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On almost sure convergence of some free energies *

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Abstract

We investigate the problem of the almost sure (a.s.) behavior of some normalized free energies. More precisely, let $w_{ij}, 1 \leq i < j)$ and $X = (X_i, 1 \leq i \leq n)$ be two independent centered sequences of i.i.d. random variables. Let moreover $\beta > 0$. We study in some cases the a.s. convergence of the quantity

$$\frac{1}{n} \log \mathbf{E}_X \exp \left\{ \frac{\beta}{\sqrt{n}} \sum_{1 \leq i < j \leq n} w_{ij} X_i X_j \right\}$$

where $\mathbf{E}_X$ denotes partial expectation with respect to X and identify the limit when X is Gaussian.

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1 Introduction

The terms "free energy" come from physics and denote a quantity that probabilists and analysts would rather call cumulant generating function or log-Laplace transform. The problem of finding an almost sure limit is not simple and as far as we know, there are few results about it. We try here to give basic properties, as well as examples to illustrate these.

Consider two *independent centered* sequences $(w_{i,j})_{1\leq i<j}$ and $X = (X_i)_{i\geq 1}$ of *i.i.d.* random variables with *unit variance* on a complete probability space $(\Omega, \mathcal{A}, \mathbf{P})$ and a real $\beta > 0$. The free energy studied here is of the form

$$F_n(\beta) = \frac{1}{n}\log \mathbf{E}_X \exp\left\{\frac{\beta}{\sqrt{n}}\sum_{1\leq i<j\leq n} w_{i,j}X_iX_j\right\} \tag{1}$$

where $\mathbf{E}_X$ denotes partial expectation with respect to X. What is the almost sure behavior of (1) as n tends to infinity? Observing the exponent in (1) one may wonder if the possible limit is non random or (and) continuous, at least in some cases. We will always assume that the condition $\mathbf{E}w_{i,j}^4 < \infty$ is satisfied.

The aim of the second section is to recall known results about symmetric random matrices. The third section underscores the basic properties of the random function $\limsup F_n$ in our general setting and the latter one deals with an example as simple as possible.

2 Real Wigner Matrices

This section is devoted to recall some of the well-known results about Wigner matrices, that satisfy by definition the following conditions:

1. symmetric;
2. entries above the main diagonal are i.i.d. random variables;
3. entries on the diagonal are i.i.d. random variables;
4. diagonal entries are independent of non-diagonal entries.

Let $W_n = (w_{i,j})_{1\leq i,j\leq n}$ be a $n\times n$ Wigner matrix. The eigenvalues of the *normalized* matrix $W_n/\sqrt{n}$ will be denoted by $\lambda_1(n), \ldots, \lambda_n(n)$. In the sequel, we will use two important asymptotic results about the empirical measure

$$\mu_n := \frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i(n)} \tag{2}$$

and the largest eigenvalue

$$\lambda_{\max}(n) := \max_{1\leq i\leq n}\lambda_i(n) \tag{3}$$

The precise statements are now given:

Theorem 1 (semi-circle law, e.g.[2]) *Let W_n be a real Wigner matrix. Assume that $\mathbf{E}w_{1,2}^2 = 1$. Then, with probability one, the spectral empirical measure μ_n converges weakly as n tends to infinity to a probability measure μ defined by*

$$\mu(dt) = \frac{1}{2\pi}\sqrt{4-t^2}\mathbf{I}_{|t|\leq 2}\,dt \tag{4}$$

Several versions and proofs (for instance by Wigner, Arnold, Bai, Pastur, Girko) of this famous theorem are available in the literature (see e.g. [9] and the references therein). Since for all x, the events $\{\mu_n(]-\infty, x]) = 1\}$ and $\{\lambda_{\max}(n) \leq x\}$ are identical, a straightforward consequence of this theorem is that

$$\liminf_n \lambda_{\max}(n) \geq 2 \tag{5}$$

almost surely. Actually, Bai [3] proved much more:

Theorem 2 *Let W_n be a real Wigner matrix. Then, with probability one,*

$$\lim_n \lambda_{\max}(n) = 2 \tag{6}$$

if and only if the following four conditions are satisfied

(i) $\mathbf{E}(w_{1,1}^+)^2 < \infty$,

(ii) $\mathbf{E}w_{1,2} \leq 0$,

(iii) $\mathrm{Var}(w_{1,2}) = 1$,

(iv) $\mathbf{E}w_{1,2}^4 < \infty$.

In the particular case where the $w_{i,j}$ are centered Gaussian and $w_{i,i} \equiv 0$, we can give a new proof of Theorem 2, avoiding Graph Theory. Using Gaussian analysis, we shall show that

$$\limsup_n \lambda_{\max}(n) \leq 2,$$

almost surely which together with (5) gives the result. The key point is

Lemma 3 *The two following inequalities hold:*

(i) For all integer n,

$$\mathbf{E}\lambda_{\max}(n) \leq 2 \tag{7}$$

(ii) Let $\varepsilon > 0$. There exist two constants $K(\varepsilon)$ and $N(\varepsilon)$ such that

$$\forall N \geq N(\varepsilon), \qquad \mathbf{E}\sup_{n\geq N} \lambda_{\max}(n) \leq 2 + \frac{K(\varepsilon)}{N^{1/2-\varepsilon}} \tag{8}$$

Actually, from the definition of $\lambda_{\max}(n)$ as

$$\lambda_{\max}(n) = \max_{\sum_i t_i^2=1} \frac{2}{\sqrt{n}} \sum_{1\le i<j\le n} w_{ij}t_it_j,$$

it is easily seen that $\limsup \lambda_{\max}(n)$ is almost surely constant by the Kolmogorov 0-1 Law and (8) gives then the a.s. inequality

$$\limsup_n \lambda_{\max}(n) \le 2$$

which allows to conclude.

Proof of Lemma 3. The scalar product in $\mathbf{R}^n$ will be denoted by $\langle\cdot,\cdot\rangle_n$ and the corresponding norm $||\cdot||_n$. If $s=(s_i)_{i\ge1}$ is a vector of ℓ_2, it will be understood that s^2 is the vector $(s_i^2)_{i\ge1}$ and that $||s||_n^2$ is the partial sum $\sum_{i=1}^n s_i^2$. Let B_2 be the unit ball of ℓ_2 and set for all (m,s) and $(n,t)\in\mathbf{N}\times\mathrm{B}_2$

$$X(n,t) := \frac{2}{\sqrt{n}} \sum_{1\le i<j\le n} w_{ij}t_it_j,$$

$$d_X^2(m,s;n,t) := \mathbf{E}|X(m,s)-X(n,t)|^2.$$

We have then for $m\le n$

$$d_X^2(s,m;t,n) = \frac{2}{m}(1-||s^2||_m^2) + \frac{2}{n}(1-||t^2||_n^2) - \frac{4}{\sqrt{mn}}(\langle s,t\rangle_m^2 - \langle s^2,t^2\rangle_m) \quad (9)$$

and we notice in particular that

$$d_X^2(n,s;n,t) = \frac{2}{n}\Big(||s-t||_n^2(1+<s,t>_n) - ||s^2-t^2||_n^2\Big) \le \frac{4}{n}||s-t||_n^2 \quad (10)$$

The right-hand side of the last inequality in (10) is the mean square increment of a Gaussian process Y_n on B_2 defined by

$$Y_n(t) := \frac{2}{\sqrt{n}}\sum_{i=1}^n t_ig_i \qquad (t\in\mathrm{B}_2)$$

where the g_i's are i.i.d. $\mathcal{N}(0,1)$. By the Comparison Theorem ([7],[8] or [5]) and a simple estimation of the supremum of Y_n, we get for each $n\ge1$

$$\mathbf{E}\sup_{t\in\mathrm{B}_2} X(t,n) \le \mathbf{E}\sup_{t\in\mathrm{B}_2} Y_n(t) \le 2\mathbf{E}\Big(\frac{1}{n}\sum_{i=1}^n g_i^2\Big)^{1/2} \le 2 \quad (11)$$

which proves (7).

To prove (8), we will use an inequality which allows, roughly speaking, to move the supremum outside the expectation $\mathbf{E}\sup_{n\ge N}\lambda_{\max}(n)$ with a small correction (see [5] p 157):

$$\mathbf{E}\sup_{n\ge N}\sup_{t\in\mathrm{B}_2} X(n,t) \le \sup_{n\ge N}\mathbf{E}\sup_{t\in\mathrm{B}_2} X(n,t) + \frac{\pi}{2}\sup_{f\in\mathrm{B}_2^{[N,+\infty[}}\mathbf{E}\sup_{n\ge N} X(n,f(n)) \quad (12)$$

Because of (7), it remains to estimate the second term of the right-hand side of (12). Inspecting (9), we note that

$$d_X^2(m,f(m);n,f(n)) \leq \frac{2}{m}+\frac{2}{n}-\frac{4}{\sqrt{mn}}\mathbf{I}_{m=n}$$

which is the mean square increment of a Gaussian process Z on $\mathbf{N}$ defined by

$$Z(n) := \sqrt{\frac{2}{n}}g_n \quad (n \geq 1).$$

Again, by the Comparison Theorem, it suffices to estimate trough basic inequalities the quantity

$$\begin{aligned}
2^{-1/2}\mathbf{E}\sup_{n\geq N} Z(n) &= \int_0^\infty \big(1-\prod_{n\geq N}(1-\mathbf{P}(g_n > t\sqrt{n}))\big)dt \\
&= \int_0^\infty \Big(1-\exp\big\{\sum_{n\geq N}\ln(1-\mathbf{P}(g_n > t\sqrt{n}))\big\}\Big)dt \\
&\leq \int_0^\infty \Big(1-\exp\big\{\sum_{n\geq N}\ln(1-e^{-t^2n/2})\big\}\Big)dt \\
&\leq N^{-1/2+\varepsilon}+\int_{N^{-1/2+\varepsilon}}^\infty \sum_{n\geq N} -\ln(1-e^{-t^2n/2})dt \\
\text{(for N large enough)} &\leq N^{-1/2+\varepsilon}+\int_{N^{-1/2+\varepsilon}}^\infty \sum_{n\geq N} 2e^{-t^2n/2}dt \\
&\leq K(\varepsilon)N^{-1/2+\varepsilon} \qquad (13)
\end{aligned}$$

where we used the following facts:

$$-\ln(1-u) \leq 2u \quad (u \in]0,1/2]),$$

$$1-e^{-u} \leq u \quad (u>0),$$

$$\sqrt{\frac{\alpha}{2\pi}}\int_u^\infty e^{-\alpha t^2/2}dt \leq e^{-u^2\alpha/2} \quad (\alpha, u > 0).$$

Combining (7), (12) and (13), we get (8). ///

3 Continuous and non random limits?

The existence of the almost sure limit of F_n is a hard question. In special cases like the SK model [10], there is no definitive result. The problem treated here is rather to known how a possible limit looks like. We begin by easy properties of the random function $\beta \mapsto F_n(\beta)$ for each $n \geq 1$ that stem from Jensen and Hölder's inequalities:

1. F_n is non negative,
2. F_n is increasing. More precisely,

$$0 \leq \beta \leq \beta' \Rightarrow F_n(\beta) \leq \frac{\beta}{\beta'} F_n(\beta'),$$

3. F_n is convex.

An straightforward consequence of these properties is that $\limsup_n F_n$ is a non decreasing convex random function from $\mathbf{R}^+$ into $\mathbf{R}^+ \cup \{+\infty\}$. For the sequel, it is convenient to set for $k, n \geq 1$

$$F_{n,k}(\beta) := \frac{1}{n} \log \mathbf{E}_X \exp\Big\{ \frac{\beta}{\sqrt{n}} \sum_{\substack{1 \leq i < j \leq n \\ j > k}} w_{i,j} X_i X_j \Big\} \tag{14}$$

To study the a.s. behavior of F_n, we introduce the following "critical" value

$$\beta_c := \sup\{\beta \geq 0 : \mathbf{P}\{\limsup_n F_n(\beta) < +\infty\} = 1\} \tag{15}$$

Proposition 4 *If β_c is positive, then, with probability one,* $\limsup_n F_n$ *is a non random and convex (continuous) function on* $[0, \beta_c[$ *into* $\mathbf{R}^+$.

Proof. Via the conditional Jensen's inequality, the assumption $\beta_c > 0$ and the fact that $\mathbf{P}\{w_{1,2} > 0\} > 0$ imply that for some real $\alpha > 0$,

$$\mathbf{E} \exp(\alpha X_1 X_2) < +\infty \tag{16}$$

Let $\beta < \beta_c$ and $\Omega_\beta := \{\limsup_n F_n(\beta) < +\infty\}$ of probability one. Since $\limsup_n F_n$ is a convex random function, then for each $\omega \in \Omega_\beta$, $\limsup_n F_{n,\omega}$ is a continuous function on $[0, \beta]$ into $\mathbf{R}^+$ and the continuity on $[0, \beta_c[$ follows by considering the event

$$\bigcap_{n \geq 1} \Omega_{\beta_c - \beta_c/n}.$$

The non randomness of $\limsup_n F_n$ relies upon the fact that for all $\beta \in [0, \beta_c[$ and all $k \geq 2$,

$$\mathbf{P}\{\limsup_n F_n(\beta) = \limsup_n F_{n,k}(\beta)\} = 1 \tag{17}$$

Indeed, (17) implies that $\limsup_n F_n$ is measurable with respect to the tail σ-field

$$\bigcap_{k \geq 2} \bigvee_{\substack{j > i \geq 1 \\ j > k}} \sigma(w_{i,j}).$$

Using continuity, we note that (17) is an immediate consequence of the following assertion: if $0 < \beta' < \beta < \beta'' < \beta_c$ and $k \geq 2$, then

$$\frac{\beta}{\beta'} \limsup_n F_n(\beta') \leq \limsup_n F_{n,k}(\beta) \leq \frac{\beta'}{\beta''} \limsup_n F_n(\beta'') \tag{18}$$

which is a consequence of Hölder inequality:

$$\mathbf{E}_X \exp\Big\{\frac{\beta'}{\sqrt{n}}\sum_{1\le i<j\le n} w_{i,j}X_iX_j\Big\} \le \mathbf{E}_X^{\beta'/\beta}\exp\Big\{\frac{\beta}{\sqrt{n}}\sum_{\substack{1\le i<j\le n\\ j>k}} w_{i,j}X_iX_j\Big\}$$
$$\times\, \mathbf{E}_X^{1-\beta'/\beta}\exp\Big\{\frac{\beta'}{(1-\beta'/\beta)\sqrt{n}}\sum_{1\le i<j\le k} w_{i,j}X_iX_j\Big\},$$

where the last factor tends to 1 when n tends to infinity because of the integrability property (16). This proves left-hand side of (18). The right-hand side is analogous and thus the proof of Proposition 4 is easily completed./ / /

The problem raised by the above proposition is the determination of the "critical" value β_c. Actually, we can give an optimal lower bound for it:

Lemma 5 *Recall the definition (15). The critical value β_c is >0 if and only if there exists $\beta>0$ such that $\mathbf{E}e^{\beta X_1^2}<+\infty$. Moreover,*

$$\beta_c \ge \sup\{\beta\ge 0 : \mathbf{E}e^{\beta X_1^2}<+\infty\} \tag{19}$$

Remark. The optimality of (19) will be shown by considering X_1 Gaussian in the next section.

Proof. We first prove (19). Recall that $\lambda_1(n),\dots,\lambda_n(n)$ denote the eigenvalues of the symmetric random matrix $(w_{i,j})_{1\le i,j\le n}/\sqrt{n}$ where we have set $w_{j,i}=w_{i,j}$ for $i<j$ and $w_{i,i}=0$. Recall also (3). Let moreover $(u_{i,j})_{1\le i,j\le n}$ be the orthogonal random matrix which diagonalizes $(w_{i,j})/\sqrt{n}$. We have then, for each $\beta\ge 0$,

$$\begin{aligned}\frac{\beta}{\sqrt{n}}\sum_{1\le i<j\le n} w_{i,j}X_iX_j &= \frac{\beta}{2}\sum_{k=1}^n \lambda_k(n)\big(\sum_{i=1}^n u_{k,i}X_i\big)^2,\\ &\le \frac{\beta}{2}\lambda_{\max}(n)\sum_{k=1}^n\big(\sum_{i=1}^n u_{k,i}X_i\big)^2.\end{aligned}$$

We introduce the orthogaussian sequence $(g_k)_{k\ge1}\sim\mathcal{N}(0,1)^{\otimes\mathbf{N}}$, independent of all the previous random sequences. We note that $\lambda_{\max}(n)$ is non-negative and using the rotational invariance of $\mathcal{N}(0,1)^{\otimes n}$, we get

$$\begin{aligned}
\exp\Big\{\frac{\beta}{2}\lambda_{\max}(n)\sum_{k=1}^n\big(\sum_{i=1}^n u_{k,i}X_i\big)^2\Big\} &= \mathbf{E}_g\exp\Big\{\sqrt{\beta\lambda_{\max}(n)}\sum_{k=1}^n\big(\sum_{i=1}^n u_{k,i}X_i\big)g_k\Big\}\\
&= \mathbf{E}_g\exp\Big\{\sqrt{\beta\lambda_{\max}(n)}\sum_{i=1}^n\big(\sum_{k=1}^n u_{k,i}g_k\big)X_i\Big\}\\
&= \mathbf{E}_g\exp\Big\{\sqrt{\beta\lambda_{\max}(n)}\sum_{i=1}^n g_i\,X_i\Big\}\\
&= \mathbf{E}_g\exp\Big\{\big(\beta\lambda_{\max}(n)\sum_{i=1}^n X_i^2\big)^{1/2}g_1\Big\}\\
&= \exp\Big\{\beta\lambda_{\max}(n)\sum_{i=1}^n X_i^2/2\Big\}.
\end{aligned}$$

It follows that

$$F_n(\beta)\le\ln\mathbf{E}_X\exp\Big\{\beta\lambda_{\max}(n)X_1^2/2\Big\}.$$

Now, since $\lim_n\lambda_{\max}(n)=2$ a.s., we obtain by the dominated convergence theorem that

$$\forall\beta<\beta_c,\ \limsup_n F_n(\beta)\le\ln\mathbf{E}_X e^{\beta X_1^2} \tag{20}$$

This proves (19).

It remains to prove that if β_c is >0, then there exists $\beta>0$ such that $\mathbf{E}e^{\beta X_1^2}<+\infty$. But it is easy to see that this is equivalent to condition (16)./ / /

The previous proof leads to some comparison with a work by Comets [4] which gives the following bound in the particular case where $w_{1,2}$ is Gaussian and X_1 takes values ±1 with probability 1/2 (the SK model): if $\beta\ge1$,

$$\limsup_n F_n(\beta)\le\beta-\frac{1}{2}\ln(\beta)-\frac{3}{4}.$$

It is interesting to note that (20) gives in this case for all $\beta\ge0$,

$$\limsup_n F_n(\beta)\le\beta,$$

a less precise bound but more general since the Gaussianity of $w_{1,2}$ is not needed but only the existence of the fourth moment required in Theorem 2.

4 The Gaussian case: $X_1\sim\mathcal{N}(0,1)$

The example treated in this section is certainly the most simple in our frame. As we shall see, it provides nevertheless a non trivial problem at $\beta=\beta_c$.

Theorem 6 *The critical value β_c is $1/2$ and the free energy has an almost sure limit for $\beta \neq 1/2$:*

(i) If $\beta < 1/2$, then

$$\lim_n F_n(\beta) = \frac{1}{4} - \frac{1}{2}\ln\Big(\frac{1+\sqrt{1-4\beta^2}}{2}\Big) - \frac{1-\sqrt{1-4\beta^2}}{8\beta^2} \qquad (21)$$

(ii) If $\beta > 1/2$, then

$$\lim_n F_n(\beta) = +\infty \qquad (22)$$

The proof of Theorem 6 is rather easy and relies upon the asymptotic property the spectral radius of Wigner matrices and the Semi-circle Law introduced in section 2 (for more details see e.g. [2] and the references therein). These tools arise very naturally in this setting. Indeed, consider the real symmetric matrix whose entries are the $w_{i,j}$ above the diagonal, and zero on the diagonal. As before, let $\lambda_1(n), \ldots, \lambda_n(n)$ denote the eigenvalues of $(w_{i,j}/\sqrt{n})_{1\le i,j\le n}$. It follows from the rotational invariance of the law of the Gaussian vector $(X_1, \ldots, X_n)$ that

$$F_n(\beta) = \frac{1}{n}\ln \mathbf{E}_X \exp\Big\{\frac{\beta}{2}\sum_{1\le i\le n}\lambda_i(n)X_i^2\Big\} \qquad (23)$$

The first step is to know whether the positive quantity $F_n(\beta)$ is a.s. finite:

Lemma 7

(i) $\beta < 1/2 \Longrightarrow \mathbf{P}(\exists N, \forall n \ge N, F_n(\beta) < +\infty) = 1;$

(ii) $\beta > 1/2 \Longrightarrow \mathbf{P}(\exists N, \forall n \ge N, F_n(\beta) = +\infty) = 1.$

Proof of Lemma 7. We only prove (i) ((ii) is analogous). Since X_1 is normal standard random variable, the expectation $\mathbf{E}\exp\{\frac{a}{2}X_1^2\}$ is finite and equal to $(1-a)^{-1/2}$ if and only if $a < 1$. Together with the independence of the X_i's, the condition

$$\beta\lambda_{\max}(n) < 1 \qquad (24)$$

is equivalent via (23) to

$$F_n(\beta) = \frac{1}{n}\sum_{i=1}^n \ln\frac{1}{\sqrt{1-\beta\lambda_i(n)}} < +\infty.$$

It remains to show that, almost surely, condition (24) is true for n sufficiently large. But we know by Theorem 2 that $\lim_n \lambda_{\max}(n) = 2$, almost surely. With the assumption $\beta < 1/2$, we get that

$$\mathbf{P}(\exists N, \forall n \ge N, \beta\lambda_{\max}(n) < 1) = 1.$$

This proves Lemma 7. ///

Proof of Theorem 6. We only prove (21). Recall the definitions of the spectral empirical measure μ_n and its limit μ in section 2. Assume that $\beta < \beta_0 < 1/2$ and consider the measurable set

$$\Omega_0 := \left\{\mu_n \xrightarrow{\text{weakly}} \mu,\ \max_{1\le i\le n} |\lambda_i(n)| \to 2,\ \exists N, \forall n \ge N, F_n(\beta) < +\infty\right\}$$

of probability one, by Theorems 1, 2 and Lemma 7. Pick $\omega \in \Omega_0$. Then, there exists an integer N_ω such that for all $n \ge N_\omega$, the spectral radius $\max_{1\le i\le n} |\lambda_{i,\omega}(n)|$ is less than $1/\beta_0$ and

$$\begin{aligned} F_{n,\omega}(\beta) &= -\frac{1}{2n}\sum_{i=1}^{n} \ln(1-\beta\lambda_{i,\omega}(n)) \\ &= -\frac{1}{2}\int_{[-\beta_0^{-1},\beta_0^{-1}]} \ln(1-\beta x)\,\mu_{n,\omega}\,(dx). \end{aligned}$$

To obtain (21), note that the integrand $x \mapsto \ln(1-\beta x)$ on $[-\beta_0^{-1}, \beta_0^{-1}]$ extends to a continuous and bounded function on $\mathbf{R}$ and thus,

$$\begin{aligned} \lim_n F_{n,\omega}(\beta) &= -\frac{1}{2}\int \ln(1-\beta x)\,\mu\,(dx) \\ &= \frac{1}{4} - \frac{1}{2}\ln\left(\frac{1+\sqrt{1-4\beta^2}}{2}\right) - \frac{1-\sqrt{1-4\beta^2}}{8\beta^2} \end{aligned}$$

which is the desired result (see [4] for the computation of the last integral).///

Besides, for the critical value $\beta_c = 1/2$, it seems rather difficult to guess the a.s. behavior of F_n. Although the integral $\int \ln(1-\beta_c\, x)\,\mu\,(dx)$ converges, we do *not* have that, almost surely,

$$\lim_n F_n(\beta_c) = -\frac{1}{2}\int \ln(1-\beta_c x)\,\mu\,(dx);$$

actually, we have

Proposition 8 *At the critical value $\beta_c = 1/2$,*

$$\mathbf{P}(\limsup_n F_n(\beta_c) = +\infty) > 0. \tag{25}$$

Proof. It suffices to prove that

$$\mathbf{P}(\limsup_n\{\lambda_{\max}(n) \ge 2\}) \ge \lim_n \mathbf{P}(\lambda_{\max}(n) \ge 2) > 0.$$

To this end, we shall use a (strong) asymptotic result on the rescaled largest eigenvalue ([11], 1.19):

$$\lim_n \mathbf{P}[n^{2/3}(\lambda_{\max}(n) - 2) \le x] = \exp\left(-\frac{1}{2}\int_x^{+\infty} q(t) + (t-x)q^2(t)dt\right) \tag{26}$$

where q is the solution of the Painlevé II differential equation $y'' = ty + 2y^3$ with the asymptotic $y(t) \sim_{t\to+\infty} \mathrm{Ai}(t)$ where Ai is the Airy function. The existence and the properties of q can be found in [6]; we only need the fact that $q > 0$. Indeed, from (26) at $x = 0$, we get that

$$\lim_n \mathbf{P}[\lambda_{\max}(n) \geq 2] = 1 - \exp\left(-\frac{1}{2}\int_0^{+\infty} q(t) + tq^2(t)dt\right) > 0,$$

which completes the proof./ / /

Proposition 25 shows that there is a discontinuity at the critical value β_c. We do not know if $\limsup_n F_n(\beta_c)$ is random or not.

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