* \text{Map}_{\Lambda \text{ Op}_P}(R, \Gamma_\bullet(N_{(n)}))
\]
for the first page of our spectral sequence (2).

2.2.4. The spectral sequence of the space of functions on the cotriple resolution of an operad. We now consider the case of the space of functions on a cofibrant \( \Lambda \)-operad such that \( R = |\text{Res}_\bullet(P)| \), where we take the geometric realization of the cotriple resolution of a connected non-unitary \( \Lambda \)-operad \( P \in s\text{Set} \Lambda \text{Op}_{\emptyset} \). We just require that \( P \) is a cofibrant as a symmetric sequence (we also say that \( P \) is \( \Sigma \)-cofibrant) in order to ensure that this object \( R = |\text{Res}_\bullet(P)| \) is cofibrant as a (connected) non-unitary \( \Lambda \)-operad in simplicial sets (see §2.2.1).

We also assume that the operad which we take as a target object in our function space has the form \( Q = \Gamma_\bullet(N) \), where \( N \) is a connected non-unitary \( \Lambda \)-operad in the additive category of chain graded dg-modules (as in the decompositions of §2.2.2). We assume that this object \( N \) satisfies the same fibration requirement as the terms \( N = N_{(n)} \) of the decomposition of a connected non-unitary \( \Lambda \)-operad \( Q = Q^\sim \) in §2.2.2.

We soon assume that \( N \) is, as a \( \Lambda \)-sequence, cofreely generated by a symmetric sequence, which we denote by \( S N \). We will use this structure in order to use Künneth isomorphism formulas and to determine the second page of our spectral sequence in terms of the homology of our objects. We tackle this question in the next section.

For the moment, we just need the fibration requirement in order to ensure that our target object \( Q = \Gamma_\bullet(N) \) defines a fibrant object of the category of (connected) non-unitary \( \Lambda \)-operad in simplicial sets. We then have an obvious identity:

\[
(1) \quad \text{Map}_{\Lambda \text{Op}}(|\text{Res}_\bullet(P)|, \Gamma_\bullet(N)) = \text{Tot Map}_{\Lambda \text{Op}}(\text{Res}_\bullet(P), \Gamma_\bullet(N)),
\]

where we consider the totalization of the cosimplicial space \( \text{Map}_{\Lambda \text{Op}}(\text{Res}_\bullet(P), \Gamma_\bullet(N)) \) defined by applying our function space functor to the simplicial object \( \text{Res}_\bullet(P) \) dimension-wise. We also easily check, by using our fibration requirement on the object \( N \) and the axioms of a simplicial model category structure, that this cosimplicial space \( \text{Map}_{\Lambda \text{Op}}(\text{Res}_\bullet(P), \Gamma_\bullet(N)) \) is fibrant in the Reedy sense. We can therefore use the general construction of §2.1.5 to get a spectral sequence such that:

\[
(2) \quad E^r(\text{Map}_{\Lambda \text{Op}}(\text{Res}_\bullet(P), \Gamma_\bullet(N))) \Rightarrow \pi_\ast(\text{Map}_{\Lambda \text{Op}}(|\text{Res}_\bullet(P)|, \Gamma_\bullet(N))),
\]

from the cosimplicial structure of our function space \( \text{Map}_{\Lambda \text{Op}}(\text{Res}_\bullet(P), \Gamma_\bullet(N)) \).

We have \( E^r_t(\text{Map}_{\Lambda \text{Op}}(\text{Res}_\bullet(P), \Gamma_\bullet(N))) = \pi^\ast \pi_t \text{Map}_{\Lambda \text{Op}}(\text{Res}_\bullet(P), \Gamma_\bullet(N)) \) for every \( t \geq s \geq 0 \), according to the general result of Proposition 2.1.12. We just see that the homotopy class sets \( \pi_t(\text{Map}_{\Lambda \text{Op}}(\text{Res}_\bullet(P), \Gamma_\bullet(N))) \) considered in this expression inherit a module structure in our case, for all \( t \geq 0 \). We can accordingly determine the cohomotopy of this cosimplicial object as the cohomology of a conormalized complex of cosimplicial modules in the sense of §5.2.1. We prove in the next chapter that this cohomology, which determines the second page of our spectral sequence, is identified with a natural cohomology theory, defined by a derived functor of derivations, on the category of operads.

We use the above spectral sequence (2) to get an approximation of the first page of the function space spectral sequence §2.2.3(2) associated to an operad \( Q^\sim \) equipped with a decomposition \( Q^\sim = \lim_n Q_{(n)} \) of the form considered in §2.2.2.

2.2.5. Tower decomposition of \( E_2 \)-operads. In what follows, we mostly deal with the case where \( Q^\sim \) is the rationalization of an \( E_n \)-operad \( Q = E_n \) in the
spectral sequence of §2.2.3. We then have an explicit definition of our tower decompositions §2.2.2(1-2). We give a brief reminder on these constructions to complete our account.

We review the case of $E_2$-operads first. We explained in §1 that we take the classifying space of the chord diagram operad $Q = B(\hat{CD})$ as a model for the rationalization of $E_2$-operads.

We use the decomposition $\hat{CD} = \lim_m \hat{CD}/F_{m+1} \hat{CD}$ of the chord diagram operad $\hat{CD}$ to get the required operadic Postnikov decomposition at the classifying space level:

\[
B(\hat{CD}) = \lim_m B(\hat{CD}/F_{m+2} \hat{CD}).
\]

We just shift the indexing of the tower decomposition of the chord diagram operad by one, and set $Q_{(m)} := B(\hat{CD}/F_{m+2} \hat{CD})$, for any $m \geq -1$, when we pass to this operad in simplicial sets.

We checked in §12.1 that these operads $Q_{(m)} := B(\hat{CD}/F_{m+2} \hat{CD})$ fit in pull-backs of the form §2.2.2(2), and where

\[
N_{(m)} := \Sigma E^0_{m+1} \mathfrak{p}
\]

is the arity-wise suspension of the component of weight $m+1$ of the Drinfeld-Kohno Lie algebra operad $E^0_{m+1} \mathfrak{p}$, for any $m \geq 0$. We then regard the (ungraded) modules $E^0_{m+1} \mathfrak{p}(r) > 0$, as dg-modules concentrated in degree 0, and we get an additive operad in dg-modules concentrated in degree 1 when we perform this suspension operation. Recall that we also write $E^0_{m+1} \mathfrak{p}[1] = \Sigma E^0_{m+1} \mathfrak{p}$ for this object.

We basically consider the tower defined in this paragraph when we tackle the applications of our spectral sequence construction to $E_2$-operads. We therefore basically compute function spaces $\text{Map}_\Lambda \mathfrak{op}(R, Q)$ with $Q = B(\hat{CD})$ as target object, and we just use the identity $B(\hat{CD}) = E_2$ to interpret our result in terms of $E_2$-operads. Let us still mention that we retrieve the Eilenberg-MacLane space of the input $K(E^0_{m+1} \mathfrak{p}, 1) = \Gamma_{\bullet}(E^0_{m+1} \mathfrak{p}[1])$ when we consider the fiber of the map $\Gamma_{\bullet}(\text{CY}L E^0_{m+1} \mathfrak{p}[1]) \to \Gamma_{\bullet}(\text{CY}L E^0_{m+1} \mathfrak{p}[1])$ occurring in the pullback diagram associated to this decomposition of the operad $Q = B(\hat{CD})$. We go back to this correspondence in §2.2.3.

2.2.6. Formality and tower decomposition of $E_n$-operads. We observed in §12 that the classifying space of the chord diagram operad $B(\hat{CD})$ is identified with the operad in simplicial sets $Q = L G_{\bullet} H^*(D_2)$ which we deduce from the cohomology of the little 2-discs operad by using the (derived functor of the) left adjoint $G_{\bullet} : dg^* \mathcal{H} \mathcal{opf} \Lambda \mathcal{O} P_{\mathcal{SG}1} \to sSet \Lambda \mathcal{O} P_{\mathcal{SG}1}$ of the operadic upgrading of the Sullivan functor $\Omega^* : sSet \Lambda \mathcal{O} P_{\mathcal{SG}1} \to dg^* \mathcal{H} \mathcal{opf} \Lambda \mathcal{O} P_{\mathcal{SG}1}$. We more precisely proved that the Chevalley-Eilenberg complex on the (completed) Drinfeld-Kohno Lie algebra operad $C_{CE}(\hat{\mathfrak{g}})$ gives a cofibrant resolution of the object $H^*(D_2)$ in the category of connected non-unitary Hopf $\Lambda$-cooperads, and that we have an identity:

\[
B(\hat{CD}) = G_{\bullet} C_{CE}(\hat{\mathfrak{g}})
\]

in the category of non-unitary $\Lambda$-operads in simplicial sets. We also explained that the existence of a weak-equivalence from the rationalization of an $E_2$-operad to the classifying space of the chord diagram operad $B(\hat{CD})$, which we deduce from the existence of rational Drinfeld’s associators, can be interpreted as a formality result for rational $E_2$-operads in simplicial sets.
We use a generalization of this model in the \(E_n\)-operad case. We then consider a Chevalley-Eilenberg complex on a graded version of the Drinfeld-Kohno Lie algebra operad \(p_n\), which reduces to the ordinary Drinfeld-Kohno Lie algebra operad in the case \(n = 2\). We explicitly have \(p = p_2\). (We just drop the notation index from the notation of the Drinfeld-Kohno Lie algebra operad when we specifically consider this case \(n = 2\) of the construction.) We still have:

\[
L\mathcal{G}_\bullet \mathcal{H}^\bullet(D_n) = G_\bullet C^\bullet_{CE}(\hat{p}_n) = \lim_m G_\bullet C^\bullet_{CE}(\frac{p_n}{F_{m+2}p_n}), =: \hat{Q}_{(m)}
\]

and we take the additive operads in dg-modules such that:

\[
N_{(m)} = \Sigma E_{m+1}^0 p_n
\]

to form the pullback diagrams §2.2.2(2), associated to this decomposition of the operad \(L\mathcal{G}_\bullet \mathcal{H}^\bullet(D_n)\).

We briefly mentioned in §12 that the formality of \(E_2\)-operads extends to \(E_n\)-operads, for any \(n \geq 2\), so that, under our conventions, we have the relation

\[
E_n^\wedge = L\mathcal{G}_\bullet \mathcal{H}^\bullet(D_n) = G_\bullet C^\bullet_{CE}(\hat{p}_n).
\]

The formality result of Kontsevich [130] implies that such a relation holds provided that we pass to the real numbers \(k = \mathbb{R}\). But the intrinsic formality theorem of [78] implies this resolution goes down to \(k = \mathbb{Q}\).

We consider the operad \(Q^\wedge = L\mathcal{G}_\bullet \mathcal{H}^\bullet(D_n)\), together with the tower defined in this paragraph, when we tackle the applications of our spectral sequence to \(E_n\)-operads. We therefore actually compute function spaces \(\text{Map}_\Lambda^\wedge O_F(R, Q)\) with \(Q^\wedge = L\mathcal{G}_\bullet \mathcal{H}^\bullet(D_n)\) as target object, and we just use the formality relation \(G_\bullet C^\bullet_{CE}(\hat{p}_n) = E_n^\wedge\) to interpret our result in terms of \(E_n\)-operads.
CHAPTER 3

Applications of the Cotriple Cohomology of Operads

The purpose of this chapter is to prove that the cohomotopy class sets $E^2 = \pi^*\pi_* \text{Map}_{\Lambda \text{Op}}(\text{Res}_*(P), \Gamma_*(N))$, which define the second page of the homotopy spectral sequence

$$E' \Rightarrow \pi_* \text{Map}_{\Lambda \text{Op}}(|\text{Res}_*(P)|, \Gamma_*(N))$$

in §2.2.4, are identified with the result of a natural cohomology theory, defined in terms of a derived functor of derivations, on the category of operads.

We use the cotriple resolution of operads to give a first definition of this cohomology theory, and we therefore adopt the expression of the cotriple cohomology to refer to this particular construction. We more precisely consider the cotriple resolution $R_* = \text{Res}_* (H_*(P))$ of the operad in graded modules $H_*(P) \in \text{gr}_\Lambda \text{Op}_{p/1}$, which we form by taking the homology of the operad in simplicial sets $P \in \text{sSet} \Lambda \text{Op}_{p/1}$ considered in our spectral sequence. We get a simplicial object of the category of operads in graded modules when we perform this construction $R_* = \text{Res}_* (H_*(P)) \in \text{sgr}_\Lambda \text{Op}_{p/1}$, and the application of our module of derivations to this simplicial object returns a cosimplicial object in the category of graded modules. We form the conormalized complex of this cosimplicial object and we take the cohomology of this complex in order to define the cotriple cohomology of the operad $H_*(P)$.

Before addressing the definition of this cohomology complex, we explain the definition of the modules of operadic derivations which we use in this construction. We first consider the case of an augmented connected non-unitary $\Lambda$-operad $R \in \text{dg}_\Lambda \text{Op}_{p/1} / \text{Com}$, which we form by taking the homology of the operad in chain graded $\Lambda$-modules $H_*(P) \in \text{gr}_\Lambda \text{Op}_{p/1}$.

The objects $N$, which define the coefficients of these dg-modules of derivations, consist of $\Lambda$-sequences equipped with an additive action of the operad $R$. We use the name of an abelian bimodule over $R$ to refer to this structure. We explain the general definition of this notion together with the definition of our modules of derivations in §3.1.

We also consider modules derivations associated to (augmented non-unitary) $\Lambda$-operads in the category of chain graded modules $\text{gr}_\Lambda \text{Mod}$ which we identify with chain graded dg-modules equipped with a trivial differential $\delta = 0$. We get an object of the category of graded modules $\text{gr Mod}$ in this case. We also have a trivial identity:

$$\text{Der}_{\text{dg}_\Lambda \text{Op}}(R, N)_\delta = \text{Der}_{\text{gr}_\Lambda \text{Op}}(R_\delta, N_\delta)$$
when we forget about differentials. Recall that $(-)_p$ denotes the obvious extension, to our categories of structured objects, of the natural forgetful functor $(-)_p : \text{dgMod} \to \text{grMod}$, from the category of dg-modules $\text{dgMod}$ towards the category of graded modules $\text{grMod}$.

We explicitly define the cohomology of an augmented non-unitary $\Lambda$-operad in graded modules $R \in \text{gr} \Lambda \text{Op}_\otimes$ with coefficients in an abelian $R$-bimodule $N$ by the formula:

$$H^*_{\text{gr} \Lambda \text{Op}_\otimes}(R, N) = \mathbb{H}^* \text{Der}_{\text{gr} \Lambda \text{Op}_\otimes}(\text{Res}_\ast(R), N),$$

where we apply the functor $\text{Der}_{\Lambda \text{Op}_\otimes}(-, -)$ to the cotriple resolution of our operad $\text{Res}_\ast(R) \in s \text{gr} \ast \Lambda \text{Op}_\otimes$. We then get a cosimplicial object in the category of abelian graded modules $\text{Der}_{\Lambda \text{Op}_\otimes}(\text{Res}_\ast(R), N)$, of which we take the conormalization $\mathbb{H}^*(-)$ and the cohomology $\mathbb{H}^n(-)$ to eventually obtain the cotriple cohomology module $H^*_{\text{gr} \Lambda \text{Op}_\otimes}(R, N)$ associated to our operad $R$.

We are going to see that the homotopy class sets occurring in the conormalized complex of the spectral sequence of §2.2.4 are equivalent to modules of derivations associated to (augmented non-unitary) $\Lambda$-operads in graded modules. We use this observation to establish that the second page of our spectral sequence is identified with the cotriple cohomology of our operad. We explain this result, together with the definition of the cotriple cohomology of operads, in §3.2.

We use modules of derivations associated to operads in dg-modules, in graded modules, and in simplicial modules. We therefore take these categories $\mathcal{M} = \text{dgMod}, \text{grMod}, s\text{Mod}$ as our main instances of base categories (besides the category of plain modules $\mathcal{M} = \text{Mod}$) from now on. We regard the category of chain graded dg-modules $\text{dg}_\ast \text{Mod}$ as a subcategory of the category of all dg-modules $\text{dgMod}$, and similarly when we deal with the category of chain graded modules $\text{gr}_\ast \text{Mod}$. We define our modules of derivations as sub-modules of hom-objects in these base categories $\mathcal{M} = \text{dgMod}, \text{grMod}, s\text{Mod}$. We devote an appendix section §3.3 to the definition of these hom-objects, and to a brief reminder on the definition of model structures on $\Lambda$-sequences which we use in our study.

We only retain the prefix of our module categories $\mathcal{M} = \text{dgMod}, \text{grMod}, s\text{Mod}$ in the notation of the categories of operads. We accordingly write $\text{dg} \Lambda \text{Op}_\otimes / \text{Com}$ for the category of augmented non-unitary $\Lambda$-operads in dg-modules $\mathcal{M} = \text{dgMod}$, and similarly, when we deal with operads in graded modules $\mathcal{M} = \text{grMod}$, and in simplicial modules $\mathcal{M} = s\text{Mod}$. We specify the base category in our notation otherwise. We accordingly write $s\text{Set} \Lambda \text{Op}_{\otimes 1}$ for the category of connected non-unitary $\Lambda$-operads in simplicial sets. We still drop the ambient category from our notation, however, when we deal with a general notion (like the general definition of the modules of derivations associated to an operad) which at least makes sense the main instances of base categories considered in this section $\mathcal{M} = \text{dgMod}, \text{grMod}, s\text{Mod}$. We adopt similar conventions in the case of $\Lambda$-sequences.

We do not assume any operadic connectedness assumption first, and we explain the definition of derivation modules on the category of general augmented non-unitary $\Lambda$-operads. We just restrict ourselves to connected augmented non-unitary $\Lambda$-operads when we address the definition of the cotriple cohomology complex, because we have a specific cotriple resolution construction in the connected setting, while the definition of our modules of derivations does not change.

For simplicity, we have assumed that we take the field of rational numbers $\mathbb{k} = \mathbb{Q}$ as a ground ring of our module categories all through this part. We may
observe, however, that our constructions remain valid over any ground ring for the moment. We would just need to assume that this ground ring is a characteristic zero field when we tackle the applications of our constructions to the computation of our homotopy spectral sequences (see §§3.2.3-3.2.5).

3.0. Multi-complex structures

The general purpose of this chapter is to prove that the terms of the homotopy spectral sequences of §2.2.4 can be computed by using objects in the category of dg-modules. We then deal with dg-modules graded in two directions in order to reflect the natural bi-grading of the terms of a spectral sequence. We may also assume, in general, that our dg-module is equipped with a differential in each grading direction.

The purpose of this preliminary section is to revisit the definition of such multiple dg-module structures and to explain our conventions in this subject.

3.0.1. Chain and cochain complexes of dg-modules. In practice, we often have
chain complexes
\[\cdots \to C_m(A) \overset{\partial}{\to} C_{m-1}(A) \overset{\partial}{\to} \cdots \to C_0(A)\]
whose components are dg-modules \(C_m(A) \in \mathcal{d}g\text{-}Mod\), and of which boundary operator \(\partial\) is defined by a collection of dg-module morphisms \(\partial : C_m(A) \to C_{m-1}(A)\) such that we have the relation \(\partial^2 = 0\) whenever defined.

We also deal with a cochain variant of these complex structures, where we index the terms of our complex \(C^m(A) \in d\mathcal{g}\text{-}\mathcal{M}od\) by an upper grading \(m \geq 0\), and we assume that the boundary operator satisfies \(\partial : C^m(A) \to C^{m+1}(A)\) for all \(m \geq 0\). We may also consider unbounded complexes of dg-modules, of which components \(C_m(A) \in d\mathcal{g}\text{-}\mathcal{M}od\) are indexed by all \(m \in \mathbb{Z}\). We readily check that a cochain complex of dg-modules \((C^\ast(A), \partial)\) is equivalent to a chain complex \((C_\ast(A), \partial)\) such that \(C_m(A) = C^{-m}(A)\) for \(m \leq 0\) and \(C_m(A) = 0\) for \(m > 0\).

We often drop the boundary operator \(\partial\) in the notation of a chain (respectively, cochain) complex of dg-modules. We also use the notation \(d\mathcal{g}_\ast d\mathcal{g}\text{-}\mathcal{M}od\) for the category of chain complexes of dg-modules, and we adopt similar conventions for the other variants of this category considered in this paragraph.

The bar complex of a (cochain) dg-algebra (see §6.3) naturally forms a chain complex in the category of (cochain) graded dg-modules, and we similarly use chain (and cochain) complexes of dg-modules in our study of the bar and Koszul duality of operads (see §§C.2-C.3).

We usually associate a complex of dg-modules to a class of structured objects (diagrams, operads, algebras) which we define in the category of dg-modules instead of the category of plain (ungraded) modules. We assume that \(A\) refers to an object of this form in the above expression, and \(C_\ast(-)\) denotes a generic functor that assigns a complex of dg-modules to this object \(A\). We usually use an algebraic construction within the category of dg-modules (for instance, the tensor structure) in order to produce the components of our complex \(C_m(A) \in d\mathcal{g}\text{-}\mathcal{M}od, m \geq 0\), from the dg-module structure underlying \(A\), while the boundary operator \(\partial\) is determined by the extra structure of our object \(A\).

3.0.2. Bi-complex structures. For the moment, we add a second lower-script in the expression of the components of our complex \(C_m(A) = \bigoplus_n C_{m,n}(A), n \in \mathbb{Z}\), when we need to specify the internal grading of these dg-modules \(C_m(A) \in d\mathcal{g}\text{-}\mathcal{M}od, m \geq 0\). In most applications, we just go back to the explicit expression of these
objects $C_m(A) \in dgMod$, $m \in \mathbb{N}$, when we need to perform operations involving the internal dg-module structure of our complex alone.

The concept of a chain complex of dg-modules is natural in most of our constructions. We may see, on the other hand, that the structure of a complex of dg-modules ($\ast$) is equivalent to a classical bi-complex structure, where we have a bigraded collection of plain (ungraded) modules and two commuting differentials. We generally take the internal grading $n \in \mathbb{Z}$ of each dg-module $C_m(A) = \bigoplus_n C_{m,n}(A)$ to define the vertical grading of this bi-complex associated to $C_\ast(A)$, while the index of these complex components $C_m(A)$, $m \geq 0$, determines the horizontal grading in the bi-complex representation. We also use the internal differential of each dg-module $C_m(A)$ to get a vertical differential $\delta : C_{m,n}(A) \to C_{m,n-1}(A)$, while the boundary operator of the complex gives the horizontal differential $\partial : C_{m,n}(A) \to C_{m-1,n}(A)$ whenever the definition of these differentials make sense. We may just observe that, since the boundary operator of our complex defines a dg-module morphism $\partial : C_m(A) \to C_{m-1}(A)$, we have the homogeneity relation $\partial(C_{m,n}(A)) \subset C_{m-1,n}(A)$ with respect to the internal grading of the dg-modules $C_m(A) \in dgMod$, $m \geq 0$, and the commutation relation $\partial\delta = \delta\partial$ with respect to the internal differentials $\delta : C_m(A) \to C_m(A)$, $m \geq 0$.

To a complex of dg-modules ($\ast$), we also associate a twisted dg-module

\begin{equation}
C_\ast(A) = \bigoplus_{m=0}^{\infty} \Sigma^m C_m(A), \partial,
\end{equation}

which we form by taking the sum of suspensions of the components of our complex $\Sigma^m C_m(A) \in dgMod$, $m \geq 0$, together with a twisting homomorphism $\partial : C_\ast(A) \to C_\ast(A)$ which we determine by the boundary operator of our complex componentwise. This twisted dg-module is nothing but the classical total complex of our object $C_\ast(A)$ when we use our correspondence between complexes of dg-modules and bi-complex structures. In what follows, we generally use the expression of total dg-module, rather than total complex, to refer to this object. We also use the expression of total degree to refer the grading of this twisted dg-module. We generally specify this total degree $n$ by an external lower-script in the notation of the complex $C_\ast(A)$. We may also write:

\begin{equation}
C_m(A)_n = C_{m,n-m}(A)
\end{equation}

for the components of the dg-module $C_m(A)$ regarded as summands of this twisted dg-module $C_\ast(A)$.

We generally omit the suspension factors in the expression of the summands of the twisted dg-module associated to a complex of dg-modules $C_\ast(A)$. We do not mark the difference of context in the notation of these objects. We simply use the position of the degree, as in the above relation, in order to distinguish one graded structure from the other. We adopt similar conventions when we deal with cochain complexes of dg-modules. We simply have to mark the opposition of sign between upper degrees and lower degrees when we combine both forms of graded structures. We follow the same conventions when the components of our complexes are graded modules, which we just regard as dg-modules equipped with a trivial differential $\delta = 0$.

3.0.3. Normalized complexes of simplicial dg-modules. We notably use the conventions of the previous paragraph when we apply the normalized complex functor construction to a simplicial object in the category of dg-modules $K \in s dgMod$. 
We can define the components of the normalized complex by the same formula as the ordinary normalized complex of simplicial (ungraded) modules:

\[ N_m(K) = \frac{K_m}{s_0(K_{m-1}) + \cdots + s_{m-1}(K_{m-1})}, \]

but we now perform our construction in the category of dg-modules, so that we naturally get an object in this category \( N_m(K) \in dgMod \), with an internal differential \( \delta : N_m(K) \to N_m(K) \) yielded by the differential of the components of our simplicial module \( K_m \in dgMod \), for all \( m \geq 0 \). We still take the alternate sum of face operators \( \partial = \sum_{i=0}^m (-1)^i d_i \) to define the normalized complex differential \( \partial : N_m(K) \to N_{m-1}(K) \), in any dimension \( m > 0 \). We just see that this map defines a morphism of dg-modules since this is the case of the face maps of our object \( K \in s dgMod \). We therefore get the structure of a chain complex of dg-modules in the sense of our definition in §3.0.1.

We also have the relation \( N_n(K) = N_n(K_\bullet) \), when we fix a degree \( n \in \mathbb{Z} \), where we consider the normalized complex of the simplicial module \( K_\bullet \in s Mod \) defined by the components of degree \( n \in \mathbb{Z} \) of the simplicial object \( K \in s dgMod \). We accordingly have the formula

\[ N_m(K)_n = N_m(K_{\bullet n-m}), \]

for any dimension \( m \in \mathbb{N} \), when we consider the total grading of our object \( n \in \mathbb{Z} \), and we follow the conventions of the previous paragraph.

We have similar observations for the conormalized complex \( N^*(K) \) of a cosimplicial object in the category of dg-modules \( K \in c dgMod \). We then get a cochain complex of dg-modules such that

\[ N^m(K)_n = N^m(K_{\bullet n+m}), \]

for any dimension \( m \in \mathbb{N} \), and any degree \( n \in \mathbb{Z} \), where we consider the conormalized complex of the cosimplicial module \( K_{\bullet} \in s Mod \) defined by the components of degree \( d = n + m \in \mathbb{Z} \) of our object \( K \in s dgMod \).

### 3.1. Modules of derivations associated to operads

The main purpose of this section is to explain the definition of the module of derivations which we associate to an operad. In a preliminary step, we explain the definition of the notion of an abelian bimodule over an operad which we take as coefficients for these modules of derivations. (We tackle the study of the modules of derivations themselves afterwards.)

We assume for simplicity that \( R \) is an augmented non-unitary \( \Lambda \)-operad (not necessarily connected) in one of our main base categories of this part \( \mathcal{M} = dgMod, grMod, sMod \), or in the category of plain modules \( \mathcal{M} = Mod \). We therefore work in any of these categories all through this section.

#### 3.1.1. The notion of an abelian bimodule over an operad

We explicitly define an abelian \( R \)-bimodule as a non-unitary \( \Lambda \)-sequence (in our working base category \( \mathcal{M} \))

\[ N = \{ N(r), r > 0 \} \in \Lambda Seq \]
equipped with an action of the operad $R$, which we determine by composition operations:
\[ o_k : R(m) \otimes N(n) \to N(m + n - 1), \]
\[ o_k : N(m) \otimes R(n) \to N(m + n - 1), \]
defined for all $m, n \geq 1$, each $k = 1, \ldots, m$, and so that an obvious extension of the usual equivariance, unit, and associativity relations of the composition structure of a $\Lambda$-operad hold. We also assume, by convention, that $N$ is equipped with an augmentation $\epsilon : N \to 0$ given by the zero map
\[ \epsilon : N(r) \to 0, \]
in any arity $r > 0$.

We basically replace one factor $R$ by $N$ in the domain of the equivariance, unit, and associativity relations of $\Lambda$-operads In order to get the expression of the extended equivariance, unit, and associativity relations for the action of $R$ on $N$.

3.1.2. The operad associated to an abelian bimodule. To any abelian bimodule $N$ over an operad $R$, we can also associate an augmented non-unitary $\Lambda$-operad $R \ltimes N$ defined by:
\[ (R \ltimes N)(r) = R(r) \oplus N(r), \]
for any arity $r > 0$, and where:
1. The restriction operation
\[ R(n) \oplus N(n) \xrightarrow{u^*} R(m) \oplus N(m), \]
associated to any injective map $u \in \text{Mor}_\Lambda(m, n)$, $m, n > 0$, is given by the internal restriction operation of the operad $R$ on the summand $R(n)$ and by the internal restriction operation of our collection $N$ on $N(n)$. (Thus, the underlying $\Lambda$-sequence of the operad $R \ltimes N$ is identified with the direct sum $R \oplus N$ of the objects $R$ and $N$ in the category of $\Lambda$-sequences.)
2. The augmentation
\[ R(r) \oplus N(r) \xrightarrow{\epsilon} \text{Com}(r) \]
is given, for any $r > 0$, by the augmentation associated with our operad $\epsilon : R \to \text{Com}$ on the summand $R(r)$ and by the null map $0 : N \to 0$ on the summand $N(r)$.
3. The composition product
\[ (R \ltimes N)(m) \otimes (R \ltimes N)(n) = \begin{cases} R(m) \otimes R(n) & \oplus R(m) \otimes N(n) \oplus N(m) \otimes R(n) \\ \oplus N(m) \otimes N(n) & \end{cases} \xrightarrow{o_k} R(m + n - 1) \oplus N(m + n - 1) \]
is defined, for any $m, n > 0$, and $k = 1, \ldots, m$, by the internal composition products of our operad on the summand $R(m) \otimes R(n)$, by the composition operations of our bimodule structure on the summands $R(m) \otimes N(n)$, $N(m) \otimes R(n)$, and by the null map on $N(m) \otimes N(n)$.

The extended equivariance, unit, and associativity relations, which we consider in the definition of an abelian $R$-bimodule structure, are actually equivalent to the equivariance, unit, and associativity relations for the composition products which
we form in the definition of this operad $R \ltimes N$ and which extend the composition products of the operad $R$.

We consider the comma category, denoted by $\Lambda \mathcal{O}_p / R$, of which objects are the augmented non-unitary $\Lambda$-operads $R$ equipped with a morphism $\epsilon : R \to R$ that factorizes the canonical augmentation (over the commutative operad $\text{Com}$) of our objects in $\Lambda \mathcal{O}_p / \text{Com}$. We immediately see that the morphism $\phi : R \ltimes N \to R$, given by the obvious projection onto $R$ in each arity $r > 0$, defines such an augmentation on the object $R \ltimes N \in \Lambda \mathcal{O}_p / \text{Com}$ which we associate to an abelian $R$-bimodule $N$. We have the following proposition, which gives a conceptual interpretation of the category of abelian bimodules associated to an operad:

**Proposition 3.1.3.** The augmented non-unitary $\Lambda$-operad $R \ltimes N$, which we associate to an abelian $R$-bimodule $N$, forms an abelian group object in the sense that this operad inherits morphisms

$$R \xymatrix{\rightarrow & R \ltimes N,} (R \ltimes N) \times_R (R \ltimes N) \xymatrix{\rightarrow & (R \ltimes N),} \quad R \ltimes N \xymatrix{\rightarrow & R \ltimes N}$$

which fulfill the same relations, in the category $\Lambda \mathcal{O}_p / R$, as the unit, multiplication, and inversion morphisms of an ordinary abelian group structure.

**Explanations.** We refer to [181] for the general definition of an abelian group object in a category. We essentially use that the unit, associativity, commutativity, and inversion relations of an ordinary abelian group structure can be expressed in terms of diagrams, involving the cartesian product which we can form in any ambient category equipped with limits. We just take the cartesian product of the comma category $\Lambda \mathcal{O}_p / R$, which is defined by the fiber product over the operad $R$, to express the relations satisfied by the structure morphisms which we associate to our object $R \ltimes N$.

Recall that limits of operads are created arity-wise in the base category. We have $(R \ltimes N)(r) \times_{R(r)} (R \ltimes N)(r) = R(r) \oplus N(r) \oplus N(r)$, for any arity $r > 0$ and our multiplication morphism is given arity-wise by the map $m : R(r) \oplus N(r) \oplus N(r) \to R(r) \oplus N(r)$ such that $m(p, x, y) = (p, x + y)$, for any $(p, x, y) \in R(r) \oplus N(r) \oplus N(r)$. We define the inversion morphism by $s(p, x) = (p, -x)$, for any $(p, x) \in R(r) \oplus N(r)$, for $r > 0$, and we define the unit morphism by the canonical embedding $e(p) = (p, 0)$, for $p \in R(r)$, $r > 0$.

We may see that every abelian group object $S$ of the category $\Lambda \mathcal{O}_p / R$ is given by a construction of this form $S = R \ltimes N$, for an abelian $R$-bimodule such that $N = \ker(\epsilon : S \to R)$. We deduce from this observation that our mapping $\Lambda \mathcal{O}_p / R \to \text{Com}$ actually yields an equivalence of categories between the category of abelian modules associated an operad $R$ and the category of abelian group objects in the comma category $\Lambda \mathcal{O}_p / R$. We also refer to [181] for an analogous statement in the context of bimodules over associative algebras. The operad case follows from the same arguments, and we leave the verification of this statement, which we only mention as a remark, to interested readers.

$\Box$

3.1.4. The abelian bimodule structure associated to an additive operad structure. In the applications of the operadic cotriple cohomology to our spectral sequences, we use the additive non-unitary $\Lambda$-operads in dg-modules $N = N_{(n)}$, which we consider in our operadic decompositions (see §2.2.2), are equivalent to abelian bimodules over the commutative operad $\text{Com}$. We have a a similar result for the
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We basically have an equivalence between the additive composition operations
\[ \circ_k : N(m) \oplus N(n) \to N(m + n - 1), \]
which define the composition structure of an additive operad in \( M \), for \( m, n > 0 \),
and \( k = 1, \ldots, m \), and the composition operations of an action of the commutative operad
\[ \circ_k : \text{Com}(m) \otimes N(n) \to N(m + n - 1), \]
\[ \circ_k : N(m) \otimes \text{Com}(n) \to N(m + n - 1). \]

We may also note that, in the particular case \( R = \text{Com} \), the construction of the operad \( R \ltimes N \) in Proposition 3.1.3 is dual to the coaugmented \( \Lambda \)-cooperad construction \( \text{Com}^c \ltimes N \) considered in the proof of Theorem 12.1.5.

3.1.5. The modules of derivations associated to an augmented \( \Lambda \)-operad. We now use the internal hom-object bifunctor \( \text{Hom}(-, -) = \text{Hom}_M(-, -) \) which we associate to our main base categories \( M = \text{dgMod}, \text{grMod}, \text{sMod} \), and the hom-objects of \( \Lambda \)-sequences \( \text{Hom}_{\Lambda S\text{eq}}(-, -) = \text{Hom}_{\text{Mod}_{\Lambda S\text{eq}}}(\text{Com}, -) \) which we deduce from this enriched category structure (see §3.3). We also consider the case of plain modules, for which we trivially have \( \text{Hom}(-, -) = \text{Mor}_{\text{Mod}}(-, -) \) and \( \text{Mor}_{\Lambda S\text{eq}}(-, -) = \text{Mor}_{\text{Mod}_{\text{Mod}}}(\text{Com}, -) \). We still essentially use that our base category forms a closed symmetric monoidal category equipped with an additive structure for the moment (in order to formulate our definitions).

The module of derivations \( \text{Der}_{\Lambda \text{Op}}(R, N) \), where \( R \) is an augmented non-unital \( \Lambda \)-operad and \( N \) is an abelian \( R \)-bimodule in the sense of §3.1.1, is a sub-object of the module of homomorphisms \( \text{Der}_{\Lambda \text{Op}}(R, N) \subset \text{Hom}_{\text{Mod}_{\text{Mod}}}(\text{Com}, -) \) of which elements
\[ \theta \in \text{Der}_{\Lambda \text{Op}}(R, N), \]
are the homomorphisms of \( \Lambda \)-sequences satisfying the derivation relation
\[ \theta(p \circ_k q) = \theta(p) \circ_k q + p \circ_k \theta(q) \]
for every composition product of the operad \( p \circ_k q \in R(m + n - 1) \), where \( p \in R(m), q \in R(n), m, n > 0, k = 1, \ldots, m \). We basically have \( \theta(p) \in N(m), \theta(q) \in N(n) \), and when we form the terms \( \theta(p) \circ_k q, p \circ_k \theta(q) \in N(m + n - 1) \) in the expression of this derivation relation, we just consider the composition operations of the action of the operad \( R \) on the object \( N \).

When we work in a general context, we assume that \( p \) and \( q \) denote abstract variables, and we use this formula (2) to express a formal combination of operations in the hom-object of our base category. We notably use the symmetry isomorphism...
of our symmetric monoidal structure when we permute the variables $p$ and $\theta$ in our derivation formula.

When we work in the category of plain modules, we can evaluate this derivation relation on actual elements without change. When we work in the category of dg-modules (respectively, graded modules), we have to mark an extra sign $\pm$, produced by our commutation relation, once we replace the factors $p$ and $\theta$ by actual elements. We easily see that the differential of a derivation is still a derivation whenever the operad $R$ and the abelian $R$-bimodule $N$ are defined in the category of dg-modules (and come equipped with an internal differential). We accordingly get that $\text{Der}_{\Lambda \odot p}(R, N) = \text{Der}_{\text{dg} \Lambda \odot p}(R, N)$ inherits a dg-module structure from $\text{Hom}_{\Lambda \text{Seq}}(R, N) = \text{Hom}_{\text{dg} \Lambda \text{Seq}}(R, N)$ and defines a functor towards the category of dg-modules therefore. We also have the already mentioned trivial identity:

$$\text{Der}_{\Lambda \odot p}(R, N)_b = \text{Der}_{\text{gr} \Lambda \odot p}(R, N)_b$$

when we consider the forgetful functor $(-)_b : \text{dg Mod} \to \text{gr Mod}$.

When we work in the category of simplicial modules $\mathcal{M} = s \text{Mod}$, we similarly easily check that the module of derivations $\text{Der}_{\Lambda \odot p}(R, N) = \text{Der}_{s \Lambda \odot p}(R, N)$ forms a simplicial sub-module of the hom-object $\text{Hom}_{\Lambda \text{Seq}}(R, N) = \text{Hom}_{s \Lambda \text{Seq}}(R, N)$. We therefore get a functor towards the category of simplicial modules.

The following proposition gives an interpretation of the module of derivations in the case $\mathcal{M} = \text{Mod}$:

**Proposition 3.1.6.** We consider the case of the base category of plain modules $\mathcal{M} = \text{Mod}$, where we have $\text{Hom}_{\text{Mod}}((-,-)) = \text{Mor}_{\text{Mod}}((-,-))$. We then have a bijection:

$$\text{Der}_{\Lambda \odot p}(R, N) = \text{Mor}_{\Lambda \odot p}/R(R, R \times N),$$

for any augmented non-unitary $\Lambda$-operad $R$, and every abelian $R$-bimodule $N$, where we consider the morphism set associated to the objects $R$ and $R \times N$ in the comma category of augmented non-unitary $\Lambda$-operads over $R$.

**Proof.** We immediately see that any morphism $\phi : R \to R \times N$ in the category of augmented non-unitary $\Lambda$-operads over $R$ is given by an expression of the form $\phi = (id, \theta)$, for some morphism of $\Lambda$-sequences $\theta : R \to N$. We readily check that the commutation of the morphism $\phi$ with operadic composites is equivalent to the derivation relation for this map $\theta$. □

We may also note that the abelian group structure associated to the object $R \times N \in \Lambda \odot p/R$ corresponds, at the morphism set level, to the natural underlying abelian group structure of our module of derivations. We mainly give this proposition to provide a first explanation on the definition of the module of derivations. We actually use a simplicial variant of this statement which we will state later on in this section. We are also going use the following theorem:

**Theorem 3.1.7.** We work in the category of plain modules $\mathcal{M} = \text{Mod}$, or in any one of our usual base categories $\mathcal{M} = \text{dg Mod}, \text{gr Mod}, s \text{Mod}$, where we can define operad derivations.

We consider a free augmented connected non-unitary $\Lambda$-operad $R = \mathcal{G}(M)$ associated to an augmented connected $\Lambda$-sequence $M \in \Lambda \text{Seq}_{>1}/\text{Com}$. We then have an identity:

$$\text{Der}_{\Lambda \odot p}(\mathcal{G}(M), N) = \text{Hom}_{\Lambda \text{Seq}}(M, N),$$
for any abelian \( R \)-bimodule \( N \) which is connected as a \( \Lambda \)-sequence.

We have a similar result for general (possibly non-connected) augmented non-unitary \( \Lambda \)-operads, but we only use the connected version of this result in our subsequence computations. We therefore focus on the case of connected augmented non-unitary \( \Lambda \)-operads.

**Proof.** In the case \( M = \text{Mod} \), we may deduce the relation of this theorem from the statement of Proposition 3.1.6. Indeed, if we assume \( R = \mathcal{O}(M) \), for some augmented connected non-unitary \( \Lambda \)-sequence \( M \in \Lambda \text{Seq}_{>1}/\operatorname{Com} \), then we have identities:

\[
\text{Mor}_{\Lambda \mathcal{O} / \mathcal{O}}(\mathcal{O}(M), R \ltimes N) = \text{Mor}_{\Lambda \text{Seq}_{>1}/\operatorname{Com}}(M, R \ltimes N) = \text{Mor}_{\Lambda \text{Seq}}(M, N).
\]

which we deduce from the definition of free objects together with the observation that the functor \( \Lambda \mathcal{O} / \mathcal{O} \to \Lambda \mathcal{O} / \operatorname{Com} \) is fully faithful (and remains so when we pass to categories of augmented objects over \( R \)). But we give another more explicit definition of this correspondence in order to get a bijection which works in other instances of base categories, and when hom-objects do not necessarily reduce to morphism sets.

In short, we prove that any derivation

\[
\theta = \theta_f \in \operatorname{Hom}_{\Lambda \text{Seq}}(\mathcal{O}(M), N)
\]

is uniquely determined by an associated homomorphism

\[
f \in \operatorname{Hom}_{\Lambda \text{Seq}}(M, N)
\]

such that \( f = \theta_{|M} \), where we consider the restriction of the map \( \theta : \mathcal{O}(M) \to P \) through the canonical embedding \( M \hookrightarrow \mathcal{O}(M) \) associated to the free operad \( \mathcal{O}(M) \).

The derivation relation clearly enables us to determine the derivation \( \theta = \theta_f \) on the operadic composites which span the free operad \( \mathcal{O}(M) \) and our main purpose consists in proving that this construction returns a coherent definition, which does not depend on the choice of a particular decomposition into operadic composition operations.

For this aim, we rely on the formalism of trees, and we consider a tree-wise extension of the definition of the composition structure of an abelian bimodule over an operad.

The tree-wise definition of the composition structure of abelian bimodules. We assume that \( R \) is any augmented connected non-unitary \( \Lambda \)-operad (not necessarily free) for the moment, and we consider an abelian bimodule \( N \) over this operad \( R \). We deduce the definition of the tree-wise composition operations associated to this abelian bimodule \( N \) from an analysis of the tree-wise composition operations of the operad \( R \ltimes N \). We use the identity \((R \ltimes N)(r) = R(r) \oplus N(r)\), for any \( r > 0 \).

We assume that \( \Sigma \) is a reduced \( r \)-tree with \( m \) vertices. We have

\[
\mathcal{O}_{\Sigma}(R \ltimes N) = \bigoplus_{l=0}^{m} \mathcal{O}_{\Sigma}(R, N)^{(l)},
\]
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where \( \Theta_T(R, N)^{(l)} \) is the sub-object of the tree-wise tensor product \( \Theta_T(R \times N) = \Theta_T(R \oplus N) \) spanned by tensors where \( l \) vertices of the tree \( T \) are labeled by a factor in \( N \), while \( m - l \) vertices are labeled by a factor in \( R \). We have by definition:

\[
\Theta_T(R, N)^{(0)} = \Theta_T(R),
\]

and we may also write:

\[
\Theta_T(R, N)' = \Theta_T(R, N)^{(1)},
\]

because we regard this additive functor in \( N \) as the first derivative of the functor \( \Theta_T(-) \) taken at the object \( R \) in the category of symmetric sequences. We may also interpret the decomposition (4) as a Taylor expansion formula.

We readily check that the tree-wise composition \( \lambda : \Theta_T(R \times N) \rightarrow (R \times N)(r) \), associated to the operad \( R \times N \), vanishes over the summands \( \Theta_T(R, N)^{(l)} \subset \Theta_T(R, N) \) such that \( l > 1 \) and reduces to:

- the composition operation of our operad \( \Theta_T(R) \rightarrow (R(r) \subset (R \times N)(r) \) on the summand \( \Theta_T(R, N)^{(0)} = \Theta_T(R) \);
- a morphism \( \Theta_T(R, N)' \rightarrow (R(r) \subset (R \times N)(r) \) involving the composition operation of our abelian bimodule on \( \Theta_T(R, N)^{(1)} = \Theta_T(R, N)' \).

To give a simple example, we consider a tree-wise tensor of the same shape as in our illustration of the definition of a free operad:

\[
(\ast)
\]

but where we take one factor in our bimodule \( \xi \in N(3) \), while the other factors belong the operad \( a \in R(2), b \in R(2), c \in R(3) \), in order to get an element of the summand \( \Theta_T(R, N)' \) as stated in our picture.

We choose a sequence of edge contractions \( T \mapsto T/e \mapsto T/e/f \mapsto T/e/f/g \) and we perform the corresponding partial composition operations, either within the operad or on the abelian bimodule factor, in order to get the element \( \lambda(\pi) \in N(6) \) associated to our tree-wise tensor \( \pi \in \Theta_T(R, N)' \). In our example, we explicitly get \( \lambda(\pi) = (1 5) \cdot ((a \circ_1 \xi) \circ_4 b) \circ_3 c \). If we make another choice of edge contraction, like \( T \mapsto T/f \mapsto T/e/f \mapsto T/e/f/g \), then we get the equivalent expression \( \lambda(\pi) = (1 5) \cdot ((a \circ_2 b) \circ_1 \xi) \circ_3 c \), where the composition product \( a \circ_2 b \) is formed within the operad \( R \), while the other operations involve the action of the operad on the abelian bimodule \( N \).

In §A.2, we use the associativity (and the equivariance) relations of the composition structure of operads to check that the tree-wise composition operation associated to an operad do not depend on choices, and fulfill a general form of associativity relations, which we formulate in terms of composition structures on trees (see also Lemma A.3.5). This general statement applies to the tree-wise composition operations of the operad \( R \times N \) associated to an abelian bimodule \( N \), and, as a by-product, to these tree-wise composition operations \( \lambda : \Theta_T(R, N)' \rightarrow N \) which we associate to an abelian bimodule structure.
In §A.4.6 (see also Lemma I.3.4.3 for an introduction to this construction), we explain that the restriction operations, giving the Λ-sequence structure of a free operad on an augmented (connected) Λ-sequence, are induced by natural transformations \( u^* : \Theta_{T}(-) \to \Theta_{u^* T}(-) \), which we associate to the morphisms \( u \) of the category \( \Lambda_{>0} \), on the tree-wise tensor products defining the free operad. We also use this observation to express the preservation of the restriction operations by the tree-wise composition operations associated to an augmented non-unitary (connected) Λ-operad. In the case of an operad of the form \( R \rtimes N \), we easily check that these tree-wise restriction operations, which we define by the process of §A.4.6, preserve the summands \( \Theta_{T}(R, N)\{l\} \), \( l \geq 0 \), of our decomposition (4) and give rise to a natural morphisms \( u^* : \Theta_{T}(R, N)\{l\} \to \Theta_{u^* T}(R, N)\{l\} \), for all \( l \geq 0 \).

Recall that our construction of the tree-wise restriction operations in §A.4.6 involves the internal restriction operations of the given Λ-sequence as well as the augmentation attached to our object. We just take the convention that the abelian bimodule \( N \) is endowed with a zero augmentation. We go back to our tree-wise tensor example (∗) in order to illustrate our computation process for \( l = 1 \). We use the conventions of §I.3.2.9 to materialize our restriction operations. We first consider the case of the map \( u : 3 \to 6 \) such that \( u(1) = 5, u(2) = 3, u(3) = 6 \). We then get the following expression:

\[
\begin{equation}
\left(7\right) \quad u^*(\pi) = \begin{array}{c}
1 \\
2 \\
3
\end{array} \Rightarrow \begin{array}{c}
1 \\
2 \\
3
\end{array} = \epsilon(b)\epsilon(c).
\end{equation}
\]

for the element \( u^*(\pi) \in \Theta_{u^* T}(R, N) \). If we consider the map \( v : 4 \to 6 \) such that \( v(1) = 1, v(2) = 4, v(3) = 3, v(4) = 6 \), then we get:

\[
\begin{equation}
\left(8\right) \quad v^*(\pi) = \begin{array}{c}
1 \\
2 \\
3
\end{array} \Rightarrow \begin{array}{c}
1 \\
2 \\
3
\end{array} = 0
\end{equation}
\]

as our process involves the application of an augmentation map on the factor \( \xi \in N(3) \) in this case in order to reduce the composition pattern in the outcome of our operation.

The preservation of the restriction operations by the composition structure of an abelian bimodule is equivalent to the commutativity of the diagrams:

\[
\begin{equation}
\begin{array}{c}
\Theta_{T}(R, N)' \\
\uparrow \quad u^* \\
\Theta_{u^* T}(R, N)' \\
\downarrow \quad u^*
\end{array} \xrightarrow{\lambda} \begin{array}{c}
N(n) \\
N(m)
\end{array},
\end{equation}
\]

for all \( u \in \text{Mor}_{\Lambda}(m, n) \), \( m, n > 1 \), and where we consider any reduced \( n \)-tree \( T \in \text{Tree}(n) \).
The definition of derivations on tree-wise tensors. We now explain the definition of derivations on the tree-wise tensor products $\Theta_T(M)$, which span the free operad $\mathcal{O}(M)$ associated to an augmented non-unitary $\Lambda$-sequence $M \in \Lambda Seq_{\geq 1}/Com$.

To make our construction more conceptual, we consider a generalization of the setting of our theorem which also fits the practical applications of our correspondence between derivations and homomorphisms. Namely, we set $Q = \mathcal{O}(M)$, and we assume that this free operad is equipped with a morphism $\phi : Q \to R$, where $R$ is the augmented non-unitary $\Lambda$-operad which naturally acts on the abelian bimodule $N$. The object $N$ then inherits an abelian $Q$-bimodule structure by restriction through the morphism $\phi : Q \to R$. We merely take $R = Q$ and $\phi = id$ to retrieve the case considered in the statement of our theorem.

Let $f \in \text{Hom}_{\Lambda Seq}(M, N)$. We use that the tree-wise tensor product $\Theta_T(M)$, where $T$ is any reduced $r$-tree, $r \in \mathbb{N}$, represents a composition scheme in the free operad $\mathcal{O}(M)$ to determine the expression of a derivation $\theta_f$ extending the homomorphism $f$ on this object $\Theta_T(M) \subset \mathcal{O}(M)$. In the case $M = \text{Mod}, dg \text{Mod}, gr \text{Mod}$, where our homomorphisms are identified with module maps (satisfying homogeneity constraints), we eventually get that the homomorphism $\theta_f \in \text{Hom}_M(\Theta_T(M), N)$, which we define on any tree-wise tensor product $\Theta_T(M)$, is given by a composite:

$$
\mathcal{O}_T(M) \xrightarrow{\partial_\phi(f)} \mathcal{O}_T(R, N) \xrightarrow{\lambda} N(r),
$$

where $\partial_\phi(f)$ denotes the map obtained by applying the homomorphism $f$ to one factor in the tree-wise tensor product $\Theta_T(M)$, the operad morphism $\phi$ to the other factors, and by summing over all positions of this distinguished factor.

For instance, if we consider a tree-wise tensor of the same shape as in $(\ast)$:

$$(\ast\ast)$$

$$
\pi = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
5
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\in \Theta_T(M),
\end{array}
$$

but where we now assume $x \in \text{M}(2)$, $y \in \text{M}(3)$, $z \in \text{M}(2)$, $t \in \text{M}(2)$, then we get the expression:

$$(11)$$

$$
\theta_f(\pi) = \pm \lambda \begin{pmatrix}
\phi(y) & \phi(t) & \phi(z) & 0 \\
5 & 2 & 3 & 4 & 1 & 6 \\
\end{pmatrix} + \pm \lambda \begin{pmatrix}
\phi(x) & \phi(t) & \phi(z) & 0 \\
5 & 2 & 3 & 4 & 1 & 6 \\
\end{pmatrix} + \pm \lambda \begin{pmatrix}
\phi(y) & \phi(t) & \phi(z) & 0 \\
5 & 2 & 3 & 4 & 1 & 6 \\
\end{pmatrix} + \pm \lambda \begin{pmatrix}
\phi(x) & \phi(t) & \phi(z) & 0 \\
5 & 2 & 3 & 4 & 1 & 6 \\
\end{pmatrix}
$$

for this tree-wise derivation, with extra-signs, occurring in the differential graded and in the graded contexts, and arising from possible tensor commutations involved in the evaluation of the map $f$ on the factors of this tensor product. Recall that
the above tree-wise tensor (***) represents the formal composite operation \( \pi = (1 \, 5) \cdot (((x \circ_1 y) \circ_2 z) \circ_3 t) \) in the free operad \( \mathfrak{O}(M) \). The definition of our map \( \theta_f \) in (11) reflects the relation:

\[
\theta_f(\pi) = \pm(1 \, 5) \cdot (((f(x) \circ_1 \phi(y)) \circ_2 \phi(z)) \circ_3 \phi(t)) \\
+ \pm(1 \, 5) \cdot (((\phi(x) \circ_1 f(y)) \circ_2 \phi(z)) \circ_3 \phi(t)) \\
+ \pm(1 \, 5) \cdot (((\phi(x) \circ_1 \phi(y)) \circ_2 f(z)) \circ_3 \phi(t)) \\
+ \pm(1 \, 5) \cdot (((\phi(x) \circ_1 \phi(y)) \circ_2 \phi(z)) \circ_3 f(t))
\]

which we deduce from the evaluation of the derivation formula §3.1.5(2) in the free operad.

We essentially perform the same construction in the simplicial module case \( \mathcal{M} = s \text{Mod} \). In this setting, we may use that a homomorphism \( f \) of dimension \( n \geq 0 \) in the simplicial hom-object \( \text{Hom}_{\Lambda \text{Seq}}(M, N) \) is defined as a morphism of \( \Lambda \)-sequences towards the object \( N^{\Delta^n} \) such that \( N^{\Delta^n}(r) = N(r)^{\Delta^n} \), for any \( r > 1 \), where we consider the canonical simplicial framing of the simplicial modules \( N(r) \), \( r > 1 \). We easily check that this simplicial framing \( N^{\Delta^n} \in s(\Lambda \text{Seq}_{\gamma 1}) \) inherits the structure of an abelian bimodule over a similarly defined simplicial object of the category of operads \( R^{\Delta^n} \in s(\Lambda \text{Op}_\gamma / \text{Com}) \), and over the plain operad \( R \in s(\Lambda \text{Op}_\gamma / \text{Com}) \) by restriction through the canonical morphism \( \eta : R \to R^{\Delta^n} \) of our simplicial framing construction. We use the action of the operad \( R \) on the object \( N^{\Delta^n} \) to adapt the definition of our map \( \theta_f \) associated to \( f \).

In all cases, we just take the sum of the tree-wise derivations \( \theta_f : \mathfrak{O}_f(M) \to N \), such as defined in (10), to get the derivation \( \theta_f \in \text{Der}_{\Lambda \text{Op}}(\mathfrak{O}(M), N) \) associated to our homomorphism \( f \in \text{Hom}_{\Lambda \text{Seq}}(M, N) \) on the free operad \( Q = \mathfrak{O}(M) \). The proof that this homomorphism \( \theta_f \) satisfies the derivation relation follows from the coherence of our tree-wise composition operations with respect to composition operations on trees. We also readily deduce, from the preservation of the tree-wise restriction operations in (9), that the derivation \( \theta_f \) preserves the restriction operations associated with the free operad, and hence, forms a homomorphism of \( \Lambda \)-sequence well. This observation finishes the proof of our theorem. \( \square \)

### 3.2. The definition and the applications of the cotriple cohomology

We now explain the definition of the cotriple cohomology for operads, and we explain the relationship between this cotriple cohomology theory and the homotopy spectral sequence of the function space \( \text{Map}_{\Lambda \text{Op}}(\lfloor \text{Res}_o(P) \rfloor, \Gamma_o(N)) \) associated to the cotriple resolution of an operad in simplicial sets \( P \).

#### 3.2.1. The definition of the cotriple cohomology

We actually consider a version of the cotriple cohomology which we define for connected augmented non-unitary \( \Lambda \)-operads in graded modules. We basically deal with the cotriple resolution functor:

\[
\text{Res}_o : gr \Lambda \text{Op}_\gamma / \text{Com} \to s \text{gr } \Lambda \text{Op}_\gamma / \text{Com},
\]

defined on this category of operads \( gr \Lambda \text{Op}_\gamma / \text{Com} \), and which we deduce from the extension, to augmented \( \Lambda \)-objects, of the free operad functor \( \mathfrak{O} : M \to \mathfrak{O}(M) \).

To be explicit, we now assume that \( R \) is any object of the category of connected augmented non-unitary \( \Lambda \)-operads in graded modules \( gr \Lambda \text{Op}_\gamma / \text{Com} \). We consider an abelian \( R \)-bimodule \( N \), defined in the same base category, and which
will give the coefficients of our cohomology theory. We also assume that this abelian $R$-bimodule $N$ is connected as a $\Lambda$-sequence.

The cotriple resolution which we consider in (1) is defined in §B.1.2 (in the general case of a connected non-unitary operad in a symmetric monoidal category). Recall simply that this simplicial object $Q = \text{Res}_*(R)$ is defined, in dimension $n \in \mathbb{N}$, by the application of a composite functor:

$$\text{Res}_n(R) = \bigtriangledown \circ \bigtriangledown \circ \cdots \circ \bigtriangledown(R),$$

where we consider the augmentation ideal of our operad $R$ and of the free operad $\Theta(-)$. We use the adjunction between the free operad $\Theta : M \mapsto \Theta(M)$ and the augmentation ideal functor $\overline{\Theta} : R \mapsto \overline{\Theta}(R)$ to determine the structure maps of this simplicial object. We use the extension of this adjunction to augmented $\Lambda$-objects in order to provide each object $\text{Res}_n(R)$, $n \in \mathbb{N}$, with an augmented non-unitary $\Lambda$-operad structure, and to get a simplicial object in the category of connected augmented non-unitary $\Lambda$-operads from our construction. We refer to §B.1.11 for more details on this definition.

We also consider the morphism

$$\epsilon : \text{Res}_*(R) \rightarrow R,$$

defined by the adjunction augmentation of the free operad $\lambda : \Theta(R) \rightarrow R$ in dimension 0, in order to provide this simplicial object $\text{Res}_*(R)$ with a canonical augmentation over the operad $R$ (see again §B.1.2).

The object $N$ inherits the structure of an abelian bimodule over the operad $\text{Res}_n(R)$, for any $n \in \mathbb{N}$, by restriction through this augmentation morphism (3). We consider the cosimplicial object of the category of graded modules:

$$\text{Der}_{\text{gr} \Lambda \text{Op}}(\text{Res}_*(R), N) \in \text{gr Mod}$$

given by the expression $\text{Der}_{\text{gr} \Lambda \text{Op}}(\text{Res}_*(R), N)^n = \text{Der}_{\text{gr} \Lambda \text{Op}}(\text{Res}_n(R), N)$, for any $n \in \mathbb{N}$, where we form our graded module of derivations with coefficients in this abelian bimodule structure over the operad $\text{Res}_n(R) \in \text{gr} \Lambda \text{Op}_{\text{gr}1} / \text{Com}$. We apply the conormalization functor of §5.2.1 degree-wise (see §3.0.3) in order to get a cochain complex of graded modules $N^* \text{Der}_{\text{gr} \Lambda \text{Op}}(\text{Res}_*(R), N) \in \text{dg}^* \text{gr Mod}$ from this cosimplicial object $\text{Der}_{\text{gr} \Lambda \text{Op}}(\text{Res}_*(R), N) \in \text{gr Mod}$.

We define the cotriple cohomology of our operad $R$ with coefficients in the abelian $R$-bimodule $N$ by the cohomology of this cochain complex:

$$H^*_{\text{gr} \Lambda \text{Op}}(R, N) = H^* N^* \text{Der}_{\text{gr} \Lambda \text{Op}}(\text{Res}_*(R), N).$$

We use that this cotriple cohomology construction naturally forms a bigraded object. We have an upper (non-negative) grading $s \in \mathbb{N}$, which arises from the simplicial grading of the cotriple resolution and corresponds to the grading of the conormalized complex construction at the level of our object, together with an internal lower grading $t \in \mathbb{Z}$, which arises from the natural grading of the module of derivations $\text{Der}_{\text{gr} \Lambda \text{Op}}(-, -) = \bigoplus_t \text{Der}_{\text{gr} \Lambda \text{Op}}(-, -)_t$. We follow our general conventions on multiple graded structures (see §§3.0.2-3.0.3). We basically take the difference $n = t - s$ as total (lower) grading for the cohomology module $H^*_{\text{gr} \Lambda \text{Op}}(R, N)$. We mark this total degree by an external (post-fix) lower-script in the expression of the cohomology. We use the upper-script $*$ in the notation of our cohomology functor.
$H^*_{\text{gr } \Lambda \odot p}(-)$ to refer to the conormalized complex grading $s \in \mathbb{N}$. We accordingly write:

$$H^*_{\text{gr } \Lambda \odot p}(R, N) = \bigoplus_{s,n} H^*_{\text{gr } \Lambda \odot p}(R, N)_n,$$

for the decomposition of the cotriple cohomology module, with the external lower-script $n \in \mathbb{Z}$ giving the total degree.

3.2.2. Recollections on the homotopy cosimplicial spectral sequence associated to function spaces of operads. We go back to the setting of the spectral sequence §2.2.4, where we consider a function space $\text{Map}_{\Lambda \odot p}(\text{Res}_s(P), \Gamma_\bullet(N))$ on the cotriple resolution of a connected non-unitary $\Lambda$-operad in simplicial sets $P \in \text{sSet } \Lambda \odot p_{\odot 1}$. We use the homology functor $H_\bullet : \text{sSet} \to \text{gr}_\bullet \text{Mod}$, which we apply to operads arity-wise, in order to associate an augmented connected non-unitary $\Lambda$-operad in chain graded modules $R = H_\bullet(P) \in \text{gr}_\bullet \Lambda \odot p_{\odot 1} / \text{Com}$ to any such object $P \in \text{sSet } \Lambda \odot p_{\odot 1}$. We consider the homology with rational coefficients $H_\bullet = H_\bullet(-, \mathbb{Q})$ all through this construction since we take the field of rational numbers $k = \mathbb{Q}$ as ground ring for our module categories.

We still consider an additive connected non-unitary $\Lambda$-operad in dg-modules $N$ and the additive connected non-unitary $\Lambda$-operad in simplicial modules $E = \Gamma_\bullet(N)$ associated to this object. We observed in §3.1.2 that such additive operad structures are equivalent to abelian bimodules over the commutative operad $\text{Com}$. We readily get that the object $N = H_\bullet(N)$ which we form by applying the homology functor $H_\bullet : \text{dgMod} \to \text{gr}_\bullet \text{Mod}$ to our additive operad $N$ arity-wise forms an abelian bimodule over $\text{Com}$ in the category of graded module. We can moreover use a restriction of structure through the canonical augmentation $\epsilon : R \to \text{Com}$ in order to provide this object $H_\bullet(N)$ with the structure of an abelian bimodule over the operad $R = H_\bullet(P) \in \text{gr}_\bullet \Lambda \odot p_{\odot 1} / \text{Com}$ which we consider in our spectral sequence.

In §2.2.4, we just assume that the additive connected non-unitary $\Lambda$-operad $N$ is fibrant as a $\Lambda$-sequence. We now need to assume that $N$ is cofreely generated by a symmetric sequence in dg-modules $S N \in \text{dg } \Sigma \text{Seq}$, in the sense that we have an end-formula:

$$N(r) = \int_{n \in \Sigma} S N(n)^{\text{Mor} \Lambda(n, r)},$$

in any arity $r > 0$, when we forget about the additive composition structure and focus on the $\Lambda$-sequence structure attached to our object. We give more details on this notion in the appendix section §3.3. We mainly use that a $\Lambda$-sequence equipped with such a cofree structure is automatically fibrant (see Proposition 3.3.8), and that the hom-object with a target object of this form in the category of $\Lambda$-sequences reduces to a hom-object in the category of symmetric sequences (see Proposition 3.3.7).

We check the following proposition, which gives an interpretation, at the simplicial set level, of the module of derivations associated to an operad:

**Proposition 3.2.3.**

(a) For any connected non-unitary $\Lambda$-operad in simplicial sets $P \in \text{sSet } \Lambda \odot p_{\odot 1}$, and any additive connected non-unitary $\Lambda$-operad in chain graded dg-modules $N$, we have an identity:

$$\text{Map}_{\Lambda \odot p}(P, \Gamma_\bullet(N)) = \text{Der}_\bullet(\Lambda \odot p(\mathbb{Q}[P], \Gamma_\bullet(N))),$$
3.2. THE DEFINITION AND THE APPLICATIONS OF THE COTRIPLE COHOMOLOGY

where we consider the module of simplicial derivations associated to the augmented non-unitary Λ-operad in simplicial modules

\[ \mathbb{Q}[P] \in s \Lambda \mathcal{O}_P \mathcal{G} / \text{Com} \]

which we obtain by applying the free module functor \( \mathbb{Q}[-] : s\text{Set} \rightarrow s\text{Mod} \) arithmetic-wise to our operad in simplicial sets \( P \).

(b) If we assume that \( P \) is a free connected non-unitary Λ-operad \( P = \Theta(M) \), for some connected non-unitary Λ-sequence in simplicial sets \( M \in s\text{Set} \Lambda \text{Seq}_{\geq 1} \), and if the object \( N \) has, as a Λ-sequence, a cofree structure of the form §3.2.2(1), then we moreover have a canonical bijection:

\[
\tau_\ast \text{Der}_\Lambda(P, \text{Gr}_\ast(N)) \cong \tau_\ast \text{Der}_\Lambda(P, \text{Gr}_\ast(N))
\]

where we consider the (non-negative degree truncation \( \tau_\ast \)) of graded derivations associated to the augmented non-unitary Λ-operad in graded modules

\[
\text{Gr}_\ast(P) = \text{Gr}_\ast(\Theta(M)) \in \text{Gr}_\ast \Lambda \mathcal{O}_P \mathcal{G} / \text{Com}
\]

obtained by applying the homology functor \( \text{Gr}_\ast(-) : s\text{Set} \rightarrow \text{Gr}_\ast\text{Mod} \) to our free operad \( P = \Theta(M) \) arithmetic-wise.

Let us mention that the first assertion of this proposition remains valid over any ground ring, while the second assertion still holds when the ground ring is a characteristic zero field. We just need this characteristic assumption in order to apply the Künneth isomorphism formula of §3.3.13.

EXPLANATIONS AND PROOFS. The identity of assertion (a). The operadic function space in assertion (a) can be defined as the space of operad morphisms towards the operad in simplicial sets \( \text{Gr}_\ast(N)^{\Delta^o} \) which we form by taking the canonical function objects of simplicial sets \( \text{Gr}_\ast(N(r))^{\Delta^o} \) in each arity \( r > 0 \) (see §2.3). We actually consider the same simplicial framing when we define the module of simplicial homomorphisms towards the abelian \( \text{Com} \)-bimodule underlying \( \text{Gr}_\ast(N) \) (see §3.3.11). We then use that the function objects \( \text{Gr}_\ast(N(r))^{\Delta^o} \) inherit a module structure from \( \text{Gr}_\ast(N(r)) \), for all \( n \in \mathbb{N} \), and for any arity \( r > 0 \).

Every morphism of \( \Lambda \)-sequences in simplicial sets \( \phi : P \rightarrow \text{Gr}_\ast(N)^{\Delta^o} \) extends to a morphism of augmented \( \Lambda \)-sequences in simplicial modules \( \phi_\ast : \mathbb{Q}[P] \rightarrow \text{Gr}_\ast(N)^{\Delta^o} \), for any \( n \in \mathbb{N} \). We easily check that the preservation of operadic composition structures by the given morphism \( \phi \) is equivalent to the derivation relation for the associated morphism of augmented \( \Lambda \)-sequences in simplicial modules \( \phi_\ast \), and this correspondence gives the identity of our proposition.

The comparison map of assertion (b). We define the comparison map of the second assertion of our proposition in two steps.

We first observe that the comparison map of hom-objects in Proposition 3.3.12 (see also Proposition 5.4.6 for the explicit definition of this map) preserves submodules of derivations, and hence, induces a morphism of dg-modules:

\[
\tau_\ast \text{Der}_\Delta(P, \text{Gr}_\ast(N)) \rightarrow \tau_\ast \text{Der}_\Delta(P, \text{Gr}_\ast(N)).
\]

We also have an identity \( \tau_\ast \text{Gr}_\ast(N) = N \) by the equivalence assertion of the Dold-Kan correspondence.
We use, in a second step, that the Künneth map of §3.3.13(1) restricts to a canonical morphism on the dg-modules of derivations:

\[ \tau_*: H_* \text{Der}_{gr \Lambda OP}(R, N) \to H_* \text{Der}_{gr \Lambda OP}(\mathcal{H}_*(R), \mathcal{H}_*(N)), \]

for any augmented non-unitary \( \Lambda \)-operad in dg-modules \( R \in dg \Lambda OP, \) and any abelian bimodule \( N \) over this operad \( R \). To be explicit, recall that we usually write \([z] \in H_\bullet(C)\) for the homology class of a cycle \( z \in Z_\bullet(C)\) in a dg-module \( C \in dg \text{Mod} \). To a homology class \([\theta] \in H_\bullet \text{Hom}_{dg \Lambda Seq}(R, N)\), represented by a cycle \( \theta \in Z_\bullet \text{Hom}_{dg \Lambda Seq}(R, N)\) in the dg-module of homomorphisms \( \text{Hom}_{dg \Lambda Seq}(R, N)\), we associate the homomorphism \( \theta_*: H_\bullet(R) \to H_\bullet(N) \) such that \( \theta_*([\xi]) = [f(p)] \), for any \([p] \in H_\bullet(R(r)), \) \( r > 0 \). We just check that this homomorphism \( \theta_* \) satisfies the derivation relation on the homology of our operad and hence, defines an element in the graded module of homology derivations \( \theta_* \in \text{Der}_{gr \Lambda OP}(H_\bullet(R), H_\bullet(N)) \), when we assume \( \theta \in Z_\bullet \text{Der}_{dg \Lambda Seq}(R, N) \).

We compose the homology morphism induced by the simplicial comparison map (1) with this homology comparison map (2), where we take \( R = N_\bullet(P) \) and \( N = N_\bullet \Gamma_\bullet(N) = N \), to get the comparison map of our statement:

\[ H_* \text{Der}_{s \Lambda OP}(Q[P], \Gamma_\bullet(N)) \xrightarrow{(1)} H_* \text{Der}_{s \Lambda OP}(N_\bullet(P), N_\bullet \Gamma_\bullet(N)) \xrightarrow{(2)} H_* \text{Der}_{gr \Lambda OP}(H_\bullet(P), H_\bullet(N)). \]

We aim to prove that this composite map defines a bijection when the assumptions of our proposition are fulfilled.

*The identity of assertion (b).* We then assume that \( P \) is the free operad \( P = \Theta(M) \) associated to a connected symmetric sequence \( M \in sSet \Lambda Seq_{>1} \). We have an inclusion relation \( Q[M] \subseteq Q[\Theta(M)] \) in the category of symmetric sequence in simplicial modules, and similarly \( N_\bullet(M) \subseteq N_\bullet(\Theta(M)) \) when we consider the normalized complexes associated to our objects. We also have an inclusion at the homology level \( H_\bullet(M) \subseteq H_\bullet(\Theta(M)) \) since \( M \) is naturally identified, as a symmetric sequence, with a direct summand of the free operad \( \Theta(M) \).

The idea is to rely on the correspondence of Theorem 3.1.7 in order to reduce the proof of our statement to a result about hom-objects of \( \Lambda \)-sequences. Let \( E = \Gamma_\bullet(N) \Leftrightarrow N = N_\bullet(E) \). Formally, we use that our comparison maps (1-2) fit in a commutative diagram:

\[
\begin{array}{ccc}
H_* \text{Der}_{s \Lambda OP}(Q[\Theta(M)], E) & \xrightarrow{\approx} & H_* \text{Hom}_{s \Lambda Seq}(Q[\Theta(M)], E) \\
\downarrow (1) \quad & & \downarrow (1') \\
H_* \text{Der}_{dg \Lambda OP}(N_\bullet(\Theta(M)), N_\bullet(E)) & \xrightarrow{\approx} & H_* \text{Hom}_{dg \Lambda Seq}(N_\bullet(\Theta(M)), N_\bullet(E)) \\
\downarrow (2) \quad & & \downarrow (2') \\
\text{Der}_{gr \Lambda OP}(H_\bullet(\Theta(M)), H_\bullet(E)) & \xrightarrow{\approx} & \text{Hom}_{gr \Lambda Seq}(H_\bullet(\Theta(M)), H_\bullet(E))
\end{array}
\]

where the horizontal maps are induced by the restriction of derivations through the canonical symmetric sequences inclusions \( Q[M] \subseteq Q[\Theta(M)] \), \( N_\bullet(M) \subseteq N_\bullet(\Theta(M)) \), \( H_\bullet(M) \subseteq H_\bullet(\Theta(M)) \), and the vertical maps \((1'\sim2')\) are the natural counterpart, on hom-objects, of our comparison maps on derivations \((1\sim2)\). (We omit to mark truncation functors to simplify the expression of the objects occurring in this diagram.)
In short, when we form the morphism (1'), we basically retrieve the homology of the Eilenberg-MacLane map of the simplicial hom-object associated to the Λ-sequences \( M \) and \( E = \Gamma_*(N) \). We prove in Proposition 3.3.12 that this comparison map automatically defines a bijection at the homology level. In the case of the morphism (2'), we retrieve the Künneth map \( \mathcal{K} \) of the Λ-sequences \( N^\ast(M) \) and \( N = \Gamma_*(E) \). We checked in §3.3.13 that this Künneth map defines an isomorphism as soon as we assume that our ground ring \( (k = \mathbb{Q} \text{ under our conventions}) \) is a characteristic zero field and that our object \( N \) is cofreely generated by a symmetric sequence. We therefore get that the vertical maps of our diagrams are both bijections, as marked in the diagram, when the assumptions of our proposition hold.

We trivially have \( \mathbb{Q}[\mathcal{C}(M)] = \mathcal{C}(\mathbb{Q}[M]) \), and the result of Theorem 3.1.7 accordingly implies that the upper horizontal map of our diagram defines a bijection, as marked in the diagram. By the Künneth formula, we also have \( H_\ast(\mathcal{C}(M)) = \mathcal{C}(H_\ast(M)) \) (at least when the ground ring is a field), and we use Theorem 3.1.7 again (in the graded module context) to obtain that the lower horizontal map of our diagram defines a bijection as well. We conclude that the composite morphism (1-2), which defines the comparison map of our proposition, is also a bijection as required, and this observation finishes the proof of our statement.

We apply the result of this proposition to the function space of our cosimplicial homotopy spectral sequence §2.2.4 which we associate to the cotriple resolution \( R_\ast = \text{Res}_\ast(P) \) of a connected non-unitary Λ-operad in simplicial sets \( P \in \text{sSet}_{\Lambda \mathcal{O}_{\emptyset 1}} \). We get the following statement:

**Proposition 3.2.4.** We consider the spectral sequence of §2.2.4. We assume that the additive connected non-unitary Λ-operad \( N \) in our spectral sequence construction has, as a Λ-sequence, a cofree structure of the form \( \mathcal{K} \) of the operator \( R = H_\ast(P) \) with coefficients in the abelian \( \text{Com-binodule} \) \( \mathcal{H}_\ast(N) \) (see §3.2.1).

This statement is actually valid over any characteristic zero field (like the result Proposition 3.2.3).

**Proof.** The Künneth formula implies that we have an identity:

\[
\mathcal{N}^\ast \pi_\ast \text{Map}_{\Lambda \mathcal{O}_{\emptyset}}(\text{Res}_\ast(P), \Gamma_*(N)) = \mathcal{N}^\ast \tau_\ast \text{Der}_{\text{gr}_{\Lambda \mathcal{O}_{\emptyset}}}(\text{Res}_\ast(H_\ast(P)), H_\ast(N)),
\]

where, on the left-hand side, we consider the conormalized homotopy complex which determines the second page of our cosimplicial spectral sequence (see §2.2.4), and, on the right-hand side, we consider the (truncation in non-negative internal degree of the) conormalized cochain complex of derivations which determines the cotriple cohomology of the operad \( R = H_\ast(P) \) with coefficients in the abelian \( \text{Com-binodule} \) \( \mathcal{H}_\ast(N) \) (see §3.2.1).

This statement is actually valid over any characteristic zero field (like the result Proposition 3.2.3).

**Proof.** The Künneth formula implies that we have an identity:

\[
\mathcal{H}_\ast \text{Res}_\ast(P) = \text{Res}_\ast(H_\ast(P)),
\]

where we consider the cotriple resolution of the homology of our operad \( H_\ast(P) \in \text{gr}_{\Lambda \mathcal{O}_{\emptyset 1} / \text{Com}} \) in the category of augmented connected non-unitary Λ-operads in graded modules. The results of Proposition 3.2.3 imply that we have an identity of cosimplicial objects in the category of graded modules:

\[
\pi_\ast \text{Map}_{\Lambda \mathcal{O}_{\emptyset}}(\text{Res}_\ast(P), \Gamma_*(N)) = \tau_\ast \text{Der}_{\text{gr}_{\Lambda \mathcal{O}_{\emptyset}}}(\text{Res}_\ast(H_\ast(P)), H_\ast(N)),
\]

by functoriality of our comparison map. We just take the conormalized cochain complex of these objects to get our statement. \( \square \)
The result of this proposition has the following immediate corollary, which was the main objective of this section:

**Theorem 3.2.5.** We consider the spectral sequence of §2.2.4. We assume that the additive connected non-unitary Λ-operad \( N \) in our spectral sequence construction is, as a Λ-sequence, cofreely generated by a symmetric sequence in dg-modules as in §3.2.2(1).

Then the second page of this function space spectral sequence associated to the cotriple resolution of a connected non-unitary operad in simplicial sets \( P \in sSet \Lambda Op_{\geq 1} \) satisfies:

\[
\pi^* \pi_* \Map_{sSet \Lambda Op}(\Res_{\bullet}(P), \Gamma_\bullet(N)) = H_*^{\Der_{\gr \Lambda Op}(\Res_{\bullet}(H_\bullet(P)), H_\bullet(N))} = E^2_{\gr \Lambda Op}(H_\bullet(P), H_\bullet(N))
\]

where we consider the operadic cotriple cohomology \( H_*^{\Lambda Op}_{\gr}(\cdot) \) of the augmented connected non-unitary Λ-operad in graded modules \( H_\bullet(P) \in \gr \Lambda Op_{\geq 1} / \Com \) with coefficients in the abelian \( \Com \)-bimodule \( H_\bullet(N) \).

We still implicitly assume that we take the field of rational numbers as ground ring \( \mathbb{k} = \mathbb{Q} \), and we consider the homology with rational coefficients in this statement (see §3.2.2). We may again check, however, that this theorem remains fully valid as soon as we work over a characteristic zero field as ground ring.

Let us mention that the bigrading of the cotriple cohomology module occurring in this theorem, namely \( H_*^{\Lambda Op}(H_\bullet(P), H_\bullet(N)) \), corresponds, under the relation of our statement, to the bi-grading of our function space spectral sequence. To be explicit, we have the more precise relation \( E_*^{2t} = H_*^{\Lambda Op}(H_\bullet(P), H_\bullet(N))_{t-s} \), for every \( t \geq s \geq 0 \).

### 3.3. Appendix: Hom-objects on the category of Λ-sequences

The first purpose of this appendix section is to explain the definition of the hom-objects on the category of (non-unitary) Λ-sequences which we use in this chapter. We still mostly deal with the case of Λ-sequences in dg-modules \( \mathcal{M} = \dg Mod \), in graded modules \( \mathcal{M} = \gr Mod \), in simplicial modules \( \mathcal{M} = sMod \), and we consider hom-object bifunctors with values in these base categories. We rely on the definition of internal hom-objects of dg-modules, graded modules and simplicial modules in §5.4. We basically define our hom-objects of (non-unitary) Λ-sequences by an end, over the category of (non-empty) finite ordinals and injective maps, of these hom-objects associated to our base categories. We also consider the basic case of Λ-sequences in the category of plain modules \( \mathcal{M} = \Mod \), but we will just record that, in this setting, our hom-objects reduce to the standard morphism sets of the category of (non-unitary) Λ-sequences, with the obvious module structure inherited from the morphism sets of the base category \( \Mor_{\Mod}(-, -) \).

We also examine general homotopical properties of the hom-objects of (non-unitary) Λ-sequences in this section. We use the model category of simplicial modules, the model category of dg-modules, and the model structure associated to Λ-sequences in these base categories in order to formulate our results (we give a brief survey of our conventions on these model structures in the next paragraph). We prove that the hom-objects of (non-unitary) Λ-sequences satisfies the same homotopy invariance properties, with respect to the weak-equivalences, the cofibrant objects, and the fibrant objects of the category of Λ-sequences, as the hom-objects
of the base category (see Proposition 5.4.4). We also check that the Eilenberg-Zilber equivalence of the hom-objects of simplicial modules (see Proposition 5.4.6) extends to the \( \Lambda \)-sequence setting.

We examine the definition of hom-objects on the category of symmetric sequences in parallel to our study of the hom-objects of \( \Lambda \)-sequences. We also review the definition of the cofree \( \Lambda \)-sequence associated to a symmetric sequence. We prove that a hom-object of \( \Lambda \)-sequences whose target is equipped with such a cofree structure reduces to a hom-object in the category of symmetric sequences. We use this result to establish the Künneth isomorphism formulas for the hom-objects of \( \Lambda \)-sequences which we use in our spectral sequence constructions.

We assume, all through this chapter, that the category of symmetric sequences is equipped with the (projective) model structure of §8.1, and the category of \( \Lambda \)-sequences is equipped with the (Reedy) model structure of §8.3. We give a brief reminder on our conventions regarding model categories before tackling the main subject of this appendix section.

3.3.1. Conventions on model categories. In this chapter, we mainly work in the base model category of chain graded dg-modules (§5.0.3), in the base model category of simplicial modules (§5.0.8), and in the base model category of all dg-modules (§5.4.1). We still mainly regard the category of all dg-modules as an auxiliary model category, which we use to express the homotopy invariance properties of our hom-objects in the differential graded setting, while our symmetric sequences (and \( \Lambda \)-sequences) are more naturally defined either in the category of chain graded dg-modules or in the category of simplicial modules.

We can however use the category embedding \( \iota : dg_{\ast} \Mod \hookrightarrow dg \Mod \) to transport any object of the category of chain graded dg-modules to the category of all dg-modules. Recall that this functor fits in a Quillen adjunction \( \iota : dg_{\ast} \Mod \rightleftarrows dg \Mod : \tau_{\ast} \), where \( \tau_{\ast} : dg \Mod \to dg_{\ast} \Mod \) denotes a truncation on the category of dg-modules (see §5.4.2). We therefore rather deal with objects defined in this category of dg-modules \( dg \Mod \) when we want to formulate the general homotopy invariance properties associated to our hom-objects in the differential graded setting.

We apply the general definitions of §8.1, and §8.3 to the category of dg-modules (respectively, simplicial modules) to get the definition of the model category of (non-unitary) symmetric sequences and (non-unitary) \( \Lambda \)-sequences in this base category \( \mathcal{M} = dg \Mod \) (respectively, \( s \Mod \)).

In both cases (symmetric sequences and \( \Lambda \)-sequences), we assume that the weak-equivalences are created arity-wise in the base category. The fibrations of the category of (non-unitary) symmetric sequences are created in the base category arity-wise too, while we use a notion of matching object to define the class of fibrations in the category of \( \Lambda \)-sequences (see §8.3.1).

We can still characterize the cofibrations associated with these model structures by the right-lifting-property with respect to the class of acyclic fibrations. We also have an explicit set of generating (acyclic) cofibrations in the category of symmetric sequences and in the category of \( \Lambda \)-sequences which we can use to get an effective definition of these cofibrations, in terms of (retracts of) morphisms equipped with a cell structure. On the other hand, we observed (in Theorem 8.3.20) that a morphism defines a cofibration in the category of \( \Lambda \)-sequences if and only if this morphism defines a cofibration in the category of symmetric sequences, after forgetting about
the action of the extra operators of the category $\Lambda$. We give brief recollections on
the definition of the generating (acyclic) cofibrations of symmetric sequences and
$\Lambda$-sequences in the course of our verifications.

3.3.2. The general definition of hom-objects on $I$-sequences, $I = \Lambda, \Sigma$. To de-
dfine our hom-objects in a general setting, we use that a $\Lambda$-sequence consists, by
definition, of a (contravariant) diagram over the category $\Lambda$ with the finite ordinals
as objects $\underline{n} = \{1 < \cdots < n\}, n \in \mathbb{N}$, and the injective maps as morphisms. Recall
that a symmetric sequence has a similar interpretation as a diagram over the sub-
category $\Sigma \subset \Lambda$ with the same objects as the category $\Lambda$, but where we only retain
the bijective maps (equivalent to permutations) in our morphism sets. Simply, we
rather deal with left-actions when we use the symmetric sequence language, while
we rather consider contravariant diagram structures, and hence right-actions, when
we regard the category $\Sigma$ as a subcategory of the category $\Lambda$.

Let $I = \Sigma, \Lambda$ be any of these categories. Let $IS_{eq} = \Sigma_{eq}, \Lambda_{eq}$ denote the
corresponding category of $I$-diagrams in a fixed base category $M$. To keep our
conventions, we also use the expression of an $I$-sequence to refer to these particular
diagram structures, and we set $M(\underline{n}) = M(n)$ for the components of an object of
these categories $M \in IS_{eq}$. We assume that $M$ is (a symmetric monoidal category)
edwarded with an internal hom-object bifunctor $\text{Hom}_M(\cdot, \cdot) : M^{op} \times M \to M$. We
generally define the hom-object $\text{Hom}_{M_{IS_{eq}}}(M, N)$ associated to $I$-sequences $M, N \in
M_{IS_{eq}}$ by the end:

$$
\text{Hom}_{M_{IS_{eq}}}(M, N) = \int_{\underline{n} \in \mathcal{I}} \text{Hom}_M(M(\underline{n}), N(n)),
$$

where we consider the hom-objects $\text{Hom}_M(M(\underline{n}), N(n))$ associated to the objects
$M(\underline{n})$ and $N(n)$ in the base category $M$, for $n \in \mathbb{N}$.

We explain soon that this end splits and that we retrieve the standard definition
of equivariant hom-objects over the symmetric groups when we take $I = \Sigma$. We
therefore mostly consider the case of the category $I = \Lambda$ when we use the end
representation. But the statements which we formulate in this framework are valid
for both $I = \Sigma, \Lambda$. We therefore consider these cases together $I = \Sigma, \Lambda$ when we
examine the application of general definitions.

In what follows, we always restrict ourselves to the case where our $I$-sequences
are non-unitary, so that we only consider ordinals $\underline{n} = \{1 < \cdots < n\}$ such that
$n > 0$. In principle, we have to form our end over the full sub-category $I_{>0}$ generated
by these non-empty ordinals $\underline{n} = \{1 < \cdots < n\}, n > 0$, when we work in the non-
unitary setting. But in what follows, we only apply this definition in the context of
a category of modules, where a non-unitary $\Lambda$-sequence $M$ can be identified with a
$\Lambda$-sequence satisfying $M(0) = 0$. Furthermore, we get the same result if we perform
our end over the sub-category $I_{>0}$, or if we consider the extension of the non-unitary
$\Lambda$-sequences by zero to form our end over the whole category $I$. We therefore do
not mark the restriction to the sub-category $I_{>0}$ in our end formula.

In our spectral sequence constructions, we also consider the case of hom-objects
on connected $I$-sequences. We similarly use that, in the context of a category of
modules, we can identify these structures with $I$-sequences satisfying $M(0) =
M(1) = 0$, and there is again no need to change our end formula to define hom-
objects on this sub-category of the category of $I$-sequences.
3.3.3. Remarks: categorical properties of hom-objects. The internal hom-object bifunctors $\text{Hom}_M(-,-)$ which we consider in general satisfy the same distribution properties with respect to limits and colimits as the morphism set bifunctors $\text{Mor}_M(-,-)$ of our base categories $M$ (see §I.0.14 and the introduction of §5.4). We deduce from general interchange formulas between limits and ends that our hom-object bifunctor on $\Lambda$-sequences $\text{Hom}_M \mathcal{J}_{\mathcal{Seq}} (-,-) : \mathcal{M} \mathcal{J}_{\mathcal{Seq}}^{\text{op}} \times \mathcal{M} \mathcal{J}_{\mathcal{Seq}} \to \mathcal{M}$ inherits the same distribution properties with respect to limits and colimits as the morphism set bifunctors.

In most cases, the internal hom-object bifunctors $\text{Hom}_M(-,-)$ which we associate to our base categories are obtained by adjunction from the tensor product operation $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ providing $\mathcal{M}$ with the structure of a symmetric monoidal category (see §I.0.14). In this setting, we may see that our hom-object bifunctor on the category of (non-unitary) $\Lambda$-sequences is characterized by an adjunction relation of the form

$$\text{Mor}_{\mathcal{M} \mathcal{J}_{\mathcal{Seq}}} (K \otimes M, N) = \text{Mor}_M(K, \text{Hom}_{\mathcal{M} \mathcal{J}_{\mathcal{Seq}}} (M, N)),$$

for any $K \in \mathcal{M}$, and $M, N \in \mathcal{M} \mathcal{J}_{\mathcal{Seq}}$, where $K \otimes M \in \mathcal{M} \mathcal{J}_{\mathcal{Seq}}$ denotes the obvious (non-unitary) $\Lambda$-sequence such that:

$$\text{(2)} \quad (K \otimes M)(r) = K \otimes M(r),$$

for any arity $r > 0$. In general, we directly work with the hom-objects $\text{Hom}_M(-,-)$ and we therefore do not consider this external tensor product operation (2) further in this book, though we may relate some of our subsequent constructions to axiomatic properties of such structures (see for instance [76, §I.1 and §II.3]).

3.3.4. The explicit definition of homomorphisms of $\Lambda$-sequences, $\mathcal{J} = \Lambda, \Sigma$. We use the expression of a homomorphism of (non-unitary) $\Lambda$-sequences to refer to the elements of our hom-objects $f \in \text{Hom}_{\mathcal{M} \mathcal{J}_{\mathcal{Seq}}} (M, N)$ (whenever the notion of an element make sense). By definition of an end, such a homomorphism $f \in \text{Hom}_{\mathcal{M} \mathcal{J}_{\mathcal{Seq}}} (M, N)$ consists of a collection of homomorphisms $f \in \text{Hom}_M(M(n), N(n))$ of the objects $M(n)$ and $N(n)$ in the base category, for $n > 0$, such that we have the relation $u^* f = f u^*$ in $\text{Hom}_M(M(n), N(m))$, for every map $u : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$ in the category $\mathcal{J} = \Sigma, \Lambda$. We then consider the morphism $u^*$ given by the action of this map $u \in \text{Mor}_\mathcal{J}(m, n)$ on the objects $M, N \in \mathcal{J}_{\mathcal{Seq}}$. In principle, we also have to take the image of this morphism under the bi-functor $\text{Hom}_M(-,-)$ when we form our relation, but in our examples of base categories, where the morphisms are identified with a subset of the homomorphisms, this image is given by the obvious composition operation, and we may more basically define a homomorphism of (non-unitary) $\Lambda$-sequences $f \in \text{Hom}_{\mathcal{M} \mathcal{J}_{\mathcal{Seq}}} (M, N)$ as a collection of homomorphisms $f \in \text{Hom}_M(M(n), N(n))$, $n > 0$, that intertwine the morphisms of the $\Lambda$-diagram structure attached to our objects $M, N \in \mathcal{J}_{\mathcal{Seq}}$.

To give a first example, in the context of plain modules $\mathcal{M} = \text{Mod}$, we have $\text{Hom}_{\text{Mod}}(-,-) = \text{Mor}_{\text{Mod}}(-,-)$, and we retrieve the definition of the morphism set $\text{Hom}_{\text{Mod} \mathcal{J}_{\mathcal{Seq}}} (-,-) = \text{Mor}_{\text{Mod} \mathcal{J}_{\mathcal{Seq}}} (-,-)$ in our end construction (1). We examine...
3.3.5. Hom-objects of symmetric sequences. In the case of the category \( J = \Sigma \), we easily see that our end \( \prod_{r>0} \text{Hom}_\Sigma(M(r), N(r)) \) splits as a direct product:

\[
\text{Hom}_\Sigma \Sigma_{eq}(M, N) = \prod_{r>0} \text{Hom}_\Sigma(M(r), N(r)),
\]

for every pair of (non-unitary) symmetric sequences \( M, N \in \Sigma_{eq>0} \) in our base category \( \mathcal{M} \), and where \( \text{Hom}_\Sigma(M(r), N(r)) \) denotes the hom-object of \( \Sigma_r \)-equivariant homomorphisms from \( M(r) \) to \( N(r) \) in the category \( \mathcal{M} \). We can define these equivariant hom-objects by the same end-formula as the hom-object of symmetric sequences. We just restrict the range of our end to the subcategory of the category \( \Sigma \) which has \( f = \{1 < \cdots < r\} \) as unique object and the symmetric group \( \Sigma_r \) as morphism set, for any \( r > 0 \).

We see that, whenever the notion of an element makes sense in our hom-objects, a \( \Sigma_r \)-equivariant homomorphism \( f \in \text{Hom}_\Sigma(M(r), N(r)) \) explicitly consists of a homomorphism in our base category \( \mathcal{M} \) which satisfies the equivariance relation \( f \sigma = \sigma f \) with respect to the action of permutations \( \sigma \in \Sigma_r \) on our objects \( M(r), N(r) \in \mathcal{M} \). We obviously have \( f \sigma = \sigma f \iff \sigma f \sigma^{-1} = f \), so that we have an identity:

\[
\text{Hom}_\Sigma(M(r), N(r)) = \text{Hom}_\Sigma(M(r), N(r))^{\Sigma_r},
\]

for each \( r > 0 \), where we consider the invariants of the hom-object \( \text{Hom}_\Sigma(M(r), N(r)) \) under the adjoint action of the symmetric group \( \sigma(f) = \sigma f \sigma^{-1} \), for \( \sigma \in \Sigma_r \). This relation actually holds in a general abstract setting.

In the case \( \mathcal{M} = \text{Mod} \), we have \( \text{Hom}_\Sigma(M(r), N(r)) = \text{Mor}_\Sigma(M(r), N(r)) \), and we just retrieve the usual notion of a morphism of \( \Sigma_r \)-modules in our definition of an equivariant homomorphism of \( \Sigma_r \)-objects.

3.3.6. Recollections on the definition of a cofree \( \Lambda \)-sequence over a symmetric sequence. We introduced the notion of a cofree \( \Lambda \)-sequence over a symmetric sequence in our study of the model of \( E_n \)-operads, where we proved that the Drinfeld-Kohno Lie algebra operad \( p_n \), as well as the higher dimensional versions of this operad \( p_n \), have such a cofree structure (see \( \S \S \)). We basically say that a non-unitary \( \Lambda \)-sequence \( N \in \Lambda \Sigma_{eq>0} \) is cofreely generated by a symmetric sequence \( S N \in \Sigma_{eq>0} \) when we have an end-formula:

\[
N(n) = \int_{\sigma \in \Sigma} S N(r)^{\text{Mor}_\Lambda(\sigma, \varnothing)},
\]

for all \( n > 0 \). We still assume that we consider an extension of our objects by zero \( N(0) = S N(0) = 0 \), which makes sense in any of the base categories considered in this section, when we perform this end over the whole category \( \Sigma \).

Recall that we set \( X^S = \prod_{X \in S} X \), for any object \( X \in \mathcal{M} \) in a category equipped with finite limits \( \mathcal{M} \), and any set \( S \in \text{Set} \). In the module context \( \mathcal{M} = \text{Mod} \), and in the context of the categories \( \mathcal{M} = \text{dg} \text{Mod}, \text{gr} \text{Mod}, \text{sMod} \) similarly, we can also identify the object \( S N(r)^{\text{Mor}_\Lambda(\varnothing, \varnothing)} \) occurring in our end with the module of set-theoretic maps \( \alpha : \text{Mor}_\Lambda(\varnothing, \varnothing) \to S N(r) \), for any \( r, n > 0 \). In our end, we just consider the module spanned by the maps \( \alpha : \text{Mor}_\Lambda(\varnothing, \varnothing) \to S N(r) \) which preserve the natural action of the symmetric groups \( \Sigma_r \) on our objects, for \( r > 0 \). We explicitly assume \( \alpha(u \sigma) = \sigma \alpha(u) \), for any \( \sigma \in \Sigma_r, u \in \text{Mor}_\Lambda(\varnothing, \varnothing) \). We also require that the \( \Lambda \)-sequence
structure of our object $N$ is yielded by the translation action of the category $\Lambda$ on the representable functors $\text{Mor}\Lambda(-,n)$, $n > 0$.

We have the following adjunction relation:

**Proposition 3.3.7.** If $N$ has the structure of a cofree $\Lambda$-sequence over a symmetric sequence $S_N$, then we have a natural isomorphism:

$$\text{Hom}_{\Lambda\text{-seq}}(M, N) \simeq \text{Hom}_{\Sigma\text{-seq}}(M, S_N),$$

for any $M \in \Lambda\text{-seq}$, where, on the right-hand side, we consider the hom-object on the underlying symmetric sequence of this $\Lambda$-sequence $M$.

**Proof.** We have a morphism of symmetric sequences $N \to S N$ defined by the canonical projection

$$N(n) = \int_{r \in \Sigma} S N(r)^{\text{Mor}\Lambda(r,n)} \to S N(n)$$
on the summands associated to the identity morphisms $id \in \text{Mor}\Lambda(n,n)$ in our ends §3.3.6(1). We basically consider the map $\text{Hom}_{\Lambda\text{-seq}}(M, N) \to \text{Hom}_{\Sigma\text{-seq}}(M, S N)$ induced by this symmetric sequence morphism $N \to S N$, to get a map from $\text{Hom}_{\Lambda\text{-seq}}(M, N)$ to $\text{Hom}_{\Sigma\text{-seq}}(M, S N)$. We then use standard end and limit manipulations to check that this map defines an isomorphism. (The definition of a cofree $\Lambda$-sequence which we give in §3.3.6 is actually a particular case of a general construction of the right-adjoint of a restriction functor on diagrams $i^*: \mathcal{C}^{\beta} \to \mathcal{C}^{\beta'}$ associated to a morphism of small categories $i: \beta \to \beta'$.) □

In our applications, we often deal with hom-objects of $\Lambda$-sequences whose target is cofreely generated by a symmetric sequence. We therefore use the result of this proposition to get a reduced representation of such hom-objects.

**Proposition 3.3.8.** If $N$ has the structure of a cofree $\Lambda$-sequence over a symmetric sequence $S N$, and we assume that $S N$ forms a fibrant object in the model category of (non-unitary) symmetric sequences §8.1.1, then $N$ forms a fibrant object in the (Reedy) model category of $\Lambda$-sequence §8.3.4.

Recall that the fibrations of the category of symmetric sequences are created arity-wise in the base category (see §8.1.1), and that every object of the category of dg-modules (respectively, simplicial modules) is automatically fibrant (see §5.4.1). The fibration assumption of this proposition on the object $S N$ is therefore trivially satisfied when we work in any one of the main base model categories of this chapter $\mathcal{M} = dg\text{-Mod}, s\text{-Mod}$.

**Proof.** We use the decomposition $\Lambda = \Lambda^+ \Sigma$ of the category $\Lambda$ (see §1.3.2.3) to get that the end, in our definition of a cofree object over a symmetric sequence §3.3.6(1), admits a reduced expression of the form:

$$N(n) = \int_{r \in \Sigma} S N(r)^{\text{Mor}\Lambda(r,n)} = \prod_{0 < r} S N(r)^{\text{Mor}\Lambda^+(r,n)},$$

in any arity $n > 0$, where we consider cartesian products over the morphism sets of the category $\Lambda^+_0 \subset \Lambda_{>0}$. We moreover easily check, by using the reduced definition of the matching objects of $\Lambda$-sequences in Proposition 8.3.2, that we have a formula:

$$M(N)(n) = \prod_{0 < r < n} S N(r)^{\text{Mor}\Lambda^+(r,n)},$$
for any arity \( n > 0 \), and the matching map \( \mu : N(n) \to M(N)(n) \) associated to our cofree \( \Lambda \)-sequence \( N \) is given by the projection of our object onto the factors of this reduced cartesian product. We therefore readily obtain that this morphism defines a fibration if the symmetric sequence \( N \) is fibrant arity-wise, and this result completes the verification of our proposition. \( \Box \)

We now revisit the definition of the hom-objects associated to the main instances of base categories \( M = d\text{g}\text{-}\text{Mod}, \text{gr}\text{-}\text{Mod}, s\text{Mod} \) considered in this chapter.

3.3.9. Hom-objects in the \( \text{dg}\text{-}\text{module and graded module framework}. \) In the case \( M = d\text{g}\text{-}\text{Mod} \), we apply our end-formula §3.3.2(1) to the internal hom-bifunctor of the category of \( \text{dg}\text{-}\text{modules} \) \( \text{Hom}_{d\text{g}\text{-}\text{Mod}}(\_, \_) \) such as defined in §5.4.3. We also consider the restriction of this bifunctor to the category of chain-graded \( \text{dg}\text{-}\text{modules} \) \( d\text{g}_s\text{-}\text{Mod} \subset d\text{g}\text{-}\text{Mod} \) when we work in this framework \( M = d\text{g}_s\text{-}\text{Mod} \). We accordingly assume that the hom-object which we assign to any pair of (non-unitary) \( \mathcal{I} \)-sequences of \( \text{dg}\text{-}\text{modules} \) \( M, N \in d\text{g}\mathcal{I}\text{Seq}_{\geq 0}, \mathcal{I} = \Lambda, \Sigma, \) is a \( \text{dg}\text{-}\text{module} 

\[
\text{Hom}_{d\text{g}\mathcal{I}\text{Seq}}(M, N) \in d\text{g}\text{-}\text{Mod}
\]

with components in all degrees \( * \in \mathbb{Z} \) (in all cases). If necessary, then we just use the truncation functor \( \tau_* : d\text{g}\text{-}\text{Mod} \to d\text{g}_s\text{-}\text{Mod} \), right-adjoint to the category embedding \( \iota : d\text{g}_s\text{-}\text{Mod} \to d\text{g}\text{-}\text{Mod} \) (see §5.4.2), in order to get a chain graded \( \text{dg}\text{-}\text{module} \) from this hom-object.

Recall that a homomorphism of (chain graded) \( \text{dg}\text{-}\text{modules} \) of (lower) degree \( d \) is a module morphism \( f : C \to D \) which raises (lower) degrees by \( d \). Thus, according to our definition, a homomorphism of degree \( d \) in the category of (non-unitary) \( \mathcal{I} \)-sequences in (chain graded) \( \text{dg}\text{-}\text{modules} \) consists of a collection of module morphisms \( f : M(n) \to N(n) \) which raise degrees by \( d \) and intertwine the action of the morphisms of the category \( \mathcal{I} = \Lambda, \Sigma \) on our objects \( M, N \in d\text{g}\mathcal{I}\text{Seq} \). The differential of this homomorphism is also defined term-wise by the commutator formula \( \delta(f) = \delta f - \pm f \delta \), where we consider the internal differentials of the \( \text{dg}\text{-}\text{modules} \) \( M(n) \) and \( N(n) \), for every arity \( n > 0 \).

In the case \( M = \text{gr}\text{-}\text{Mod} \), we similarly assume that the hom-object which we assign to (non-unitary) \( \mathcal{I} \)-sequences of graded modules \( M, N \in \text{gr}\mathcal{I}\text{Seq}_{> 0} \) is a graded module

\[
\text{Hom}_{\text{gr}\mathcal{I}\text{Seq}}(M, N) \in \text{gr}\text{-}\text{Mod}
\]

with components in all degrees \( * \in \mathbb{Z} \) (in all cases). The elements of this graded module have the same explicit definition as in the \( \text{dg}\text{-}\text{module} \) setting (we just forget about the differential). We accordingly have an identity

\[
\text{Hom}_{d\text{g}\mathcal{I}\text{Seq}}(M, N)_\delta = \text{Hom}_{\text{gr}\mathcal{I}\text{Seq}}(M, N)_\delta,
\]

for all (non-unitary) \( \mathcal{I} \)-sequences in the category of (chain-graded) \( \text{dg}\text{-}\text{modules} \) \( M, N \in d\text{g}_s\mathcal{I}\text{Seq}_{> 0} \), when we apply the natural forgetful functor \( (\_)_\delta : d\text{g}\text{-}\text{Mod} \to \text{gr}\text{-}\text{Mod} \) to our objects.

In the case of symmetric sequences \( \mathcal{I} = \Sigma \), we just retrieve the usual equivariance constraints in our definition of a homomorphism. We can also use the observations of §3.3.5 to get an expression of our hom-objects in terms of modules of invariant elements.

We checked in §5.4 that the internal hom-object bifunctor on the category of \( \text{dg}\text{-}\text{modules} \) satisfies the same homotopy invariance properties as the function
space bifunctor of a simplicial model category (see Proposition 5.4.4). We have the following analogue of this statement for our hom-object bifunctor on the category of \( J \)-sequences in dg-modules:

**Proposition 3.3.10.** Let \( J = \Sigma, \Lambda \).

(a) If \( M \) is a fibrant object in the model category of (non-unitary) \( J \)-sequences, then the functor \( \text{Hom}_{dg\,J\,\text{Seq}}(-, M) \) carries any weak-equivalence of cofibrant (non-unitary) \( J \)-sequences \( f : K \sim \rightarrow L \) to a weak-equivalence in the category of dg-modules:

\[
    f^* : \text{Hom}_{dg\,J\,\text{Seq}}(L, M) \sim \rightarrow \text{Hom}_{dg\,J\,\text{Seq}}(K, M).
\]

(b) If \( K \) is a cofibrant object in the model category of (non-unitary) \( J \)-sequences, then the functor \( \text{Hom}_{dg\,J\,\text{Seq}}(K, -) \) carries any weak-equivalence of fibrant (non-unitary) \( J \)-sequences \( f : M \sim \rightarrow N \) to a weak-equivalence in the category of dg-modules:

\[
    f_* : \text{Hom}_{dg\,J\,\text{Seq}}(M, K) \sim \rightarrow \text{Hom}_{dg\,J\,\text{Seq}}(N, K).
\]

Recall that all objects of the model category of (non-unitary) \( J \)-sequences, \( J = \Sigma, \Lambda \), are cofibrant if the ground ring is a field of characteristic zero (as we generally assume throughout this chapter). The cofibration requirement of this proposition is therefore void in this context. In the case \( J = \Sigma \), we obtain that all objects of our category \( dg\,J\,\text{Seq} = dg\,\Sigma\,\text{Seq} \) are fibrant. We can therefore forget about the fibration assumption in this context.

**Proof (outline).** In this proof, we use the short notation \( \text{Hom}_{J\,\text{Seq}}(-, -) = \text{Hom}_{dg\,J\,\text{Seq}}(-, -) \) for the hom-objects on the category of \( J \)-sequences in dg-modules, and also write \( \text{Hom}(-, -) = \text{Hom}_{dg\,\text{Mod}}(-, -) \) for the internal hom-object which we associate to this base category. We rely on the proof of the homotopy invariance of hom-objects of dg-modules in Proposition 5.4.4. We use the same general argument line, and the crux of our proof still lies in the verification that our hom-object bifunctor \( \text{Hom}_{-,-} : dg\,J\,\text{Seq}^{op} \times dg\,J\,\text{Seq} \rightarrow dg\,\text{Mod} \) satisfies an analogue of the pullback-corner property of function spaces (see §2.1.6).

To be explicit, we consider a cofibration in the category of \( J \)-sequences \( i : K \rightarrow L \), a fibration \( p : M \rightarrow N \), and the pullback-corner morphism

\[
(1) \quad \text{Hom}_{J\,\text{Seq}}(L, M) \overset{(i^*, p_*)}{\rightarrow} \text{Hom}_{J\,\text{Seq}}(K, M) \times_{\text{Hom}_{J\,\text{Seq}}(K, N)} \text{Hom}_{J\,\text{Seq}}(L, N),
\]

which we obtain by filling out the diagram

\[
(2)
\]

\[
\text{Hom}_{J\,\text{Seq}}(L, M) \quad \overset{(i^*, p_*)}{\rightarrow} \quad \text{Hom}_{J\,\text{Seq}}(K, M) \times_{\text{Hom}_{J\,\text{Seq}}(K, N)} \text{Hom}_{J\,\text{Seq}}(L, N) \quad \overset{p_*}{\rightarrow} \quad \text{Hom}_{J\,\text{Seq}}(K, N).
\]

We claim that this morphism defines a fibration of dg-modules, and that this fibration is also acyclic when \( i \) or \( p \) is so. We already observed that our hom-objects on \( J \)-sequences satisfy the same distribution properties with respect to colimits and
limits as the morphism sets of our base categories (see §3.3.3). We can therefore apply usual patching arguments, and the standard stability properties of cofibrations and fibrations under categorical operations in model categories (see §1.1), to reduce the verification of our pullback-corner property to the case where \( i \) is a generating (acyclic) cofibration of the category of \( \mathcal{I} \)-sequences.

In the case \( \mathcal{I} = \Sigma \), we accordingly consider morphisms of the form \( i \otimes \Sigma F' : C \otimes \Sigma F' \to D \otimes \Sigma F' \), where \( i : C \to D \) is an (acyclic) cofibration of the category of dg-modules, and the tensor product \( - \otimes \Sigma F' \), \( r > 0 \), refers to a free object functor from dg-modules to symmetric sequences (see §8.1.2). We easily check that, under the adjunction relation of Proposition 8.1.5, our pullback-corner morphism (1) reduces in this case to the pullback-corner morphism

\[
\text{Hom}(D, M(r)) \xrightarrow{(i^*, (p, \mu))} \text{Hom}(C, M(r)) \times_{\text{Hom}(C, N(r))} \text{Hom}(D, N(r)),
\]

of the internal hom-objects of dg-modules, where we consider the arity \( r \) component \( p : M(r) \to N(r) \) of our symmetric morphism \( p : M \to N \). We can therefore deduce our claim from the classical pullback-corner property of the internal hom-object of dg-modules, which we use in the proof of Proposition 5.4.4.

In the case \( \mathcal{I} = \Lambda \), we deal with morphisms \( (i_*, \lambda_*) : C \otimes \Lambda F' \otimes_{C \otimes \Lambda F'} D \otimes \partial \Lambda F' \to D \otimes \Lambda F' \) such that \( i : C \to D \) is an (acyclic) cofibration of the category of dg-modules, and \( (i_*, \lambda_*) \) refers to a pushout-corner morphism associated to a free object functor from dg-modules to \( \Lambda \)-sequences \( - \otimes \Lambda F' \) together with a boundary map \( \lambda_* : - \otimes \partial \Lambda F' \to - \otimes \Lambda F' \), \( r > 0 \) (see §§8.3.6-8.3.10). We easily check that, under the adjunction relation of Proposition 8.3.9, our pullback-corner morphism (1) reduces in this case to the pullback-corner morphism

\[
\text{Hom}(D, M(r)) \xrightarrow{(i^*, (p, \mu))} \text{Hom}(C, M(r)) \times_{\text{Hom}(C, S(r))} \text{Hom}(D, S(r)),
\]

where we set \( S(r) = \mathcal{H}(M)(r) \times_{\mathcal{H}(N)(r)} N(r) \) and we consider the \( r \)th relative matching morphism \( (p, \mu) : M(r) \to \mathcal{H}(M)(r) \times_{\mathcal{H}(N)(r)} N(r) \) of our fibration \( p : M \to N \). We can therefore deduce our claim from the classical pullback-corner property of the internal hom-object of dg-modules again, and from the assertion that this relative matching morphism \( (p_*, \mu_*) \) defines a fibration (respectively, an acyclic fibration) by definition (respectively, by Proposition 8.3.5) if \( p \) is so in the Reedy model category of \( \Lambda \)-sequences.

We finish the proof of this proposition by the same argument line as in Proposition 5.4.4. Namely, we apply our pullback-corner property to a trivial map \( p : M \to 0 \) to get that the functor \( \text{Hom}_{\mathcal{S}eq}(-, M) \) carries (acyclic) cofibrations to (acyclic) fibrations when \( M \) is fibrant, and we use the Brown Lemma (Lemma 1.2.1) to get the homotopy invariance property of assertion (a). The argument is symmetrical in the case of assertion (b). \( \square \)

3.3.11. Hom-objects in the simplicial module framework. In the case of the category \( \mathcal{M} = s\mathcal{M}od \), we apply our definition of the hom-object on the category of \( \mathcal{I} \)-sequences in §3.3.2(1) to the internal hom-bifunctor of the category of simplicial modules \( \text{Hom}_{s\mathcal{M}od}(-, -) \), such as defined in §5.4.5. We accordingly associate a simplicial module:

\[
\text{Hom}_{s\mathcal{S}eq}(M, N) \in s\mathcal{M}od
\]

to any pair of (non-unitary) \( \mathcal{I} \)-sequences of simplicial modules \( M, N \in s\mathcal{S}eq_{>0}, \mathcal{I} = \Sigma, \Lambda \).
Recall that the simplicial hom-bifunctor of the category of simplicial modules is defined by the identity \( \text{Hom}_{s\text{Mod}}(K, L) \otimes \text{Mor}_{s\text{Mod}}(K, L) \cdot \), for any \( K, L \in s\text{Mod} \), where \( L^\Delta^* \) denotes the canonical simplicial framing of the object \( L \) in the category \( s\text{Mod} \), which is basically defined, in dimension \( n \in \mathbb{N} \), by the module of set-theoretic simplicial maps \( \sigma : \Delta^n \to L \) together with the module structure inherited from the simplicial module \( L \in s\text{Mod} \). We also have an obvious identity:

\[
\text{Hom}_{s\text{Seq}}(M, N) = \text{Mor}_{s\text{Seq}}(M, N^\Delta^*),
\]

where \( N^\Delta^* \) denotes the simplicial object of the category of \( J \)-sequences in simplicial modules such that \( N^\Delta^*(r) = N(r)^\Delta^* \) for any arity \( r > 0 \). We do not use this interpretation of our construction in this book, but we may observe that the collection \( N^\Delta^* \) defines a simplicial framing of the object \( N \in s\text{Seq} \) in the model category of \( J \)-sequences (without any further assumption in the case \( J = \Sigma \), provided that \( N \) is fibrant in the case \( J = \Lambda \)). The simplicial module \( \text{Hom}_{s\text{Seq}}(M, N) \) therefore represents a simplicial function space bifunctor on the model category of \( J \)-sequences in simplicial modules in the sense of §3.2.11.

We still have the following comparison result:

**Proposition 3.3.12.** Let \( J = \Sigma, \Lambda \). We have a canonical weak-equivalence of dg-modules:

\[
\tau_* \text{Hom}_{s\text{Seq}}(M, N) \otimes \text{Hom}_{s\text{Seq}}(N, (N)),
\]

induced by the comparison map of Proposition 5.4.6, and which relates the normalized chain complex of the simplicial module of homomorphisms of (non-unitary) \( J \)-sequences in simplicial modules \( M, N \in s\text{Seq}_{>0} \) to the (non-negative degree truncation of the) dg-module of homomorphisms associated to the normalized chain complexes of these (non-unitary) \( J \)-sequences \( N_*(M), N_*(N) \in dg\text{Seq}_{>0} \).

**Proof.** We use the functoriality of our comparison map in Proposition 5.4.6 to check that this map passes to our end §3.3.2(1) well. We similarly check that the right inverse of this comparison map in the proof of Proposition 5.4.6, as well as the simplicial homotopy making this map a weak-equivalence, induce well-defined maps on our hom-objects of \( J \)-sequences. We therefore get that our comparison map of hom-objects of \( J \)-sequences defines a weak-equivalence well, as claimed in our statement, and this result holds without any assumption on the objects \( M, N \in s\text{Seq}_{>0} \), and for both diagrammatic shapes \( J = \Sigma, \Lambda \) as well.

### 3.3.13. Künneth formulas.

We also have a natural transformation at the homology level:

\[
(\cdot)_* : H_* \text{Hom}_{s\text{Seq}}(M, N) \to H_* \text{gr}_{s\text{Seq}}(H_*(M), H_*(N)),
\]

which is induced by the Künneth map of §5.4.7, and relates the homology of the dg-module of homomorphisms of (non-unitary) \( J \)-sequences in dg-modules \( M, N \in dg\text{Seq}_{>0} \), \( J = \Sigma, \Lambda \), to the graded module of homomorphisms on the homology of these (non-unitary) \( J \)-sequences \( H_*(M), H_*(N) \in gr\text{Seq}_{>0} \). We just use the functoriality of our construction in §5.4.7 to check that our Künneth map passes to the end §3.3.2(1) well. We can still compose this natural transformation with the homology isomorphism induced by the weak-equivalence of Proposition 3.3.12 to get an analogous Künneth map

\[
(\cdot)_* : H_* \text{Hom}_{s\text{Seq}}(M, N) \to H_* \text{gr}_{s\text{Seq}}(H_*(M), H_*(N)),
\]
for the homology of the simplicial module of homomorphisms of (non-unitary) I-sequences in simplicial modules \( M, N \in s I \text{Seq}_{>0}, J = \Sigma, \Lambda \).

In the context of symmetric sequences \( J = \Sigma \), we can use the representation of the hom-object \( \text{Hom}_M \Sigma \text{Seq}(M, N) \) in terms of invariant modules (see §3.3.5), to establish that our hom-object satisfies a Künneth isomorphism formula. We explained in §5.4.7 that the Künneth map associated to the hom-object of dg-modules is an iso when the ground ring is a field. We also already used (in the particular case of symmetric algebras) that the homology preserves coinvariant modules when the ground ring is a field of characteristic 0 (see §6.1.8), and we may check, by the same argument, that the homology preserves invariant modules as well when we are in this situation. We therefore readily conclude that our Künneth map on the homology of the hom-object of (non-unitary) symmetric sequences in dg-modules \( M, N \in dg \Sigma \text{Seq}_{>0} \) defines an isomorphism

\[
\text{H}^* \text{Hom}_{dg \Sigma \text{Seq}}(M, N) \cong \text{Hom}_{gr \Sigma \text{Seq}}(\text{H}_*(M), \text{H}_*(N))
\]

when the ground ring is a field of characteristic 0, and we get the same result when we consider the hom-object of (non-unitary) symmetric sequences in simplicial modules \( M, N \in s \Sigma \text{Seq}_{>0} \).

In the context of \( \Lambda \)-sequences \( J = \Lambda \), we use the adjunction relation of Proposition 3.3.7 to get that the same result holds

\[
\text{H}_* \text{Hom}_{dg \Lambda \text{Seq}}(M, N) \cong \text{Hom}_{gr \Lambda \text{Seq}}(\text{H}_*(M), \text{H}_*(N)),
\]

for any \( M \in dg \Lambda \text{Seq}_{>0} \), when \( N \) has the structure of a cofree \( \Lambda \)-sequence over a symmetric sequence \( SN \) (and, still, under the assumption that the ground ring is a field of characteristic 0). We simply use the reduced representation of the end §3.3.6(1) in the proof of Proposition 3.3.8 to establish that the homology \( \text{H}_*(N) \) inherits the same cofree structure in the category of \( \Lambda \)-sequences in graded modules as our object \( N \in dg \Lambda \text{Seq}_{>0} \). We have the same Künneth isomorphism theorem for the hom-object of (non-unitary) symmetric sequences in simplicial modules.
CHAPTER 4

Applications of the Koszul Duality of Operads

We now aim to compute the cotriple cohomology modules $H^*_{gr \Lambda OP}(H_*(P), H_*(N))$ which, according to the results of the previous chapter, determine the second page of the homotopy spectral sequences of §2.2.4.

We prove in a first step that the cochain complex of the cotriple cohomology $N^*_{Der_{gr \Lambda OP}(Res \cdot (R), M)}$, where we set $R = H_*(P)$ and $M = H_*(N)$, admits a reduction into a quasi-isomorphic complex which we form by considering a subcomplex of maximal simplicial chains inside the cotriple resolution $Res \cdot (R)$. This reduced complex is identified with a complex of derivations $Der_{dg \Lambda OP}(Res^* (R), M)$ where we take a resolution of the operad $R \in gr \Lambda OP$ within the category of augmented connected non-unitary $\Lambda$-operads in dg-modules $Res^* (R) \in dg \Lambda OP_{a1}/Com$ instead of the simplicial resolution given by the cotriple construction. We more specifically consider the resolution functor $Res^*: R \mapsto Bc_B(R)$ which arises from the bar duality of operads and of which we explain the definition in §C.2. We explicitly have $Res^* (R) = B^c B(R)$, for every $R \in gr \Lambda OP_{a1}/Com$, where we consider the operadic bar construction functor $B: R \mapsto B(R)$, from operads to cooperads, and the operadic cobar construction functor $B^c: C \mapsto B(C)$, which goes the other way round, from cooperads to operads.

In §C.3, we explain that, in good cases, the homology of the bar construction $B(R)$ reduces to certain components of maximal weight, and that these components define a cooperad in graded modules $K(R) \subset B(R)$, referred to as the Koszul dual of the operad $R$. We moreover observe that the object $Res^* (R) = B^c K(R)$, which we obtain by applying the cobar construction to this cooperad $K(R)$, defines a minimal model of our operad $R$. We say that the operad $R$ is Koszul when these statements hold.

We prove that, in the case of a Koszul operad, the cochain complex of the cotriple cohomology $N^*_{Der_{gr \Lambda OP}(Res^* (R), M)}$ admits a second reduction which we obtain by replacing the cobar-bar resolution $Res^* (R) = B^c B(R)$ by the Koszul resolution $Res^* (R) = B^c K(R)$ in the outcome of our first reduction process. We give, besides, an explicit description of these reduced complexes of derivations $Der_{dg \Lambda OP}(Res^* (R), M)$ which we associate to our differential graded resolutions $Res^* (R) = B^c K(R), B^c B(R)$. In the next chapter, we will compute the complex of derivations on the Koszul resolution of Gerstenhaber operad in order to get our result about the homotopy spectral sequence of the space of functions on $E_2$-operads.

We mostly consider the case of operads in graded modules when we address the applications to our spectral sequences. We worked in this setting in the previous chapter, when we explain the definition of the cotriple cohomology of operads. However, the definition of the complex of the cotriple cohomology admits an obvious extension to operads in dg-modules, and this setting is more natural when we tackle
our reductions to derivations over our differential graded resolutions of operads. Therefore, we are going to assume for a while that \( R \) is a connected augmented non-unitary \( \Lambda \)-operad in \( \text{dg-modules} \ \text{dg} \Lambda \mathcal{O}_\mathcal{P}_{\emptyset 1} / \text{Com} \). We similarly consider that the abelian \( R \)-bimodules \( \mathcal{N} \), which define the coefficients of our complexes of derivation, lie in the base category of \( \text{dg-modules} \). We also pass to the \( \text{dg-module} \) setting when we form the cotriple resolution of our operad \( \text{Res}_\bullet (R) \) and when we take the module of derivations associated to this object. We then get a cosimplicial object in the category of \( \text{dg-modules} \) \( \text{Der}_{\text{dg} \Lambda \mathcal{O}_\mathcal{P}} (\text{Res}_\bullet (R), \mathcal{N}) \in \mathcal{C}_{\text{dg}} \mathcal{M} \odot \) of which we take the conormalized complex (see \( \S 3.0.3 \)) to eventually obtain a cochain complex in the category of \( \text{dg-modules} \) \( \text{N}^* \text{Der}_{\text{dg} \Lambda \mathcal{O}_\mathcal{P}} (\text{Res}_\bullet (R), \mathcal{N}) \in \text{dg}^* \text{dg} \mathcal{M} \odot \).

We go back to the graded module setting in the second part of this chapter only, when we address the applications of our reduction methods to the spectral sequence of \( \S 2.2.4 \).

We now need to take care of homotopy invariance properties associated with our complexes. We mainly use the Reedy model category of \( \Lambda \)-sequences in \( \text{dg-modules} \), and the homotopy invariance of hom-objects on this model category in order to prove the validity of our reduction methods. We refer to \( \S 3.3 \) for a brief reminder on the definition of this model structure on \( \Lambda \)-sequences, and for a full account of the homotopy invariance statements used in this chapter. We do not, but we could go further into this study by using a counterpart, in the category of \( \text{dg-modules} \), of the Reedy model category of non-unitary \( \Lambda \)-operads in simplicial sets (see \( \S 8.4 \)). We can precisely establish that the \( \text{dg-module} \) of operadic derivations \( \text{Der}_{\text{dg} \Lambda \mathcal{O}_\mathcal{P}} (Q, \mathcal{N}) \) satisfy homotopy invariance properties when we restrict ourselves to operads \( Q \) which form a cofibrant object in the (Reedy) model category of augmented non-unitary \( \Lambda \)-operads in \( \text{dg-modules} \), and when the abelian bimodules of coefficients of our modules of derivations \( \mathcal{N} \) form a fibrant object in the (Reedy) model category of \( \Lambda \)-sequences. We leave the verification of such results to interested readers.

We may similarly see that the cobar-bar resolution \( \text{Res}_\bullet (R) = B^* B(R) \) (and the Koszul resolution as well whenever this construction make sense) forms a cofibrant object in the model category of augmented non-unitary \( \Lambda \)-operads in \( \text{dg-modules} \), at least when the operad \( R \) is cofibrant as a symmetric sequence. (Recall that we also say that \( R \) is \( \Sigma \)-cofibrant when this condition holds.) We do not need this observation in our work. Nonetheless, we implicitly use the \( \Sigma \)-cofibration property in the proofs of this chapter. The \( \Sigma \)-cofibration condition is just void in our setting.

To be more explicit, recall that we take the field of rational numbers \( k = \mathbb{Q} \) as ground ring all through this part. For simplicity, we still follow this convention in this chapter. However, our results remain valid without change as soon as we work over a field of characteristic zero. This assumption implies that any symmetric sequence in \( \text{dg-modules} \) is automatically cofibrant, and this is this property which ensures the validity of our constructions in general.

We devote the first section of the chapter (\( \S 4.1 \)) to the definition of our comparison map between the cotriple cohomology complex and the complex of derivations on the cobar-bar (respectively, Koszul) resolution of operads. We address the applications to homotopy spectral sequence in a second section (\( \S 4.2 \)). We give an explicit definition of bi-graded structures on the complex of derivations of the cobar-bar (respectively, Koszul) resolution of our operads, and we check that this bi-graded structure corresponds the homotopy spectral sequence bi-grading. We
give by the way an explicit description of the derivation complex associated to the Koszul construction which we use in our subsequent computations.

4.1. The application of modules of derivations on the cobar-bar and Koszul constructions

We rely on the explicit description of the cotriple resolution, in terms of composite tree-wise tensors (see §B.1), in order to define our comparison map between the cotriple cohomology complex and the complex of derivations on the cobar-bar and the Koszul resolution of operads. We give brief recollections on this tree-wise tensor construction first. We also briefly review the definition of the cobar-bar resolution of operads. We explain the definition of the comparison map from the cotriple cohomology complex to the complex of derivations on the cobar-bar construction afterwards.

We examine the reduction of our complex of derivations to the module of derivations on the Koszul resolution in the second part of this section. We still give short recollections on the Koszul construction before tackling this second reduction process.

4.1.1. Reminder on the tree-wise description of the cotriple resolution for operads.

We basically check, in §B.1, that the cotriple resolution admits an expansion of the form:

\[ \text{Res}_n(R)(r) = \operatorname{colim} \Theta \left( \Sigma \bar{R} \right), \]

in any simplicial dimension \( n \in \mathbb{N} \), and for any arity \( r > 0 \), where we take a colimit over chains of morphisms of reduced \( r \)-trees, modulo a natural correspondence yielded by the action of \( r \)-tree isomorphisms on such chains and tree-wise tensors. We represent an element of the operad \( \text{Res}_n(R) \), \( n \in \mathbb{N} \), by a pair \((\lambda, \pi)\), where \( \lambda \) denotes such a chain of tree morphisms \( \overrightarrow{T}_0 \leftarrow \overrightarrow{T}_1 \leftarrow \ldots \leftarrow \overrightarrow{T}_n \) and we assume \( \pi \in \Theta \left( \Sigma \bar{R} \right) \).

The face morphism \( d_i : \text{Res}_n(R) \to \text{Res}_{n-1}(R) \), \( i = 0, \ldots, n \), is given by the omission of the term \( \overrightarrow{T}_i \), \( i = 0, \ldots, n \), in any such chain (combined with the tree-wise composition operation \( \lambda : \Theta \left( \Sigma \bar{R} \right) \to \Theta \left( \Sigma \bar{R} \right) \) in the case \( i = n \)), while the degeneracy \( s_j : \text{Res}_n(R) \to \text{Res}_{n+1}(R) \), \( j = 0, \ldots, n \), is given by the repetition of the term \( \overrightarrow{T}_j \). We also trivially have \( \text{Res}_0(R) = \Theta(R) \) and the augmentation \( \epsilon : \text{Res}_0(R) \to R \), which we associate to the definition of this resolution, is given by the tree-wise composition product term-wise. We refer to §B.1 for further details on this construction.

4.1.2. Reminder on the cobar-bar resolution. In §§C.2.2-C.2.3, we define the bar construction of a connected non-unitary operad in dg-modules \( R \in dgOP_{\emptyset} \) as a quasi-cofree cooperad of the form:

\[ B(R) = (\Theta^c(\Sigma R), \partial), \]

where we consider the cofree cooperad on a suspension of the augmentation ideal of our operad \( R \in Seq_{>1} \). The twisting coderivation \( \partial : \Theta^c(\Sigma R) \to \Theta^c(\Sigma R) \), which we associate to this object is also determined by the composition products of our operad. The cobar construction is defined dually, by a quasi-free operad of the form:

\[ B^c(C) = (\Theta(\Sigma^{-1} \bar{C}), \partial), \]
and where the twisting derivation $\partial : \Theta((\Sigma^{-1} \mathcal{C}) \to \Theta((\Sigma^{-1} \mathcal{C})$ is determined by the coproducts of the given cooperad $\mathcal{C}$.

The cobar-bar construction

$$R_{\ast}(R) = B^c B(R),$$

which we associate to an operad $R \in dg \circ p_{\varnothing 1}$, is also endowed with an augmentation $\epsilon : B^c B(R) \to R$ which is defined, on the generating symmetric sequence $\Sigma^{-1} B(R) \subset \Theta((\Sigma^{-1} B(R))$ of this quasi-free operad $B^c B(R) = (\Theta((\Sigma^{-1} B(R)), \partial)$, by the natural projection map $\Sigma^{-1} \Theta(\Sigma R) \to \Sigma^{-1} \Sigma R = R$ of the cofree cooperad $\mathcal{B}(R)$ = $\Theta(\Sigma R)$. We check in $\S C.2$ (see more precisely Theorem C.2.9) that this augmentation $\epsilon : B^c B(R) \to R$ defines a weak-equivalence, for any operad $R \in dg \circ p_{\varnothing 1}$. We moreover check that the cobar-bar construction $R_{\ast}(R) = B^c B(R)$ inherits an augmented non-unitary (connected) $\Lambda$-operad structure when $R$ is so. Therefore get that $R_{\ast}(R) = B^c B(R)$ represents a quasi-free resolution of the object $R$ in this category of operads $dg \Lambda \circ p_{\varnothing 1} / \text{Com}$. We also trivially have $B^c B(R) \in dg_{\ast} \Lambda \circ p_{\varnothing 1} / \text{Com}$ when our operad is chain graded $R \in dg_{\ast} \Lambda \circ p_{\varnothing 1} / \text{Com}$.

4.1.3. Recollections on the comparison map from the cobar-bar construction to the cotriple resolution. In Proposition C.2.16, we prove that the cobar-bar construction is equipped with a canonical weak-equivalence towards the normalized chain complex of the cotriple resolution:

$$R_{\ast}(R) \xrightarrow{\sim} N_{\ast} \text{Res}_{\ast}(R)$$

To get this result, we use that the cobar-bar construction admits an expansion of the form:

$$B^c B(R)(r) = \colim_{\Sigma^{-1}} \Lambda (\mathcal{T} / \Sigma) \otimes \Theta_{\mathcal{T}}(R),$$

for any arity $r > 0$, where the colimit runs over morphisms of reduced $r$-trees, modulo the natural correspondence yielded by the action of $r$-tree isomorphisms again. The extra factor $\Lambda (\mathcal{T} / \Sigma)$ encodes a sign (and a degree shift) which reflects the position of suspension and desuspension symbols in the tree-wise expression of the symmetric sequence $B^c B(R)_{\varnothing} = \Theta((\Sigma^{-1} \Theta(\Sigma R))$ underlying the operad $B^c B(R)$.

The comparison morphism (1) is defined term-wise by the mapping which carries the term associated to any morphism $\mathcal{S} \leftarrow \mathcal{T}$ in this expansion (2) to the terms associated to the maximal non-degenerate chains of tree morphisms $\mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_n$ satisfying $\mathcal{S} = \mathcal{T}_0$ and $\mathcal{T} = \mathcal{T}_n$ in the expansion of the cotriple resolution $\S 4.1.1(1)$. We take a sum, over all decompositions of this form, of a morphism which is given by the identity on the tree-wise tensor product $\Theta_{\mathcal{T}}(R)$ and carries the factor $\Lambda (\mathcal{T} / \Sigma)$ to a sign which we determine from the orientation of the simplex $\mathcal{T}_0 \leftarrow \cdots \leftarrow \mathcal{T}_n$ (see the explanations of Proposition C.2.16 for details).

The operad $R_{\ast}(R) = B^c B(R)$ is equipped with an extra grading and forms, like the normalized complex of the cotriple resolution, a chain complex in the category of dg-modules. In our expansion (2), we integrate this extra grading in the factor $\Lambda (\mathcal{T} / \Sigma)$, which forms a graded module concentrated in degree card $V(\mathcal{T}) - \text{card} V(\mathcal{S})$, where $V(-)$ denotes the vertex sets of our trees. The comparison map (1) clearly preserves the extra grading associated to our objects.

Recall that the object $N_{\ast} R_{\ast}(R)$ inherits the structure of an operad in dg-modules, and our comparison map (1) is actually a morphism of operads (see Proposition C.2.16). In the case where $R$ is equipped with the structure of an augmented connected non-unitary $\Lambda$-operad, we still get that our comparison map...
preserves the $\Lambda$-diagram structure on our objects, and hence, defines a morphism in the category of augmented connected non-unitary $\Lambda$-operads in dg-modules (see Proposition C.2.19).

If we pass to derivation modules, then we obtain the following comparison result:

**Theorem 4.1.4.** We have a natural weak-equivalence

$$N^\ast \text{Der}_{dg\Lambda \circ p}(\text{Res}_*(R), N) \xrightarrow{\sim} \text{Der}_{dg\Lambda \circ p}(B^c B(R), N),$$

for any augmented connected non-unitary $\Lambda$-operad $R \in \text{dg} \Lambda \撬 \text{Com}$ (under the convention that we take $k = \mathbb{Q}$ as ground ring), and any abelian $R$-bimodule $N$ which forms a fibrant object in the model category of $\Lambda$-sequences (see §3.3.1).

This theorem holds without change as long as we take a characteristic zero field as ground ring $k$. If we work in a more general setting, then we have to assume that the operad $R$ is cofibrant as a symmetric sequence, and our proof is actually valid as soon as this convention is fulfilled.

In applications, we mainly consider the case where the abelian bimodule $N$ forms, as a $\Lambda$-sequence, a cofree object over a symmetric sequence $S N$ (see §3.3.6). Simply recall that the fibration condition of our statement is automatically satisfied in this context (see Proposition 3.3.8). We give more details on the proof of our theorem in this cofree case. We only give short indications on the extension of our arguments in the general fibrant $\Lambda$-sequence case.

**Explanations and Proof.** We explain the definition of this comparison map on derivation modules in a first step. We then use a spectral sequence argument to reduce our verification to the case of a comparison map between hom-objects on derivation modules in a first step. We then use a spectral sequence argument in the general fibrant $\Lambda$-sequence case.

**First Step. The definition of the comparison map.** We define our map by a composite:

1. $$N^\ast \text{Der}_{dg\Lambda \circ p}(\text{Res}_*(R), N) \xrightarrow{\sim} \text{Der}_{dg\Lambda \circ p}(N_\ast \text{Res}_*(R), N)$$
2. $$\Rightarrow \text{Der}_{dg\Lambda \circ p}(B^c B(R), N),$$

where (1) arises from the definition of the normalized complex $N_\ast \text{Res}_*(R)$ as a quotient object in the category of $\Lambda$-sequences, and the map (2) is the morphism induced by the comparison map $B^c B(R) \to N_\ast \text{Res}_*(R)$ of §4.1.3 on derivation modules.

To be more explicit, we consider an element of our conormalized complex $\theta \in N^\ast \text{Der}_{dg\Lambda \circ p}(\text{Res}_*(R), N)$, where we fix a dimension $n \in \mathbb{N}$. This element is a derivation $\theta \in \text{Der}_{dg\Lambda \circ p}(\text{Res}_n(R), N)$ satisfying $\theta s_j = 0$ for every degeneracy map $s_j$ on the simplicial object $\text{Res}_*(R)$. We consider the obvious homomorphism $\theta : N_\ast(\text{Res}_*(R)) \to N$ induced by our derivation $\theta : \text{Res}_n(R) \to N$ on the quotient module $N_\ast(\text{Res}_*(R)) = \text{Res}_n(R)/s_0 \text{Res}_{n-1}(R) + \cdots + s_{n-1} \text{Res}_{n-1}(R)$, and which vanishes on the components of degree $* \neq n$ of the normalized complex. We easily check that this homomorphism defines a derivation $\theta \in \text{Der}_{dg\Lambda \circ p}(N_\ast \text{Res}_*(R), N)$.

(To perform this verification, we essentially use that the operad $N_\ast \text{Res}_*(R)$ acts on $N$ through the augmentation $\varepsilon : N_0 \text{Res}_*(R) \to R$ which the normalized complex $N_\ast \text{Res}_*(R)$ inherits from the simplicial object $\text{Res}_*(R)$, and which vanishes in degree $* \neq 0$ by construction.) We accordingly have a mapping (1). We may also
check that this mapping, which we basically deduce from the identities of hom-objects $N^n \text{Hom}_{dg \Lambda S eq}(\text{Res}_n(R), N) = \text{Hom}_{dg \Lambda S eq}(N_n \text{Res}_n(R), N)$, for any $n \in \mathbb{N}$, is bijective (as marked in our construction), but we do not really use this observation.

Second Step. The spectral sequence for the module of derivations on the cotriple resolution. By definition of the cotriple resolution, we have an identity of operads $\text{Res}_n(R) = \mathcal{O}(\nabla^n(R))$ where we set:

$$(3) \quad \nabla^n(R) = \circledast \circ \cdots \circ \circledast(R),$$

for any $n \in \mathbb{N}$. In the expansion of §4.1.1(1), we can identify this $\Lambda$-sequence $\nabla^n(R)$, $n \in \mathbb{N}$, with the sub-object of the free operad $\text{Res}_n(R) = \mathcal{O}(\nabla^n(R))$ consisting of terms associated to the chains of tree morphisms $\mathcal{T}_n \leftarrow \cdots \leftarrow \mathcal{T}_0$ such that $\mathcal{T}_0 = Y$. Recall that these objects $\nabla^n(R) \subset \mathcal{O}(\nabla^n(R))$, $n \in \mathbb{N}$, are preserved by the action of degeneracies $s_j$ on the cotriple resolution $\text{Res}_n(R)$, by the faces $d_i$ such that $i > 0$ as well, but not by the 0-th coface $d_0$ (see §B.1.8). We can however provide the collection of these objects $\mathcal{O}(\nabla(R))$ with a simplicial structure such that $d_0 = 0$ (see §C.2.12).

By Theorem 3.1.7, we have $\text{Der}_{dg \Lambda \circ \mathcal{O}p}(\text{Res}_n(R), N) = \text{Hom}_{dg \Lambda S eq}(\nabla^n(R), N)$ in any dimension $n \in \mathbb{N}$, and since the isomorphism that gives this relation (which we abusively regard as an identity) is natural, we moreover have an identity:

$$(4) \quad N^* \text{Der}_{dg \Lambda \circ \mathcal{O}p}(\text{Res}_n(R), N)_b = N^* \text{Hom}_{dg \Lambda S eq}(\nabla^0(R), N)_b$$

between the conormalized complex of the cotriple cohomology on the one-hand, and the conormalized complex of the cosimplicial object $\text{Hom}_{dg \Lambda S eq}(\nabla^0(R), N)$ on the other hand, at least, if we neglect differentials. We are more precisely going to see that the action on our cosimplicial modules of the cofaces $d_i$ such that $i > 0$ are preserved by this component-wise relation, but not the action of the 0-th coface $d_0$. We therefore add a $b$ lower-script in our formula, to mark the neglect of the conormalized complex differentials, for the moment. We also use, in what follows, that the cochain complex occurring in this reduction process (4) is identified with a hom-object:

$$(5) \quad N^* \text{Hom}_{dg \Lambda S eq}(\nabla^0(R), N) = \text{Hom}_{dg \Lambda S eq}(N_\bullet \mathcal{O}(\nabla(R), N)$$

which we associate to the normalized complex of the simplicial object $\mathcal{O}(\nabla(R))$ in the category of $\Lambda$-sequences in dg-modules.

The functoriality of the relation of Theorem 3.1.7 implies that the maps

$$(6) \quad N^{n-1} \text{Der}_{dg \Lambda \circ \mathcal{O}p}(\text{Res}_n(R), N) \xrightarrow{d_i} N^n \text{Der}_{dg \Lambda \circ \mathcal{O}p}(\text{Res}_n(R), N)$$

given by the composition with the face morphisms of the cotriple resolution for $i > 0$ correspond, under our relation (4-5), to the maps

$$(7) \quad \text{Hom}_{dg \Lambda S eq}(N_{n-1} \mathcal{O}(\nabla(R), N) \xrightarrow{(d_i)^*} \text{Hom}_{dg \Lambda S eq}(N_n \mathcal{O}(\nabla(R), N)$$

given by the composition with the restriction of these face operators to the generating object $\mathcal{O}(\nabla(R)) \subset \text{Res}_n(R)$. The idea is to define a filtration in order to isolate these terms $d_i^*$, $i > 0$ of the differential $\delta = \sum_{i=0}^{n}(-1)^i d_i$ in the conormalized complex of the cotriple cohomology $N^* \text{Der}_{dg \Lambda \circ \mathcal{O}p}(\text{Res}_n(R), N)$. In fact, we rather deal with a decomposition of our cochain complex into the limit of a tower of degree-wise surjections of dg-modules, which is equivalent to such a filtration (see §I.7.3.2). We go back to this correspondence later on, when we address the definition of a spectral sequence associated to this filtration.
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We first adapt the definition of the arity filtration of non-unitary $\Lambda$-operads in simplicial sets (in the proof of Theorem 8.4.12) to equip the cotriple resolution with a filtration

$$ l = \ar\underleftarrow{\leq}^d \Res_s(R) \subset \cdots \subset \ar\underleftarrow{\leq}^s \Res_s(R) \subset \cdots \subset \Res_s(R) $$

such that $\Res_s(R) = \colim_i \ar\underleftarrow{\leq}^i \Res_s(R)$. We define this arity filtration at the level of the generating $\Lambda$-sequence of our object $\tilde{\Theta}^\bullet(R)$. We explicitly set:

$$ \ar\underleftarrow{\leq}^s \tilde{\Theta}^\bullet(R)(r) = \begin{cases} \tilde{\Theta}^\bullet(R)(r), & \text{if } r \leq s, \\ 0, & \text{otherwise}, \end{cases} $$

for any $r > 0$, and we take:

$$ \ar\underleftarrow{\leq}^s \tilde{\Theta}^\bullet(R)_y = \Theta(\ar\underleftarrow{\leq}^s \tilde{\Theta}^\bullet(R)_y), $$

to define the term of level $s$ in the filtration of our operad $\Res_s(R) = \Theta(\tilde{\Theta}^\bullet(R))$. We just check that this sub-operad (10), which we define dimension-wise (after forgetting about the action of simplicial operators), is preserved by the action of the 0-face $d_0$ on the cotriple resolution, while we have faces $d_i, i > 0$, and degeneracies $s_j$, inherited from the generating object $\ar\underleftarrow{\leq}^s \tilde{\Theta}^\bullet(R) \subset \tilde{\Theta}^\bullet(R)$. We accordingly get that these sub-operads (8) inherit a simplicial structure and form a nested sequence of sub-objects of the cotriple resolution in the category of simplicial augmented non-unitary $\Lambda$-operads in dg-modules.

If we work modulo degeneracies, then we have the more precise relation:

$$ d_0(\ar\underleftarrow{\leq}^s \tilde{\Theta}^\bullet(R)) \subset \Theta(\ar\underleftarrow{\leq}^{s-1} \tilde{\Theta}^\bullet(R)) \pmod{\text{degeneracies}} $$

since:

- the non-degenerate elements of the object $\ar\underleftarrow{\leq}^s \tilde{\Theta}^\bullet(R)(r) = \tilde{\Theta}^\bullet(R)(r)$, for any given arity $r \leq s$, are represented by tree-wise tensors $\pi \in \Theta\underline{\underline{1}}_n(R)$ associated to chains of reduced $r$-trees $Y = \underline{1}_0 \leftarrow \underline{1}_1 \leftarrow \cdots \leftarrow \underline{1}_n$ satisfying $\underline{1}_i \neq \underline{1}_{i+1}$ for any $i < n$ in the expansion of §4.1.1(1),
- the face $d_0$ carries such a tree-wise tensor to a decomposable element of the free operad $\Theta(\tilde{\Theta}^{n-1}(R))$, with the tree $\underline{S} = \underline{1}_1 \in \text{Tree}(r)$ as underlying composition scheme,
- and the relation $\#I_y \geq 2$ for the input set of any vertex $v \in V(\underline{S})$ implies that the arities of the factors of this composite, given by the cardinal of the input sets $\underline{S}_v$, $v \in V(\underline{S})$, are strictly less than the initially given arity $r \leq s$ (we then use the non-degeneracy assumption $\underline{S} = \underline{1}_1 \neq Y$).

We now consider the tower of derivation modules:

$$ N^* \Der_{dg} \Lambda \circ p(\Res_s(R), N) \to \cdots $$

$$ \cdots \to N^* \Der_{dg} \Lambda \circ p(\ar\underleftarrow{\leq}^s \Res_s(R), N) \to \cdots $$

$$ \cdots \to N^* \Der_{dg} \Lambda \circ p(\ar\underleftarrow{\leq}^1 \Res_s(R), N) = 0, $$

which we deduce from the filtration of the operad $\Res_s(R)$. We clearly have the relation:

$$ N^* \Der_{dg} \Lambda \circ p(\Res_s(R), N) = \lim_s N^* \Der_{dg} \Lambda \circ p(\ar\underleftarrow{\leq}^s \Res_s(R), N) $$
when we take the limit of this tower. We still have a level-wise analogue of relations (4-5):

\[(14) \quad N^s \text{Der}_{dg} \Lambda \Omega p(\text{ar}^\leq_{\leq s} \text{Res}_*(R), N)_b = \text{Hom}_{dg} \Lambda S \text{Seq}(N, \text{ar}^\leq_{\leq s} \tilde{\Theta}^*(R), N)_b,\]

for any \(s \geq 1\). The morphisms of our tower (12) moreover fit in commutative diagrams:

\[(15) \quad \begin{array}{c}
\text{Hom}_{dg} \Lambda S \text{Seq}(N, \text{ar}^\leq_{\leq s} \tilde{\Theta}^*(R), N)_b \\
\downarrow \\
N^s \text{Der}_{dg} \Lambda \Omega p(\text{ar}^\leq_{\leq s} \text{Res}_*(R), N)_b \\
\end{array}
\rightarrow \begin{array}{c}
N^s \text{Der}_{dg} \Lambda \Omega p(\text{ar}^\leq_{\leq s-1} \text{Res}_*(R), N)_b \\
\downarrow \\
\text{Hom}_{dg} \Lambda S \text{Seq}(N, \text{ar}^\leq_{\leq s-1} \tilde{\Theta}^*(R), N)_b
\end{array}
\]

by functoriality of this correspondence, where the lower horizontal mapping is the obvious restriction map, induced by the embedding \(\text{ar}^\leq_{\leq s-1} \tilde{\Theta}^*(R) \hookrightarrow \text{ar}^\leq_{\leq s} \tilde{\Theta}^*(R)\), on our hom-objects. Recall simply that the maps, which give the vertical arrows in the above diagram (15) and our level-wise identity (14), are defined by the restriction of derivations on free operads to generating objects.

We now consider the simplifying case of an abelian \(R\)-bimodule \(N\) which, as a \(\Lambda\)-sequence, admits a cofree structure over a symmetric sequence \(S N\) (see §3.3.6). We then get (by Proposition 3.3.7) that the dg-hom of \(\Lambda\)-sequences of our formula (14) admits a further reduction:

\[(16) \quad \text{Hom}_{dg} \Lambda S \text{Seq}(N, \text{ar}^\leq_{\leq s} \tilde{\Theta}^*(R), N) = \text{Hom}_{dg} \Sigma S \text{Seq}(N, \text{ar}^\leq_{\leq s} \tilde{\Theta}^*(R), S N).\]

We immediately deduce from this relation that the lower horizontal map in our diagram (15) defines a degree-wise surjection for each \(s > 1\). Then we trivially obtain that the same result holds when we pass to derivation modules.

We consider the kernels of these maps occurring in our tower of derivation modules:

\[(17) \quad F_s = \ker(N^s \text{Der}_{dg} \Lambda \Omega p(\text{ar}^\leq_{\leq s} \text{Res}_*(R), N) \rightarrow N^s \text{Der}_{dg} \Lambda \Omega p(\text{ar}^\leq_{\leq s-1} \text{Res}_*(R), N)).\]

The notation \(F_s\) is motivated by the analogy between these kernels and the fibers occurring in a tower of fibrations of simplicial sets, such as defined in §2.1.1. We neglect differentials in a first step. We then readily deduce from our relations (14-16) that these kernels satisfy the relation:

\[(18) \quad F_s = \text{Hom}_{dg} \Sigma N \text{Mod}(N, \tilde{\Theta}^*(R)(s), S N(s)).\]

We claim that this identity (18) determines \(F_s\) not only as a graded object (after forgetting about differentials), but as a dg-module yet. To check this observation, we basically prove that the coface operator \(d^0\) of our cosimplicial modules in (17) vanishes on \(F_s\). This vanishing result essentially follows from the inclusion relation (11). Indeed, if we have a homomorphism \(f \in \text{Hom}_{dg} \Lambda S \text{Seq}(N, \text{ar}^\leq_{\leq s} \tilde{\Theta}^*(R), N) = N^s \text{Hom}_{dg} \Lambda S \text{Seq}(\text{ar}^\leq_{\leq s} \tilde{\Theta}^*(R), N)\) such that \(f|_{\text{ar}^\leq_{\leq s-1} \tilde{\Theta}^*(R)} = 0\), then relation (11) implies that the corresponding derivation \(\theta_f \in N^s \text{Der}_{dg} \Lambda \Omega p(\text{ar}^\leq_{\leq s} \text{Res}_*(R), N)\), such as defined in the proof of Theorem 3.1.7, satisfies \(d^0(\theta_f) = \theta_f d_0 = 0\) on \(N, \text{ar}^\leq_{\leq s} \tilde{\Theta}^*(R)\), and this argument proves our claim. We therefore obtain that the differential of our conormalized complex reduces, on \(F_s\), to the differential inherited from the normalized complex of the symmetric sequence \(\tilde{\Theta}^*(R) \subset \text{Res}_*(R)\), and our relation (18) therefore holds in the category of dg-modules.
We may readily adapt the definition of the spectral sequence of a tower of fibrations in §2.1 to the tower of dg-module morphisms which we are now considering. We get a spectral sequence abutting to the homology of our dg-module of derivations. We can, on the other hand, consider the filtration such that:

\[(19) \quad F_s N^s \text{Der}_{dg} \Lambda \circ p(\text{Res}_s(R), N) = \ker(N^s \text{Der}_{dg} \Lambda \circ p(\text{Res}_s(R), N) \to N^s \text{Der}_{dg} \Lambda \circ p(\text{ar}_{<s}^s \text{Res}_s(R), N)),\]

for any \(s \geq 1\), and our tower spectral sequence of dg-modules is actually equivalent to the spectral sequence which is classically associated to a filtration. The kernels (17) are identified with the sub-quotients \(E_s^0 = F_s / F_{s-1}\) of this filtration and the homology of these objects defines the first page of our spectral sequence. We therefore have a spectral sequence such that:

\[(20) \quad E_s^0 = H_s \text{Hom}_{dg} \Sigma, \text{Mod}(N, \bar{\delta}^s(R)(s), \text{S N}(s)) \Rightarrow H_s N^s \text{Der}_{dg} \Lambda \circ p(\text{Res}_s(R), N),\]

and of which first page is determined by the homology of a dg-hom on the normalized complex of the generating collection of the cotriple resolution \(N, \bar{\delta}^s(R)\).

In the general case of an abelian \(R\)-bimodule \(N\) which is just fibrant as a \(\Lambda\)-sequence (but not necessarily cofreely cogenerated by a symmetric sequence), we still get that the morphisms of our tower (12) are degree-wise surjections (equivalent to fibrations) in the category of dg-modules, and we still have a spectral sequence which we can deduce from this tower decomposition. We moreover have an identity:

\[(21) \quad F_s = \text{Hom}_{dg} \Sigma, \text{Mod}(N, \bar{\delta}^s(R)(s), \ker(N(s) \to \mathcal{M}(N)(s))),\]

generalizing the above relation (18), where we just consider the kernel of the matching maps \(\mu : N(s) \to \mathcal{M}(N)(s)\) instead of a cogenerating symmetric sequence. We accordingly have an extension of the above spectral sequence (20) to the case of general fibrant \(\Lambda\)-sequence. We just have to take \(\text{S N}(s) = \ker(\mu : N(s) \to \mathcal{M}(N)(s))\), for any \(s > 1\), in this general setting.

Third Step. The spectral sequence for the module of derivations on the cobar-bar resolution. We now define an analogue of the previous spectral sequence for the module of derivations on the cobar-bar resolution.

We have by definition \(B^c B(R)_s = \bar{\delta}(\Sigma^{-1} B(R)_s)\), when we forget about differentials, and we regard the desuspended bar construction \(\Sigma^{-1} B(R)\) as a generating object of the operad underlying \(\text{Res}_s(R) = B^c B(R)\). In the expansion of §4.1.3(2), we identify this generating symmetric sequence \(\Sigma^{-1} B(R)_s \subset \bar{\delta}(\Sigma^{-1} B(R)_s)\) with the sub-object of the cobar-bar construction consisting of terms associated to tree morphisms \(\pi \leftarrow \pi\) such that \(\pi = \pi\). Recall that the desuspended bar construction \(\Sigma^{-1} B(R)\) can also be identified with the indecomposable quotient of the cobar-bar resolution (see §C.2). By Theorem 3.1.7, we now get an identity of graded modules:

\[(22) \quad \text{Der}_{dg} \Lambda \circ p(B^c B(R), N)_s = \text{Hom}_{dg} \Lambda, \text{Seq}(\Sigma^{-1} B(R)_s, N_s).\]

We may see that this relation preserves the differential of the bar construction, but we need to add an extra twisting differential to our hom-object on the right hand side in order to keep track of the action of the differential of the cobar construction on the left hand side. We again define a filtration on our dg-module of derivations in order to isolate the part of the differential which comes from the bar construction. We follow the same plan as in the case of the dg-module of derivations on the cotriple resolution.
In a first step, we adapt the definition of the arity filtration of non-unitary 
Λ-operads to get a filtration of the cobar-bar construction
\[ I(23) \]
\[ \Lambda \text{-operads to get a filtration of the cobar-bar construction} \]
such that \( B^c B(R) = \text{colim}_s \ar_{\leq s}^c B^c B(R) \). We still define this arity filtration at the
level of the generating Λ-sequence \( \Sigma^{-1} \bar{B}(R) \) of our object. We explicitly set:
\[ \ar_{\leq s} \Sigma^{-1} \bar{B}(R)(r) = \begin{cases} 
\Sigma^{-1} \bar{B}(R)(r), & \text{if } r \leq s, \\
0, & \text{otherwise,} 
\end{cases} \]
for any \( r > 0 \), and we take:
\[ \ar_{\leq s}^c B^c B(R) = \mathcal{O}(\ar_{\leq s} \Sigma^{-1} \bar{B}(R)), \]
to define the term of level \( s \) in the filtration of our operad \( B^c B(R) = \mathcal{O}(\Sigma^{-1} \bar{B}(R)) \).
We just check that this sub-operad (25), which we form in the category of graded
modules (after forgetting about the action of differentials), is preserved by the
action of the twisting differential of the cobar construction, while the twisting
differential of the bar construction (and the internal differential of the operad
\( R \)) trivially gives a differential on the generating collection of our operad
\( \ar_{\leq s} \Sigma^{-1} \bar{B}(R) \subset \Sigma^{-1} \bar{B}(R) \) We accordingly get that these sub-operads (25)
inherit a natural differential graded structure and form a nested sequence of sub-
objects of the cobar-bar resolution in the category of augmented non-unitary Λ-
operads in dg-modules.

The twisting differential of the cobar construction actually satisfies the more
precise relation
\[ \partial(\ar_{\leq s} \Sigma^{-1} \bar{B}(R)) \subset \mathcal{O}(\ar_{\leq s-1} \Sigma^{-1} \bar{B}(R)), \]
because this map carries any element \( \pi \in \Sigma^{-1} \bar{B}(R)(r) \) to a composite element in
the free operad, and the relation \( B(R)(0) = \bar{B}(R)(1) = 0 \) implies that the factors
of this composite necessarily have an arity less than \( r \), and hence, less or equal to
\( s - 1 \) when we assume \( \pi \in \ar_{\leq s} \Sigma^{-1} \bar{B}(R)(r) \Rightarrow r \leq s. \)

We now consider the tower of derivation modules:
\[ \text{Der}_{dg} \Lambda \mathcal{O}_p(B^c B(R), N) \rightarrow \cdots \\
\cdots \rightarrow \text{Der}_{dg} \Lambda \mathcal{O}_p(\ar_{\leq s}^c B^c B(R), N) \rightarrow \cdots \\
\cdots \rightarrow \text{Der}_{dg} \Lambda \mathcal{O}_p(\ar_{\leq 1}^c B^c B(R), N) = 0, \]
which we deduce from our filtration of the operad \( B^c B(R) \). We again have a formal
identity:
\[ \text{Der}_{dg} \Lambda \mathcal{O}_p(B^c B(R), N) = \lim_s \text{Der}_{dg} \Lambda \mathcal{O}_p(\ar_{\leq s}^c B^c B(R), N) \]
when we take the limit of this tower, and we still have a level-wise analogue of our
reduction relation (22):
\[ \text{Der}_{dg} \Lambda \mathcal{O}_p(\ar_{\leq s}^c B^c B(R), N)_b = \text{Hom}_{dg} \Lambda \mathcal{O}_p(\ar_{\leq s} \Sigma^{-1} \bar{B}(R), N)_b, \]
for any \( s \geq 1 \). The morphisms of our tower (27) moreover fit in commutative diagrams:

\[
\begin{array}{ccc}
\text{Der}_{dg} \Lambda \circ p(\ar_{\leq s} \Sigma^{-1} \bar{B}(R), N) & \longrightarrow & \text{Der}_{dg} \Lambda \circ p(\ar_{\leq s-1} \Sigma^{-1} \bar{B}(R), N) \\
\text{Hom}_{dg} \Lambda S_{eq}(\ar_{\leq s} \Sigma^{-1} \bar{B}(R), N) & \longrightarrow & \text{Hom}_{dg} \Lambda S_{eq}(\ar_{\leq s-1} \Sigma^{-1} \bar{B}(R), N)
\end{array}
\]

where the lower horizontal mapping is the obvious restriction map, induced by the canonical embedding \( \ar_{\leq s-1} \Sigma^{-1} \bar{B}(R) \hookrightarrow \ar_{\leq s} \Sigma^{-1} \bar{B}(R) \), on our hom-objects.

We again consider the simplifying case of an abelian \( R \)-bimodule \( N \) which, as a \( \Lambda \)-sequence, admits a cofree structure over a symmetric sequence \( S N \). We still get, in this context, that the dg-hom of \( \Lambda \)-sequences of our formula (29) admits a further reduction:

\[
\text{Hom}_{dg} \Lambda S_{eq}(\ar_{\leq s} \Sigma^{-1} \bar{B}(R), N) = \text{Hom}_{dg} \Lambda S_{eq}(\ar_{\leq s} \Sigma^{-1} \bar{B}(R), S N).
\]

This relation also implies that the lower horizontal map of our diagram (30) defines a degree-wise surjection for each \( s > 1 \), and we trivially obtain that the same result holds then when we pass to derivation modules. We consider the kernels of these maps occurring in our tower of derivation modules:

\[
F_s = \ker(\text{Der}_{dg} \Lambda \circ p(\ar_{\leq s} \Sigma^{-1} \bar{B}(R), N) \rightarrow \text{Der}_{dg} \Lambda \circ p(\ar_{\leq s-1} \Sigma^{-1} \bar{B}(R), N)).
\]

We deduce from the above relations (29-31) that these kernels are equal to the following hom-objects:

\[
F_s = \text{Hom}_{dg} \Sigma_{\mathcal{M}_{od}}(\Sigma^{-1} \bar{B}(R)(s), S N(s)),
\]

and as a dg-module yet, because the filtration relation (26) implies that the twisting differential of the cobar construction vanishes over this kernel, while the isomorphism giving our identity (29) preserves the twisting differential of the bar construction (as well as the internal differentials of our objects).

We equip the dg-module of derivations on the cobar-bar resolution with the filtration such that:

\[
F_s = \text{Der}_{dg} \Lambda \circ p(\Sigma^{-1} \bar{B}(R), N)
\]

\[
= \ker(\text{Der}_{dg} \Lambda \circ p(\Sigma^{-1} \bar{B}(R), N) \rightarrow \text{Der}_{dg} \Lambda \circ p(\ar_{\leq s} \Sigma^{-1} \bar{B}(R), N)),
\]

for any \( s \geq 1 \), and we consider the spectral sequence associated to a filtration. The kernels (32) are identified with the sub-quotients \( E^0_s = F_s / F_{s-1} \) and the homology of these objects defines the first page of a spectral sequence abutting to the homology of our dg-module of derivations on the cobar-bar resolution. We more explicitly get:

\[
E^1_s = H_s \text{Hom}_{dg} \Sigma_{\mathcal{M}_{od}}(\Sigma^{-1} \bar{B}(R)(s), S N(s)) = H_s \text{Der}_{dg} \Lambda \circ p(\Sigma^{-1} \bar{B}(R), N).
\]

when we use relation (33) to compute this homology.

In the general case of an abelian \( R \)-bimodule \( N \) which is just fibrant as a \( \Lambda \)-module (but not necessarily cofreely cogenerated by a symmetric sequence), we still get that the morphisms of our tower (27) are degree-wise surjections (equivalent to fibrations) in the category of dg-modules, and we still have a spectral sequence, deduced from this tower decomposition, abutting to the homology of our dg-module of derivations. We moreover get an expression of the above form (35) for the first
page of this spectral sequence. We just take $sN(s) = \ker(\mu : N(s) \to \mathcal{H}(N)(s))$, for any $s > 1$, in this general setting.

**Fourth Step. The spectral sequence argument.** We immediately see that our comparison map $\mathbb{B}^c \mathbb{B}(R) \to \mathbb{N}, \text{Res}^*(R)$ in §4.1.3(1) preserves the filtration associated to our operads. We deduce from this observation that our comparison map on derivation modules preserve the filtrations which we define at this level, and which we use in our spectral sequence construction. This comparison map accordingly induces a morphism of spectral sequences:

$$E^r N^* \text{Der}_{dg} \Lambda \circ p(\text{Res}^*(R), N) \to E^r \text{Der}_{dg} \Lambda \circ p(\mathbb{B}^c \mathbb{B}(R), N),$$

which on the first page is identified with the morphism

$$H_* \text{Hom}_{dg} \Sigma, \text{S}_{eq}(\mathbb{N}, \mathcal{O}^*(R), S N) \to H_* \text{Hom}_{dg} \Sigma, \text{S}_{eq}(\Sigma^{-1} \mathbb{B}(R), S N)$$

induced by the restriction of our operadic comparison map §4.1.3(1) to the $\Lambda$-sequences occurring in these hom-objects:

$$\Sigma^{-1} \mathbb{B}(R) \to \mathbb{N}, \mathcal{O}^*(R).$$

We checked in Theorem C.2.17 (see also Proposition C.2.19) that this map defines a weak-equivalence of augmented $\Lambda$-sequences. We deduce from this statement and the homotopy invariance claims of Proposition 3.3.10 that our spectral sequence argument readily extends to the general case of an abelian $R$-bimodule $N$ which is just fibrant as a $\Lambda$-sequence. We essentially have to consider the generalization of the spectral sequences which we sketched in the course of this proof.

4.1.5. **Reminder on the Koszul construction associated to an operad.** Recall that the bar construction $\mathbb{B}(R) = (\mathcal{O}^c(\Sigma R), \partial)$ of an operad $R$ naturally forms a chain complex of dg-modules with a grading $\mathbb{B}(R) = \bigoplus_n \mathbb{B}_n(R)$ inherited from the weight decomposition of the cofree operad $\mathcal{O}^c(\Sigma R) = \bigoplus_n \mathcal{O}^c_n(\Sigma R)$ (see §C.3.2).

In §C.3.2, we observe that the components of the bar construction $\mathbb{B}_n(R)(r)$ vanish when $n \geq r$ and $R$ is a connected non-unitary operad. We moreover check that the weight graded objects $K(R)(r) \subset \mathbb{B}(R)(r)$ such that

$$K_{r-1}(R)(r) = \ker(\mathbb{B}_{r-1}(R)(r) \to \mathbb{B}_{r-2}(R)(r))$$

and $K_{n}(R)(r) = 0$ for $n \neq r-1$, form a sub-cooperad of the bar construction $\mathbb{B}(R)$. In our definition, we also regard each $K(R)(r)$ as a sub-complex of the bar construction $\mathbb{B}(R)(r)$ concentrated in degree $n = r - 1$, and equipped with a trivial twisting differential $\partial = 0$. The cooperad structure on $K(R)$ obviously preserves the complex grading inherited from $\mathbb{B}(R)$ so that $K(R)$ naturally forms a cooperad in the category of dg-modules equipped with a non-negative weight grading $M = gr_*(dg \text{Mod})$.

In §C.3 (see more precisely Theorem C.3.6), we also explain that the dual operad of this cooperad $K(R)^\vee$ has an explicit description by generators and relations which we can easily determine from the composition structure of the operad $R$. We go back
to the definition of this presentation in §5.1, where we determine the image of the
$n$-Gerstenhaber operad $R = \text{Gerst}_n$ under this cooperad construction $K : R \mapsto K(R)$.

We consider the quasi-free operad $B^c K(R) = (\mathfrak{B}([\Sigma^{-1} K(R)], \partial)$, which we obtain
by taking the image of this cooperad $K(R)$ under the cobar construction, and which
naturally forms a sub-object of the cobar-bar resolution:

\[ B^c K(R) \subset B^c B(R). \]

We also consider the augmentation

\[ \epsilon : B^c K(R) \to R \]

defined by the restriction of the canonical weak-equivalence of the cobar-bar reso-

lution $\epsilon : B^c B(R) \to R$ to this sub-operad $B^c K(R) \subset B^c B(R)$. We can identify this

augmentation map with the operad morphism $\epsilon = \phi(\kappa)$ induced by the homomor-

phism of symmetric sequences

\[ \kappa = \epsilon|_{[\Sigma^{-1} K(R)]} : [\Sigma^{-1} K(R)] \to R \]

given by the identity:

\[ [\Sigma^{-1} K(R)](2) = \Sigma^{-1} B(R)(2) = R(2) \]

in arity $r = 2$, and which vanishes in arity $r \neq 2$.

We now say that a connected non-unitary operad $R$ is Koszul when the canonical

embedding $K(R) \to B(R)$ forms a weak-equivalence (see §C.3.3). Then we have

a weak-equivalence at the level of the cobar construction $\iota : B^c K(R) \simeq \to B^c B(R)$,

and the operad $\text{Res}_*(R) = B^c K(R)$, equipped with the above augmentation, conse-

quently defines a resolution of our operad $R$. We also say that $K(R)$ is the Koszul
dual cooperad of the operad $R$, while we refer to this resolution $\text{Res}_*(R) = B^c K(R)$,

obtained by applying the cobar construction to the Koszul construction, as the

Koszul resolution.

In §C.3 (see more precisely Proposition C.3.5) we observe that $B^c K(R)$ inherits

the structure of an augmented connected non-unitary $\Lambda$-operad, and forms a sub-

object of the cobar-bar construction $B^c K(R)$ in this category $dg\Lambda\text{Op}_{\mathcal{O}(1)} / \text{Com}$, when

our operad does so $R \in dg\Lambda\text{Op}_{\mathcal{O}(1)} / \text{Com}$. If we assume in addition that our operad

$R$ is Koszul, then we conclude that the Koszul resolution $\text{Res}_*(R) = B^c K(R)$ defines a resolution of our operad within the category augmented connected non-unitary $\Lambda$-operads $dg\Lambda\text{Op}_{\mathcal{O}(1)} / \text{Com}$. We then have the following statement:

**Theorem 4.1.6.** Let $R$ be an augmented non-unitary connected $\Lambda$-operad in $\text{der}$-modules. If we assume that this operad is Koszul, then the Koszul duality weak-

equivalence $\iota : B^c K(R) \simeq \to B^c B(R)$ induces a weak-equivalence at the level of derivation

modules:

\[ \text{Der}_{dg\Lambda\text{Op}}(B^c B(R), N) \simeq \to \text{Der}_{dg\Lambda\text{Op}}(B^c K(R), N), \]

under the convention that we take $k = \mathbb{Q}$ as ground ring, and when we have an

abelian $R$-bimodule of coefficients $N$ which forms a fibrant object in the model cat-

egory of $\Lambda$-sequences.

**Proof.** We easily check that the definition of our spectral sequence in the third

step of the proof of Theorem 4.1.4 works in the same way when we replace the cobar-

bar construction $\text{Res}_*(R) = B^c B(R)$ by the Koszul resolution $\text{Res}_*(R) = B^c K(R)$. 
We therefore have a spectral sequence of the form:

\[ E_1^{s} = H_s \text{Hom}_dg \Sigma_\text{Mod}(\Sigma^{-1} \mathcal{C}(R)\langle s \rangle, S \mathcal{N}(s)) \Rightarrow H_* \text{Der}_{dg \text{ operad}} (B^e \mathcal{C}(R), N), \]

for any of the cooperads \( \mathcal{C}(R) = B(R), K(R) \) and the associated resolutions \( \text{Res}_* (R) = B^e \mathcal{C}(R) \) which we consider in our statement, while \( S \mathcal{N} \) denotes the symmetric sequence such that \( S \mathcal{N}(s) = \text{ker}(\mu : \mathcal{N}(s) \rightarrow \mathcal{N}(\mathcal{N}(s))) \), for any \( s > 0 \) (we refer to the proof of Theorem 4.1.4 for further explanations on this definition).

We also have a spectral sequence morphism:

\[ E^r \text{Der}_{dg \text{ operad}} (B^e B(R), N) \xrightarrow{E^r} E^r \text{Der}_{dg \text{ operad}} (B^e K(R), N), \]

yielded by our operad morphism \( \iota : B^e K(R) \xrightarrow{\sim} B^e B(R) \), and which abuts to the morphism induced by the comparison map of our theorem in homology. This spectral sequence morphism is given, on the first page, by the homology of the morphism of hom-objects

\[ \text{Hom}_dg \Sigma_\text{Mod}(\Sigma^{-1} \mathcal{B}(R)\langle s \rangle, S \mathcal{N}(s)) \xrightarrow{\iota^*} \text{Hom}_dg \Sigma_\text{Mod}(\Sigma^{-1} \mathcal{K}(R)\langle s \rangle, S \mathcal{N}(s)) \]

induced by the canonical embedding \( \iota : \mathcal{K}(R) \hookrightarrow \mathcal{B}(R) \). We use the assumption that this map defines a weak-equivalence \( \iota : \mathcal{K}(R) \xrightarrow{\sim} \mathcal{B}(R) \) (and the result of Proposition 3.3.10) to conclude that our spectral sequence morphism defines an iso from the first page on, and the claim of our theorem follows from standard spectral sequence comparison arguments again. \( \square \)

### 4.2. The structure and the applications of the Koszul derivation complex

The dg-module of derivations \( N_* \text{Der}_{dg \text{ operad}} (\mathcal{R}_*(R), M) \) in Theorem 4.1.4 is the total complex of a cochain complex in the category of dg-modules by definition. We aim to retrieve an analogous structure on the dg-module of derivations \( \text{Der}_{dg \text{ operad}} (\mathcal{R}_*(R), M) \) which we associate to the cobar-bar resolution \( \text{Res}_* (R) = B^e B(R) \), and similarly when we replace the cobar-bar resolution by the Koszul resolution \( \text{Res}_* (R) = B^e K(R) \). In a preliminary step, we check that the conormalized cochain complex \( N_* \text{Der}_{dg \text{ operad}} (\mathcal{R}_*(R), M) \) which we associate to the cotriple resolution \( \text{Res}_* (R) \) admits a reduced representation, as a twisted dg-module on the hom-object \( \text{Hom}_{dg \text{ operad}} (N_* \mathcal{B}^* (R), M) \) which we consider in the spectral sequence argument of Theorem 4.1.4.

We prove that our modules of derivations on the differential graded resolutions \( \text{Res}_* (R) = B^e B(R), B^e K(R) \) admit a similar reduced representation and we use this result to get our definition of a bi-complex structure on these modules of derivations. We obtain by the way that our comparison morphisms in Theorem 4.1.4 and Theorem 4.1.6 define morphisms of chain complexes in dg-modules. We use this observation and the result of the previous chapter to prove that the module of derivations on the Koszul resolution of the homology of an operad in simplicial sets \( R = H_*(P) \) inherits a bi-graded structure which corresponds to the natural bi-grading of the spectral sequence computing the homotopy of the function space \( \text{Map}_{dgo}(\langle \mathcal{R}_*(P), \mathcal{T}_*(N) \rangle) \) associated to the cotriple resolution of this operad \( P \in s\text{Set} \Lambda \text{operad} \).
4.2.1. The reduced form of the complex of derivations on the cotriple resolution.

In what follows, we write \( N^*_n(C_\bullet) \) for the collection of dg-modules \( N_n(C_\bullet), n \in \mathbb{N} \), which underlies the normalized complex of a simplicial object in dg-modules \( C_\bullet \in s dgMod \), but where we forget about the differential of the normalized complex construction and we only retain the component-wise differential \( \delta : N_n(C_\bullet) \to N_{n+1}(C_\bullet), n \in \mathbb{N} \), induced by the internal differential of our object \( C_\bullet \in s dgMod \). We adopt a similar convention \( N^*_n(C^*) \) for the collection of dg-modules \( N^*_{n}(C^*), n \in \mathbb{N} \), which underlies the conormalized complex of a cosimplicial object in dg-modules \( C^* \in c dgMod \).

In (the second step of the proof of) Theorem 4.1.4, we establish that the conormalized cochain complex of derivations on the cotriple resolution is isomorphic to a conormalized cochain complex of \( \Lambda \)-sequence homomorphisms degree-wise. This identity reads:

\[
N^*_n \text{Der}_{dg \Lambda \mathcal{O}_P}(\text{Res}_\bullet(R), N) = \text{Hom}_{dg \Lambda \text{Seq}}(N^*_n \tilde{\mathcal{O}}^*(R), N)
\]

when we use our \( b \) notation. This relation implies that our conormalized cochain complex of derivations is equivalent to a twisted dg-module of the form:

\[
N^*_n \text{Der}_{dg \Lambda \mathcal{O}_P}(\text{Res}_\bullet(R), N) = (\text{Hom}_{dg \Lambda \text{Seq}}(N^*_n \tilde{\mathcal{O}}^*(R), N), \partial),
\]

and where the twisting differential \( \partial \) has a component

\[
\text{Hom}_{dg \Lambda \text{Seq}}(N^*_s \tilde{\mathcal{O}}^*(R), N)_{k} \xrightarrow{\partial} \text{Hom}_{dg \Lambda \text{Seq}}(N^*_s \tilde{\mathcal{O}}^*(R), N)_{k-1}
\]

defined in each cochain degree \( s \in \mathbb{N} \) and each total degree \( k \in \mathbb{Z} \). We just transport the differential of the conormalized complex \( N^*_n \text{Der}_{dg \Lambda \mathcal{O}_P}(\text{Res}_\bullet(R), N) \) through our graded object identity (1) to determine the expression of this twisting differential (3). Recall that we deduce this relation (1) from the correspondence of Theorem 3.1.7, where we determine the module of derivations on a free operad (we refer to the proof of Theorem 4.1.4 for details on this application of Theorem 3.1.7). Let us observe that we can still decompose the twisting differential of our complex (3) in two parts \( \partial = \partial' + \partial'' \), where \( \partial' \) corresponds to the alternated sum \( \partial' = \sum_{i>0} (-1)^i d_i \) of the cofaces \( d_i \) such that \( i > 0 \) on the conormalized complex of derivations \( N^*_n \text{Der}_{dg \Lambda \mathcal{O}_P}(\text{Res}_\bullet(R), N) \), while \( \partial'' \) corresponds to the map induced by the 0-coface \( d_0 \). In the spectral sequence argument of Theorem 4.1.4, we actually isolate this first component \( \partial' \) of the twisting differential \( \partial = \partial' + \partial'' \), and we observe that this map \( \partial' \) is identified with the image of the differential of the normalized complex \( N_* \tilde{\mathcal{O}}^*(R) \) in the dg-hom \( \text{Hom}_{dg \Lambda \text{Seq}}(N_* \tilde{\mathcal{O}}^*(R), N) \). We may therefore rewrite our complex (2) as a twisted object of the form:

\[
(\text{Hom}_{dg \Lambda \text{Seq}}(N^*_s \tilde{\mathcal{O}}^*(R), N), \partial) = (\text{Hom}_{dg \Lambda \text{Seq}}(N_* \tilde{\mathcal{O}}^*(R), N), \partial'),
\]

to retrieve the dg-module \( \text{Hom}_{dg \Lambda \text{Seq}}(N_* \tilde{\mathcal{O}}^*(R), N) \), which we consider in the proof of Theorem 4.1.4. When we use this representation, we integrate the twisting differential \( \partial' \) in the internal differential of this dg-module \( \text{Hom}_{dg \Lambda \text{Seq}}(N_* \tilde{\mathcal{O}}^*(R), N) \).

The second component \( \partial'' \) of the twisting differential \( \partial = \partial' + \partial'' \) is determined by the relation:

\[
\theta(\partial''(f)) = d_0(\theta(f)) = \theta(f)d_0 \Leftrightarrow \partial''(f) = \theta(f)d_0|_{\tilde{\mathcal{O}}^*(R)}.
\]
for any homomorphism $f \in \text{Hom}_{\text{dg} \Lambda S_{eq}}(N, \hat{\Theta}^\bullet(R), N)$, where $\theta_f$ denotes this derivation which we associate to $f$ and which we deduce from the correspondence of Theorem 4.1.4.

4.2.2. Recollections on the twisted dg-module structure of the cobar-bar and Koszul resolutions. We already recalled that the bar construction $B(R)$ is naturally equipped with the structure of a chain complex in the category of dg-modules. We also explained that the Koszul construction $K(R)$ forms a sub-chain complex of the bar construction on which the twisting differential reduces to zero.

In what follows, we also use the notation $B(R) = B_\ast(R)$, with a * mark added, to refer to this underlying chain complex of the bar construction, and we adopt a similar convention for the Koszul construction $K(R) = K_\ast(R)$. Let $C_\ast(R) = B_\ast(R), K_\ast(R)$ denote any of these chain complex in dg-modules which we associate to $R$. Recall that we use the notation $\partial'$ to refer to the twisting differential of the bar construction $B(R)$. We use the same notation for the twisting differential of our complex $C_\ast(R)$ with the convention that we have $\partial' = 0$ in the case $C_\ast(R) = K_\ast(R)$. In what follows, we also adopt the notation $C_\ast(R)$ for the collection of dg-modules $C_n(R), n \in \mathbb{N},$ underlying our complex $C_\ast(R)$, where we forget about the twisting differential $\partial' : C_n(R)_* \to C_{n-1}(R)_{*−1}, n \in \mathbb{N}$, and we only retain the component-wise differential $\delta : C_n(R)_* \to C_{n}(R)_{*−1}, n \in \mathbb{N}$, determined by the internal differential of our operad $R$.

We similarly write $\text{Res}_\ast^c(R)$ for the object defined by forgetting about the twisting differential in the resolution $\text{Res}_\ast(R) = B^cC(R)$ which we associate to any of our constructions $C(R) = B(R), K(R)$. We then have the expression:

\[(1) \quad \text{Res}_\ast^c(R) = \bigoplus \Sigma^{-1} \check{C}_\ast^c(R),\]

where we consider the free operad on the graded object $\Sigma^{-1} \check{C}_\ast^c(R)$ underlying the symmetric sequence $\Sigma^{-1} \check{C}(R) = \Sigma^{-1} \check{C}_\ast(R)$ which we take in the definition of the cobar construction $\text{Res}_\ast(R) = B^cC(R)$. Recall that we use the notation $\partial''$ to refer to the twisting differential of the cobar construction $B^c(C)$ of a cooperad $\mathcal{C} \in \text{dg} \Lambda p_{\varnothing 1}$. In the case $\mathcal{C} = C(R)$, we write $\partial = \partial' + \partial''$ for the total twisting differential of our object, which we obtain by adding the twisting differential of the cobar construction

\[(2) \quad \partial'' : \bigoplus (\Sigma^{-1} \check{C}_\ast^c(R)) \to \bigoplus (\Sigma^{-1} \check{C}_\ast^c(R))\]

to the free operad derivation

\[(3) \quad \partial' : \bigoplus (\Sigma^{-1} \check{C}_\ast^c(R)) \to \bigoplus (\Sigma^{-1} \check{C}_\ast^c(R))\]

induced by the twisting differential of the complex

\[(4) \quad \Sigma^{-1} \check{C}_\ast(R) = (\Sigma^{-1} \check{C}_\ast^c(R), \partial').\]

We then have the following expressions:

\[(5) \quad \text{Res}_\ast(R) = (\bigoplus (\Sigma^{-1} \check{C}_\ast(R)), \partial' + \partial'') = (\bigoplus (\Sigma^{-1} \check{C}_\ast^c(R)), \partial' + \partial'')\]

for our resolution $\text{Res}_\ast(R)$.

In the next paragraph, we examine the structure of the module of derivations on the cobar-bar and Koszul resolutions which we deduce from these twisted object decomposition. We keep the same notation and conventions as in this paragraph. In particular, we write $C(R) = B(R), K(R)$ for any of the constructions which we associate to our operad $R \in \text{dg} \Lambda \text{Op}_{p\varnothing 1} / \text{Com}$, with an extra * mark when we want to refer to the underlying chain complex structure of our object $C(R) = C_\ast(R)$. 

4.2.3. The reduced form of the complex of derivations on the cobar-bar and Koszul resolutions. In (the third step of the proof of) Theorem 4.1.4, we establish that the cochain complex of derivations on the cobar-bar resolution is isomorphic to a cochain complex of Λ-sequence homomorphisms degree-wise, and we have an analogous statement for the Koszul resolution which we use in (the proof of) Theorem 4.1.6. In both cases \( \text{Res}_*(R) = B^\circ C(R) \), where \( C(R) = B(R), K(R) \), the obtained identity reads:

\[
\text{Der}_{dg \Lambda \text{Op}}(\text{Res}^*_* (R), M) = \text{Hom}_{dg \Lambda S \text{eq}}(\Sigma^{-1} \mathcal{C}^*_*(R), M)
\]

when we use our \( \mathcal{v} \) notation. Recall that we still deduce this relation from the correspondence of Theorem 3.1.7 where we determine the module of derivations on a free operad (as in the case of the conormalized complex of derivations on the cotriple resolution).

The degree-wise relation (1) implies again that our cochain complex of derivations on the differential graded resolution \( \text{Res}_*(R) = B^\circ C(R) \) is equivalent to a twisted dg-module of the form:

\[
\text{Der}_{dg \Lambda \text{Op}}(\text{Res}_*(R), M) = (\text{Hom}_{dg \Lambda S \text{eq}}(\Sigma^{-1} \mathcal{C}^*_*(R), M), \partial).
\]

To define the twisting differential of this dg-module, we now use that the differential of a derivation \( \theta : \text{Res}_*(R) \to M \) in the dg-module \( \text{Der}_{dg \Lambda \text{Op}}(\text{Res}_*(R), M) \) is given by a difference \( (\delta - \partial)(\theta) = \delta(\theta) - \partial(\theta) \), where \( \delta(\theta) : \text{Res}_*(R) \to M \) denotes the commutator of our homomorphism \( \theta \) with the differential components yielded by the internal differential of our \( \text{dg} \)-operad \( R \) and of our abelian \( R \)-bimodule \( M \), while we write \( \partial(\theta) : \text{Res}_*(R) \to M \) for the component of the differential given by the composite of the map \( \theta \) with the twisting differential of the operad \( \text{Res}_*(R) \).

We explicitly set \( \partial(\theta) = \pm \theta \partial \), with an additional sign \( \pm \) reflecting the permutation of the factors \( \partial \) and \( \theta \) which we use when we form this expression. We just check that these maps \( \delta(\theta), \partial(\theta) : \text{Res}_*(R) \to M \) define derivations well as soon as we assume \( \theta \in \text{Der}_{dg \Lambda \text{Op}}(\text{Res}_*(R), M) \). We then transport this twisting differential \( \partial(\theta) = \pm \theta \partial \) on the module of derivations \( \text{Der}_{dg \Lambda \text{Op}}(\text{Res}_*(R), M) \) through our degree-wise identity (1) to get the twisting differential associated to our dg-module of homomorphisms in (2). We explicitly determine the twisting differential of a homomorphism \( f \in \text{Hom}_{dg \Lambda S \text{eq}}(\Sigma^{-1} \mathcal{C}^*_*(R), M) \) by the relation:

\[
\theta_{\partial(f)} = \pm \theta_{f} \partial \Leftrightarrow \partial(f) = \pm \theta_{f} \partial|_{\Sigma^{-1} \mathcal{C}^*_*(R)},
\]

where we again write \( \theta_{f} \) for this derivation which we associate to \( f \) by the correspondence of Theorem 3.1.7.

We can still use the decomposition \( \partial = \partial' + \partial'' \) of the twisting differential of the resolution \( \text{Res}_*(R) \) to decompose this twisting differential of homomorphisms in two parts. We readily deduce from the relation \( \partial'(|\Sigma^{-1} \mathcal{C}(R)) \subset \Sigma^{-1} \mathcal{C}(R) \) that the term \( \partial'(f) \) of the twisting differential of a homomorphism \( \partial(f) = \partial'(f) + \partial''(f) \) is identified with the signed composite \( \pm f \partial' \) which we form in the dg-hom \( \text{Hom}_{dg \Lambda S \text{eq}}(\Sigma^{-1} \mathcal{C}^*_*(R), M) \), and hence, with the components of the differential of the dg-module \( \text{Hom}_{dg \Lambda S \text{eq}}(\Sigma^{-1} \mathcal{C}^*_*(R), M) \) yielded by the twisting differential of the complex \( \Sigma^{-1} \mathcal{C}^*_*(R) \). We just retrieve the correspondence used in the third step of the proof of Theorem 4.1.4 in this case. We may again integrate this component of the twisting differential of our dg-module (2) in the internal differential of the object \( \text{Hom}_{dg \Lambda S \text{eq}}(\Sigma^{-1} \mathcal{C}^*_*(R), M) \). We then get an identity:

\[
(\text{Hom}_{dg \Lambda S \text{eq}}(\Sigma^{-1} \mathcal{C}^*_*(R), M), \partial) = (\text{Hom}_{dg \Lambda S \text{eq}}(\Sigma^{-1} \mathcal{C}^*_*(R), M), \partial''),
\]
where we put the second component of the twisting differential apart.

We now use our relation (3), and we unravel the definition of the derivation \(\theta_f\) associated to a homomorphism \(f \in \text{Hom}_{dgAOp}(\Sigma^{-1} \mathcal{C}_s(R), M)\) in order to determine the component of the twisting differential \(\partial''(f) = \theta_f \partial''|_{\Sigma^{-1} \mathcal{C}_s(R)}\) yielded by the twisting differential of the cobar construction \(\partial'' : \mathbb{B}^c \mathcal{C}(R) \to \mathbb{B}^c \mathcal{C}(R)\). Recall that this map \(\partial''|_{\Sigma^{-1} \mathcal{C}(R)}\) is defined by the sum of the partial coproducts of our cooperad \(D = \mathcal{C}(R)\). Thus, if we apply the construction of Theorem 3.1.7 to the composite elements returned by this operation, then we get the following expression:

\[
\partial''(f)(\gamma) = \sum_{\rho_{\Gamma}(\gamma) \in \text{Tree}_2(r)} \left\{ \pm \lambda \left( \begin{array}{cccc}
\vdots & \ldots & \ldots & \vdots \\
\vdots & \ldots & \ldots & \vdots \\
\vdots & \ldots & \ldots & \vdots \\
\end{array} \right) + \pm \lambda \left( \begin{array}{cccc}
\vdots & \ldots & \ldots & \vdots \\
\vdots & \ldots & \ldots & \vdots \\
\vdots & \ldots & \ldots & \vdots \\
\end{array} \right) \right\},
\]

for any element \(\gamma \in \mathcal{C}(R)(r), r > 0\), where we write

\[
\rho_{\Gamma}(\gamma) = \sum_{\rho_{\Gamma}(\gamma)} \left\{ \begin{array}{cccc}
\vdots & \ldots & \ldots & \vdots \\
\vdots & \ldots & \ldots & \vdots \\
\vdots & \ldots & \ldots & \vdots \\
\end{array} \right\}
\]

for the expansion of the coproduct of our element over any composition scheme defined by a reduced tree with two vertices \(\Gamma \in \text{Tree}_2(r)\). In the formula of our twisting differential (5), we pick a set of representatives of isomorphism classes of reduced \(r\)-trees with two vertices, and we actually take a sum of coproducts running over this set \([\Gamma] \in \pi_0 \text{Tree}_2(r)\). (For simplicity, we just omit to mark this choice in our formula.) We apply the homomorphism \(f : \Sigma^{-1} \mathcal{C}(R) \to M\) to one factor of the tree-wise tensor (6) and we apply the map \(\epsilon = \epsilon|_{\Sigma^{-1} \mathcal{C}(R)} : \Sigma^{-1} \mathcal{C}(R) \to M\), which induces the augmentation of our resolution \(\epsilon : \text{Res}_s(R) \to M\), to the other factor. This operation returns an element of the tree-wise tensor product \(\Theta_{\epsilon}(R, M)'\), and we just perform the tree-wise composition operation \(\lambda : \Theta_{\epsilon}(R, M)' \to M(r)\) associated to our abelian bimodule structure to complete the definition of our map \(\partial''(f) : \gamma \mapsto \partial''(f)(\gamma)\).

4.2.4. The complex structure of the module of derivations on the cobar-bar and Koszul resolutions. We use the reduced form of the module of derivations on the cobar-bar and Koszul resolutions in §4.2.3(2-4) to define a cochain complex structure by the following formula:

\[
\text{Der}_{dgAOp}(\text{Res}_s(R), M) = \text{Hom}_{dgASeq}(\Sigma^{-1} \mathcal{C}_s(R), M),
\]

where we consider the (desuspension of the) component of degree \(s + 1\) of the chain complex \(\mathcal{C}_s(R)\), in the hom-object of §4.2.3(2-4). This shift on the grading of the chain complex \(\mathcal{C}_s(R)\) can be explained by the desuspension operation occurring in our construction. In short, we have to consider a bi-graded desuspension which decreases both the horizontal and the total grading of our object while the vertical degree (see §3.0.2) is fixed by the operation.
The first component of the twisting differential $\partial'$, which the above hom-object inherits from $\mathcal{C}_0(R)$, trivially increases the grading associated to this cochain structure by one. We claim that the second component of the twisting differential $\partial''$ satisfies the same homogeneity relation. We can readily deduce this result from the expression of this twisting differential in §4.2.4 together with the observation that the coproduct of our cooperads $\mathcal{C}(R) = \mathcal{B}(R), \mathcal{K}(R)$ preserves the extra chain grading associated with our objects while the map $\epsilon|_{\mathcal{C}_s(R)} : \mathcal{C}_s(R) \to R$ vanishes on the chain components of degree $s \neq 1$. The total twisting differential $\partial = \partial' + \partial''$ associated to our object in §4.2.3 therefore admits a component such that:

$$\text{Hom}_{\mathrm{dg}} \Lambda \otimes \mathcal{S}_{eq}(\mathcal{C}'_s(R), \mathcal{M}) \ni \partial \to \text{Hom}_{\mathrm{dg}} \Lambda \otimes \mathcal{S}_{eq}(\mathcal{C}'_{s+1}(R), \mathcal{M}) \otimes \mathcal{M}_{k-1},$$

for each cochain degree $s \in \mathbb{N}$ and each total degree $k \in \mathbb{Z}$, as required in the definition of a cochain complex structure in the category of dg-modules.

We use these cochain complex structures to make the comparison results of Theorem 4.1.4 and Theorem 4.1.6 more precise:

**Proposition 4.2.5.** The comparison maps of Theorem 4.1.4 and Theorem 4.1.6 define morphisms of cochain complexes in the category of dg-modules when:

- we equip the conormalized complex of the module of derivations on the cotriple resolution with its natural cochain complex structure (see §3.0.3),
- we equip the module of derivations on the cobar-bar and Koszul resolutions $\mathcal{R}_s(R) = \mathcal{B}_s \mathcal{B}(R), \mathcal{B}_s \mathcal{K}(R)$ with the grading and the cochain complex structure of §4.2.4.

**Explanations.** In the course of our verifications (in the proof of Theorem 4.1.4 and Theorem 4.1.6), we already used that our comparison morphisms reduce to morphisms of hom-objects

$$(1) \quad \text{Hom}_{\mathrm{dg}} \Lambda \otimes \mathcal{S}_{eq}(\mathcal{N}_s \mathcal{O}^s(R), \mathcal{M}) \ni \to \text{Hom}_{\mathrm{dg}} \Lambda \otimes \mathcal{S}_{eq}(\mathcal{O}'(R), \mathcal{M}) \otimes \mathcal{M}_{k-1},$$

which we associate to morphisms

$$(2) \quad \mathcal{O}'(R) \to \mathcal{O}(R)$$

in the category of dg-modules.

The embedding $\mathcal{N}_s \mathcal{O}^s(R) \to \mathcal{O}'(R)$, which we consider in this sequence, defines a morphism of chain complexes of dg-modules by definition of the Koszul construction (see §4.1.5). The second morphism of the sequence $\mathcal{O}'(R) \to \mathcal{O}_s \mathcal{O}^s(R)$ defines a morphism of chain complexes of dg-modules too. We refer to §C.2 (see more precisely the proof of Proposition C.2.16 and of Theorem C.2.17), where we explain the definition of this morphism with full details, for this verification (see also our short reminder in §4.1.3).

From these statements, we get that our comparison morphisms preserve the cochain grading of our modules of derivations since we deduce this grading from the natural chain complex structure of these objects (2) in our hom-objects (1). We moreover obtain that our comparison morphisms commute with the internal differentials of our cochain complexes of derivations, and with the component $\partial'$ of
the twisting differential as well since this term $\partial'$, according to our observations, arises from the twisting differentials of our objects in (2). The assertion that our comparison morphisms commute with the remaining component $\partial''$ of the twisting differential on the modules of derivations is then equivalent to the preservation of the total differential, which we obviously implicitly use when we define our comparison morphisms in the proof of Theorem 4.1.4 and Theorem 4.1.6. We therefore get that these morphisms define morphisms of cochain complexes of dg-modules, as claimed in our proposition.

We go back to the setting of §3.2.2, where we examine the applications of the cotriple cohomology to the spectral sequence of function spaces on the category of operads. We aim to express the second page of our spectral sequence in terms of the dg-module of derivations on the Koszul resolution which we consider in this chapter. We elaborate on the cochain complex constructions of §§4.2.1-4.2.4 to define a bi-grading at the level of the homology of this dg-module of derivations, and we use the result of Proposition 4.2.5 to relate this bi-grading to the natural bi-graded structure which we associate with the cotriple cohomology, and with the second page of our spectral sequence.

4.2.6. The applications to the spectral sequence of operadic function spaces. Recall that in our definition of the cotriple cohomology

$$H^*_{\Lambda \circ p} (R, N) = H^* \text{Der}_{\Lambda \circ p} (\text{Res}_* (R), N)$$

we consider the cosimplicial module of derivations $\text{Der}_{\Lambda \circ p} (\text{Res}_* (R), N)$ on the cotriple resolution $\text{Res}_* (R)$ of an augmented non-unitary connected $\Lambda$-operad in graded modules $R \in \text{gr } \Lambda \circ p_{q \geq 1} / \text{Com}$. We assume that the abelian $R$-bimodules $N$, which we take as coefficients for this module of derivations, is defined in the category of graded modules too.

In §3.2.1, we explain that this cotriple cohomology naturally forms a bi-graded object

$$H^*_{\Lambda \circ p} (R, N) = \bigoplus_{s,t} H^s_{\Lambda \circ p} (R, N)_{t-s},$$

with an upper (cochain) grading $s \in \mathbb{N}$ arising from the cosimplicial structure and the conormalized complex construction $H^s(-)$, while the lower (unbounded) degree $t \in \mathbb{Z}$ arises from the internal grading of the module of derivations $\text{Der}_{\Lambda \circ p} (-, -) = \bigoplus_t \text{Der}_{\Lambda \circ p} (-, -)_t$ (before we perform our conormalized cochain complex construction). We take the difference of these degrees $k = t - s$ as total (lower) grading for this cohomology module $H^*_{\Lambda \circ p} (R, N)$, and we still follow our general convention to mark this degree as a postfix lower-script in the expression of our object. Recall that the shift in the expression of the total degree $k = t - s$ reflects the performance of the conormalized cochain complex construction in the category of dg-modules (see §3.0.3). In Theorem 3.2.5, we check that this bi-grading on the cotriple cohomology corresponds to the bi-grading of the cosimplicial homotopy spectral sequence of function spaces on the category of operads.

We revisit the definition of this bi-grading on the cotriple cohomology in order to retrieve the counterpart of this structure for the module of derivations on the cobar-bar resolution and on the Koszul resolution which we consider in this chapter. We use that a graded module is equivalent to a dg-module equipped with a trivial differential in order to relate the objects which we consider in our comparison theorems to the objects which we consider in our first definition of
the cotriple cohomology in the previous chapter. We therefore regard our operad in graded modules \( R \in gr \Lambda O_{\mathcal{A}}/\textbf{Com} \) and the associated abelian bimodule \( N \) as objects defined in the category of dg-modules, but where the internal differential is trivial. We then have an identity \( N^* \text{Der}_{dg \Lambda O_p}(\text{Res}_s(R), N) = N^* \text{Der}_{gr \Lambda O_p}(\text{Res}_s(R), N) \), asserting that this general conormalized cochain complex of dg-modules \( N^* \text{Der}_{dg \Lambda O_p}(\text{Res}_s(R), N) \), which we consider in the comparison theorems of the present chapter, reduces the conormalized cochain complex of graded modules \( N^* \text{Der}_{gr \Lambda O_p}(\text{Res}_s(R), N) \), which we consider in the definition of the cotriple cohomology, and where the differential is only given by the twisting differential of the conormalized complex construction.

We use the cochain complex structures of the previous paragraphs \( \S\S 4.2.1-4.2.4 \) to formulate the analogue of this statement for the dg-module of derivations \( \text{Der}_{dg \Lambda O_p}(\text{Res}_s(R), N) \) which we associate to the cobar-bar resolution \( \text{Res}_s(R) = \mathcal{B}^s B(R) \), and to the Koszul resolution \( \text{Res}_s(R) = \mathcal{B}^s K(R) \) of our operad \( R \). We basically get that these cochain complexes of derivations \( \text{Der}_{dg \Lambda O_p}(\text{Res}_s(R), N) = \text{Der}_{dg \Lambda O_p}(\text{Res}_s(R), N) \) are equivalent to cochain complexes of graded modules

\[
\text{Der}_{dg \Lambda O_p}(\text{Res}_s(R), N) \in gr \text{Mod}, \quad s \in \mathbb{N},
\]
equipped with a trivial internal differential, so that the total differential of our complex reduces to the twisting differential

\[
\text{Der}_{dg \Lambda O_p}(\text{Res}_s(R), N)_k \xrightarrow{\partial} \text{Der}_{dg \Lambda O_p}(\text{Res}_s(R), N)_{k-1},
\]
defined for each \( s \in \mathbb{N}, k \in \mathbb{N}, \) and which we determine from the structure of our resolution \( \text{Res}_s(R) = \mathcal{B}^s B(R), \mathcal{B}^s K(R) \).

The vanishing of the internal differential implies that the homology of this complex inherits a decomposition, which we write:

\[
H_* \text{Der}_{dg \Lambda O_p}(\text{Res}_s(R), N) = \bigoplus_{s,t} H^t \text{Der}_{dg \Lambda O_p}(\text{Res}_s(R), N)_{t-s},
\]
and where \( s \in \mathbb{N} \) corresponds to the grading of our cochain complex of derivations \( \text{Der}_{dg \Lambda O_p}(\text{Res}_s(R), N), s \in \mathbb{N} \). We still follow our convention to mark the total grading of our homology module \( k = t-s \) as a post fixes lower-script in the expression of the homogeneous components of this decomposition \( H^* \text{Der}_{dg \Lambda O_p}(\text{Res}_s(R), N) \).

We immediately get from the observation of Proposition 4.2.5 that the comparison morphisms of Theorem 4.1.4 and Theorem 4.1.6 induce term-wise isomorphisms of bi-graded objects at the homology level:

\[
H_{gr \Lambda O_p}(R, N)_{t-s} = H^* \text{Der}_{dg \Lambda O_p}(\mathcal{B}^s B(R), N)_{t-s} = H^* \text{Der}_{dg \Lambda O_p}(\mathcal{B}^s K(R), N)_{t-s}
\]
when we assume that our operad \( R \), as well as the associated abelian bimodule \( N \), are defined in the base category of graded modules. We get the following statement when we apply this result to our homotopy spectral sequence:

**THEOREM 4.2.7.** If we assume that the homology of our operad \( R = H_*(P) \) forms a Koszul operad in the spectral sequence of \( \S 2.2.4 \)

\[
E' \Rightarrow \pi_* \text{Map}_{\mathcal{S}et \Lambda O_p}(\mathcal{B}^s \text{Res}_s(P), \Gamma_*(N)),
\]
while the additive connected non-unitary $\Lambda$-operad $N$ is, as a $\Lambda$-sequence, cofreely generated by a symmetric sequence in dg-modules (as in §3.2.2), then we have an identity:

$$E_2^{st} = H^s \text{Der}^*_{dg \Lambda \text{Op}}(B^c K(H_*(P)), H_*(N))_{t-s}$$

for each $t \geq s \geq 0$, where on the right-hand side of this equation, we consider the homology of the module of derivations on the Koszul resolution of our operad §4.1.5 together with the bi-grading defined in the previous paragraph §4.2.6.

Recall that we give a reduced representation of this cochain complex of derivations $\text{Der}^*_{dg \Lambda \text{Op}}(B^c K(H_*(P)), H_*(N))$ in §4.2.3(4), and the differential which we associate to this object reduces to the map $\partial = \partial''$ of which we make the expression explicit in §4.2.3(5-6).
CHAPTER 5

The Applications of the Koszul Duality for $E_n$-operads

We determine the homotopy spectral sequence of the function spaces associated to the simplicial resolution of $E_2$-operads in this chapter. We first examine the applications of our previous constructions in the general case of $E_n$-operads, where we take any $n \geq 2$. We address the applications to $E_2$-operads in a second step.

We use that the $n$-Gerstenhaber operad $\text{Gerst}_n$, which represents the homology of an $E_n$-operad (see §I.4.2), forms an instance of a Koszul operad. We actually have $K(\text{Gerst}_n) = \Lambda^{-n} \text{Gerst}_n^\vee$, where $\Lambda$ refers to a suspension functor on operads, and $(-)^\vee$ refers to the duality functor of graded modules, which we apply arity-wise to operads and to cooperads. We review these results in the first section of the chapter (§5.1).

We use this Koszul duality result to compute the homotopy spectral sequence of function spaces associated to $E_n$-operads, $n \geq 2$. We devote the second section of the chapter (§5.2) to this subject. We more precisely consider function spaces $\text{Map}_{\text{Set} \Lambda \text{Op}}(\lvert \text{Res}_\ast (E_m) \rvert, \Gamma_\ast (\Sigma E^0 p_n))$, where the source object $R = \lvert \text{Res}_\ast (E_m) \rvert$ is our cotriple resolution of an $E_m$-operad in simplicial sets $P = E_m$, and the target object is the additive operad $\Gamma_\ast (N) = \Gamma_\ast (\Sigma E^0 p_n)$ on the (suspension of the) weight graded object $E^0 p_n$ associated to our $E_n$-operad analogue of the Drinfeld-Kohno Lie algebra operad $p_n$, $n \geq 2$ (see §§2.2.5-2.2.6). We make explicit the complex of derivations which we associate to the Koszul construction of the $m$-Gerstenhaber operad $\text{Res}_\ast (\text{Gerst}) = B_c K(\text{Gerst}_n)$ and where we take the additive operad in graded modules $N = \Sigma E^0 p_n$ as coefficients. We get, according to the result of Theorem 4.2.7, a chain complex that determines the second page of the homotopy spectral sequence:

$$E^2 = H_{gr \Lambda \text{Op}}(\text{Gerst}_m, \Sigma E^0 p_n) \Rightarrow \pi_\ast (\text{Map}_{\text{Set} \Lambda \text{Op}}(\lvert \text{Res}_\ast (E_m) \rvert, \Gamma_\ast (\Sigma E^0 p_n)))$$

We then focus on the case $n = 2$ of this construction. We write $\text{Gerst} = \text{Gerst}_2$ for short when we address this case. We similarly set $p = p_2$. Recall that this operad $p$ is defined within the category of plain (ungraded) modules $\text{Mod}$ (unlike the higher dimensional Drinfeld-Kohno Lie algebras operad $p_n$, $n > 2$). We also have $\Sigma E^0 p = E^0 p[1]$ (see §2.2.5), where we use the notation $p[1]$ for the additive operad of the category of graded modules $gr \text{Mod}$ formed by putting each component of the Drinfeld-Kohno Lie algebra operad $p$ in degree 1. We compute the homology (in non-negative degrees) of the dg-module of derivations $\text{Der}_{\Lambda \text{Op}}(B^c K(\text{Gerst}), E^0 p[1])$, which we associate to the Koszul resolution of the 2-Gerstenhaber operad $\text{Res}_\ast (\text{Gerst}) = B^c K(\text{Gerst})$. We determine the second page of the spectral sequence of Theorem C from the result of this computation.

For simplicity, we keep our convention to take the field of rational number as ground ring $k = \mathbb{Q}$ all through this chapter. Let us mention, nonetheless, that
the statements of the first section, where we examine the Koszul duality of the Gerstenhaber operads, are valid over any ground ring.

5.1. The Koszul dual of the Gerstenhaber operads

The purpose of this section is to determine the Koszul dual of the Gerstenhaber Operads $Gerst_n$. We explicitly check the relation $k(Gerst_n) = (\Lambda^n Gerst_n)^\vee$ alluded to in the introduction of this chapter. This statement is actually a result of E. Getzler, J. Jones [91] and M. Markl [159] (see Theorem 5.1.7 for a more precise bibliography on this subject), and we mainly survey results of the literature, which we use in our subsequent computations, through this section. We also refer to the bibliography for details on the proof of the statements which we recall in this account.

Recall that the arity-wise dual of a cooperad in the category of Q-modules forms an operad. In §C.3 (see more precisely Theorem C.3.6), we explain that the definition of the Koszul dual $k(R)$ of an operad $R \in \mathcal{O}_{p,q}$, in terms of a kernel of the bar complex (see §4.1.5), can be used to get a presentation of the dual operad of this cooperad $k(R)^\vee$.

We use this approach to establish the identity $k(Gerst_n) = (\Lambda^n Gerst_n)^\vee$, which is given by the formula $k(Gerst_n)^\vee = \Lambda^n Gerst_n$. We explain the general definition of the operadic suspension functor $\Lambda : R \mapsto \Lambda R$, and we give a short reminder on the definition of the $n$-Gerstenhaber operad, before tackling the verification of this Koszul duality relation.

5.1.1. The operadic suspension functor. Recall that $\mathbb{Q}[1] \in gr \mathcal{M}od$ denotes, under our conventions, the graded module of rank one which has a single component in degree one.

We form the (non-unitary) endomorphism operad associated to this graded module. Let $\Lambda = \text{End}_{\mathbb{Q}[1]}$. The component of arity $r$ of this operad is identified with a graded module of rank one $\Lambda(r) = \mathbb{Q} \varphi_r$, for each $r > 0$, with a canonical generating element $\varphi_r \in \Lambda(r)$, corresponding to the obvious homomorphism $\mathbb{Q}[1]^{\otimes r} = \mathbb{Q}[r] \to \mathbb{Q}[1]$, in degree $1 - r$. The action of any permutation $w \in \Sigma_r$ on this element $\varphi_r \in \Lambda(r)$ is given by the formula $w \varphi_r = sgn(w) \varphi_r$. The signature corresponds to the action of the permutation on the tensor product $\mathbb{Q}[1]^{\otimes r}$. We also have the composition formula $\varphi_m \circ_k \varphi_n = (-1)^{(k-1)(n-1)} \varphi_{m+n-1}$, for any $m, n > 0$ and $k = 1, \ldots, m$.

We define the suspension $\Lambda R$ of a non-unitary operad $R \in dg \mathcal{O}_{p,q}$ by the arity-wise tensor product $\Lambda R = \Lambda \boxtimes R$ in the category of operads (see §2.2.3). We accordingly have an identity:

$$\Lambda R(r) = \mathbb{Q} \varphi_r \otimes R(r) = \Sigma^{1-r} R(r),$$

for any arity $r > 0$, where we consider the $1-r$ fold suspension of the dg-module $R(r)$ underlying our operad $R$. We assume that the symmetric group acts diagonally on each term of this operad and we define our operadic composition products factor-wise. We explicitly have $w \cdot (\varphi_r \otimes p) = sgn(w) \varphi_r \otimes (wp)$, for every permutation $w \in \Sigma_r$, and any element $p \in R(r)$, $r > 0$. We similarly get the formula $(\varphi_m \otimes p) \circ_k (\varphi_n \otimes q) = \pm (-1)^{(k-1)(n-1)} \varphi_{m+n-1} \otimes (p \circ_k q)$ for any composition operation, with $p \in R(m), q \in R(n)$. The sign $\pm$ reflects the permutation of the factors $p$ and $\varphi_n$ involved in this composition process.
The $s$-fold suspension operation is equivalent to an arity-wise tensor product $\Lambda^s R = \Lambda^s \boxtimes R$, where $\Lambda^s$ represents the $s$-fold tensor power of our operad $\Lambda$. We now have $\Lambda^s(r) = \mathbb{Q} p_r^s$, for any arity $r > 0$, where we write $p_r^s$ for the element of degree $s(1 - r)$ defined by the $s$-fold tensor power of the generating element $p_r$ in the graded module $\Lambda^s(r) = \Lambda(r) \otimes \cdots \otimes \Lambda(r)$. We get the formula $w \cdot p_r^s = \text{sgn}(w)^s p_r^s$, for the action of a permutation $w \in \Sigma_r$, in any arity $r > 0$, and we obtain $p_n^{s} o_\epsilon p_n^{s} = (-1)^{(k-1)(n-1)} p_{m+n-1}^s$ for a composition operation in this operad $\Lambda^s$. We also get the expression:

$$\Lambda^s R = \mathbb{Q} p_r^s \otimes R(r) = \Sigma^{s(1-r)} R(r)$$

for the components of the operad $\Lambda^s R$, and we can determine the symmetric and composition structure of this operad in the same way as in the case of a one-fold suspension.

5.1.2. **Reminder on the $n$-Gerstenhaber operad.** Recall that the $n$-Gerstenhaber operad $\text{Gerst}_n$ is defined by the presentation:

\[
\begin{align*}
(1) \quad \text{Gerst}_n &= \mathbb{Q}(\mu(x_1, x_2) \oplus \mathbb{Q} \lambda(x_1, x_2)) \\
\mu(\mu(x_1, x_2), x_3) &\equiv \mu(x_1, \mu(x_2, x_3)), \\
\lambda(\lambda(x_1, x_2), x_3) + \lambda(\lambda(x_2, x_3), x_1) + \lambda(\lambda(x_3, x_1), x_2) &\equiv 0, \\
\lambda(\mu(x_1, x_2), x_3) &\equiv \mu(\lambda(x_1, x_3), x_2) + \mu(x_1, \lambda(x_2, x_3)),
\end{align*}
\]

where we have a generating operation $\mu = \mu(x_1, x_2) \in \text{Gerst}_n(2)$ in degree 0, a generating operation $\lambda = \lambda(x_1, x_2) \in \text{Gerst}_n(2)$ in degree $n - 1$, together with an action of the transposition $(1) \in \Sigma_2$ such that $(12)\mu = \mu$ and $(12)\lambda = (-1)^n \lambda$ (see §1.4.2.13). In the homology of the little 2-discs space $\text{Gerst}_n(2) = H_\ast(D_n(2)) = H_\ast(S^{n-1})$, these generating operations correspond to the fundamental classes of the point $\mu = [pt]$ and of the $(n-1)$-sphere $\lambda = [S^{n-1}]$ (see Theorem 1.4.2.15).

In §1.4.2.13, we explain that $\mu = \mu(x_1, x_2)$ represents a commutative product operation in the $n$-Gerstenhaber operad, because the relations of the above presentation include the defining relations of the commutative operad. We formally have an operad morphism $\iota : \text{Com} \to \text{Gerst}_n$, which carries the generating operation of the commutative operad, also denoted by $\mu \in \text{Com}(2)$, to our commutative product operation in the $n$-Gerstenhaber operad $\mu \in \text{Gerst}_n(2)$. We easily check that this construction gives an operad embedding, because our morphism $\iota : \text{Com} \to \text{Gerst}_n$ has a retraction $\epsilon : \text{Gerst}_n \to \text{Com}$ which carries the other generating operation of the $n$-Gerstenhaber operad $\lambda \in \text{Gerst}_n$ to zero. We equivalently get that the suboperad of the $n$-Gerstenhaber operad generated by the operation $\mu \in \text{Gerst}_n(2)$ is identified with the commutative operad.

In §1.4.2.13, we also mention that $\lambda = \lambda(x_1, x_2)$ represents a Lie bracket operation of degree $n-1$ in the $n$-Gerstenhaber operad. We may actually use the operadic suspension operation of §5.1.1 to formalize this correspondence. We basically get that the $(1-n)$-fold suspension of the Lie operad $\Lambda^{1-n}$ $\text{Lie}$ is identified with an operad generated by such an operation $\lambda \in \Lambda^{1-n}$ $\text{Lie}(2)$, of degree $1-n$, satisfying $(12)\lambda = (-1)^n \lambda$, and for which we obtain the same Jacobi relation $\lambda(\lambda(x_1, x_2), x_3) + \lambda(\lambda(x_2, x_3), x_1) + \lambda(\lambda(x_3, x_1), x_2) \equiv 0$ as in our definition of the $n$-Gerstenhaber operad. We then have an operad morphism $\iota : \Lambda^{1-n} \text{Lie} \to \text{Gerst}_n$, which carries this generating operation of the suspended Lie operad $\lambda \in \Lambda^{1-n}$ $\text{Lie}(2)$ to Lie bracket operation in the $n$-Gerstenhaber operad $\lambda \in \text{Gerst}_n(2)$. We easily check, again, that this construction gives an operad embedding. We equivalently
get that the sub-operad of the $n$-Gerstenhaber operad generated by the operation
\begin{equation}
\lambda \in \text{Gerst}_n(2)
\end{equation}
is identified with the suspended Lie operad $\Lambda^{1-n} \text{Lie}$.

5.1.3. Monomials in the $n$-Gerstenhaber operad. In what follows, we also use
the notation $\mu(x_1, x_2) = x_1 x_2$ and $\lambda(x_1, x_2) = [x_1, x_2]$ for the generating
operations of the $n$-Gerstenhaber operad, and similarly when we deal with monomials
$\pi(x_1, \ldots, x_r) \in \text{Gerst}_m(r)$ which we form by taking any operadic composite of
these generating operations. We mainly use the literal notation $\mu = \mu(x_1, x_2)$ and
$\lambda = \lambda(x_1, x_2)$ when we deal with the presentation of our operad by generators and
relations, while we use the above algebraic notation when we deal with the module
structure of each object $\text{Gerst}_m(r)$, $r > 0$.

We first note that the component $\Lambda^{1-n} \text{Lie}(r)$ of the Lie algebra operad $\Lambda^{1-n} \text{Lie}$
admits a basis consisting of monomials of the form:
\begin{equation}
\rho(x_1, \ldots, x_r) = [\ldots [[x_{j_1}, x_{j_2}], x_{j_3}], \ldots, x_{j_r}],
\end{equation}
for any $r > 0$, where $(j_1, \ldots, j_r)$ runs over the permutaions of the index set
$(1, \ldots, r)$ satisfying $j_1 = 1$. The elements of this basis correspond (up to a sign) to
the standard basis elements of the Lie operad (see §I.1.2.11) in the suspended module
$\Lambda^{1-n} \text{Lie}(r) = \Lambda^{1-n}(r) \otimes \text{Lie}(r) = \Sigma(1-n) \text{Lie}(r)$. We also consider a straightforward generalization of this basis for the objects $\Lambda^{1-n} \text{Lie}(r)$ when we have an indexing set $\tau = \{i_1, \ldots, i_r\}$ equipped with an ordering $i_1 < \cdots < i_r$. We then consider permutations $(j_1, \ldots, j_r)$ of the set $(i_1, \ldots, i_r)$ such that $j_1 = i_1$ in the definition of our basis.

The mixed relation $\lambda(\mu(x_1, x_2), x_3) \equiv \mu(\lambda(x_1, x_3), x_2) + \mu(x_1, \lambda(x_2, x_3))$ of
our presentation (1) implies that any element of the $n$-Gerstenhaber operad $\pi = \pi(x_1, \ldots, x_r) \in \text{Gerst}_n(r)$ can be expressed as a formal product:
\begin{equation}
\pi(x_1, \ldots, x_r) = \rho_1(x_{j_1}, \ldots, x_{j_{i_1}}) \cdots \rho_m(x_{j_{m1}}, \ldots, x_{j_{mr_m}}),
\end{equation}
representing a composite of the monomial element of the commutative operad
$p(x_1, \ldots, x_m) = x_1 \cdots x_m \in \text{Com}(m)$ with operations of the suspended Lie
operad:
\begin{equation}
\rho_k = \rho_k(x_{jk_1}, \ldots, x_{jk_{r_k}}) \in \Lambda^n \text{Lie}(r_k), \quad k = 1, \ldots, m,
\end{equation}
together with the application of a shuffle permutation materialized by the indexing
of the variables in this composite operation. We more precisely assume that the subsets
of variables $\{x_{jk_1}, \ldots, x_{jk_{r_k}}\}$ associated to the factors $\rho_k = \rho_k(x_{jk_1}, \ldots, x_{jk_{r_k}})$,
$k = 1, \ldots, m$, form a partition of the set $\{x_1, \ldots, x_r\}$, so that each variable $x_i$, $i = 1, \ldots, r$, occurs once and only once in our composite. We can moreover use the order relation $j_{k_1} < \cdots < j_{k_{r_k}}$ in the ambient set $\tau = \{1 < \cdots < r\}$ to provide each of these variable subsets with a canonical ordering. We use these orderings $j_{k_1} < \cdots < j_{k_{r_k}}$, $k = 1, \ldots, s$, to get a canonical expression of the factors (3) of our composite (2) in terms of the basis elements of the suspended Lie operad (1). We then have the following more precise result:

**Proposition 5.1.4.** The composite operations in §5.1.3(2), where the factors
$\rho_k = \rho_k(x_{jk_1}, \ldots, x_{jk_{r_k}})$, $k = 1, \ldots, s$, run over our basis elements of the suspended
Lie operad §5.1.3(1), form a basis of the component of arity $r$ of the $n$-Gerstenhaber operad
$\text{Gerst}_n(r)$, for any $r > 0$.  

For instance, in low arity \( r = 2, 3 \), we explicitly obtain \( \text{Gerst}_n(2) = \mathbb{Q}[x_1, x_2] \oplus \mathbb{Q}x_1x_2 \), and:

\[
\text{Gerst}_n(3) = \mathbb{Q}[[x_1, x_2], x_3] \oplus \mathbb{Q}[[x_1, x_3], x_2],
\]

\[
\oplus \mathbb{Q}[x_1, x_2]x_3 \oplus \mathbb{Q}[x_2, x_3]x_1 \oplus \mathbb{Q}[x_1, x_3]x_2 \oplus \mathbb{Q}x_1x_2x_3.
\]

**Explanations and References.** We already explained that each element \( p(x_1, \ldots, x_r) \in \text{Gerst}_n(r) \) has an expansion of the form \( \S_{\text{Gerst}_n} \). We now claim that this expansion is also unique. This result implies that the \( n \)-Gerstenhaber operad \( \text{Gerst}_n \) forms an instance of an operad defined by a distribution law in the sense of \([72, 159]\).

The work of F. Cohen \([46]\) gives, together with the identity between the \( n \)-Gerstenhaber operad and the homology of the little \( n \)-discs, a description of the free algebra over the \( n \)-Gerstenhaber operad which implies the result of this proposition. The article \([159]\) by M. Markl gives an algebraic approach for the proof of such statements, in terms of the concept of a distributive law for operads. The main result of this reference is a general coherence theorem which can be used to obtain a result of the form of our proposition from a rewriting law governing the composition of the generating operations of operads defined by a presentation. This coherence applies to the \( n \)-Gerstenhaber operad (see Example 3.3. in *loc. cit.*). We also refer to Loday-Vallette’s book \([150, \S 8.6]\) for the relationship between this notion of a distributive law and the general theory of rewriting for operads.

Let us observe that the definition of the \( n \)-Gerstenhaber operad in \( \S 5.1.2 \) makes sense over any ground ring. The references, which we cite in this verification, actually provide a proof of our proposition in this general context. \( \square \)

We now check the relation \( K(\text{Gerst}_n) = (\Lambda^n \text{Gerst}_n)^\vee \iff K(\text{Gerst}_n)^\vee = \Lambda^n \text{Gerst}_n \). We will write \( \pi(x_1, \ldots, x_r)^\vee \in \text{Gerst}_n(r)^\vee \) for the elements of the graded module \( \text{Gerst}_n(r)^\vee \) which we obtain by dualizing the basis of Proposition 5.1.4.

In \( \S C.3 \), we explain that the top components of the bar construction of an operad \( B_{r-1}(R)(r) \), \( r > 0 \), are identified with the components of a cofree cooperad \( \mathcal{O}(\Sigma M) \), where we consider the arity-wise suspension of the symmetric sequence such that \( M(2) = R(2) \) and \( M(r) = 0 \) for \( r \neq 2 \). We dually have \( B_{r-1}(R)(r)^\vee = \mathcal{O}(\Sigma^{-1} M^\vee)(r) \), for any \( r > 0 \), where we consider the free operad on the dual of this arity-wise symmetric sequence \( M \). We moreover explained that the relation \( K(R)(r)^\vee = \text{coker}(\partial^\vee : B_{r-2}(R)(r)^\vee \to B_{r-1}(R)(r)^\vee) \), for any \( r > 0 \), obtained by dualizing the definition of the Koszul construction \( K(R) \) identifies the dual of this cooperad \( K(R)^\vee \) with the quotient of the free operad \( \mathcal{O}(\Sigma^{-1} M^\vee) \) over the ideal generated by the image of the coboundary map

\[
\begin{array}{ccc}
\Sigma^{-1} R(3)^\vee & \xrightarrow{\partial^\vee} & \mathcal{O}(\Sigma^{-1} M^\vee)(3) \\
= B_1(R)(3)^\vee & = B_2(R)(3)^\vee & = B_3(R)(3)^\vee
\end{array}
\]

in arity 3.

In the case of the \( n \)-Gerstenhaber operad \( R = \text{Gerst}_n \), we have an obvious isomorphism

\[
\Sigma^{-1}(Q \mu(x_1, x_2)^\vee \oplus Q \lambda(x_1, x_2)^\vee) \xrightarrow{\sim} \text{p}_1^\vee \otimes (Q \mu(x_1, x_2) \oplus Q \lambda(x_1, x_2)) = \Sigma^{-1} \text{Gerst}_n(2)^\vee = \Lambda^n \text{Gerst}_n(2)
\]
which associates the suspended Lie bracket operation $\hat{\lambda} = \rho^n_2 \otimes \lambda$ in the suspended operad $\Lambda^n \text{Gerst}_n$ to the dual element of the product operation $\mu^\vee$ in the module $\text{Gerst}_n(2)^\vee$, and the suspended product operation $\hat{\mu} = \rho^n_2 \otimes \mu$ to the dual element of the Lie bracket $\lambda^\vee$. We easily check that this mapping matches the grading and the action of the transposition on our module. (We make this correspondence explicit in the proof of the next proposition.) Then we get the following statement:

**Proposition 5.1.5.** The above mapping $\chi: \Sigma^{-1} \text{Gerst}_n(2)^\vee \xrightarrow{\sim} \Lambda^n \text{Gerst}_n(2)$, such that $\chi(\lambda^\vee) = \rho^n_2 \otimes \mu$ and $\chi(\mu^\vee) = \rho^n_2 \otimes \lambda$, induces an isomorphism $K(\text{Gerst}_n)^\vee \xrightarrow{\sim} \Lambda^n \text{Gerst}_n$ on the dual operad of the Koszul construction $K(\text{Gerst}_n)^\vee$.

**Proof.** We refer to [91, Lemma 3.2] for the result of this proposition. We may again check that our statement remains valid over any ground ring though we still assume $k = \mathbb{Q}$ for simplicity.

We use that the $n$-fold suspension of the $n$-Gerstenhaber operad has a presentation of the same form as the $n$-Gerstenhaber operad §5.1.2(1), with our elements

$\mu = \rho^n_2 \otimes \mu \in \Lambda^n \text{Gerst}_n(2), \quad \lambda = \rho^n_2 \otimes \lambda \in \Lambda^n \text{Gerst}_n(2),$ (1)

as generating operations of degree $\text{deg}(\hat{\mu}) = -n$ and $\text{deg}(\hat{\lambda}) = -1$ respectively, together with an action of the transposition $(1 2) \in \Sigma_2$ given by the formulas:

$(1 2)\hat{\mu} = (-1)^n \hat{\mu}, \quad (1 2)\hat{\lambda} = \hat{\lambda}.$ (2)

which we readily obtain from the definition of the suspension functor on operads. We also easily retrieve the relations associated to these generating operations from the definition of the composition structure on the operadic suspension. We explicitly obtain:

$\hat{\mu}(\hat{\mu}(x_1, x_2), x_3) = (-1)^n \hat{\mu}(x_1, \hat{\mu}(x_2, x_3)),$ (3)

$\hat{\lambda}(\hat{\lambda}(x_1, x_2), x_3) + \hat{\lambda}(\hat{\lambda}(x_2, x_3), x_1) + \hat{\lambda}(\hat{\lambda}(x_3, x_1), x_2) = 0,$ (4)

$\hat{\lambda}(\hat{\mu}(x_1, x_2), x_3) = (-1)^n \hat{\lambda}(\hat{\lambda}(x_1, x_3), x_2) + (-1)^n \hat{\mu}(x_1, \hat{\lambda}(x_2, x_3)).$ (5)

For instance, we have the formulas $\hat{\mu}(\hat{\mu}(x_1, x_2), x_3) = \rho^n_2 \otimes \mu(\mu(x_1, x_2), x_3)$ and $\hat{\mu}(x_1, \hat{\mu}(x_2, x_3)) = (-1)^n \rho^n_2 \otimes \mu(x_1, \mu(x_2, x_3))$, from which we obtain the suspended associativity identity (3). We argue similarly for the other relations.

We mainly have to check that the image of our coboundary map

$\Sigma^{-1} \text{Gerst}_n(3)^\vee \xrightarrow{\delta^\vee} \Theta(2)(\Sigma^{-1} \text{Gerst}_n)(3) = \Theta(2)(\Sigma^{-1}(\mathbb{Q} \mu^\vee \oplus \mathbb{Q} \lambda^\vee))$ (6)

is equal to the $\Sigma_3$-module generated by these relations (3-5) in $\Theta(\mathbb{Q} \lambda^\vee \oplus \mathbb{Q} \hat{\mu})(3)$ when we perform the mapping of our proposition $\mu^\vee \mapsto \lambda^\vee$ and $\lambda^\vee \mapsto \hat{\mu}$. We use the basis of Proposition 5.1.4 for the module $\text{Gerst}_n(3)$ and the definition of the bar differential (dual to this coboundary map) in terms of the partial composition product of our operad to carry out this verification.

We basically compute the matrix of the bar differential in our bases. We determine the dual coboundary map. We readily get relations (3-5) by taking the value of the obtained mapping on the dual of the basis elements of Proposition 5.1.4, and this computation gives the proof of our claim. We just have to take care of signs involved in the definition of the bar differential and in tensor permutations...
which implicitly occur when we use the duality relation \( \Theta(\Sigma^1(Q\mu \oplus Q\lambda)) \simeq \Theta(\Sigma(Q\mu \oplus Q\lambda))^\vee \).

5.1.6. The \( \Lambda \)-sequence structure on the Koszul construction. In §4.1.2, we briefly recall that the cobar-bar resolution \( \text{Res}_*(R) = \mathcal{B}^c \mathcal{B}(R) \) inherits the structure of an augmented non-unitary \( \Lambda \)-operad, and defines a resolution of our operad \( R \) within the category of augmented connected non-unitary \( \Lambda \)-operads \( \text{dg} \Lambda \text{Op}_{\mathcal{A}_1} / \text{Com} \) when we assume \( R \in \text{dg} \Lambda \text{Op}_{\mathcal{A}_1} / \text{Com} \). (We refer to Proposition C.2.19 for further details on this result.) In §4.1.5, we explain that the operad \( \mathcal{B}^c K(R) \subset \mathcal{B}^c \mathcal{B}(R) \), which we associate to the Koszul construction \( C(R) = K(R) \), inherits the structure of an augmented non-unitary \( \Lambda \)-operad similarly, and defines a sub-object of the cobar-bar resolution \( \text{Res}_*(R) = \mathcal{B}^c \mathcal{B}(R) \) within \( \text{dg} \Lambda \text{Op}_{\mathcal{A}_1} / \text{Com} \). (We refer to Proposition C.3.5 for further details on this statement.)

In both cases \( C(R) = B(R), K(R) \), the \( \Lambda \)-structure which we define on our operad \( \text{Res}_*(R) = \mathcal{B}^c C(R) \) can be determined from an augmented \( \Lambda \)-sequence structure which we associate to the generating collection \( \Sigma^{-1} C(R)_b \subset \mathcal{B}^c C(R)_b \) of our object \( \mathcal{B}^c C(R)_b = \mathcal{O}(\Sigma^{-1} C(R)_b) \). In the case of the Koszul construction \( C(R) = K(R) \), the restriction operations \( u^* : \Sigma^{-1} \bar{K}(R)_b(n)_b \rightarrow \Sigma^{-1} \bar{K}(R)_b(m)_b \), which define the \( \Lambda \)-diagram structure associated to this symmetric sequence \( \Sigma^{-1} \bar{K}(R)_b \), are trivial as soon as \( u : \{1 < \cdots < m\} \rightarrow \{1 < \cdots < n\} \) is a proper injection (explicitly, as soon as we have \( n > m \)). The restriction operations which we define on the operad \( \mathcal{B}^c C(R)_b = \mathcal{O}(\Sigma^{-1} \bar{K}(R)_b) \) are therefore entirely determined by the action of the augmentation \( \epsilon : \Sigma^{-1} \bar{K}(R) \rightarrow \text{Com} \) and the internal symmetric structure of our collection through the reduction process of Lemma I.3.4.3.

This augmentation \( \epsilon : \Sigma^{-1} \bar{K}(R) \rightarrow \text{Com} \) represents the restriction, to our generating symmetric sequence \( \Sigma^{-1} \bar{K}(R)_b \subset \mathcal{B}^c K(R)_b \), of the augmentation morphism \( \epsilon : \mathcal{B}^c K(R) \rightarrow \text{Com} \) which we define on the operad \( \text{Res}_*(R) = \mathcal{B}^c K(R) \).

This map \( \epsilon : \mathcal{B}^c K(R) \rightarrow \text{Com} \) is also identified with the composite of the natural augmentation morphism of the Koszul construction \( \epsilon : \mathcal{B}^c K(R) \rightarrow R \) and the structure augmentation \( \epsilon : R \rightarrow \text{Com} \) of our operad \( R \in \text{dg} \Lambda \text{Op}_{\mathcal{A}_1} / \text{Com} \). The map \( \epsilon : \Sigma^{-1} \bar{K}(R) \rightarrow \text{Com} \) accordingly vanishes in arity \( r \neq 2 \) and reduces to a homomorphism \( \epsilon : K(R)_2(2) \rightarrow \text{Com}(2) \) (of degree 1) in arity 2. We have \( K(R)_2 = \Sigma R(2) \) by definition of the Koszul construction \( K(R) \), and we just take the obvious mapping \( K(R)_2 = \Sigma R(2) \rightarrow \text{Com}(2) \), where we consider the augmentation of our operad \( \epsilon : R \rightarrow \text{Com} \) to explicitly determine this arity 2 component of our augmentation morphism \( \epsilon : \Sigma^{-1} \bar{K}(R) \rightarrow \text{Com} \) (we still refer to Proposition C.3.5 for more details on this construction).

We can easily make this homomorphism for the Koszul dual \( K(\text{Gerst}_n) = \Lambda^{-n} \text{Gerst}_n \) of the Gerstenhaber operad \( R = \text{Gerst}_n \), such as determined in Proposition 5.1.5. We simply get that this homomorphism \( \epsilon : K(\text{Gerst}_n)_2(2) \rightarrow \text{Com}(2) \), which therefore determines the whole \( \Lambda \)-structure of the operad \( \text{Res}_*(\text{Gerst}_n) = \mathcal{B}^c K(\text{Gerst}_n)_b \), carries the element \( \phi_2^{-n} \otimes \lambda^\vee \in \Lambda^{-n} \text{Gerst}_n(2)^\vee \) to the product operation \( \mu \in \text{Com}(2) \), and cancels \( \phi_2^{-n} \otimes \mu^\vee \in \Lambda^{-n} \text{Gerst}_n(2)^\vee \).

We have the following statement:

**Theorem 5.1.7** (E. Getzler and J. Jones [91], M. Markl [159]). The \( n \)-Gerstenhaber operad is Koszul, with the cooperad \( K(\text{Gerst}_n) = (\Lambda^n \text{Gerst}_n)^\vee \), determined by the result of Proposition 5.1.5, as Koszul dual cooperad.
The resolution $\text{Res}_*(R) = B^c K(\text{Gerst}_n)$, which we obtain in this case, also inherits an augmented non-unitary $\Lambda$-operad structure (according to our observations of §5.1.6) and fits in a commutative diagram

$$B^c K(\text{Gerst}_n) \longrightarrow B^c B(\text{Gerst}_n) \cong \text{Gerst}_m$$

in the category of augmented connected non-unitary $\Lambda$-operads in dg-modules.

Explanations and references. The first cited article [91], by E. Getzler and J. Jones, provides a geometric proof of this Koszul duality statement, by using the Fulton-MacPherson model of $E_n$-operads (see our survey in §I.4.3), and identifying the bar construction $B(\text{Gerst}_n)$ with the first term of a spectral sequence determined by the natural structure of a manifold with corners on the spaces $FM_n(r)$, $r > 0$, that form this operad $\text{Gerst}_n$.

The second cited article [159], by M. Markl, gives an algebraic proof of Getzler-Jones theorem, by using the definition of the $n$-Gerstenhaber operad $\text{Gerst}_n$ in terms of a distributive law between the commutative operad and the (suspended) Lie operad. We also refer to [150, §8.6] for the relationship between the distributive law method and the application of rewriting methods to the theory of Koszul operads.

Let us mention that the result of this theorem is valid over any ground ring. □

5.2. The cotriple cohomology of the Gerstenhaber operads

The previous Koszul duality theorem enables us to determine the cotriple cohomology modules $H^*_{\text{gr}} \Lambda \circ \mathcal{O}_p (\text{Gerst}_m, N)$ which we associate to the $m$-Gerstenhaber operad $\text{Gerst}_m$, for any $m \geq 2$, and the second page of the function space spectral sequence which we associate to the cotriple resolution of an $E_m$-operad:

$$E^2 = H^*_{\text{gr}} \Lambda \circ \mathcal{O}_p (\text{Gerst}_m, N) \Rightarrow \pi_* \text{Map}_\Lambda \circ \mathcal{O}_p (|\text{Res}_*(E_m)|, \Gamma_*(N)),$$

for any additive connected $\Lambda$-operad in graded modules $N$. To be explicit, we have:

$$H^*_{\text{gr}} \Lambda \circ \mathcal{O}_p (\text{Gerst}_m, N) = H^* \text{Der}_{dg \Lambda \circ \mathcal{O}_p} (B^c K(\text{Gerst}_m), N)$$

by Theorem 4.2.7, where we consider the complex of derivations on the Koszul resolution $\text{Res}_*(\text{Gerst}_m) = B^c K(\text{Gerst}_m)$ of the Gerstenhaber operad $\text{Gerst}_m = H_*(E_m)$.

We use the general description of this Koszul derivation complex in §4.2.3, and the determination of the Koszul dual cooperad of the $m$-Gerstenhaber operad in Proposition 5.1.5, to tackle the problem of computing this cotriple cohomology module. We also use the definition of the bi-grading on our derivation complex, such as specified in §4.2.4, when we address the applications of our computation to the spectral sequence of function spaces. We briefly revisit these constructions in this section, and then we explicitly compute the cohomology of our complex of derivations in the case of the 2-Gerstenhaber operad $\text{Gerst} = \text{Gerst}_2$ and when we take the suspension of the weight graded object $E^0 \mathcal{P}$ underlying the Drinfeld-Kohno Lie algebra operad $\mathcal{P} = p_2$ as coefficients. We eventually get the result announced in Theorem C. The proof of this statement is actually the main objective of this section.

In the introduction of this chapter, we briefly mentioned that the statements of the previous section are valid over any ground ring. We have to assume that the
ground ring is a field of characteristic zero from now on, because we use results of the previous chapters which we establish in this setting. For simplicity, we keep our convention that we take the field of rational numbers as ground ring \( \Bbb Q \) in the rest of this chapter. We only address the extension of our results to the case of other characteristic zero fields in side remarks.

5.2.1. The derivation complex on the Koszul resolution of the \( m \)-Gerstenhaber operad, \( m \geq 2 \). Let us recap the result of our constructions. In a first stage, we consider the derivation complex of Koszul resolution of the \( m \)-Gerstenhaber operad \( \text{Gerst}_m \), \( m \geq 2 \), with coefficients in a general abelian \( \text{Gerst}_m \)-bimodule \( N \). We just assume that this bimodule of coefficients is defined within the base category of graded modules, and hence, has no internal differential. In the next paragraph, we will take \( N = \Sigma \mathbb{E}^0 p_n \), the suspension of the weight graded object \( \mathbb{E}^0 p_n \) defined by the graded Drinfeld-Kohno Lie algebra operad \( p_n \), \( n \geq 2 \), and we will prove that our derivation complex admits a further reduction in this case after reviewing the \( \Lambda \)-sequence structure of this object.

By the observations of \( \S \S 4.2.3-4.2.4 \), we have the identity:

\[
\text{Der}_{d_0}^* \Lambda \circ \circ (\text{Res}_{\Lambda} (\text{Gerst}_m), N)_{t-s} = \text{Hom}_{\Lambda \circ \circ} (\Sigma^{-1} K_{s+1} (\text{Gerst}_m), N)_{t-s}
\]

for any horizontal grading \( s \in \mathbb{N} \), and total grading \( k = t - s \in \mathbb{Z} \), where we consider the graded module of homomorphisms on the component of degree \( s+1 \) of the Koszul construction \( K(\text{Gerst}_m) = K_s (\text{Gerst}_m) \). The differential of our complex reduces to the twisting differential

\[
\text{Hom}_{\Lambda \circ \circ} (\Sigma^{-1} K_{s+1} (\text{Gerst}_m), N)_s \overset{\partial''}{\longrightarrow} \text{Hom}_{\Lambda \circ \circ} (\Sigma^{-1} K_{s+2} (\text{Gerst}_m), N)_{s-1}
\]

given by the formula of \( \S 4.2.3(5-6) \).

The Koszul construction \( K(\text{Gerst}_m) \) satisfies:

\[
K_{s+1} (\text{Gerst}_m) (r) = \begin{cases} 
\Lambda^{-m} \text{Gerst}_m (r)^{\vee}, & \text{if } r = s + 2, \\
0, & \text{otherwise},
\end{cases}
\]

by definition of the grading of this complex in \( \S 4.1.5 \) (see also \( \S C.3.3 \)), and according to our computation of (the dual operad of) this cooperad \( K(\text{Gerst}_m) \) in Proposition 5.1.5.

In \( \S 5.1.6 \), we mention that the restriction operations \( u^* : K(\text{Gerst}_m) (l) \to K(\text{Gerst}_m) (k) \) which we define on this Koszul construction \( K(\text{Gerst}_m) \) are trivial when \( u : \{ 1 < \cdots < k \} \to \{ 1 < \cdots < l \} \) is a proper injective maps (equivalently, when \( k < l \)). In fact, we may deduce this result from the above formula (3) and the observation that the decomposition \( K(\text{Gerst}_m) = \bigoplus_{s \geq 0} K_{s+1} (\text{Gerst}_m) \) holds in the category of \( \Lambda \)-sequences.

5.2.2. The case of coefficients in the graded Drinfeld-Kohno Lie algebra operad.

We now consider the case where \( N \) is the suspension of the weight graded object \( \mathbb{E}^0 p_n \) associated to the graded Drinfeld-Kohno Lie algebra operad \( p_n \), for any \( n \geq 2 \). Briefly recall that \( p_n \) is identified with the classical Drinfeld-Kohno Lie algebra operad in the case \( n = 2 \), and that \( p_n \) is defined in the category of Lie algebras in chain graded modules (unless we assume \( n = 2 \), in which case we can identify our operad with a chain graded object concentrated in degree 0). The graded Lie algebras that define this operad \( p_n \) inherit an extra weight grading, which is preserved by the composition structure of our operad, and when we write \( N = \Sigma \mathbb{E}^0 p_n \), we consider the arity-wise and component-wise suspension of the weight
graded object of the category of additive operads in graded modules that underlies our object $p_n$.

We have the obvious relation:

$$\text{Hom}_{\text{gr} \Lambda \times \text{Seq}}(\Sigma^{-1} K_{s+1} (\text{Gerst}_m), \Sigma E^0 p_n)_{t-s} = \text{Hom}_{\text{gr} \Lambda \times \text{Seq}}(K_{s+1} (\text{Gerst}_m), E^0 p_n)_{t-s-2},$$

for any $s \in \mathbb{N}$, $k = t-s \in \mathbb{Z}$, and we therefore get the following expression for the second page of our homotopy spectral sequence:

$$E^2_{st} = H^s(\text{Hom}_{\text{gr} \Lambda \times \text{Seq}}(K_{s+1} (\text{Gerst}_m), E^0 p_n), \partial^r)_{t-s-2} \Rightarrow \pi_{t-s}(\text{Map}_{\Lambda \times \text{Op}}(\text{Res}_*(E_m), E^0 p_n)),$$

where we consider the components of the cohomology of our derivation complex in horizontal degree $s \in \mathbb{N}$ and total degree $k = t-s \geq 0$. We review the expression of the twisting differential of this derivation complex $\partial^r$ in a subsequent paragraph.

We observed in §12 that the additive operad $E^0 p_n$ is, as $\Lambda$-sequence, cofreely generated by a symmetric sequence $S E^0 p_n$. To be more explicit, we define the components of this symmetric sequence $S E^0 p_n$ as modules $S E^0 p_n(r)$, $r > 0$, spanned by monomials $\pi = \pi(t_{i,j_1}, \ldots, t_{i,j_l}) \in p_n(r)$ of which support, defined by the union of the indexing pairs $\{i_k, j_k\}$ of the variables $t_{i,j}$ in the expression of these monomials, satisfies $\text{supp} \, \pi = \{1, \ldots, r\}$. To determine our cofree $\Lambda$-sequence structure, we prove that each object $E^0 p_n(r)$, $r > 0$, has a decomposition:

$$E^0 p_n(r) = \prod_{u \in \text{Hor}_{\Lambda} (m-r)} S E^0 p_n(m),$$

where the product runs over the set of increasing maps $u : \{1 < \cdots < m\} \to \{1 < \cdots < r\}$. The factor associated to any such map $u : \{1 < \cdots < s\} \to \{1 < \cdots < r\}$ in this product corresponds to the summand of the Lie algebra $p_n(r)$ spanned by monomials $\pi = \pi(t_{i,j_1}, \ldots, t_{i,j_l})$ satisfying $\text{supp} \, \pi = \{u(1), \ldots, u(m)\}$.

By Proposition 3.3.7, this cofree $\Lambda$-sequence structure implies that the above hom-objects (1) admit the following extra reduction:

$$\text{Hom}_{\text{gr} \Lambda \times \text{Seq}}(K_{s+1} (\text{Gerst}_m), E^0 p_n)_{t-s-2} = \text{Hom}_{\text{gr} \Sigma \times \text{Seq}}(K_{s+1} (\text{Gerst}_m), S E^0 p_n)_{t-s-2},$$

for any $s \in \mathbb{N}$, and $k = t-s \in \mathbb{Z}$.

In fact, the vanishing of the restriction operation associated to any proper injection on the Koszul construction $K(\text{Gerst}_m)$ implies that any homomorphism $f \in \text{Hom}_{\text{gr} \Lambda \times \text{Seq}}(K_{s+1} (\text{Gerst}_m), E^0 p_n)$ vanishes on the summands spanned by the monomials $\pi$ such that $\text{supp} \, \pi \nsubseteq \{1, \ldots, r\}$ in the expression of the object $E^0 p_n(r)$ which we deduce from our cofree representation (3). We get, therefore, that any homomorphism of $\Lambda$-sequences $f \in \text{Hom}_{\text{gr} \Lambda \times \text{Seq}}(K_{s+1} (\text{Gerst}_m), E^0 p_n)$ is identified with the extension by zero of an underlying homomorphism of symmetric sequences $f : K_{s+1} (\text{Gerst}_m) \to S E^0 p_n$, and this result gives our correspondence (4).

We unravel the symmetric sequence structure of the cooperad $K(\text{Gerst}_m) = \Lambda^{-m} \text{Gerst}_m$, to determine the above modules of homomorphisms (4). We consider the monomial basis of the module $\text{Gerst}_m(r)$, such as defined in §5.1.3 (see also Proposition 5.1.4), for any $r > 0$. We take the (desuspension of the) dual of this
basis in the module $\Lambda^{-m} \text{Gerst}_m(r)^\vee$. We write
\[ p_r^{-m} \otimes \pi(x_1, \ldots, x_r)^\vee \in \Lambda^{-m} \text{Gerst}_m(r)^\vee \]
for the element of this basis which we associate to any monomial $\pi(x_1, \ldots, x_r) \in \text{Gerst}_m(r)$ in §5.1.3(2).

We focus on the case $m = n = 2$ for our computations. Recall that we use the notation $\text{Gerst} = \text{Gerst}_2$ and $p = p_2$ in this fundamental case. We use that the module $E_0^0$ is concentrated in degree 0, and that the grading of our hom-object in (4) is therefore determined by the internal grading of the Koszul construction $K(\text{Gerst})$.

We make this grading explicit in order to determine the components of our complex in each degree $* \geq 0$. We get the following propositions:

**Proposition 5.2.3.** The graded hom-module
\[ \text{Hom}_{\text{gr}} \Lambda \text{Seq}(\Sigma^{-1} K_1(\text{Gerst}), \Sigma E_0^0 p)_s = \text{Hom}_{\text{gr}} \Lambda \text{Seq}(K_1(\text{Gerst}), E_0^0 p)_{s-2}, \]
which defines the component of our complex §5.2.2(1) in horizontal degree $s = 0$, vanishes in total degree $* > 1$, and reduces to the modules
\[ \text{Hom}_{\text{gr}} \Lambda \text{Seq}(\Sigma^{-1} K_1(\text{Gerst}), \Sigma E_0^0 p)_1 = \text{Hom}_{Q[\Sigma_2]}(\underbrace{\mathbb{Q} p_2^{-2} \otimes [x_1, x_2]^\vee, E_0^0 p(2)}_{= \Lambda^{-2} \text{Gerst}_2(2)^\vee}), \]
\[ \text{Hom}_{\text{gr}} \Lambda \text{Seq}(\Sigma^{-1} K_1(\text{Gerst}), \Sigma E_0^0 p)_0 = \text{Hom}_{Q[\Sigma_2]}(\underbrace{\mathbb{Q} p_2^{-2} \otimes (x_1 x_2)^\vee, E_0^0 p(2)}_{= \Lambda^{-2} \text{Gerst}_2(2)^\vee}), \]
in degrees $* = 0, 1$. Both modules have rank one, with a generating element such that $f(p_2^{-2} \otimes [x_1, x_2]^\vee) = t_{12}$ in degree 1, and a generating element such that $f(p_2^{-2} \otimes (x_1 x_2)^\vee) = t_{12}$ in degree 0.

**Proof.** The hom-module identities of this proposition follow from a straightforward inspection of the gradings attached to our objects (as we explained before our statement).

Recall that the transposition $(1 \ 2) \in \Sigma_2$ acts trivially on both the Lie product operation $\Lambda(x_1, x_2) = [x_1, x_2] \in \text{Gerst}_2(2)$, and the commutative product operation $\mu(x_1, x_2) = x_1 x_2 \in \text{Gerst}_2(2)$. We similarly have $(1 \ 2)t_{12} = t_{12}$ in the Drinfeld-Kohno Lie algebra $p(2)$, and the formulas of the proposition therefore return equivariant maps which span our modules of homomorphisms in degree $* = 0, 1$. \[ \square \]

**Proposition 5.2.4.** The graded hom-module
\[ \text{Hom}_{\text{gr}} \Lambda \text{Seq}(\Sigma^{-1} K_2(\text{Gerst}_2), \Sigma E_0^0 p)_s = \text{Hom}_{\text{gr}} \Lambda \text{Seq}(K_2(\text{Gerst}_2), E_0^0 p)_{s-2}, \]
which defines the components of our complex §5.2.2(1) in horizontal degree $s = 1$, vanishes in total degree $* > 0$, and reduces to the module
\[ \text{Hom}_{\text{gr}} \Lambda \text{Seq}(\Sigma^{-1} K_2(\text{Gerst}_2), \Sigma E_0^0 p)_0 = \text{Hom}_{Q[\Sigma_2]}(\underbrace{\mathbb{Q} p_3^{-2} \otimes (Q[x_1, x_2] \cdot x_3)^\vee \oplus Q[x_1, x_3, x_2]^\vee, E_0^0 p(3)}_{= \Lambda^{-2} \text{Gerst}_2(3)^\vee}, \]
in total degree $* = 0$. We moreover have $SE_m^0 p(3) = L_m(t_{12}, t_{23})$, for every weight $m > 1$, while $SE_m^0 p(3) = 0$ for $m = 1$, and any equivariant homomorphism $f \in$
where \( h(t_{12}, t_{23}) \in \mathbb{L}_m(t_{12}, t_{23}) \) is a solution of the anti-involution and hexagon equations of the graded Grothendieck-Teichmüller Lie algebra \( \mathfrak{g} \mathfrak{t} \mathfrak{t} \) (in weight \( m \)).

**Proof.** We again obtain our identity of hom-modules from a straightforward inspection of the gradings attached to our objects.

Recall that the element \( z = t_{12} + t_{23} + t_{13} \) lies in the center of the Drinfeld-Kohno Lie algebra \( p(3) \). We can therefore identify the generating elements \( t_{12}, t_{23}, t_{13} \) of this Lie algebra \( p(3) \) with variables satisfying \( t_{12} + t_{23} + t_{13} = 0 \) when we work in the components of weight \( m > 1 \) in \( E^0 p(3) \). We may also eliminate this variable to get the relation \( SE^0_m p(3) = \mathbb{L}_m(t_{12}, t_{23}) \) used in our statement for any \( m > 1 \). We have by equivariance:

\[
\begin{align*}
\pi f(\rho_3^{-2} \otimes [[x_1, x_2], x_3]) & = h(t_{12}, t_{23}) \Rightarrow \pi f(\rho_3^{-2} \otimes [[x_1, x_3], x_2]) = h(t_{13}, t_{23}),
\end{align*}
\]

and from the relations

\[
\begin{align*}
(13) & : [[x_1, x_2], x_3] = [[x_3, x_2], x_1] = -[[x_1, x_2], x_3] - [[x_1, x_3], x_2] \\
(13) & : [[x_1, x_3], x_2] = [[x_3, x_1], x_2] = [[x_1, x_3], x_2]
\end{align*}
\]

we get that the equivariance relation \( f((13) - (13)) : f(-) \) for our homomorphism \( f \in \text{Hom}_{\Sigma_3 \text{Mod}}(\Lambda^{-2} \text{Gerst}_2(3)^\vee, SE^0_m p(3)) \) is equivalent to the equations:

\[
\begin{align*}
& h(t_{23}, t_{12}) = -h(t_{12}, t_{23}) \\
& h(t_{13}, t_{12}) = -h(t_{12}, t_{23}) + h(t_{13}, t_{23}),
\end{align*}
\]

for our Lie algebra element \( h(t_{12}, t_{23}) \in \mathbb{L}_m(t_{12}, t_{23}) \). Hence, we exactly retrieve the anti-involution and hexagon equations of the graded Grothendieck-Teichmüller Lie algebra up to a trivial rewriting of the terms in the hexagon equation (see §I.77). □

**Proposition 5.2.5.** The graded hom-modules

\[
\text{Hom}_{\Lambda \text{Seq}}(\Sigma^{-1} K_{s+1}(\text{Gerst}_2), E^0 p)_s = \text{Hom}_{\Lambda \text{Seq}}(K_s(\text{Gerst}_2), E^0 p)_{s-2},
\]

which defines the components of our complex §5.2.2(1) entirely vanish in total degree \( s \geq 0 \) when \( s > 1 \).

**Proof.** We simply use that any monomial \( \pi(x_1, \ldots, x_r) \in \text{Gerst}_2(r) \) has degree \( \deg(\pi) \leq r - 1 \), to obtain that the module \( \Lambda^{-2} \text{Gerst}_2(r)^\vee_k = (\text{Gerst}_2(r)_{k+2(r-1)})^\vee \), giving the components of total degree \( s = k + 2 \Rightarrow k = s - 2 \) of our graded hom-module \( \text{Hom}_{\Lambda \text{Seq}}(\Sigma^{-1} K_{s+1}(\text{Gerst}_2), E^0 p)_s \), vanishes when \( k > 1 - r \Rightarrow s > 3 - r \), and the result of the proposition follows (since \( r = s + 2 \)). □

To complete our computation, we just have to determine the component of the twisting differential of our complex on the homomorphisms of Proposition 5.2.4. We review the definition of our twisting differential, for the derivation complex on the Koszul construction of the \( m \)-Gerstenhaber operad \( \text{Gerst}_m \), for any \( m \geq 2 \), and...
we go back to the general case of a graded Drinfeld-Kohno Lie algebra operad $p_n$, with $n \geq 2$, before tackling this computation.

Let us simply observe that, when we consider the homogeneous component of weight 1 of the graded object $E^0 p$ as coefficients in our complex §5.2.1(1), we get a derivation complex which is concentrated in horizontal degree $s = 0$ and reduces to the modules considered in Proposition 5.2.3. The homomorphisms given in this Proposition 5.2.3 have therefore a trivial twisting differential for degree reason.

5.2.6. The expression of the twisting differential on the derivation complex for the Gerstenhaber operad. We apply the general definition of the twisting differential in §4.2.3(5.6) to the particular case of the $m$-Gerstenhaber operad $Gerst_m$.

We use the basis of the module $\Lambda^{-m} Gerst_m(r)^\vee$ defined by the (desuspension of the) dual of the basis elements of the $m$-Gerstenhaber operad in §5.1.3. We basically dualize the map

\[
\bigoplus_{\Gamma \in \mathcal{T}ree_2(r)} \big\{ \Lambda^m Gerst_m(l) \big\} \big\{ \Lambda^m Gerst_m(k) \big\} \big/ iso \big\{ \Lambda^m Gerst_m(k+l-1) \big\},
\]

giving the composition structure of the operad $\Lambda^m Gerst_m$, to determine the tree-wise coproducts

\[
\rho_r^{\vee}(\varphi^{-m} \otimes \pi(x_1, \ldots, x_r)^\vee) = \sum_{\rho_l^{\vee}(\varphi^{-m} \otimes \pi^l)} \big\{ \begin{array}{c}
\Lambda^m Gerst_m(l) \\
\Lambda^m Gerst_m(k) \\
\end{array} \big\} \big\{ \begin{array}{c}
\varphi^{-m} \otimes \pi(x_1, \ldots, x_k)^\vee \\
\varphi^{-m} \otimes \pi(x_1, \ldots, x_k)^\vee \\
\end{array} \big\},
\]

for our basis elements $\varphi^{-m} \otimes \pi(x_1, \ldots, x_r)^\vee \in \Lambda^{-m} Gerst_m(r)^\vee$, and in turn, to get the expression of our twisting differential:

\[
\partial'(f)(\varphi^{-m} \otimes \pi(x_1, \ldots, x_r)^\vee)
\]

\[
= \sum_{\rho_l^{\vee}(\varphi^{-m} \otimes \pi^l)} \begin{pmatrix}
\pm \lambda \\
\end{pmatrix} \bigg\{ \begin{array}{c}
\varphi^{-m} \otimes \pi^l \otimes \pi'(x_{j_1}, \ldots, x_{j_l})^\vee \\
\varphi^{-m} \otimes \pi'(x_{j_1}, \ldots, x_{j_l})^\vee \\
\end{array} \bigg\} \bigg\{ \begin{array}{c}
f\varphi^{-m} \otimes \pi(x_{i_1}, \ldots, x_{i_k})^\vee \\
f\varphi^{-m} \otimes \pi'(x_{i_1}, \ldots, x_{i_k})^\vee \\
\end{array} \bigg\},
\]

\[
+ \pm \lambda \bigg\{ \begin{array}{c}
\varphi^{-m} \otimes \pi^l \otimes \pi'(x_{j_1}, \ldots, x_{j_l})^\vee \\
\varphi^{-m} \otimes \pi'(x_{j_1}, \ldots, x_{j_l})^\vee \\
\end{array} \bigg\} \bigg\{ \begin{array}{c}
f\varphi^{-m} \otimes \pi(x_{i_1}, \ldots, x_{i_k})^\vee \\
f\varphi^{-m} \otimes \pi'(x_{i_1}, \ldots, x_{i_k})^\vee \\
\end{array} \bigg\},
\]
for any homomorphism \( f : \Lambda^{-m} \text{Gerst}^\vee \to E^0 p \). (We just have to take of extra signs arising from the position of suspension symbols when we take this twisting differential.)

Recall that the augmentation \( \epsilon : \Lambda^{-m} \text{Gerst}^\vee \to \text{Com}(r) \) maps the element \( \rho_2^{-m} \otimes \lambda(x_1, x_2)^\vee \) to the commutative product operation \( \mu \in \text{Com}(2) \) and vanishes otherwise. We therefore only need to determine the composites involving this factor \( \rho_2^{-m} \otimes \lambda(x_1, x_2)^\vee \) in the expression of our tree-wise coproduct (2). Then we use the abelian \( \text{Com} \)-bimodule structure on the graded object \( E^0 p_n \) underlying the Drinfeld-Kohno Lie algebra operad \( p_n \), and the action of the operation \( \mu \in \text{Com}(2) \) through this bimodule structure on \( E^0 p_n \), to perform to the tree-wise composition operations occurring in the expression of our twisting differential. By definition of this action, we get:

\[
\begin{align*}
t_{ij} \circ_k \mu &= \begin{cases} 
  t_{i+1,j+1}, & \text{if } k < i < j, \\
  t_{ij+1} + t_{i+1,j+1}, & \text{if } k = i < j, \\
  t_{ij+1}, & \text{if } i < k < j, \\
  t_{ij} + t_{ij+1}, & \text{if } i < k = j, \\
  t_{ij}, & \text{if } i < j < k,
\end{cases}
\end{align*}
\]

\[\mu \circ_k t_{ij} = t_{i+k-1,j+k-1}\]

for any \( t_{ij} \in p_n(r), \ r > 1, \ k = 1, \ldots, r \), and we have

\[
\pi(t_{i,j_1}, \ldots, t_{i,j_l}) \circ_k \mu = \pi(t_{i,j_1} \circ_k \mu, \ldots, t_{i,j_l} \circ_k \mu),
\]

\[
\mu \circ_k \pi(t_{i,j_1}, \ldots, t_{i,j_l}) = \pi(\mu \circ_k t_{i,j_1}, \ldots, \mu \circ_k t_{i,j_l}),
\]

for any \( \pi = \pi(t_{i,j_1}, \ldots, t_{i,j_l}) \in p_n(r) \).

**Proposition 5.2.7.** The boundary relation \( \partial^\nu(f) = 0 \) for a homomorphism \( f \in \text{Hom}_{\text{gr} \Lambda^s \Sigma \equiv} (\Sigma^{-1} \text{Ker}(\text{Gerst}_2), \Sigma E^0 p)_0 \) of degree 0 is equivalent to the pentagon equation of the graded Grothendieck-Teichmüller Lie algebra \( \text{grt} \) for the Lie polynomial \( h(t_{12}, t_{23}) \in \mathbb{L}(t_{12}, t_{23}) \) which we associate to such a homomorphism in Proposition 5.2.4.

**Proof.** In Proposition 5.2.4, we basically observed that our homomorphism \( f \in \text{Hom}_{\text{gr} \Lambda^s \Sigma \equiv} (\Sigma^{-1} \text{Ker}(\text{Gerst}_2), \Sigma E^0 p)_0 \) is given by an equivariant map

\[
f : \Lambda^{-2}(\Lambda^{-1} \text{Lie}(3))^\vee \to E^0 p(3)
\]

defined on the dual of the desuspended Lie operad \( \Lambda^{-1} \text{Lie} \), which we identify with a direct summand of the symmetric sequence \( \text{Ker}(\text{Gerst}_2) = \Lambda^{-2} \text{Gerst}^\vee_2 \).

We readily check that the composition product of the 2-Gerstenhaber operad preserves the desuspended Lie operad \( \Lambda^{-1} \text{Lie} \) as a quotient object. We deduce from this observation that we can reduce the computation of our twisting differential in §5.2.6 to the elements of this symmetric sequence \( (\Lambda^{-1} \text{Lie})^\vee \subset \text{Gerst}^\vee_2 \) (which also forms a sub-cooperad of the dual of the 2-Gerstenhaber operad \( \text{Gerst}^\vee_2 \)).

We use that the module \( \Theta_2(\Lambda^2(\Lambda^{-1} \text{Lie}))(4) \) has a basis consisting of the tree-wise tensors represented on the left-hand side of Figure 5.2.6, where the marks \( i, j, k \ldots \) run over appropriate index permutations. We determine the tree-wise composition products associated to these elements in our basis of the module \( \Lambda^2(\Lambda^{-1} \text{Lie}))(4) \subset \Lambda^2 \text{Gerst}_2(4) \). We readily deduce from this computation that our twisting differential
satisfies:
\[ \partial''(f)(\rho_4^{-2} \otimes [[[x_1, x_2], x_3], x_4]^\vee) \]
\[ = f(\rho_3^{-2} \otimes [[[x_1, x_2], x_3]^\vee) \circ_1 \mu - f(\rho_3^{-2} \otimes [[[x_1, x_2], x_3]^\vee) \circ_2 \mu + f(\rho_3^{-2} \otimes [[[x_1, x_2], x_3]^\vee) \circ_3 \mu - \mu \circ_1 f(\rho_3^{-2} \otimes [[[x_1, x_2], x_3]^\vee) - \mu \circ_2 f(\rho_3^{-2} \otimes [[[x_1, x_2], x_3]^\vee) \]
\[ = h(t_{12}, t_{23}) \circ_1 \mu - h(t_{12}, t_{23}) \circ_2 \mu + h(t_{12}, t_{23}) \circ_3 \mu - h(t_{12}, t_{23}) - h(t_{12}, t_{23}) \]
\[ = h(t_{13} + t_{23}, t_{34}) - h(t_{12} + t_{13}, t_{24} + t_{34}) + h(t_{12}, t_{23} + t_{24}) - h(t_{12}, t_{23}) - h(t_{23}, t_{34}). \]

Figure 5.1. The composites of the basis of tree-wise tensors of arity 4 in the suspended Lie operad
and hence, we retrieve the pentagon condition in the definition of the graded Grothendieck-Teichmüller Lie algebra $\text{grt}$ when we take the boundary equation $\partial^p(f)(p_4^{-2} \otimes ([x_1, x_2], x_3) x_4) = 0$. We just note that the validity of this boundary relation on the element $p_4^{-2} \otimes \pi(x_1, x_2, x_3, x_4) = p_4^{-2} \otimes \pi(x_1, x_i, x_j, x_k)$ implies that our relation holds for all basis elements $p_4^{-2} \otimes \pi(x_1, x_i, x_j, x_k)$ by equivariance.

We record the outcome of our cohomology computations and we summarize the applications of these results to our homotopy spectral sequence in the following statement. We go back to the notation of Theorem C. Recall that we have $\Sigma E^0 p = E^0 p[1]$ (see §2.2.5) and $K(E^0 p, 1) = \Gamma_\ast(E^0 p)$ is identified with the additive operad in simplicial modules consisting of the collection of Eilenberg-MacLane spaces $K(E^0 p(r), 1) = \Gamma_\ast(E^0 p(r), 1)$.

**Theorem 5.2.8.** We consider the homotopy spectral sequence associated to the function space of the cotriple resolution of an $E_2$-operad $\text{Res}_\ast(E_2)$, and where we take the Eilenberg-MacLane space operad $K(E^0 m p, 1)$ on a homogeneous component $E^0_m p$ of a fixed weight $m \geq 1$ of the Drinfeld-Kohno Lie algebra operad $p = \{p(r), r > 1\}$ as target object:

$$ I^t = E^r(\text{Map}_{sSet \Lambda Op}(\text{Res}_\ast(E_2), K(E^0_m p, 1))). $$

(a) In the case $m = 1$, we have $I_{00}^t = I_{01}^0 = Q t_{12}$ and $I_{12}^s = 0$ for $t > 1$ or $s > 0$. The cycles $z \in I_{0t}^2$ which span these spectral sequence terms and which we associate to elements $c \in Q t_{12}$ are given by the homomorphisms of Proposition 5.2.3 through the correspondence of Theorem 4.1.4 and Theorem 4.1.6.

(b) In the case $m > 1$, we have $I_{11}^t = \text{grt}_{m-1}$, the homogeneous component of weight $m$ of the graded Grothendieck-Teichmüller Lie algebra $\text{grt}$, and we have $I_{ad}^s = 0$ for $(s, t) \neq (1, 1)$. The cycles $z \in I_{ad}^t$ which span these spectral sequence terms and which we associated to elements $h = h(t_{12}, t_{23}) \in \text{grt}_{m-1}$ are given by the homomorphisms of Proposition 5.2.4 through the correspondence of Theorem 4.1.4 and Theorem 4.1.6.

Recall that we take $k = Q$ as ground ring by convention. Let us mention, nonetheless, that the above theorem (as well as our intermediate propositions) remains valid as long as we take a characteristic zero field as ground ring.

We now have to interpret the result of this spectral sequence computation. We tackle this task in the next chapter.
The Interpretation of the Result of the Spectral Sequence in the $E_2$-operad Case

The goal of this chapter is to complete the proof of our main theorems. Namely: we check that the Grothendieck-Teichmüller group maps bijectively onto the group of homotopy classes of (rational) homotopy automorphisms of $E_2$-operads, while the connected component of the identity of this homotopy automorphism space is weakly-equivalent to the (rationalization of the) group of rotations. We also check that the whole homotopy automorphism space of a rational $E_2$-operad is weakly-equivalent, as a simplicial monoid, to the semi-direct product of the Grothendieck-Teichmüller group and of this classifying space.

We examine the correspondence between the Grothendieck-Teichmüller group and the group of homotopy classes of homotopy automorphisms first (§ 6.1). We address the equivalence between the rationalization of the group of rotations and the connected component of the identity in our homotopy automorphism space afterwards (§6.2).

For simplicity, we still take the field of rational numbers as ground ring all through this chapter $k = \mathbb{Q}$, so that we can entirely interpret our results in the setting of §§9-10, where we define our model for the rational homotopy of operads. Nonetheless, we use explicit models of the rationalization in the $E_2$-operad case rather than the general construction of the model, which we could also use to give a generalization of our result in the case where the ground ring is a field of characteristic zero.

To be explicit, we take the classifying space of the parenthesized braid operad $E_2 = B(PaB)$ as a working model of an $E_2$-operad. We still use the cotriple resolution construction to get a cofibrant resolution of this operad in the category of simplicial sets:

$$R = |\text{Res}_\bullet (B(PaB))|.$$  

Recall that we write $PaB^\sim$ for the operad in groupoids which we obtain by applying the Malcev completion process arity-wise to the parenthesized braid operad $PaB$. We replace the parenthesized braid operad $PaB$ by this completion $PaB^\sim$ to get our model for the rationalization of $E_2$-operads in the category of simplicial sets. We accordingly set $E_2 = B(PaB^\sim)$ and we similarly take:

$$R^\sim = |\text{Res}_\bullet (B(PaB^\sim))|$$

when we need a cofibrant resolution of this operad.

In what follows, we mainly use the interpretation of the parenthesized braid operad $PaB$ as the operad in groupoids that governs the category of braided monoidal
categories. In §1.6.2, we more precisely explain that a morphism over the parenthesized braid operad $\nu: PaB \to Q$, where $Q$ is an operad in the category of categories, is determined by a multiplication operation $m \in \text{Ob } Q(2)$ together with the isomorphisms of a braided monoidal structure in the morphism sets of the operad $Q$. We use this correspondence and the functoriality of our constructions to define our mapping from the Grothendieck-Teichmüller group towards the space of homotopy automorphisms of a rational $E_2$-operad in simplicial sets.

In a preliminary step, we give short recollections on the definition of the Grothendieck-Teichmüller group, and we explain with more details the definition of this mapping from the Grothendieck-Teichmüller group towards the space of homotopy automorphisms of a rational $E_2$-operad in simplicial sets.

### 6.0. Recollections on the Grothendieck-Teichmüller group

We actually prove our statement indirectly, by using a natural action of the Grothendieck-Teichmüller group on the set of the Drinfeld associators, and by associating an element of the function space $\text{Map}_{\Lambda OP}(\vert \text{Res}_* (B(PaB)) \vert, B(\hat{CD}))$ to any Drinfeld associator. We then use the spectral sequences of the previous chapters to check that this mapping induces a bijection at the homotopy level. We therefore review the definition of both the Grothendieck-Teichmüller group and the set of Drinfeld’s associators first. We also explain the definition of counterparts, on the set of Drinfeld’s associators, of structures which we use in the definition of our homotopy spectral sequences.

#### 6.0.1. The definition of the Grothendieck-Teichmüller group and associators

We consider the Malcev completion of the operad of Parenthesized braids as we briefly mention in the chapter introduction. In §I.10.1, we formally define the Grothendieck-Teichmüller group $GT(Q)$ as the group formed by the automorphisms of the non-unitary $\Lambda$-operad in Malcev complete groupoids $PaB^\sim$ which are given by the identity mapping on objects.

Recall that any morphism $\phi^\sim: PaB^\sim \to PaB^\sim$ is the extension of a morphism defined on the parenthesized braid operad $PaB$ (see §I.9.2.7). We may therefore more effectively define an element of the Grothendieck-Teichmüller group $GT(Q)$ as a morphism of non-unitary $\Lambda$-operads in groupoids $\phi: PaB \to PaB^\sim$ which is the identity mapping on objects and induces an isomorphism on the morphism sets of our operads when we pass to the completion (see §I.10.1.2).

The set of Drinfeld’s associators, denoted by $Ass(Q)$, and which we therefore use in our study of homotopy automorphisms, is formally defined as the set of arity-wise equivalences of non-unitary $\Lambda$-operads in Malcev complete groupoids $\nu^\sim: PaB^\sim \to CD^\sim$ from the completed parenthesized braid operad $PaB^\sim$ to the operad of chord diagrams $CD^\sim$. We may still use that any such morphism $\nu^\sim: PaB^\sim \to CD^\sim$ is the extension of a morphism defined on the parenthesized braid operad $PaB$. We mostly use this representation in what follows, and therefore, we rather define an element of the set of Drinfeld’s associators $Ass(Q)$ as a morphism of non-unitary $\Lambda$-operads in groupoids $\nu: PaB \to CD^\sim$ which induces an arity-wise equivalence of operads in groupoids when we pass to the completion (see §I.10.2.15).

We have a natural action of the Grothendieck-Teichmüller group $GT(Q)$ on the set of the Drinfeld associators $Ass(Q)$ by translation on the right. To be explicit, we assume that we have a morphism $\nu: PaB \to CD^\sim$ which represents an element of the set of associators $\nu \in Ass(Q)$. The morphism, denoted by $\nu \circ \gamma: PaB \to CD^\sim$,
which represents the action of an element of the Grothendieck-Teichmüller group \( \gamma \in GT(\mathbb{Q}) \) on this object \( v \in \text{Ass}(\mathbb{Q}) \), can be defined as the composite

\[
\begin{align*}
\gamma : & \quad \text{PaB} \xrightarrow{\gamma} \text{PaB}^\sim \\
& \quad \downarrow \quad \uparrow \quad \exists \gamma^*\text{PaB} \\
& \quad \text{PaB} \xrightarrow{\exists} \text{CD}^\sim,
\end{align*}
\]

where we consider the extension of our morphism \( v : \text{PaB} \to \text{CD}^\sim \) to the completion of the parenthesized braid operad \( \text{PaB}^\sim \).

6.0.2. The tower decomposition of the set of Drinfeld’s associators. We use that the set of Drinfeld’s associators admits a decomposition as the limit of a tower of quotient objects. This tower represents a discrete of the tower of function spaces which we consider in the definition of our homotopy spectral sequence.

We still rely on the natural decomposition of the chord diagram operad \( \text{CD}^\sim = \lim_m \text{q}_m \text{CD}^\sim \) such that \( \text{q}_m \text{CD}(r)^\sim = G \mathbb{U}(p(r)/F_{m+1} p(r)) \), for each arity \( r > 0 \), where we consider the filtration of the Drinfeld-Kohno Lie algebra \( p(r) \) by the weight (see §12.1.2). Recall that the chord diagram operad actually forms an operad in the category of groups, which are just identified with groupoids with a single object.

We have the relation

\[
\text{Mor}_{\text{Grd}_2 \text{op}}(\text{PaB}, \text{CD}^\sim) = \lim_m \text{Mor}_{\text{Grd}_2 \text{op}}(\text{PaB}, \text{q}_m \text{CD}^\sim)
\]

at the morphism set level, and we just set

\[
\text{q}_m \text{Ass}(\mathbb{Q}) = \text{im}(\text{Ass}(\mathbb{Q}) \to \text{Mor}_{\text{Grd}_2 \text{op}}(\text{PaB}, \text{q}_m \text{CD}^\sim))
\]

for any \( m \geq 0 \), to define our tower decomposition of the set of Drinfeld’s associators.

We are going to use that the parenthesized braid operad \( \text{PaB} \) represents the operad in groupoids that governs the category of a braided monoidal categories. We go back to this interpretation in a subsequent paragraph. We just use for the moment that the component of arity 2 of an operad morphism \( v : \text{PaB} \to \text{CD}^\sim \) is determined by an assignment \( v(\tau) = \exp(kt_{12}) \), where \( \tau \in \text{Mor} \text{PaB}(2) \) is a morphism of the parenthesized braid operad which models the symmetry isomorphism of braided monoidal categories, and \( k \in \mathbb{Q} \) is a multiplicative parameter. We have a similar formula \( v(\tau) = \exp(kt_{12}) \) for the morphisms \( v : \text{PaB} \to \text{q}_m \text{CD}^\sim \) occurring at a level \( m \geq 1 \) of our tower (1-2). In the case \( m = 1 \), this correspondence \( v \mapsto v(\tau) = \exp(kt_{12}) \) gives a bijection:

\[
\text{Mor}_{\text{Grd}_2 \text{op}}(\text{PaB}, \text{q}_1 \text{CD}^\sim) = \exp(\mathbb{Q} t_{12}),
\]

because all basis elements of weight 1 of the Drinfeld-Kohno Lie algebra operad \( t_{ij} \in p(r) \) are identified with the image of the arity 2 element \( t_{12} \in p(2) \) under a restriction operation.

We observed in Proposition I.10.2.13 that an operad morphism \( v : \text{PaB} \to \text{CD}^\sim \) defines a categorical equivalence arity-wise (and hence, an element of the set of Drinfeld’s associators) if and only if the corresponding scalar parameter \( k \in \mathbb{Q} \) such that \( v(\tau) = \exp(kt_{12}) \) is invertible. We accordingly have the identity:

\[
\text{q}_0 \text{Ass}(\mathbb{Q}) = \exp(\mathbb{Q}^\times t_{12}),
\]
at level $m = 0$, and we have the relation

$$q_m \text{Ass}(\mathbb{Q}) = \text{Mor}_{\mathcal{G}_\Lambda \circ p}(P\mathcal{A}, q_{m+1} \mathcal{CD}) \times_{\text{exp}(\mathbb{Q} t_{12})} \text{exp}(\mathbb{Q}^\times t_{12})$$

for any $m \geq 0$. We also obtain that each map $q_m \text{Ass}(\mathbb{Q}) \to q_{m-1} \text{Ass}(\mathbb{Q})$ forms a surjection, and we still have the relation $\text{Ass}(\mathbb{Q}) = \lim_m q_m \text{Ass}(\mathbb{Q})$ at the limit level, so that our construction (1-2) returns a tower decomposition of the set of Drinfeld’s associators well.

6.0.3. The filtration of the Grothendieck-Teichmüller group. We readily check that the action of the Grothendieck-Teichmüller group $GT(\mathbb{Q})$ on $\text{Ass}(\mathbb{Q})$ goes down to each quotient of this decomposition $\text{Ass}(\mathbb{Q}) = \lim_m q_m \text{Ass}(\mathbb{Q})$. We can moreover provide $GT(\mathbb{Q})$ with a filtration

$$(1) \quad GT(\mathbb{Q}) = F_0 GT(\mathbb{Q}) \supset \cdots \supset F_m GT(\mathbb{Q}) \supset F_{m+1} GT(\mathbb{Q}) \supset \cdots$$

such that $F_{m+1} GT(\mathbb{Q})$, $m \geq 0$, is the normal sub-group of the Grothendieck-Teichmüller group $GT(\mathbb{Q})$ that consists of elements $\gamma \in GT(\mathbb{Q})$ which act trivially on $q_m \text{Ass}(\mathbb{Q})$. We explicitly have $\gamma \in F_{m+1} GT(\mathbb{Q})$ if and only if we have the relation $\bar{v} \circ \gamma \equiv \bar{v}$ in $q_m \text{Ass}(\mathbb{Q})$, for every $\bar{v} \in q_m \text{Ass}(\mathbb{Q})$. We actually have $\gamma \in F_{m+1} GT(\mathbb{Q})$ as soon as $\gamma \in GT(\mathbb{Q})$ fixes an element $\bar{v}$ in the set $q_m \text{Ass}(\mathbb{Q})$, and $GT(\mathbb{Q})/F_{m+1} GT(\mathbb{Q})$ acts freely (and transitively) on $q_m \text{Ass}(\mathbb{Q})$.

We moreover have the relation $(F_m GT(\mathbb{Q}), F_n GT(\mathbb{Q})) \subset F_{m+n} GT(\mathbb{Q})$, for any $m, n \geq 0$, as well as $\bigcap_m F_m GT(\mathbb{Q}) = 0$ and $GT(\mathbb{Q}) = \lim_m GT(\mathbb{Q})/F_m GT(\mathbb{Q})$, so that $GT(\mathbb{Q}) = F_1 GT(\mathbb{Q})$ forms a pro-unipotent group (see §I.10.1.7). We actually have

$$(2) \quad \mathfrak{g} \mathfrak{t}_m = F_m GT(\mathbb{Q})/F_{m+1} GT(\mathbb{Q}),$$

for any $m > 0$, where $\mathfrak{g} \mathfrak{t}$ is the graded Grothendieck-Teichmüller Lie algebra (of which we just retrieved the definition in the spectral sequence computation of the previous chapter), while we get:

$$(3) \quad GT(\mathbb{Q})/F_1 GT(\mathbb{Q}) = \mathbb{Q}^\times$$

for the first sub-quotient of the filtration. In §I.??, we also take $\mathfrak{g} \mathfrak{t}_0 = \mathbb{Q}$ to extend the definition of the graded Grothendieck-Teichmüller Lie algebra $\mathfrak{g} \mathfrak{t}$ in weight 0. In short, we consider the Lie algebra of the multiplicative group which represents this first sub-quotient of our filtration on the Grothendieck-Teichmüller group. In what follows, we also use the exponential notation $e^\xi$ to refer to the element of the group $F_m GT(\mathbb{Q})/F_{m+1} GT(\mathbb{Q})$ associated to an element $\xi \in \mathfrak{g} \mathfrak{t}_m$, because we use the exponential mapping (see §I.8.1) in the definition of our identity (2).

From the observation that the group $GT(\mathbb{Q})/F_{m+1} GT(\mathbb{Q})$ acts freely (and transitively) on $q_m \text{Ass}(\mathbb{Q})$, we immediately get that elements $\bar{v}, \bar{v}_0 \in q_m \text{Ass}(\mathbb{Q})$ are identified in the set $q_{m-1} \text{Ass}(\mathbb{Q})$ if and only if we have the relation $\bar{v} = \bar{v}_0 \circ e^\xi$ in $q_m \text{Ass}(\mathbb{Q})$ for some sub-quotient group element $e^\xi \in F_m GT(\mathbb{Q})/F_{m+1} GT(\mathbb{Q})$ associated to a homogeneous element of weight $m$ of the graded Grothendieck-Teichmüller Lie algebra $\xi \in \mathfrak{g} \mathfrak{t}_m$. In §I.??, we use this relationship to get an explicit definition of the module $\mathfrak{g} \mathfrak{t}_m$ in terms of a sub-quotient of the set of Drinfeld’s associators. We revisit this correspondence in the next paragraph.

6.0.4. The graded Grothendieck-Teichmüller Lie algebra and the tower decomposition of associators. In our homotopy spectral sequence, we use the fiber of our maps of function spaces to compare the levels of our tower. We basically aim to
obtain a similar correspondence for our tower decomposition of the set of Drinfeld’s associators. We then consider the homogeneous components $\mathcal{E}_m^0 \mathfrak{p}$, $m > 0$, of the Drinfeld-Kohno Lie algebra operad $\mathfrak{p}$. We use the exponential mapping $\exp : \xi \mapsto \exp(\xi)$ to embed the additive group $\mathcal{E}_m^0 \mathfrak{p}(r) = F_m \mathfrak{p}(r)/F_{m+1} \mathfrak{p}(r)$, into the group $\mathcal{Q}_m \mathcal{C} \mathcal{D}(r)^\sim = \mathcal{G} \hat{\bigcup}(\mathfrak{p}(r)/F_{m+1} \mathfrak{p}(r))$, for any arity $r > 0$. We use that the collection of these maps $\exp : \mathcal{E}_m^0 \mathfrak{p}(r) \to \mathcal{Q}_m \mathcal{C} \mathcal{D}(r)^\sim$ defines a morphism of non-unitary $\Lambda$-operads $\exp : \mathcal{E}_m^0 \mathfrak{p} \to \mathcal{Q}_m \mathcal{C} \mathcal{D}$, for each $m \geq 1$. We then regard the additive operad $\mathcal{E}_m^0 \mathfrak{p}$ as an additive operad in the category of abelian groups. We also use, in what follows, that these exponential maps define an isomorphism between the abelian groups $\mathcal{E}_m^0 \mathfrak{p}(r)$ and the kernels of the quotient maps of our tower $\mathcal{Q}_m \mathcal{C} \mathcal{D}(r)^\sim \to \mathcal{Q}_{m-1} \mathcal{C} \mathcal{D}(r)^\sim$, in any arity $r > 0$, which moreover lie in the center of the groups $\mathcal{Q}_m \mathcal{C} \mathcal{D}(r)^\sim = \mathcal{G} \hat{\bigcup}(\mathfrak{p}(r)/F_{m+1} \mathfrak{p}(r))$, for all $m > 1$.

We consider the morphism sets

$$(1) \quad \mathcal{E}_m^0 \text{Ass}(\mathcal{Q}) = \text{Mor}_{\mathcal{G}_r \mathcal{D} \Lambda \mathcal{O}_P}(\mathcal{P} B \mathcal{E}_m^0 \mathfrak{p})$$

for all $m \geq 0$, where we again regard the additive operad $\mathcal{E}_m^0 \mathfrak{p}$ as an operad in (abelian) groups. We easily see that this object $\mathcal{E}_m^0 \text{Ass}(\mathcal{Q})$ inherits an additive group structure on the target, for each $m \geq 0$, and acts on the set $\mathcal{Q}_m \text{Ass}(\mathcal{Q})$ which defines the $m$th level of our tower decomposition of the set of Drinfeld’s associators, when we assume $m \geq 1$. We then use the arity-wise action of the additive group $\mathcal{E}_m^0 \mathfrak{p}(r)$ on $\mathcal{Q}_m \mathcal{C} \mathcal{D}(r)^\sim = \mathcal{G} \hat{\bigcup}(\mathfrak{p}(r)/F_{m+1} \mathfrak{p}(r))$ through the exponential map, for $r > 0$. We also get that elements $\bar{v}, \bar{v}_0 \in \mathcal{Q}_m \text{Ass}(\mathcal{Q})$ have the same image in $\mathcal{Q}_{m-1} \text{Ass}(\mathcal{Q})$, for $m \geq 1$, if and only if we have the relation $\bar{v} = e^{\xi} \cdot \bar{v}_0$ in $\mathcal{Q}_m \text{Ass}(\mathcal{Q})$ for some $\xi \in \text{Mor}_{\mathcal{G}_r \mathcal{D} \Lambda \mathcal{O}_P}(\mathcal{P} B \mathcal{E}_m^0 \mathfrak{p})$.

We now have the identity:

$$(2) \quad \mathcal{G}_r \mathcal{M}_m = \text{Mor}_{\mathcal{G}_r \mathcal{D} \Lambda \mathcal{O}_P}(\mathcal{P} B \mathcal{E}_m^0 \mathfrak{p})$$

for each $m \geq 1$, since we can characterize the homogeneous components of the Grothendieck-Teichmüller Lie algebra by the same relation with respect to the action of the Grothendieck-Teichmüller group on the tower underlying the set of Drinfeld’s associators. We also have

$$(3) \quad \mathcal{G}_r \mathcal{M}_0 = \text{Mor}_{\mathcal{G}_r \mathcal{D} \Lambda \mathcal{O}_P}(\mathcal{P} B \mathcal{E}_0 \mathfrak{p})$$

just because we have $\mathcal{E}_0 \mathfrak{p} = q_1 \mathcal{C} \mathcal{D}^\sim$, and we already mentioned in §6.0.2 that any morphism of non-unitary $\Lambda$-operads $\nu : \mathcal{P} B \to q_1 \mathcal{C} \mathcal{D}^\sim$ is uniquely determined by the assignment $\nu(\tau) = k t_{12}$, $k \in \mathcal{Q}$, in arity 2.

We already briefly recalled, in the introduction of this chapter, that the parenthesized braid operad $\mathcal{P} B$ is identified with the operad governing braided monoidal structures. We also use this result to get an explicit description of the Lie algebra $\mathcal{G}_r$ from the relation $\mathcal{G}_r \mathcal{M}_m = \text{Mor}_{\mathcal{G}_r \mathcal{D} \Lambda \mathcal{O}_P}(\mathcal{P} B \mathcal{E}_m^0 \mathfrak{p})$. We give a brief reminder on this correspondence in the next paragraph in order to complete our recollections on the Grothendieck-Teichmüller group.

6.0.5. The explicit definition of morphisms on the parenthesized braid operad.

We use, to be more precise, that giving a morphism $\nu : \mathcal{P} B \to \mathcal{Q}$ towards any non-unitary $\Lambda$-operad in the category of categories $\mathcal{Q} \in \mathcal{C} \mathcal{A} \mathcal{T} \mathcal{O}_P$ amounts to giving:

1. a multiplication operation $\mu \in \mathcal{O} \mathcal{B} \mathcal{Q}(2)$;
2. a symmetry isomorphism $\tau \in \text{Mor}_{\mathcal{Q}(2)}(\mu(x_1, x_2), \mu(x_2, x_1))$;
3. and an associativity isomorphism $\alpha \in \text{Mor}_{\mathcal{Q}(3)}(\mu(\mu(x_1, x_2), x_3), \mu(x_1, \mu(x_2, x_3)))$
so that the usual unit, hexagon, and pentagon constraints of braided monoidal categories holds within the morphism sets of our operad \( Q \). The parenthesized braid operad \( PaB \) can equivalently be identified with a universal object equipped with such an operation \( \mu \in \text{Ob} \ PaB \), and morphisms \( \tau, \alpha \in \text{Mor} \ PaB \) that formalize the definition of a braided monoidal structure in a category.

In what follows, we also use that the 1-simplex of the simplicial set \( B(PaB) \) associated to the morphism \( \tau^2 : \mu \to \mu \) gives a representative of the Lie algebra bracket of the 2-Gerstenhaber operad in the homology module \( H_*(B(PaB)(2)) = \text{Gerst}_2(2) \), while the 0-simplex defined by the object \( \mu \in \text{Ob} \ PaB(2) \) gives a representative of the multiplication operation in the homology of this operad.

In the case of the additive operad \( E_0^m \ p, m \geq 1 \), which we identify with an operad in groupoids such that \( \text{Ob} \ E_0^m \ p = pt \), the structure encoded by a morphism \( v : PaB \to E_0^m \ p \) is fully determined by giving an arity 2 element \( c \in E_0^m \ p(2) \) such that \( c = v(\tau) \) together with an arity 3 element \( a \in E_0^m \ p(3) \) such that \( a = v(\alpha) \). If \( m = 1 \), then the unit constraints imply that we have \( a = 0 \). We also have \( E_0^1 \ p(2) = \mathbb{Q} t_{12} \) and the constraints of braided monoidal categories are satisfied for any choice of a braiding element \( c \in \mathbb{Q} t_{12} \) completing the trivial associativity isomorphism \( a = 0 \) in this case. If \( m > 1 \), then we have \( E_0^m \ p(2) = 0 \), and we therefore get that our structure is determined by the choice of an element \( a \in E_0^m \ p(3) \). We already used that the module \( E_0^m \ p(3) \) is identified with the homogeneous component of weight \( m \) of the free Lie algebra \( L(t_{12}, t_{23}) \) when \( m > 1 \), because the element \( z = t_{12} + t_{23} + t_{13} \) satisfies \( [t_{12}, z] = [t_{23}, z] = [t_{13}, z] = 0 \). We therefore get that an element \( a \in E_0^m \ p(3) \) is defined by a homogeneous Lie polynomial \( a = h(t_{12}, t_{23}) \in \mathbb{L}_m(t_{12}, t_{23}) \) and when we make the hexagon and pentagon constraints of braided monoidal categories explicit for this associativity isomorphism and the braiding \( c = 0 \), we obtain the equations which we retrieved in the computations of §5.2.

Note however that the constructions of the previous chapter give a correspondence of a different nature than the plain definition of morphisms on the parenthesized braid operad. The main purpose of our verifications is to check that we retrieve this definition when we take the image of our maps in the homotopy spectral sequence of our function spaces.

6.0.6. The mapping between the Grothendieck-Teichmüller group and the homotopy automorphism space of \( E_2 \)-operads. We have obvious mappings:

\[
\begin{align*}
(1) \quad & \text{Mor}_{\text{Grd} \Lambda \text{Op}}(PaB^\gamma, PaB^\gamma) \to \text{Mor}_{s\text{Set} \Lambda \text{Op}}(B(PaB^\gamma), B(PaB^\gamma)) \\
& \quad \to \text{Mor}_{s\text{Set} \Lambda \text{Op}}(|\text{Res}^\bullet(B(PaB^\gamma)|, |\text{Res}^\bullet(B(PaB^\gamma)|)
\end{align*}
\]

by functoriality of the classifying space construction and of our definition of cofibrant resolutions \( P \to |\text{Res}_{\bullet}(P)| \) on the category of non-unitary \( \Lambda \)-operads in simplicial sets \( s\text{Set} \Lambda \text{Op}_\prec \). We use this correspondence to associate an isomorphism (and hence a weak-equivalence of operads in simplicial sets)

\[
\phi_\gamma : |\text{Res}^\bullet(B(PaB^\gamma)| \xrightarrow{\cong} |\text{Res}^\bullet(B(PaB^\gamma)|)
\]

to any element of the Grothendieck-Teichmüller group \( \gamma \in GT(\mathbb{Q}) \). We accordingly get a map of simplicial monoids

\[
\rho : GT(\mathbb{Q}) \to \text{hAut}_{s\text{Set} \Lambda \text{Op}}(|\text{Res}^\bullet(B(PaB^\gamma)|)
\]
where we regard the group $GT(Q)$ as a discrete space and we consider the homotopy automorphism space associated to the object $R^\sim = |Res_*(B(PaB))|$ which defines our working cofibrant model of a rational $E_2$-operad in the category of simplicial sets.

In the introduction of this chapter, we briefly explained that we use a natural action of the space of homotopy automorphisms on function spaces to gain our result about the correspondence between $GT(Q)$ and the homotopy automorphism classes of rational $E_2$-operads. We first have an obvious action

(4) $GT(Q) \xrightarrow{\sigma} hAut_{\Lambda OP}(|Res_*(B(PaB))|) \cong Map_{\Lambda OP}(|Res_*(B(PaB))|, B(CD'))$

which we deduce from the composition operation of the function space bifunctor on the simplicial category of operads in simplicial sets. We also have a weak-equivalence

(5) $Map_{\Lambda OP}(|Res_*(B(PaB))|, B(CD')) \xrightarrow{\sim} Map_{\Lambda OP}(|Res_*(B(PaB))|, B(CD'))$

given by the composition with the morphism $|Res_*(B(PaB))| \rightarrow |Res_*(B(PaB))|$, because this morphism is identified with a rationalization map in the category of operads (see §12.1) and the operad $B(CD')$, which defines the target object of our function spaces, is rational too (see again §12.1). We now consider the obvious mappings:

(6) $Mor_{\grd \Lambda OP}(PaB, CD') \rightarrow Mor_{\Set \Lambda OP}(B(PaB), B(CD'))$

\[
\rightarrow Mor_{\Set \Lambda OP}(\underbrace{|Res_*(B(PaB))|}_{=Map_{\Lambda OP}(|Res_*(B(PaB))|, B(CD'))}, |B(CD')|)
\]

to get a map

(7) $\sigma : Ass(Q) \rightarrow Map_{\Lambda OP}(\underbrace{|Res_*(B(PaB))|}_{=Map_{\Lambda OP}(|Res_*(B(PaB))|, B(CD'))}, B(CD'))$

from the set of Drinfeld’s associators $Ass(Q)$, regarded as a discrete space, towards our function space $Map_{\Lambda OP}(\underbrace{|Res_*(B(PaB))|}_{=Map_{\Lambda OP}(|Res_*(B(PaB))|, B(CD'))}, B(CD'))$.

We readily check that the action of the Grothendieck-Teichmüller group on the function space $Map_{\Lambda OP}(\underbrace{|Res_*(B(PaB))|}_{=Map_{\Lambda OP}(|Res_*(B(PaB))|, B(CD'))}, B(CD'))$ in (4) goes down to the space $Map_{\Lambda OP}(\underbrace{|Res_*(B(PaB))|}_{=Map_{\Lambda OP}(|Res_*(B(PaB))|, B(CD'))}, B(CD'))$ through this mapping and the natural action of the Grothendieck-Teichmüller group on the set of Drinfeld’s associators:

(8) $GT(Q) \xrightarrow{\sigma} Ass(Q) \xrightarrow{\sim} Map_{\Lambda OP}(\underbrace{|Res_*(B(PaB))|}_{=Map_{\Lambda OP}(|Res_*(B(PaB))|, B(CD'))}, B(CD'))$.

We actually use this correspondence in our proof that our mapping (3) induces a group isomorphism in homotopy. We also use an extension of our mapping (7) to the morphism sets which occur in our tower decomposition of the set of Drinfeld’s associators in §6.0.2. The idea is to compare this tower with the spectral sequence constructions which we introduced in the previous chapter to compute the homotopy of our function space. We address this subject in the next section.

6.1. The degree zero homotopy of the space of homotopy automorphisms

We examine the correspondence between the Grothendieck-Teichmüller group, the elements of the set of Drinfeld’s associators, and the homotopy of our function spaces more thoroughly in this section. We essentially prove that the mapping:

$$\sigma : Ass(Q) \rightarrow Map_{\Lambda OP}(\underbrace{|Res_*(B(PaB))|}_{=Map_{\Lambda OP}(|Res_*(B(PaB))|, B(CD'))}, B(CD')),$$
defined in §6.0.6, induces a bijection onto the set formed by the homotopy classes of operad morphisms $\phi : |\text{Res}_*(\text{B}(PaB))| \to \text{B}(CD)$ which extend to a weak-equivalence on the rationalization of the operad $R = |\text{Res}_*(\text{B}(PaB))|$. We deduce our main theorem (about the isomorphism between the Grothendieck-Teichmüller group and the group of homotopy classes of homotopy automorphisms of rational $E_2$-operads) from this statement. In fact, we also use the action of the Grothendieck-Teichmüller group on associators in the verification of our weak-equivalence claim at the associator level.

The idea is to compare the tower underlying the set of Drinfeld’s associators with the tower of homotopy class sets arising from the spectral sequence constructions of the previous chapters. We then deal with a generalization of the above mapping for morphisms of operads in groupoids $\nu : PaB \to q_m CD$ with values in the quotients of the chord diagram operad $q_m CD$, $m \geq 1$, which occur at the levels of our tower. We also consider the morphisms $\xi : PaB \to E_m^0 \mathbb{p}$ with values in the homogeneous summands of the Drinfeld-Kohno Lie algebra operad $Q$ and the quotients of the chord diagram operad $q_m CD$, $m \geq 1$, which we use to compare the morphisms occurring at different levels.

We just use that the definition of our mapping in §6.0.6 makes sense for any set of morphisms $\nu : PaB \to Q$ with an operad in the category of categories as target object $Q \in \text{Cat}\Lambda\text{Op}$. Let us briefly review our construction. We first use the functoriality of the classifying space construction to associate a morphism of operads in simplicial sets $\phi = \phi_\nu : \text{B}(PaB) \to \text{B}(Q)$ to any morphism of operads in the category of categories $\nu : PaB \to Q$. We compose this morphism with the augmentation $\epsilon : |\text{Res}_*(\text{B}(PaB))| \to \text{B}(PaB)$ afterwards to get a morphism defined on the cofibrant resolution $R = |\text{Res}_*(E_2)|$ which we consider in our function spaces. We generally do not make this second restriction operation explicit. We just implicitly consider that our morphisms of operads in simplicial sets $\phi : \text{B}(PaB) \to \text{B}(Q)$ are equivalent to composite morphisms of the form

\[\text{Res}_*(\text{B}(PaB)) \to \text{B}(PaB) \xrightarrow{\phi} \text{B}(Q)\]

on the cofibrant operad $R = |\text{Res}_*(\text{B}(PaB))|$. We examine the case of the morphisms towards the homogeneous components of the Drinfeld-Kohno Lie algebra operad $Q = E_m^0 \mathbb{p}$, $m \geq 1$, before explaining our level-wise comparison arguments for the elements of the set of Drinfeld’s associators. In this case, our mapping returns morphisms towards the operads in simplicial sets $K(E_m^0 \mathbb{p}, 1)$ which occur in our decomposition of the classifying space of the chord diagram operad $\text{B}(CD)$. We determine a representative cycle of any such morphism in the second page of our homotopy spectral sequence:

\[\Gamma = E^r(\text{Map}_{\text{Set}\Lambda\text{Op}}(\text{Res}^*(E_2), K(E_m^0 \mathbb{p}, 1))).\]

To be explicit, recall that we regard the object $E_m^0 \mathbb{p}$ as a collection of additive groups $E_m^0 \mathbb{p}(r)$, $r > 0$, equipped with the additive operad structure considered in §5.2.2 when we deal with morphisms of the form $\nu : PaB \to E_m^0 \mathbb{p}$. We have an obvious identity $K(E_m^0 \mathbb{p}, 1) = \text{B}(E_m^0 \mathbb{p})$. We therefore associate morphisms of operads in simplicial sets of the form $\phi = \phi_\nu : \text{B}(PaB) \to K(E_m^0 \mathbb{p}, 1)$ to these morphisms of operads in groupoids $\nu : PaB \to E_m^0 \mathbb{p}$. We use that any such morphism is determined by an arity 2 element $c \in E_m^0 \mathbb{p}(2)$, which represents the image of the braiding isomorphism of the parenthesized braid operad $\tau \in \text{Mor} PaB(2)$ in $E_m^0 \mathbb{p}$, and by an arity 3 element $h(t_{12}, t_{23}) \in E_m^0 \mathbb{p}(3)$, which represents the image of the
associator \( \alpha \in \text{Mor}\ PaB(3) \), and which satisfies the defining equations of the graded Grothendieck-Teichmüller Lie algebra, as we explained in §6.0.5. Recall that we just have \( c \in \mathbb{Q}\tau_{12} \) when \( m = 1 \) and \( E^m_0(p(2)) = 0 \Rightarrow c = 0 \) otherwise.

We essentially prove that the mapping \( \phi \mapsto c \in \mathbb{Q}\tau_{12} \) for \( m = 1 \), and the mapping \( \phi \mapsto h(t_{12}, t_{23}) \in \text{grt}_{m-1} \) for \( m > 1 \) returns a representative cycle of our morphism \( \phi = \phi_v : B(PaB) \rightarrow K(E^m_0, p, 1) \) in the description of the second page of the spectral sequence (1) given by the result of Theorem 5.2.8. We deduce from this correspondence that these morphisms of operads in simplicial sets which we associate to our morphisms of operads in groupoids \( v : PaB \rightarrow E^m_0, p \) give all degree 0 elements in the homotopy of the operadic function space \( \text{Map}_{\Lambda \text{-Op}}([\text{Res}_*(E^0_2)], K(E^0_0, p, 1)) \).

To recap, we consider a morphism of operads in simplicial sets of the form:

\[
(\ast\ast) \quad |\text{Res}_*(B(PaB))| \xrightarrow{\sim} B(PaB) \xrightarrow{\phi} K(E^m_0, p, 1)
\]

and associated to a morphism of operads in groupoids \( v : PaB \rightarrow E^m_0, p \), for some fixed weight \( m \geq 1 \). We also write \( \phi = \phi_v \) for short. We aim to determine a representing cycle of this morphism in our spectral sequence. We go through our spectral sequence construction and the whole results of the previous chapters of this part in order to make such a cycle explicit. For short, we use the notation \( B = B(PaB), R_* = \text{Res}_*(B(PaB)), E = K(E^0_0, p, 1) \) all through this verification.

6.1.1. Recollections on the information gained from the homotopy spectral sequence of a cosimplicial space. Let us recap the result of our spectral sequence construction. In the case of a cosimplicial space such that \( X^\ast = \text{Map}_{\Lambda \text{-Op}}(R_\ast, E) \), we have \( \text{Tot}_s(X) = \text{Map}_{\Lambda \text{-Op}}(sk_s \mid R_\ast \mid E) \), for any \( s \geq 0 \), because the function space bi-functor clearly carries the coend defining the \( s \)-dimensional skeleton of the geometric realization to the end defining the \( s \)-dimensional level of the totalization of our cosimplicial object. Besides, the fibrations \( p_s : \text{Tot}_s(X) \rightarrow \text{Tot}_{s-1}(X) \) are identified with the maps

\[
i_s^\ast : \text{Map}_{\Lambda \text{-Op}}(sk_s \mid R_\ast \mid E) \rightarrow \text{Map}_{\Lambda \text{-Op}}(sk_{s-1} \mid R_\ast \mid E)
\]

induced by the obvious restriction operation \( i_s^\ast(\phi) = \phi|_{sk_{s-1} \mid R_\ast} \) on our function spaces. From the general construction of the spectral sequence of a tower of fibrations in §2.1, we get a decomposition of the homotopy of the space of functions on our operadic resolution

(1) \[ \pi_s(\text{Map}_{\Lambda \text{-Op}}(\mid R_\ast \mid, E)) = \lim_s q_s \pi_s(\text{Map}_{\Lambda \text{-Op}}(\mid R_\ast \mid, E)) \]
such that

(2) \[ q_s \pi_s(\mid - \mid) = \text{im}(\pi_s(\text{Map}_{\Lambda \text{-Op}}(\mid R_\ast \mid, E)) \rightarrow \pi_s(\text{Map}_{\Lambda \text{-Op}}(sk_s \mid R_\ast \mid, E))) \]

where we consider the image of the map induced by the restriction operation \( \phi \mapsto \phi|_{sk_s \mid R_\ast} \) on the homotopy of our function space \( \text{Tot}(X) = \text{Map}_{\Lambda \text{-Op}}(\mid R_\ast \mid, E) \).

The objects \( E^m_s(\pi_s(\mid - \mid)) = \ker(q_s \pi_s(\mid - \mid) \rightarrow q_{s-1} \pi_s(\mid - \mid)) \), which we associate to such a tower of homotopy class sets in §2.1.7, consist, for any \( s \geq 0 \), of the elements \( [\phi] \in q_s \pi_s(\text{Map}_{\Lambda \text{-Op}}(\mid R_\ast \mid, E)) \) satisfying \( [\phi]|_{sk_{s-1} \mid R_\ast} \equiv 0 \). These modules \( E^m_s(\pi_s(\mid - \mid)) \) form sub-quotients of the spectral sequence terms \( T^2_s = \pi^s \pi_s(\text{Map}_{\Lambda \text{-Op}}(R_\ast, E)) \) computed in Theorem 5.2.8, and our purpose is to compute cycles \( z \in T^2_s \) giving the first non-trivial term of the morphisms \( \ast\ast \) in our tower of homotopy class sets (1-2).
6.1.2. The restriction to the 0-skeleton of the resolution. We first determine the homotopy class of the restriction of our morphisms (**) to the 0-skeleton of our resolution $\mathfrak{s}k_0 | R_\bullet | \subset | R_\bullet |$ in order to compute the image of these morphisms at the zero level. We just have $\mathfrak{s}k_0 | R_\bullet | = R_0 = \emptyset(B)$ in this case, and the restriction of any morphism $\phi : | R_\bullet | \to E$ to this operad $\mathfrak{s}k_0 | R_\bullet | \subset | R_\bullet |$ automatically yields an element in the cohomotopy class set

$$\pi^0\pi_0 \text{Map}_{\Lambda^p}(R_\bullet, E) \subset \pi^0_0 \text{Map}_{\Lambda^p}(R_0, E),$$

which represents the term of bi-degree $(0,0)$ in the second page of our spectral sequence (I). Recall that we use a conormalized complex construction in the general definition of our cohomotopy class sets, but at the zero level, we trivially have $\pi^0\pi_0(X) = \pi_0(X^0)$, for any cosimplicial space $X \in c s\text{Set}$. We therefore forget about the conormalization construction in our next verifications.

Let $[z] \in I^2_{00}$ be this bi-degree $(0,0)$ cohomotopy class which we associate to our morphism (**) We aim to make explicit the element $c \in E^0 p(2)$ that corresponds to this class $[z] \in I^2_{00}$ when we apply the result of Theorem 5.2.8. We get the following statement:

**Proposition 6.1.3.** In the representation $I^2_{00} = E^0 p(2) = Q t_{12}$ yielded by the result of Theorem 5.2.8, the class $[z] \in I^2_{00}$ which we associate to our morphism of operads in simplicial sets (**) in the second page of our spectral sequence $I^2 \Rightarrow \pi_* \text{Map}_{\Lambda^p}(| R_\bullet |, E)$ is given by (twice) the element $c = \nu(\tau) \in Q t_{12}$ associated to the morphism of operads in groupoids $\nu : \text{Pa}B \to E^0 p$ underlying our morphism $\phi = \phi_\nu$.

**Proof.** We first have to determine the image of the homotopy class of our morphism $\phi_{| \mathfrak{s}k_0 | R_\bullet |}$ under the isomorphisms

$$\pi_* \text{Map}_{\Lambda^p}(\emptyset(B), E) \xrightarrow{\sim} \text{Der}_{\text{gr} \Lambda^p}(\rho_*(\emptyset(B)), H_0(N_*(E)[1]),$$

$$\xrightarrow{\sim} \text{Hom}_{\text{gr} \Lambda^p}(H_0(B), H_0(N_*(E)[1]),$$

which give the connection between the second page of our spectral sequence and the cotriple cohomology theory of operads at the zero level. For this purpose, we simply take the morphism induced by $\phi_{| \mathfrak{s}k_0 | R_\bullet |} : R_0 \to E$ on the homology module $H_0(B) \subset H_0(\emptyset(B)) = H_0(R_0)$.

We then evaluate this morphism on the image of the element $\rho_2^2 \otimes \mu(x_1, x_2) \in K_1(\text{Gerst})(2)$ under the comparison maps

$$\Sigma^{-1} K_1(\text{Gerst}) \to \Sigma^{-1} B_1(\text{Gerst}) \to N_0(\bar{\theta}^*(\text{Gerst})) = \text{Gerst}$$

which give the connection between the cotriple cohomology and the cohomology of derivations on the Koszul construction of the Gerstenhaber operad in cohomological degree zero. If we go back to our definition, then we get the correspondence:

$$\Sigma^{-1}(\rho_2^2 \otimes (x_1, x_2))^\vee \mapsto \Sigma^{-1} x_{\lambda(x_1, x_2)} \mapsto \Sigma^{-1} x_{\lambda(x_1, x_2)}$$
at this degree 0 level.

Recall that we have the relation \( \lambda(x_1, x_2) = [\tau^2 : \mu \to \mu] \) in the homology of the classifying space of the parenthesized braid operad \( \mathcal{P} \mathcal{B} \), where we consider the 1-simplex of \( B(\mathcal{P} \mathcal{B}(2)) \) defined by the morphism \( \tau^2 \in \text{Mor} \mathcal{P} \mathcal{B}(2) \) on the object \( \mu \in \text{Ob} \mathcal{P} \mathcal{B}(2) \). We therefore get the formula \( \phi_*(\lambda(x_1, x_2)) = [\nu(\tau^2)] = [2\nu(\tau)] \) when we take the image of this class under our morphism \( \phi = \phi_\tau \), and this result gives the assertion of the proposition. \( \square \)

6.1.4. The restriction to the 1-skeleton of the resolution. We now consider a morphism \( v : \mathcal{P} \mathcal{B} \to E_0^m p \) towards a homogeneous component \( E_0^m p \) of the operad \( E^0 p \), and a morphism of operads in simplicial sets (**) which we associate to such a morphism of operads in groupoids.

We also assume \( m > 1 \) from now on. We then have \( \Gamma_{00}^2 = E_0^m p(2) = 0 \), and our result actually implies that the morphism \( \phi|_{\mathcal{S}k_2} | _{\mathcal{P} \mathcal{B}} \) which we consider in §6.1.2 is homotopically trivial in this case. We therefore take the restriction of our morphism (**) to the 1-skeleton of our resolution \( \mathcal{S}k_1 | _{\mathcal{P} \mathcal{B}} \subset | _{\mathcal{P} \mathcal{B}} \), and we aim to determine a representative cycle of this morphism in the term \( \Gamma_{11}^2 \) of our spectral sequence.

In the case of a function space \( X = \text{Map}_{\Lambda \text{Op}}(\mathcal{P} \mathcal{B}, E) \), we have an identity:

\[
F_s = \text{fib}(\text{Map}_{\Lambda \text{Op}}(\mathcal{S}k_1 | _{\mathcal{P} \mathcal{B}}, E) \to \text{Map}_{\Lambda \text{Op}}(\mathcal{S}k_{s-1} | _{\mathcal{P} \mathcal{B}}, E))
\]

\[
= \text{Map}_{\Lambda \text{Op}}(\text{cfib}(\mathcal{S}k_{s-1} | _{\mathcal{P} \mathcal{B}} \to \mathcal{S}k_s | _{\mathcal{P} \mathcal{B}}), E).
\]

Furthermore, the identity \( F_s = N^s(X)^{\Delta^s/\partial \Delta^s} \), which we use to establish the relation \( \pi_*(F_s) = N^s \pi_*(X) \) in Proposition 2.1.12, can be retrieved by applying the functor \( \text{Map}_{\Lambda \text{Op}}(-, E) \) to the diagram of cofibration sequences of operads

\[
\begin{array}{c}
\text{L}(\mathcal{P} \mathcal{B})_s \otimes \Delta^s \bigvee_{L(\mathcal{P} \mathcal{B})_s \otimes \partial \Delta^s} R_s \otimes \partial \Delta^s \Rightarrow \mathcal{S}k_{s-1} | _{\mathcal{P} \mathcal{B}} \Rightarrow \mathcal{S}k_s | _{\mathcal{P} \mathcal{B}} \Rightarrow \mathcal{S}k_s | _{\mathcal{P} \mathcal{B}} \\
R_s \otimes \Delta^s \Rightarrow \mathcal{S}k_{s-1} | _{\mathcal{P} \mathcal{B}} \Rightarrow \mathcal{S}k_s | _{\mathcal{P} \mathcal{B}} \Rightarrow \mathcal{S}k_s | _{\mathcal{P} \mathcal{B}} \\
R_s / L_s(\mathcal{P} \mathcal{B}) \otimes \Delta^s / \partial \Delta^s \Rightarrow \text{cfib}(\mathcal{S}k_{s-1} | _{\mathcal{P} \mathcal{B}} \to \mathcal{S}k_s | _{\mathcal{P} \mathcal{B}})
\end{array}
\]

which follows from the general pushout construction of §3.3 (see Proposition 3.3.11), and by using the relations

\[
\text{Map}_{\Lambda \text{Op}}(R_s / L_s(\mathcal{P} \mathcal{B}) \otimes \Delta^s / \partial \Delta^s, E)
\]

\[
= \text{Map}_{\Lambda \text{Op}}(R_s / L_s(\mathcal{P} \mathcal{B}), E)^{\Delta^s / \partial \Delta^s}
\]

\[
= \ker(\text{Map}_{\Lambda \text{Op}}(R_s, E) \to N^s \text{Map}_{\Lambda \text{Op}}(R_s, E))^{\Delta^s / \partial \Delta^s},
\]

which arise from the adjoint definition of our function space bifunctor on the category of operads (see §2.3).

To get this relation, we notably use that the function space bifunctor carries colimits of operads on the source to limits of simplicial sets and carries the latching object in our cofibration sequence \( L_s(\mathcal{P} \mathcal{B}) \) to a cosimplicial matching object in the
category of simplicial sets $\mathcal{M}^\ast(-)$. In our correspondence, we moreover use that the
color{conormalization} of a cosimplicial space is equivalent to the kernel $N^\ast(X) = \ker(\mu : X^s \to \mathcal{M}^\ast(X))$, for any dimension $s \geq 0$, where we consider the matching map
towards the $s$th matching object of this space $\mathcal{M}^\ast(X)$.

The vertices of the simplicial sets $N^\ast(\text{Map}_{\Lambda \mathcal{O}_p}(R_\ast, E))$, which therefore correspond to the vertices of our fibrations, are also equivalent to $s$-simplices $\tilde{\phi} \in N^\ast(\text{Map}_{\Lambda \mathcal{O}_p}(R_\ast, E))$, satisfying $d_i(\tilde{\phi}) = *$ for any $i = 0, \ldots, s$, and we rather use this interpretation of our result in what follows.

Recall that we have $N^\ast_\pi(X) \cong \pi_\ast N^\ast(X)$, for any $s \geq 0$, and in general, we use this relation to determine the homotopy groups $\pi_n(F_\ast) = \pi_\ast N^\ast(X)$ which define the first page of the spectral sequence of a cosimplicial space $X \in \mathcal{C}Set$. In the case of our morphism $\phi : |R_\ast| \to E$, we have $\phi|_{|sk_0| R_\ast} \sim 0 \Rightarrow \phi|_{|sk_1| R_\ast} \sim \tilde{\phi}$ for some morphism of non-unitary $\Lambda$-operads $\tilde{\phi} : sk_1 | R_\ast | \to E$ satisfying $\tilde{\phi}|_{sk_0 | R_\ast} = 0$. To get a representing cycle $z \in I_{11}$ of our morphism $\phi$ in the first page of the spectral sequence $I^\ast$, we take the homotopy class of the morphism induced by this map $\tilde{\phi}$ on the object $R_1 / L_1(R) \otimes \Delta^1 / \partial \Delta^1 \xrightarrow{\sim} \text{cofib}(sk_0 | R_\ast | \to sk_1 | R_\ast |)$, and we use the above relations.

We aim to make explicit the element $a \in \text{grt}_{m-1} \subset E_0^m \mathfrak{p}(3)$ that corresponds to the homotopy class of this cycle $[z] \in I_{11}$ in the second page of our spectral sequence $I^2 \Rightarrow \pi_* \text{Map}_{\Lambda \mathcal{O}_p}(R, E)$ when we apply the result of Theorem 5.2.8. We get the following statement:

**PROPOSITION 6.1.5.** In the representation $I_{11} = \text{grt}_{m-1}$ yielded by the result of Theorem 5.2.8, the class $[z] \in I_{11}$ which we associate to our morphism of operads in simplicial sets $(**) in the second page of our spectral sequence $I^2 \Rightarrow \pi_* \text{Map}_{\Lambda \mathcal{O}_p}(|R_\ast|, E)$ is given by the element $a = \nu(\alpha) \in \text{grt}_{m-1}$ associated to the morphism of operads in groupoids $\nu : P\mathcal{B} \to E_0^m \mathfrak{p}$ underlying our morphism $\phi = \phi^\nu$.

**PROOF.** In our construction of the class $[z] \in I_{11}$, we start with the rela-
tion $\phi|_{sk_0 | R_\ast} \sim 0$. Let $\psi : sk_0 | R_\ast | \to E\Delta^1$ be a homotopy giving this relation, so that we have $d_0 \psi = \phi|_{sk_0 | R_\ast}$ and $d_1 \psi = 0$. To carry out our construction, we formally have to pick a solution of the following lifting problem

\[
\begin{array}{c}
L_1(R_\ast) \otimes \Delta^1 \xrightarrow{\psi} E\Delta^1
\\
\downarrow \phi|_{sk_0 | R_\ast}
\\
\text{sk}_1 | R_\ast
\\
\end{array}
\]

We then set $\tilde{\phi} = d_1 \psi$ to effectively get a morphism $\phi : \text{sk}_1 | R_\ast | \to E$ satisfying $\tilde{\phi} \sim d_0 \psi = \phi|_{sk_1 | R_\ast}$ and $\phi|_{sk_0 | R_\ast} = d_1 \psi|_{sk_0 | R_\ast} = d_1 \psi = 0$.

Recall that we basically consider the value of this morphism $\phi = d_1 \psi$ on the object $R_1 \otimes \Delta^1$. This operation returns a 1-simplex:

\[\phi \in \text{Mor}_{\Lambda \mathcal{O}_p}(R \otimes \Delta^1, E) = \text{Map}_{\Lambda \mathcal{O}_p}(R, E)_1\]

and we have $\phi|_{L_1(R_\ast) \otimes \Delta^1} = 0 \Leftrightarrow s^0(\phi) = 0 \Leftrightarrow \phi \in N^1 \text{Map}_{\Lambda \mathcal{O}_p}(R, E)_1$. We moreover have $\phi|_{R_1 \otimes \Delta^1} = 0 \Leftrightarrow d_0(\phi) = d_1(\phi) = 0$ so that $\phi$ determines a 1-dimensional homotopy class for the simplicial set $X = N^1 \text{Map}_{\Lambda \mathcal{O}_p}(R, E)$. This homotopy class $[\phi] \in N^1 \pi_1 \text{Map}_{\Lambda \mathcal{O}_p}(R, E)$ defines a representative cycle of our morphism $\phi : |R_\ast| \to E\Delta^1$. 


$E$ in the first page of our spectral sequence $\Sigma^1$, and the element $[z] \in \Pi^2$ which we aim to determine is given by the class of this cycle in the cohomotopy class set $\Pi^2 = \pi_1 \pi_1 \text{Map}_A \circ p(R, E)$.

We now consider the image of the homotopy class of our morphism $\widetilde{\phi}$ under the isomorphisms

\[
\begin{align*}
N^1 \pi_* \text{Map}_A \circ p(\Theta \circ \overline{B}), E & \xrightarrow{\sim} \text{Der}_{\text{gr} \Lambda} \circ p(H_*(\Theta \circ \overline{B}), H_\ast N_\ast(E)[1]) \\
& \xrightarrow{=_{\Sigma^p}} \text{Hom}_{\text{gr} \Lambda \text{Seq}}(\Theta(H_*(\overline{B})), H_\ast N_\ast(E)[1]),
\end{align*}
\]

which give the connection between our cohomotopy class set and the cotriple cohomological theory of operads in cohomology degree one.

Recall that for a free operad such as $R_1 = \Theta(\overline{B})$, we have $R_1 \otimes \Delta^1 = \Theta(\overline{B}) \otimes \Delta^1$, where we set $\Theta(\overline{B}) \otimes \Delta^1(r) = \Theta(\overline{B})(r) \otimes \Delta^1$, for any $r > 0$. Recall also that we denote the fundamental class of the simplex $\Delta^1$ by $\imath_1 \in (\Delta^1)_0$ and we write $[\imath_1] \in N_\ast(\Delta^1)$ for the element associated to this simplex in the normalized complex $N_\ast(\Delta^1)$.

Let $f \in \text{Hom}_{\text{gr} \Lambda \text{Seq}}(\Theta(H_*(\overline{B})), H_\ast N_\ast(E))$ denote the homomorphism which we associate to our homotopy class $[\phi]$ under the above correspondence. The value of this homomorphism on the homology class $c \in H_\ast(\Theta(\overline{B}))(r) = H_\ast(\Theta(\overline{B}))(r)$ of a cycle $[\pi] \in N_\ast(\Theta(\overline{B}))(r)$, where $r > 0$ denotes any fixed arity, can be determined by the following mapping (at the normalized chain complex level):

\[
\begin{align*}
\pi & \mapsto [\pi] \otimes [\imath_1] \\
\Sigma & \mapsto \nabla([\pi] \otimes [\imath_1]) \\
\delta & \mapsto \delta(\nabla([\pi] \otimes [\imath_1])),
\end{align*}
\]

where we take the image of the tensor $[\pi] \otimes [\imath_1] \in N_\ast(\Theta(\overline{B}))(r) \otimes N_\ast(\Delta^1)$ under the Eilenberg-MacLane map $\nabla : N_\ast(\Theta(\overline{B}))(r) \otimes N_\ast(\Delta^1) \to N_\ast(\Theta(\overline{B})(r) \times \Delta^1)$, followed by the map induced by the restriction of our morphism $\phi : R_1 \otimes \Delta^1 \to E$ to the sub-object $\Theta(\overline{B}) \otimes \Delta^1 \subset \Theta(\overline{B}) \otimes \Delta^1 = R_1 \otimes \Delta^1$.

We have to evaluate this homomorphism on the image of the element $\rho_3^2 \otimes [x_1, x_2, x_3] \in K_2(\text{Gerst})(3)$ under the comparison maps

\[
\Sigma^{-1} K_2(\text{Gerst}) \to \Sigma^{-1} B_2(\text{Gerst}) \to N_1(\Theta^*(\text{Gerst}))
\]

which give the connection between the cotriple cohomology and the cohomology of derivations on the Koszul construction of the Gerstenhaber operad in cohomological degree one. If we go back to our definition, then we get:

\[
\begin{align*}
\Sigma^{-1} \rho_3^2 \otimes [x_1, x_2, x_3] & \mapsto \Sigma^{-1} \left(\begin{array}{cc}
1 & 2 \\
\Sigma \mu(x_1, x_2) & 3 \\
\Sigma \mu(x_1, x_2) & 1 \\
\Sigma \mu(x_1, x_2) & 0
\end{array}\right)
\end{align*}
\]
for the first mapping, followed by:

\[
\Sigma^{-1}(\begin{array}{c}
\Sigma \mu(x_1, x_2) \\
\Sigma s(x_1, x_2)
\end{array}) \rightarrow \begin{array}{c}
\mu(x_1, x_2) \\
\mu(x_1, x_2)
\end{array}, \in B_2(\text{Gerst})(3)
\]

for the second mapping. We consider the obvious 0-dimensional cycle

\[
\pi = \begin{array}{c}
\mu(x_1, x_2) \\
\mu(x_1, x_2)
\end{array} \in N_1(\text{Gerst})(3)
\]

of which homology class gives the result of this composite comparison map. We just regard the above tree-wise tensors as the expression of elements of the normalized complex of the simplicial set \(\text{Gerst}(\tilde{B})(3)\) in the definition of this cycle (7).

We form the following diagram in order to determine the image of this element under our mapping (3):

\[
\begin{array}{c}
\text{N}_* (\text{E}((\tilde{B})(3) \times \Delta^1)) \xrightarrow{\text{N}_* (\text{E}(3) \Delta^1)} \text{N}_* (\text{E}(3))
\end{array}
\]

To define this diagram, we basically consider, as in the definition of our mapping (3), an obvious restriction (in arity 3) of the morphisms occurring in our lifting construction (1) and we apply the normalization functor to these maps. We still prolong this diagram on the left, by using the Eilenberg-MacLane transformation

\[
\text{N}_* (\tilde{B})(3) \otimes \text{N}_* (\Delta^1) \xrightarrow{\sim} \text{N}_* (\tilde{B})(3) \times \Delta^1
\]

which we have to consider in the definition of our mapping (3). We also prolong our diagram on the right by using the comparison morphism of hom-objects of Proposition 5.4.6:
Recall that $\sigma_0 \mid R_0 \mid = R_0 = \text{Res}_0(B) = \mathcal{O}(B)$ and $\psi : \mathcal{O}(B) \to E^{\Delta^1}$ is a homotopy giving the relation $\phi|_{R_0} \sim 0$ in the category of operads.

Let $[d_0(\ell_1)]^\nu, [d_1(\ell_1)]^\nu, [\ell_1]^\nu \in N^*(\Delta^1)$ denote the dual basis elements of the canonical basis of the dg-module $N_*(\Delta^1)$ (recall that $\ell_1$ denotes the fundamental simplex of $\Delta^1$). When we form the prolongment

$$N_*(\mathcal{O}(B)(r)) \xrightarrow{\phi} N_*(E(r)^{\Delta^1}) \xrightarrow{\nabla^*} N_*(E(r)) \otimes N^*(\Delta^1),$$

for any arity $r > 0$, we get a homotopy between the composite

$$N_*(\mathcal{O}(B)(PaB)) \xrightarrow{d_0} N_*(B(PaB)) \xrightarrow{\delta_0} N_*(E^m, p, 1) = E^m_0, p[1],$$

which represents the image of our morphism under the normalization functor $N_*(-)$, and the trivial morphism in the category of dg-modules. We accordingly have $\nabla^*\psi_*(\xi) = \phi_*(\xi) \otimes [d_0(\ell_1)]^\nu + \sigma_*(\xi) \otimes [\ell_1]^\nu$, for any $\xi \in N_*(\mathcal{O}(B)(r))$, where $\sigma_* : N_*(\mathcal{O}(B)(r)) \to N_*(E(r))$ is a homomorphism of degree 1 which satisfies $\sigma_1 \delta_1(\xi) + \sigma_0 \delta_0(\xi) = \phi_*(\xi)$, and which, in the language of classical homological algebra, defines a chain-homotopy between the morphisms $\phi_*$ and the trivial map. Recall that we have $E(r) = K(E^m_0, p(r), 1) = N_*E(r) = E^m_0, p(r)[1]$, and the differential on $N_*(E(r))$ is therefore trivial in our setting, so that our chain-homotopy relation reduces to the identity $\sigma_1 \delta_1(\xi) = \phi_*(\xi)$, for any $\xi \in N_*(\mathcal{O}(B)(r))$.

We determine the image of the tensors $[\pi] \otimes [d_0(\ell_1)]^\nu, [\pi] \otimes [d_1(\ell_1)]^\nu \in N_0(\mathcal{O}(B)(3)) \otimes N_0(\partial \Delta^1) = N_0(\mathcal{O}(B)(3) \times \partial \Delta^1)$ and $[\pi] \otimes [\ell_1]^\nu \in N_0(\mathcal{O}(B)(3)) \otimes N_1(\Delta^1)$ under the horizontal arrows of our diagram. We have

$$[\pi] \otimes [d_0(\ell_1)]^\nu \in N_0(\mathcal{O}(B)(3)) \otimes N_0(\partial \Delta^1) \xrightarrow{\phi} N_0(\mathcal{O}(B)(P\mathcal{A}B)(3)) \otimes N_0(\partial \Delta^1) = \{d_0(\pi)\} \in N_*P\mathcal{A}B(3),$$

because we assume $m > 1 \Rightarrow E^m_0, p(2) = 0$ and this implies that our homotopy $\psi$ (like our morphism $\phi$) vanishes in arity 2, and on any tree-wise composite of elements of arity 2 therefore. We similarly see that the mapping $\phi_*$, which we consider on the lower row of our diagram, cancels the tensor $[\pi] \otimes [\ell_1]^\nu$.

We have on the other hand:

$$[\pi] \otimes [d_1(\ell_1)]^\nu \in N_0(\mathcal{O}(B)(3)) \otimes N_0(\partial \Delta^1) \xrightarrow{\phi} N_0(\mathcal{O}(B)(P\mathcal{A}B)(3)) \otimes N_0(\partial \Delta^1) \xrightarrow{\nabla^*\psi} \{\nabla^*\psi(\mu) \otimes \psi(\ell_1)\},$$

Indeed, we have the boundary relation $\delta [\mu \circ_1 \mu] = [\mu \circ_1 \mu] - [\mu \circ_1 \mu] \in N_0(\mathcal{O}(B)(3)) = N_0(\mathcal{O}(P\mathcal{A}B)(3))$, where we consider the 1-simplex defined by the morphism $\alpha \in \text{Mor} P\mathcal{A}B(3)$. This identity implies $\psi_*(\mu \circ_1 \mu - [\mu \circ_1 \mu]) = [\psi_*(\mu \circ_1 \mu)] - \phi_*(\mu \circ_1 \mu) \otimes [d_0(\ell_1)]^\nu + [\phi_*(\alpha : \mu \circ_1 \mu \to \mu \circ_1 \mu)] \otimes [\ell_1]^\nu$ according to our discussion on the definition of the map $\nabla^*\psi_*$. We still have $\phi_*(\mu \circ_1 \mu) = \phi_*(\mu) \circ_2 \phi_*(\mu) = 0$ and...
similarly \(\phi_*(\mu \circ \mu) = 0\) since \(\phi_*\) vanishes in arity 2. Recall that \(h(t_{12}, t_{23}) \in E^0_n p(3)\) represents the image of the associator \(\alpha \in \text{Mor } PaB(3)\) under the morphism of operads \(v\) associated to our morphism \(\phi = \phi_*\). We therefore get the above result for the image of the tensor \([\pi] \otimes [d_1(t_1)]\) under our mapping.

Now, from the relation \(\delta \psi_*(\pi \otimes [t_1]) = \psi_* \delta([\pi] \otimes [t_1]) = \psi_*(\pi \otimes [d_0(t_1)] - [\pi] \otimes [d_1(t_1)]) = \psi_*(\pi \otimes [d_0(t_1)] - [\pi] \otimes [d_1(t_1)]),\) we obtain that \(\psi_*(\pi \otimes [t_1])\) is given by an expression of the form:

\[
(11) \quad \psi_*(\pi \otimes [t_1]) = h(t_{12}, t_{23}) \otimes [d_1(t_1)]'.
\]

This formula implies that our mapping \(\phi_* = d_1 \psi_*\) carries \([\pi] \otimes [t_1]\) to \(h(t_{12}, t_{23}) \in E^0_n p(3)\). Hence, we get the correspondence claimed in our proposition well.

The results of the previous propositions give the following statement:

**Proposition 6.1.6.** The mapping \(v \mapsto \phi_v\) which associates the morphism of operads in simplicial sets \((**\)) to any morphism of operads in groupoids \(v : PaB \to E^0_{n+1} p\) induces a bijection

\[
\text{grt}_m = \text{Mor}_{\text{grd } \Lambda Op}(PaB, E^0_{n+1} p) \simeq \pi_0 \text{Map}_{\Lambda Op}([\text{Res}_*[B(PaB)]], B(\text{CD}^r))
\]

for any \(m \geq 0\) when we pass to the homotopy of our function space.

We now examine the correspondence between the elements of the set of Drinfeld’s associators \(\text{Ass}(Q)\), corresponding to the case \(Q = CD^r\) of our mapping, and the homotopy of the space \(\text{Map}_{\Lambda Op}([\text{Res}_*[B(PaB)]], B(\text{CD}^r))\).

We fix an element in the set of Drinfeld’s associators \(\nu_0 \in \text{Ass}(Q)\) and we take the morphism associated to this element \(\phi_{\nu_0} : [\text{Res}_*[B(PaB)]] \to B(\text{CD}^r)\) as a base point in \(\text{Map}_{\Lambda Op}([\text{Res}_*[B(PaB)]], B(\text{CD}^r))\). We basically use the action of the Grothendieck-Teichmüller group on \(\text{Ass}(Q)\) in order to compare the homotopy class of a morphism \(\phi : [\text{Res}_*[B(PaB)]] \to B(\text{CD}^r)\) with the homotopy class of a morphism \(\phi_{\nu_0} : [\text{Res}_*[B(PaB)]] \to B(\text{CD}^r)\) associated to an element of the set of Drinfeld’s associators \(v \in \text{Ass}(Q)\).

We assume that \(\phi : [\text{Res}_*[B(PaB)]] \to B(\text{CD}^r)\) extends to a weak-equivalence on the rationalization of the operad \(R = [\text{Res}_*[B(PaB)]]\). We precisely prove that, under this assumption, we can construct an element \(\gamma \in GT^r(Q)\) such that we have the relation \([\phi] = [\phi_{\nu_0}]\) in the homotopy of the space \(\text{Map}_{\Lambda Op}([\text{Res}_*[B(PaB)]], B(\text{CD}^r))\).

We also check that we have an implication \([\phi_{\nu_0}] = [\phi_{\nu_0}] \Rightarrow \gamma = 1\). We get by the way that the Grothendieck-Teichmüller group maps bijectively onto the group of homotopy classes of homotopy automorphisms of the rationalization of an \(E_2\)-operad as expected.

We perform these constructions level-wise, by using the tower decomposition of the set of Drinfeld’s associators on one hand, and the tower decomposition of the homotopy of our function space that arises from the spectral sequence

\[
\text{II}^r \Rightarrow \pi_* \text{Map}_{\Lambda Op}([\text{Res}_*[B(PaB)]], B(\text{CD}^r))
\]

constructed in §2.2.3 on the other hand. We also use the correspondence of the previous proposition to carry out this inductive process. We recap the construction of our spectral sequence (II) and the information which we gain from our computations before proving our statement.

For short, we still write \(B = B(PaB)\) and \(R_* = \text{Res}_*[B(PaB)]\) all along this verification. We also set \(R = [R_*]\) for the geometric realization of the simplicial
object $R_* = \text{Res}_*(B(PaB))$ which we take as a cofibrant resolution of the operad $B = B(PaB)$.

6.1.7. Recollections on the information gained from the homotopy spectral sequence of a tower of fibrations. Recall that we deduce our spectral sequence from the decomposition of the function space:

$$\Map_\Lambda \otimes_p (R, B(\mathcal{C}D)) = \lim_m \Map_\Lambda \otimes_p (R, B(q_m \mathcal{C}D)),$$

which arises from the natural decomposition of our target object

$$B(\mathcal{C}D) = \lim_m B(q_m \mathcal{C}D).$$

In §2.2.3, we also check that the fiber of each map

$$\Map_\Lambda \otimes_p (R, B(q_{m+1} \mathcal{C}D)) \xrightarrow{p_m} \Map_\Lambda \otimes_p (R, B(q_m \mathcal{C}D))$$

in this tower decomposition is identified with the function space

$$F_m = \Map_\Lambda \otimes_p (R, K(E^0_{m+1} p, 1)),$$

where we consider the Eilenberg-MacLane space on the component of weight $m + 1$ of the Drinfeld-Kohno Lie algebra operad $E^0_{m+1} p$. We moreover observe that the space $\Map_\Lambda \otimes_p (R, K(E^0_{m+1} p, 1))$ inherits an abelian group structure, for each $m \geq 0$, and acts on $\Map_\Lambda \otimes_p (R, B(q_{m+1} \mathcal{C}D))$ so that our map $p_m$ actually forms a principal fibration with this abelian group as a fiber. Recall that we deduce the group structure of the space $\Map_\Lambda \otimes_p (R, K(E^0_{m+1} p, 1))$ from the natural abelian group structure of the Eilenberg-MacLane space $K(E^0_{m+1} p, 1)$ occurring on the target, while we use the arity-wise action of the abelian group $E^0_{m+1} p(r)$, $r > 0$, on the group of group-like elements

$$q_{m+1} \mathcal{C}D(r) = \mathcal{G}(p(r)/F_{m+2} p(r))$$

through the exponential mapping

$$\exp : E^0_{m+1} p(r) \to \mathcal{G}(p(r)/F_{m+2} p(r))$$

(as in the construction of §2.2.3) to get an action of this abelian group on the function space $\Map_\Lambda \otimes_p (R, B(q_{m+1} \mathcal{C}D))$, for any $m \geq 0$ (see §2.2.3).

From the general construction of the spectral sequence of a tower of fibrations, we get a decomposition of the homotopy of our function space

$$\pi_* \Map_\Lambda \otimes_p (R, B(\mathcal{C}D)) = \lim_m q_m \pi_* \Map_\Lambda \otimes_p (R, B(\mathcal{C}D)),$$

such that:

$$q_m \pi_* (-) = \text{im}(\pi_* \Map_\Lambda \otimes_p (R, B(\mathcal{C}D)) \to \pi_* \Map_\Lambda \otimes_p (R, B(q_m \mathcal{C}D))).$$

The objects $E^0_{m} \pi_* (-) = \ker(q_m \pi_* (-) \to q_{m-1} \pi_* (-))$ which we associate to this tower of homotopy class sets form sub-quotients of the terms of our spectral sequence. Recall that we have:

$$\II^1_m = \pi_* \Map_\Lambda \otimes_p (\text{Res}_*(B(PaB)), K(E^0_{m+1} p, 1)),
= R$$

for any $m \geq 0$. 

$$\II^1_m = \pi_* \Map_\Lambda \otimes_p (\text{Res}_*(B(PaB)), K(E^0_{m+1} p, 1)),
= R$$
We immediately see that our mapping \( \sigma : \text{Ass}(\mathbb{Q}) \to \text{Map}_{\Lambda \circ p}(R, B(\mathbb{D})) \) occurs as the limit of a tower of maps:
\[
q_m \text{Ass}(\mathbb{Q}) \subset \text{Mor}_{\text{Grd} \Lambda \circ p}(\text{PaB}, q_{m+1} \mathbb{D}) \to \text{Map}_{\Lambda \circ p}(R, B(q_{m+1} \mathbb{D}))
\]
which we obtain by applying the construction of \( \S 6.0.6 \) to the operads in groupoids \( Q = q_m \mathbb{D}. \) We readily check, besides, that the action of the group \( \text{grt}_m = \text{Mor}_{\text{Grd} \Lambda \circ p}(\text{PaB}, E^0_{m+1} p) \) on the set \( q_m \text{Ass}(\mathbb{Q}) \subset \text{Mor}_{\text{Grd} \Lambda \circ p}(\text{PaB}, q_m \mathbb{D}) \) is transported to the action of the simplicial group \( \text{Map}_{\Lambda \circ p}(R, K(E^0_{m+1} p, 1)) \) on the space \( \text{Map}_{\Lambda \circ p}(R, B(q_m \mathbb{D})) \) through the comparison maps of this tower and the comparison map \( \text{Mor}_{\text{Grd} \Lambda \circ p}(\text{PaB}, E^0_{m+1} p) \to \text{Map}_{\Lambda \circ p}(R, K(E^0_{m+1} p, 1)) \) studied in the first step of our verifications. We use these observations in the proof of the following statement:

**Theorem 6.1.8.** Let \( \phi : R \to B(\mathbb{D}) \) be a morphism of non-unitary \( \Lambda \)-operads in simplicial sets, defined on \( R = |\text{Res}_q(B(\text{PaB}))| \), and which extends to a weak-equivalence \( \phi : R^\wedge \to B(\mathbb{D}) \) on the rationalization of this operad \( R^\wedge = |\text{Res}_q(B(\text{PaB}))| \).

We then have a uniquely determined element of the set of Drinfeld’s associates \( v \in \text{Ass}(\mathbb{Q}) \), given by an action \( v = v_0 \circ \gamma \) of an element of the Grothendieck-Teichmüller group \( \gamma \in GT(\mathbb{Q}) \) on our base element \( v_0 \in \text{Ass}(\mathbb{Q}) \), such that we have the identity:
\[
[\phi] = [\phi_v]
\]
in the degree zero homotopy of the function space \( \text{Map}_{\Lambda \circ p}(R, B(\mathbb{D})) \), where we consider the morphism of operads in simplicial sets
\[
R \overset{\sim}{\to} B(\text{PaB}) \overset{\phi_v}{\to} B(\mathbb{D})
\]
associated to the morphism of operads in groupoids \( v : \text{PaB} \to \mathbb{D} \) that represents our element \( v \in \text{Ass}(\mathbb{Q}) \) (see \( \S 6.0.1 \)).

**Explanations and Proof.** Recall that the rationalization morphism \( \eta : R \to R^\wedge \) induces a weak-equivalence of function spaces
\[
\text{Map}_{\Lambda \circ p}(R^\wedge, B(\mathbb{D})) \overset{\sim}{\to} \text{Map}_{\Lambda \circ p}(R, B(\mathbb{D}))
\]
and hence, a bijection when we pass to homotopy:
\[
\pi_* \text{Map}_{\Lambda \circ p}(R^\wedge, B(\mathbb{D})) \overset{\sim}{\to} \pi_* \text{Map}_{\Lambda \circ p}(R, B(\mathbb{D}))
\]
for any \( * \geq 0 \). In our statement, we precisely assume that \( [\phi] \) lies in the image of the subset of homotopy classes of morphisms \( [\phi] \) which define a weak-equivalence of non-unitary \( \Lambda \)-operads on this rationalization:
\[
\phi^\wedge : R^\wedge \overset{\sim}{\to} B(\mathbb{D}).
\]

In our proof, we actually use a simpler characterization of this subset. We rely on the following arguments. In a first step, we use the result of Proposition 6.1.3 to fix a morphism of operads in groupoids \( v : \text{PaB} \to E^0 p \), determined by the assignment \( v : \tau \mapsto k_t \tau, k \in \mathbb{Q} \), in arity 2, and so that the homotopy class of the morphism of operads in simplicial sets \( \phi_v : B(\text{PaB}) \to K(E^0 p, 1) \) associated to \( v \) is equal to the image of the homotopy class of our morphism \( \phi : R \to B(\mathbb{D}) \) when we pass to the zero level \( B(q_m \mathbb{D}) = K(E^0 p, 1) \) of our tower of classifying spaces \( \lim_m B(q_m \mathbb{D}) = B(\mathbb{D}) \).
We are going to use the identity $CD(2)^\gamma = \exp(E_1^0 p(2)) = \exp(Q t_{12})$ in arity 2 and the relations $H_\ast(B(PaB(2)^\gamma)) = H_\ast(B(PaB(2))) = H_\ast(B(CD(2)^\gamma)) = Gerst(2)$ in homology. We mentioned in the proof of Proposition 6.1.3 that the cycle $[\tau^2]$ defines a representative of the Lie bracket operation of the Gerstenhaber operad $\lambda(x_1, x_2) \in Gerst(2)$ in the homology of the classifying space $B(PaB(2))$, while the class of the cycle $[t_{12}]$ defines a representative of this Lie bracket in the homology of the space $B(CD(2)^\gamma)$. We accordingly check in the proof of Proposition 6.1.3 that the multiplicative scalar $k \in \mathbb{Q}$, which we assign to a morphism of operads in groupoids $\nu : PaB \to E_1^0 p$ can also be determined by the relation

$$\phi_\nu(\nu(x_1, x_2)) = 2k t_{12},$$

when we consider the morphisms of operads in simplicial sets $\phi_\nu : B(PaB) \to K(E_1^0 p, 1)$ associated to $\nu$. We accordingly have an identity

$$\phi_\nu(\lambda(x_1, x_2)) = k \lambda(x_1, x_2)$$

in $Gerst(2) = H_\ast(B(CD^\gamma))$, for any morphism $\phi$ homotopic to $\phi_\nu$. We immediately get that this scalar is invertible $k \in \mathbb{Q}^\times$ when we assume that our morphism $\phi$ defines a rational equivalence, because this requirement implies that $\phi$ induces an isomorphism of the Gerstenhaber operad in homology (with rational coefficients). We accordingly have $\nu \in \mathbb{Q}^\times t_{12} \Rightarrow \nu \in q_0 Ass(\mathbb{Q})$. We also have the relation $\nu \equiv \nu_0 \circ \gamma$ in $q_0 Ass(\mathbb{Q})$, for some $\gamma \in GT(\mathbb{Q})/F_1 GT(\mathbb{Q})$.

We assume by induction that we have an element $\rho_{m-1} \in q_{m-1} Ass(\mathbb{Q})$ at the $m - 1$th level of the tower of the sets of Drinfeld’s associators $q_{m-1} Ass(\mathbb{Q}) = \text{im}(Ass(\mathbb{Q}) \to \text{Mor}_{Grd\,\Lambda OP}(PaB, CD^\gamma)), m \geq 1$, such that the homotopy class of the morphism of operads in simplicial sets $\phi_{\rho_{m-1}} : B(PaB) \to B(q_m CD^\gamma)$ associated to this element $\rho_{m-1}$ is equal to the image of our morphism $\phi : R \to B(CD^\gamma)$ in the degree zero homotopy of the function space $\text{Map}_{\Lambda OP}(R, B(q_m CD^\gamma))$. We may still write $\rho_{m-1} = \nu_0 \circ \gamma_{m-1}$ for some $\gamma_{m-1} \in GT(\mathbb{Q})/F_m GT(\mathbb{Q})$. We pick an arbitrary lifting $\rho_m^0 \in q_m Ass(\mathbb{Q})$ of this element $\rho_{m-1} \in q_{m-1} Ass(\mathbb{Q})$. We use the structure of our tower of principal fibrations to obtain that the class of the morphisms $\phi$ and $\phi_{\rho_m^0}$ in the degree zero homotopy of the function space $\text{Map}_{\Lambda OP}(R, B(q_{m+1} CD^\gamma))$ differ by the action of (the homotopy class of) a morphism $\phi_{\xi_m} \in \text{Map}_{\Lambda OP}(R, K(E_1^0 m+1 CD^\gamma))$, which according to the result of Proposition 6.1.5 also corresponds to a uniquely determined element of the graded Grothendieck-Teichmüller Lie algebra $\xi_m \in grt_m$.

We have

$$[\phi] \equiv [\phi_{\xi_m}] \cdot [\phi_{\rho_m^0}] \Rightarrow [\phi] \equiv [\phi_{\rho_m}]$$

for $\rho_m = e^{\xi_m} \cdot \rho_m^0$, since we observed that our mappings in §6.0.6 preserve these actions. We can therefore continue our induction process.

We have $\rho_m = \rho_{m-1} \in q_{m-1} Ass(\mathbb{Q})$, and this element $\rho_m$ is also uniquely determined in $q_m Ass(\mathbb{Q})$ according to our construction. We moreover have $\rho_m = \nu_0 \circ \gamma_m$, for some element $\gamma_m \in GT(\mathbb{Q})/F_{m+1} GT(\mathbb{Q})$ (which is uniquely determined as well) such that $\gamma_m \equiv \gamma_{m-1}$ in $GT(\mathbb{Q})/F_m GT(\mathbb{Q})$.

We then have the identity $[\phi] = [\phi_{\nu}]$ in the homotopy of the function space $\text{Map}_{\Lambda OP}(R, B(CD^\gamma))$ for the element of the set of Drinfeld’s associator $\nu \in \text{Ass}(\mathbb{Q})$ determined by the sequence $\rho_m \in q_m \text{Ass}(\mathbb{Q}), m \geq 0$, in our tower decomposition $\text{Ass}(\mathbb{Q}) = \lim_m q_m \text{Ass}(\mathbb{Q})$. We still have $\nu = \nu_0 \circ \gamma$ for the element of
the Grothendieck-Teichmüller group \( \gamma \in GT(\mathbb{Q}) \) determined by the relation \( \gamma \equiv \gamma_m F_{m+1} GT(\mathbb{Q}), \) for \( m \geq 0. \)

We can now establish our main statement:

**Theorem 6.1.9.** The mapping \( \rho : GT(\mathbb{Q}) \to h\text{Aut}_{\Lambda OP}(\{Res_*(B(PaB))\}), \) defined in §6.0.6, induces a bijection when we pass to the degree zero homotopy of our homotopy automorphism space:

\[
\rho : GT(\mathbb{Q}) \xrightarrow{\sim} \pi_0 h\text{Aut}_{\Lambda OP}(\{Res_*(B(PaB))\}).
\]

We accordingly have an isomorphism between the Grothendieck-Teichmüller group \( GT(\mathbb{Q}) \) and the group of homotopy automorphisms of the operad \( R = \{Res_*(B(PaB))\} \) which this degree zero homotopy class set \( \pi_0 h\text{Aut}_{\Lambda OP}(R) \) represents.

**Proof.** We deduce this result from the previous theorem and from the observation that the (free and transitive) action of the Grothendieck-Teichmüller group \( GT(\mathbb{Q}) \) on the set of Drinfeld’s associators \( Ass(\mathbb{Q}) \) corresponds to the (free and transitive) action of the homotopy class group \( \pi_0 h\text{Aut}_{\Lambda OP}(\{Res_*(B(PaB))\}) \) on the homotopy class set \( \pi_0 h\text{iso}_{\Lambda OP}(\{Res_*(B(PaB))\}, B(CD)) \) considered in this statement. \( \square \)

6.2. The action of the classifying space of the additive group and the concluding result

We now examine the homotopy type of the space \( h\text{Aut}(E_2)_{id} \) formed by the connected component of the identity map \( id \) within the homotopy automorphism space \( h\text{Aut}(E_2) \). We prove that this space is weakly-equivalent, as a simplicial monoid, to the classifying space of the additive group \( B(\mathbb{Q}) \). We also check that the whole homotopy automorphism space \( h\text{Aut}(E_2) \) is weakly-equivalent to the semidirect product \( GT(\mathbb{Q}) \ltimes B(\mathbb{Q}) \) in the category of simplicial monoids (as announced in Theorem A).

We still take the operad \( E_2^c = R = \{Res_*(B(PaB))\} \) as a cofibrant model of a rational \( E_2 \)-operad in the category of simplicial sets. We first explain the definition of a map \( \rho : B(\mathbb{Q}) \to h\text{Aut}_{\Lambda OP}(R) \) from the simplicial abelian group \( B(\mathbb{Q}) \) towards our homotopy automorphism space \( h\text{Aut}_{\Lambda OP}(R) \).

We can actually define a similar map without considering a rationalization of our object. We then get a map \( \rho : B(Z) \to h\text{Aut}_{\Lambda OP}(R) \) defined on the classifying space of the additive group \( Z \) and with value in the automorphism space of the operad \( R = \{Res_*(B(PaB))\} \). We explain our process in this setting, and we simply check that our constructions have, at each stage, an obvious extension to the rational context.

6.2.1. The definition of central elements in the pure braid groups. In a first step, we use the definition of central elements in the pure braid groups \( z_r \in P_r, r > 0, \) to get a action of the additive group on the morphism sets of the groupoids \( PaB(r), r > 0, \) underlying the parenthesized braid operad \( PaB \). In what follows, we also write \( z = z_r \) for short when the order \( r > 0 \) can be omitted.

We rely on the definition of the operad \( PaB \) as a pullback of the operad of colored braids \( CoB \) (see §1.6.2), and on the group identity \( \text{Mor}_{PaB(r)}(\pi, \pi) = \text{Mor}_{PaB(r)}(\pi, \pi) = P_r, \) valid for any object \( \pi \in \text{Ob} PaB(r) \), defined by a magma element \( \pi = \pi(x_{s(1)}, \ldots, x_{s(r)}) \in \Omega(r) \) with a given underlying permutation \( s \in \Sigma_r \).
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(see §1.6.1.3). We take

\[ z = z_2 = \tau^2 = \]

the canonical generator of the pure braid group \( P_2 \), in arity \( r = 2 \).

We identify this braid with an automorphism of the object \( \mu = \mu(x_1, x_2) \) in \( \text{PaB}(2) \). We easily see that the operadic composites \( z \circ_1 z \in \text{Mor}_{\text{PaB}(3)}(\mu \circ_1 \mu, \mu \circ_1 \mu) \) and \( z \circ_2 z \in \text{Mor}_{\text{PaB}(3)}(\mu \circ_1 \mu, \mu \circ_1 \mu) \) correspond to the same element \( z_3 \), given by

\[ z_3 = \]

in the pure braid group on 3 strands \( P_3 \). We then check, by a straightforward induction, that all operadic composites of the morphism \( z \in \text{Mor}_{\text{PaB}(2)}(\mu, \mu) \) which we form in a given arity \( r > 0 \), by combining composition products \( \circ_i \) and an action of permutations \( s \in \Sigma_r \), correspond to the same element \( z_r \) of the pure braid group \( P_r \). We have for instance:

\[ z \circ_1 (z \circ_1 z) \equiv z \circ_1 (z \circ_2 z) \equiv (z \circ_2 z) \circ_1 z \equiv \cdots \equiv \]

in the group \( P_3 \).

We can use these decompositions to prove that the element \( z = z_r \in P_r \) commutes with the generators of the braid group \( B_r \), and hence, is central in \( B_r \), for any \( r > 0 \). For instance, the relation \( \tau_1 z_3 = z_3 \tau_1 \) in \( B_3 \), where \( \tau_1 \) is the generating braid

\[ \tau_1 = \]

follows from the identity \( (z \circ_1 z) \cdot (\tau \circ_1 id) = (z \cdot \tau) \circ_1 (z \cdot id) = (\tau \cdot z) \circ_1 (id \cdot z) = (\tau \circ_1 id) \cdot (z \circ_1 z) \) in the morphism set \( \text{Mor}_{\text{PaB}(3)}(\mu \circ_1 \mu, \mu \circ_2 \mu) \), where we use the decomposition \( z_3 = z \circ_1 z \) of our element \( z_3 \in P_3 \). We actually retrieve the standard generating elements of the center of the group \( B_r \) (see [122, §1.3.3]) with our operadic approach. The relation \( z = z_r \in Z(B_r) \) implies that we have a commutative diagram

\[ \]

in the groupoid \( \text{PaB}(r) \), for any \( a \in Z \), and any morphism \( \alpha \in \text{Mor}_{\text{PaB}(r)}(\pi, \rho) \), where we consider the endomorphisms of the objects \( \pi, \rho \in \text{Ob PaB}(r) \) determined by our element \( z_r \) in the pure braid group \( P_r \). This commutation relation trivially extends to the Malcev completion of the groupoid \( \text{PaB}(r) \). We then assume \( a \in Q \).
6.2.2. The definition of the action. We now have a map of simplicial sets

\[ B(\mathcal{Z}) \times B(PaB(r)) \rightarrow B(PaB(r)) \]

yielded by the translation action \( \alpha \mapsto \alpha \cdot z^a \) on the morphism sets of the operad \( PaB(r) \), for any \( r > 0 \). To be explicit, we consider an \( n \)-simplex \( \alpha \in B_n(PaB(r)) \),
given by a chain of composable morphisms \( \{ \pi_0 \xrightarrow{\alpha_1} \pi_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} \pi_n \} \) in the
groupoid \( PaB(r) \). For any \( \underline{a} = (a_1, \cdots, a_n) \in B_n(\mathcal{Z}) \), we write:

\[ \underline{a} \cdot z^a = \{ \pi_0 \xrightarrow{\alpha_1 \cdot z^{a_1}} \pi_1 \xrightarrow{\alpha_2 \cdot z^{a_2}} \cdots \xrightarrow{\alpha_n \cdot z^{a_n}} \pi_n \} \]

to denote a component-wise translation operation, defining the action of this

simplicial group element \( \underline{a} \) on our simplex \( \alpha \). In this expression, we just take the

composite of each \( \alpha_i \), with the endomorphism of the object \( \pi_{i-1} \) defined by the

pure braid group element \( z^{a_i} \). We use our commutation relations §6.2.1(5) to check that this dimension-wise action of the simplicial group \( B(\mathcal{Z}) \) on the simplicial set

\( B(PaB(r)) \) preserves the face and degeneracy operations attached to our objects.

We argue similarly to get an action of the simplicial abelian group \( B(\mathbb{Q}) \) in the

classifying space \( B(PaB(r)) \) associated to the Malcev completion of the groupoid

\( PaB(r) \), for any \( r > 0 \).

In a second step, we use the operadic decompositions of our elements \( z \in P \), to extend this translation action (1) to the cotriple resolution \( R = |\text{Res}_* B(PaB)| \) of the

operad in simplicial sets \( B(PaB) \). Recall that we have \( |\text{Res}_* (P)| = \text{Diag} \text{Res}_*(P) \), for any operad in simplicial sets \( P \in sSet \Lambda \otimes \mathbb{Q}^1 \), where we consider the diagonal complex of the bisimplicial object \( \text{Res}_*(P) \), and that any simplex in the cotriple construction \( \text{Res}_*(P) \) is represented by a pair \( (\lambda, \pi) \), where \( \lambda \) denotes a chain of

tree morphisms \( \mathbf{T}_0 \leftarrow \mathbf{T}_1 \leftarrow \cdots \leftarrow \mathbf{T}_m \) and we assume \( \pi \in \Delta \mathbf{T}_n \)(P) (see §B.1).

We proceed as follows to define a map of simplicial sets

\[ B(\mathcal{Z}) \times \text{Diag} \text{Res}_* B(PaB(r)) \rightarrow \text{Diag} \text{Res}_* B(PaB(r)) \]

giving our extension of the translation action. Let again \( \underline{a} \in B(\mathcal{Z}) \). We now assume

\( P = B(PaB) \) and \( \pi \in \emptyset \mathbf{T}_n (B(PaB)) \), for a pair \( (\lambda, \pi) \), defining an \( n \)-simplex of the

simplicial set \( \text{Diag} \text{Res}_* B(PaB(r)) \). We define the mapping \( (\lambda, \pi) \mapsto (\lambda, \pi \cdot z^a) \)

by considering a diagonal translation action \( \pi_v \mapsto \pi_v \cdot z^a \) on the factors \( \pi_v \in B(PaB(\mathcal{T}_v)) \), \( v \in \mathbf{V}(\mathbf{T}_n) \), of our tree-wise tensor \( \pi \). We easily check again that the face and degeneracy operations attached to our objects are preserved by this dimension-wise action of the simplicial group \( B(\mathcal{Z}) \) on the simplicial set \( B(PaB(r)) \).

By adjunction, the maps (3) determine morphisms

\[ \rho : B(\mathcal{Z}) \rightarrow \text{Mor}_{sSet}(R(r), R(r)^{\Delta^*}) \]

towards the function spaces \( \text{Map}_{sSet}(R(r), R(r)) = \text{Mor}_{sSet}(R(r), R(r)^{\Delta^*}) \) associated to the simplicial sets \( R(r) \) defining the terms of our operad \( R \). The coherence of our collection of central elements \( z \in P_r \), \( r > 0 \), with respect to the composition structure of the parenthesized braid operad \( PaB(r) \) implies that the collection of maps \( \rho(\underline{a}) : R(r) \rightarrow R(r)^{\Delta^*} \), \( r > 0 \), which we assign to any group element \( \underline{a} \in B(\mathcal{Z}) \),
defines a morphism of \( \Lambda \)-operads when we regard the objects \( R^{\Delta^*}(r) = R(r)^{\Delta^*} \), \( r > 0 \), as the components of the simplicial framing of the operad \( R \). This collection of maps \( \rho(\underline{a}) : R(r) \rightarrow R(r)^{\Delta^*} \), \( r > 0 \), accordingly defines an element of the function space of operads \( \text{Map}_{sSet \Lambda \otimes \mathbb{Q}^1}(R, R) = \text{Mor}_{sSet \Lambda \otimes \mathbb{Q}^1}(R, R^{\Delta^*}) \), and the mapping \( \underline{a} \mapsto \rho(\underline{a}) \)
\(\rho(u)\) defines a morphism of simplicial monoids \(\rho : B(\mathbb{Z}) \to \text{Map}_{\text{Set}}(R, R)\), which trivially lands in the connected component of the identity map in \(\text{Map}_{\text{Set}}(R, R)\), because \(B(\mathbb{Z})\) is connected.

Hence, we finally obtain a map \(\rho : B(\mathbb{Z}) \to \text{hAut}_{\text{op}}(R)\) and we similarly get:

\[
\rho : B(Q) \to \text{hAut}_{\text{op}}(\left\{\text{Res}_* B(PaB')\right\})_{\text{id}}
\]

when we work rationally. This map is a morphism of simplicial monoids by construction. We then establish the following result:

**Theorem 6.2.3.** The map of the previous paragraph defines a weak-equivalence

\[
\rho : B(Q) \xrightarrow{\sim} \text{hAut}(\left\{\text{Res}_* B(PaB')\right\})_{\text{id}}
\]

in the category of simplicial sets.

**Proof.** For short, we still use the notation \(R = |\text{Res}_* (B(PaB))|\) and \(R^\sim = |\text{Res}_* (B(PaB'))|\).

We immediately deduce, from our spectral sequence constructions and the computations of Theorem 5.2.8, that the space \(\text{Map}_{\text{op}}(R, B(CD'))\) has a trivial homotopy in degree \(\ast > 1\), and we get the same result for our homotopy automorphism space \(\text{hAut}_{\text{op}}(R)\) which has the same homotopy as this function space. We also deduce, from the outcome of our spectral sequence computations (and the rational structure of our operads), that the group \(\pi = \pi_1 \text{hAut}(R)\) forms a rational sub-module of the additive group \(Q\). We therefore have either \(\pi = 0\) or \(\pi = Q\), and we just have to check that our map \(\rho : B(Q) \xrightarrow{\sim} \text{hAut}(R)\) induces an non-trivial morphism at the fundamental group level to complete the proof of our theorem.

We pick an element of the set of Drinfeld’s associators \(v_0 \in \text{Ass}(Q)\) and we form the morphism of operads in simplicial sets \(\phi_{v_0} : B(PaB) \to B(CD')\) associated to \(v_0\) to get a base point in the space \(\text{Map}_{\text{op}}(R, B(CD'))\) after composition with the augmentation \(R^\sim \to B(PaB)\). We also consider the morphism of operads in Malcev complete groupoids \(v_0 : PaB \to CD\) extending \(v_0 : PaB \to CD\) and we form the morphism of operads in simplicial sets \(\phi_{v_0} : B(PaB') \to B(CD')\) associated to this operad morphism \(v_0\) to get a base point in the space \(\text{Map}_{\text{op}}(R^\sim, B(CD'))\) after composition with the augmentation \(R^\sim \to B(PaB')\). We have a commutative diagram:

\[
\begin{array}{ccc}
B(Q) & \xrightarrow{\rho} & \text{hAut}_{\text{op}}(R)_{\text{id}} \\
\downarrow & & \downarrow \\
\text{hAut}_{\text{op}}(R)_{\text{id}} \times \{\phi_{v_0}\} & \xrightarrow{\sim} & \text{Map}_{\text{op}}(R^\sim, B(CD'))_{\phi_{v_0}} \\
\downarrow & & \downarrow \\
\text{Map}_{\text{op}}(R, B(CD'))_{\phi_{v_0}} & \xrightarrow{\sim} & \text{Map}(R^\sim, B(CD'))_{\phi_{v_0}}
\end{array}
\]

where we consider the action of our simplicial monoid on our base point to get the mapping on the left-hand side of our diagrams, and we use the projection onto the arity 2 component of an operad to form our mappings on the right-hand side.
We have \( R(2) \cong B(PaB(2)) = B(Q) \) and \( B(CD(2)) = B(Q) \). We easily see that the fill-in mapping
\[
\hat{\rho} : B(Q) \to \text{Map}(R(2), B(CD(2)))_{\phi_0},
\]
in the above diagram corresponds to the map \( \hat{\rho} : B(Q) \times B(Q) \to B(Q) \) induced by the translation action of the additive group \( Q \) on itself when we apply these identities \( R(2) \cong B(CD(2)) = B(Q) \). We have \( \text{Map}(B(Q), B(Q))_{\phi_0} \cong B(Q) \) on the other hand, and we readily deduce from these observations that our map \( \hat{\rho} \) actually defines a weak-equivalence of simplicial sets.

We conclude that our map \( \rho : B(Q) \to h\text{Aut}_A \circ P(R)^{id} \) induces a split injection \( \rho_* : \pi_1 h\text{Aut}_A \circ P(R)^{id} \) at the fundamental group level, and this verification finishes the proof of our theorem.

We still write \( R^\gamma = |\text{Res}_\gamma(b(PaB))| \). The mapping \( \rho : \text{GT}(Q) \to h\text{Aut}_A \circ P(R) \), which yields our group isomorphism \( h\text{Aut}_A \circ P(R) \), carries the elements of the Grothendieck–Teichmüller group \( \text{GT}(Q) \) to automorphisms of the operad \( R^\gamma \). We can therefore consider a conjugation action of the group \( \text{GT}(Q) \) on the monoid \( h\text{Aut}_A \circ P(R) \), and this action clearly preserves the connected component of the identity \( h\text{Aut}_A \circ P(R)^{id} \) within this space.

Recall that the Grothendieck–Teichmüller group is, on the other hand, endowed with a morphism \( \lambda : \text{GT}(Q) \to Q^\times \) which maps any element \( \gamma \in \text{GT}(Q) \) to the scalar \( \lambda = \lambda_\gamma \), such that we have the identity \( \gamma(\tau^2) = (\tau^2)^\lambda \) in \( \text{Mor}_{PaB(2)}(\mu, \mu) = P_2^\gamma \).

We have the following observation:

**Proposition 6.2.4.** We have a commutative diagram
\[
\begin{array}{ccc}
B(Q) & \xrightarrow{\rho} & h\text{Aut}_A \circ P(R)^{id} \\
\downarrow{\lambda_\gamma} & & \downarrow{\phi_{\rho(\gamma)}} \\
B(Q) & \xrightarrow{\rho} & h\text{Aut}_A \circ P(R)^{id}
\end{array}
\]
for any element \( \gamma \in \text{GT}(Q) \), where \( \phi_{\rho(\gamma)} \) denotes the conjugation action of the operad automorphism \( \rho(\gamma) : R^\gamma \to R^\gamma \) on the space \( h\text{Aut}_A \circ P(R)^{id} \), and we consider the action of the multiplicative scalar \( \lambda_\gamma \in Q^\times \) associated to \( \gamma \) on the simplicial set \( B(Q) \).

**Proof.** We go back to the definition of our map \( \rho : B(Q) \to h\text{Aut}_A \circ P(R)^{id} \) in §6.2.2 in terms of the arity-wise action of the simplicial abelian group \( B(Q) \) on the components of our operad \( R^\gamma = \text{Diag Res}_\gamma(b(PaB)) \).

We easily see that we have a commutative diagram
\[
\begin{array}{ccc}
B(Q) \times \text{Diag Res}_\gamma(b(PaB))(r) & \xrightarrow{\rho(\gamma)} & \text{Diag Res}_\gamma(b(PaB))(r) \\
\downarrow{\lambda_\gamma \times \rho(\gamma)} & & \downarrow{\rho(\gamma)} \\
B(Q) \times \text{Diag Res}_\gamma(b(PaB))(r) & \xrightarrow{\rho(\gamma)} & \text{Diag Res}_\gamma(b(PaB))(r)
\end{array}
\]
for each arity \( r > 0 \), where we consider the morphism \( \rho(\gamma) : \text{Diag Res}_\gamma(b(PaB)) \to \text{Diag Res}_\gamma(b(PaB)) \) associated to \( \gamma \in \text{GT}(Q) \), and the multiplication by the scalar \( \lambda_\gamma \in Q^\times \) on \( B(Q) \). We deduce this result from the fact that our action of the simplicial group \( B(Q) \) on \( R^\gamma \) is defined, at the level of the groupoids \( PaB(n)^\gamma \), by a translation by operadic composites of morphisms of the form \( z^a = (\tau^2)^a \),
a ∈ ℚ, (see §6.2.1) for which we have γ((τ^2)^a) = (τ^2)^{ζ}\gamma a. Recall simply that γ is a morphism of operads in (Malcev complete) groupoids by definition. This morphism therefore preserves such translation operations, the powers z^a, a ∈ ℚ, as well as the operadic composites which we consider in our process.

Then we just pass to the map ρ : B(ℚ) → hAut_{λOp}(R^\hat) corresponding to our simplicial group action in §6.2.2 to get the commutative diagram of the proposition from the above diagrams (1).

□

This proposition implies that our maps assemble to a morphism of simplicial monoids

[\rho : GT(ℚ) ⋊ B(ℚ) → hAut_{λOp}(R^\hat) ],

where we consider the action of the Grothendieck-Teichmüller group GT(ℚ) on B(ℚ) through our character map λ : GT(ℚ) → ℚ^\times when we form the semi-direct product GT(ℚ) ⋊ B(ℚ).

Recall that the connected components of a homotopy automorphism space are all weakly-equivalent in general (see §2.2.2). Theorem 6.2.3 therefore implies that our mapping ρ : GT(ℚ) ⋊ B(ℚ) → hAut_{λOp}(R^\hat) induces a weak-equivalence from the simplicial set \{ γ \} × B(ℚ), where γ ∈ GT(ℚ), to the connected component of the morphism ρ(γ) : R^\hat → R^\hat associated to γ in the homotopy automorphism space hAut_{λOp}(R^\hat). The result of Theorem 6.1.9 implies, on the other hand, that the connected components of the space hAut_{λOp}(R^\hat) are in a one-to-one correspondence with the elements of the Grothendieck-Teichmüller group GT(ℚ). (This implication follows from general observations of §2.2.2.) Therefore, we have the following concluding statement:

**Theorem 6.2.5 (Theorem A).** The space of homotopy automorphism of a rational E_2-operad hAut_{λOp}(E_2) is weakly-equivalent, as a simplicial monoid, to the semi-direct product of the Grothendieck-Teichmüller group with the classifying space of the additive group:

[\rho : GT(ℚ) ⋊ B(ℚ) → hAut_{λOp}(E_2) ],

where we consider the natural action of the Grothendieck-Teichmüller group GT(ℚ) on the classifying space B(ℚ) through the character map λ : GT(ℚ) → ℚ^\times to form our semi-direct product GT(ℚ) ⋊ B(ℚ). □
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