Quotient Morphisms, Compositions, and Fredholm Index

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Abstract

In this paper we define a composition between two quotient morphisms and prove a multiplication formula for the index of the composition of two Fredholm quotient morphisms. Using this formula and the fact that any linear relation can be seen as a quotient morphism, we obtain a multiplication formula for the index of the composition of two Fredholm linear relations.

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1 Introduction

The importance of the quotient vector spaces and associated morphisms has been emphasized in a long series of papers by L. Waelbroeck (see, for instance, the works [14] - [17], to quote only a few). The main idea in Waelbroeck's contributions is to consider quotients of topological (or bornological) vector spaces by continuously embedded subspaces. The resulting quotient spaces, although, in general, not separated as topological spaces, have a very rich structure, allowing, in particular, the development of the spectral theory of the morphisms in one or several variables, the construction of the analytic functional calculus and other (somehow unexpected) properties, traditionally considered in the context of complete vector spaces (usually Banach or Fréchet spaces) with separated topologies (see also [11, 12]).

The Fredholm and spectral theory, developed in the framework of quotient Banach spaces (in the more restricted sense of quotients of Banach spaces by closed subspaces) also has some important advantages due to appropriate duality properties (see [1, 2], etc.).

The aim of this paper is to study linear maps defined between spaces of the form X/X_0, where X is a vector space and X_0 is a vector subspace of X. As X/X_0 is itself a vector space, a legitimate question is why to complicate the matter instead of simply using vector spaces. The answer to this question is that this framework allows us to introduce a composition of morphisms which is more complex than the usual one (see Remark 3.4), as well as other algebraic operations, which are natural only in this context. The motivation of this approach comes from the theory of linear relations, introduced many years ago by Arens in [3] (but emphasized even earlier by J. von Neumann [13]), a concept which lately has been systematically studied by many authors (see, for instance, [5, 6, 9], as well as the works cited by these authors). As any linear relation can be associated with a unique linear map with values in a quotient space, which is a particular case of what we already called a quotient morphism, an independent study of quotient morphisms is rewarding when applied to linear relations, as we shall see in the last section. We also specify that the composition of two relations is a particular case of the composition of the quotient morphisms mentioned above.

In the next section we deal with linear maps which are not everywhere defined, extending the concept of Fredhom unbounded transformation in a Banach space, and proving a multiplication of the index result in this algebraic context (see Theorem 2.1).

The third section is dedicated to the study of quotient morphisms. We introduce the composition of quotient morphisms mentioned above, as well as other algebraic operations, and prove a generalization of the result concerning the multiplication of the index (using partially Theorem 2.1) for Fredholm quotient morphisms (see Theorem 3.8).

In the last section, we apply these results to linear relations (see Theorem 4.5), obtaining more general statements than the corresponding ones from [5]. This is possible because the associated morphism of a composition of two relations is the composition, in our generalized sense, of the morphisms associated with the corresponding relations (see Proposition 4.3).

2 Fredholm applications

In this section we consider vector spaces over the field $\mathbb{K}$, which is either the real field $\mathbb{R}$ or the complex one $\mathbb{C}$ (other fields might also be considered). If X is a vector space over $\mathbb{K}$, we designate by $\dim X$ the *dimension* (finite or infinite) of X over $\mathbb{K}$. Note that if

$$0 \to X \xrightarrow{A} Y \xrightarrow{B} Z \to 0 \tag{1}$$

is an exact sequence of vector spaces and linear maps (for elementary algebraic properties see, for instance [8]), then $\dim Y = \dim X + \dim Z$.

Having in mind the case of unbounded linear operators in Banach spaces, we shall discuss linear transformations which are not, in general, everywhere defined. Specifically, we consider linear maps between two arbitrary vector spaces X, Y having the form $T : D(T) \subset X \mapsto Y$, where $D(T)$, which is itself a vector space, is the *domain of definition* of T. The *range* of T, the *kernel* (or the *null space*) of T and the *graph* of T will be denoted by $R(T)$, $N(T)$ and $G(T)$, respectively.

Given two arbitrary linear maps $T : D(T) \subset X \mapsto Y$ and $S : D(S) \subset Y \mapsto Z$, we define their *composition* $S \circ T$ in the following way. The domain $D(S \circ T) \subset D(T)$ is given by $T^{-1}(D(S))$ and $(S \circ T)(x) = S(T(x))$ for all $x \in D(S \circ T)$. Note that $R(S \circ T) = S(R(T) \cap D(S))$.

When $T : X \mapsto Y$ and $S : Y \mapsto Z$, the composition $S \circ T$ will be simply denoted by ST.

As usually (see, for instance, [7] or [2]), we say that a linear map $T : D(T) \subset X \mapsto Y$ is *Fredholm* if both $\dim N(T)$ and $\dim Y/R(T)$ are finite. In that case, the (*algebraic*) *index* of T, denoted by $\mathrm{ind}(T)$, is given by

$$\mathrm{ind}(T) = \dim N(T) - \dim Y/R(T).$$

In this section we study the index of the composition $S \circ T$ of two Fredholm single-valued maps T, S, extending the standard multiplication result for the Fredholm index (see, for instance, [7] or [2]). We have the following assertion, for which we give a direct proof.

Theorem 2.1 *Let $T : D(T) \subset X \mapsto Y$, $S : D(S) \subset Y \mapsto Z$ be linear maps. Then*

$$\begin{aligned} &\dim N(S \circ T) + \dim Y/R(T) + \dim Z/R(S) = \\ &\dim Z/R(S \circ T) + \dim N(T) + \dim N(S) + \dim Y/(R(T) + D(S)). \end{aligned} \tag{2}$$

In particular, when T, S are Fredholm maps, then $S \circ T$ is Fredholm and

$$\mathrm{ind}(S \circ T) = \mathrm{ind}(S) + \mathrm{ind}(T) + \dim Y/(R(T) + D(S)). \tag{3}$$

Proof. We first note that $N(S \circ T) = T^{-1}(N(S))$. Therefore, the following sequence of vector spaces

$$0 \to N(T) \hookrightarrow N(S \circ T) \xrightarrow{T_0} R(T) \cap N(S) \to 0 \tag{4}$$

is exact, where T_0 is the restriction of T. Moreover

$$\dim N(S \circ T) = \dim N(T) + \dim R(T) \cap N(S), \tag{5}$$

showing, in particular, that $N(S \circ T)$ is finite dimensional when T and S are Fredholm.

The inclusions $R(S \circ T) \subset R(S) \subset Z$ induce the exact sequence of spaces

$$0 \to R(S)/R(S \circ T) \hookrightarrow Z/R(S \circ T) \xrightarrow{J_0} Z/R(S) \to 0, \tag{6}$$

where J_0 is the induced map, implying the equality

$$\dim Z/R(S \circ T) = \dim R(S)/R(S \circ T) + \dim Z/R(S). \tag{7}$$

The exactness of the sequence

$$0 \to N(S) \hookrightarrow D(S) \xrightarrow{S} R(S) \to 0, \tag{8}$$

as well as the exactness of the sequence

$$0 \to R(T) \cap N(S) \hookrightarrow R(T) \cap D(S) \xrightarrow{S_0} R(S \circ T) \to 0, \tag{9}$$

where S_0 is the restriction of S, are obvious. Consequently, if

$$\hat{S} : D(S)/(R(T) \cap D(S)) \mapsto R(S)/R(S \circ T)$$

is the map induced by S and

$$J : N(S)/(R(T) \cap N(S)) \mapsto D(S)/(R(T) \cap D(S))$$

is induced by the inclusions, the sequence

$$0 \to N(S)/(R(T) \cap N(S)) \xrightarrow{J} D(S)/(R(T) \cap D(S)) \xrightarrow{\hat{S}} R(S)/R(S \circ T) \to 0 \tag{10}$$

is exact, via the 3×3 lemma (see [8], Lemma II.5.1), because of the exactness of the sequences (9) and (8) and that of the corresponding columns made of (9), (8) and (10). The space $D(S)/(R(T) \cap D(S))$ is isomorphic to the space $(D(S) + R(T))/R(T)$, which in turn is a subspace of the space $Y/R(T)$. When T is Fredholm, and so the space $Y/R(T)$ is finite dimensional, then the space $D(S)/(R(T) \cap D(S))$ is also finite dimensional, inferring by (10) that $R(S)/R(S \circ T)$ is finite dimensional. Therefore, when both T and S are Fredholm, then, by (7), $\dim Z/R(S \circ T))$ is finite and, as we already saw, $N(S \circ T)$ is finite dimensional. Hence, we deduce that if T and S are Fredholm, the map $S \circ T$ is also Fredholm.

Now, let us prove equality (2), with no restriction on T and S. Note that

$$\dim R(T) \cap D(S) = \dim R(T) \cap N(S) + \dim R(S \circ T) \tag{11}$$

by (9), and

$$\dim D(S)/(R(T) \cap D(S)) = \dim N(S)/(R(T) \cap D(S)) + \dim R(S)/R(S \circ T), \tag{12}$$

via (10). Clearly,

$$\dim N(S) = \dim R(T) \cap N(S) + \dim N(S)/(R(T) \cap N(S)) \tag{13}$$

and

$$\dim Y/R(T) = \dim(D(S)+R(T))/R(T) + \dim Y/(D(S)+R(T)). \qquad (14)$$

Therefore, by (5), (14), (12), (13) and (7) we have:

$$\dim N(S\circ T) + \dim Y/R(T) + \dim Z/R(S) =$$

$$\dim N(T) + \dim R(T)\cap N(S) + \dim(D(S)+R(T))/R(T) +$$

$$\dim Y/(D(S)+R(T)) + \dim Z/R(S) =$$

$$\dim N(T) + \dim Y/(D(S)+R(T)) + \dim N(S)/(R(T)\cap N(S)) +$$

$$\dim R(S)/R(S\circ T) + \dim R(T)\cap N(S) + \dim Z/R(S) =$$

$$\dim N(T) + \dim N(S) + \dim Y/(D(S)+R(T)) + \dim Z/R(S\circ T),$$

which is precisely equality (2), using again the isomorphism between the spaces $D(S)/(R(T)\cap D(S))$ and $(D(S)+R(T))/R(T)$.

Equation (3) is a direct consequence of (2), when T and S are Fredholm.

Remark 2.2 If $T : X \mapsto Y$ and $S : Y \mapsto Z$ are Fredholm maps, the previous theorem gives, in particular, the well known classical formula $\mathrm{ind}(ST) = \mathrm{ind}(S) + \mathrm{ind}(T)$, which is the multiplication of the Fredholm index.

Remark 2.3 Let us verify formula (3) when X, Y, Z are finite dimensional, using the previous remark. Let $T : D(T) \subset X \mapsto Y$, $S : D(S) \subset Y \mapsto Z$ be linear maps. We may assume, with no loss of generality, that $D(T) = X$. We first verify (3) when $Y = R(T) + D(S)$. Let Y_1 be a complement of $R(T)\cap D(S)$ in $R(T)$. Then Y_1 is also a complement of $D(S)$ in Y. Let $X_0 = T^{-1}(D(S))$, and let X_1 be a complement of $N(T)$ in $T^{-1}(Y_1)$, which is also a complement of X_0 in X. This implies that the map $T : X_1 \mapsto Y_1$ is bijective. Therefore $\mathrm{ind}(T) = \mathrm{ind}(T_0)$, where $T_0 = T|X_0$. But $S\circ T = ST_0$. Thus $\mathrm{ind}(S\circ T) = \mathrm{ind}(ST_0) = \mathrm{ind}(S) + \mathrm{ind}(T_0) = \mathrm{ind}(S) + \mathrm{ind}(T)$, via the previous remark, which shows that (3) is true in this case.

When Y is not necessarily equal to $R(T) + D(S)$, we replace the map T by $T' : X \mapsto R(T) + D(S)$, $T' = T$. As we have, again by the previous remark, that $\mathrm{ind}(T) = \mathrm{ind}(T') + \mathrm{ind}(I')$, where I' is the inclusion $R(T) + D(S) \subset Y$ with $\mathrm{ind}(I') = -\dim Y/(R(T) + D(S))$, and that $S\circ T = S\circ T'$, we infer easily, using the first part of the proof, that (3) holds in this case too.

If X, Y are Banach spaces and $T : D(T) \subset X \mapsto Y$ is linear, one usually sais that T is *Fredholm* if T is closed and Fredholm as above (see Definition II.1.2 from [2]). As the composition of two closed operators is not necessarily a closed operator, a consequence of Theorem 2.1 is the following.

Corollary 2.4 *Assume that X, Y, Z are Banach spaces and let $T : D(T) \subset X \mapsto Y$, $S : D(S) \subset Y \mapsto Z$ be Fredholm closed operators such that $S\circ T$ is closed. Then $S\circ T$ is Fredholm and*

$$\mathrm{ind}(S\circ T) = \mathrm{ind}(S) + \mathrm{ind}(T) + \dim Y/(R(T) + D(S)). \qquad (15)$$

Corollary 2.4 seems not to be known in the framework of unbounded operators. Note also that if X, Y, Z are Banach spaces and $T : D(T) \subset X \mapsto Y$, $S : D(S) \subset Y \mapsto Z$ are Fredholm paraclosed operators, then $S \circ T$ is paraclosed, Fredholm, and $\mathrm{ind}(S \circ T)$ satisfies the equation from the statement of Corollary 2.4. In other words, $S \circ T$ is automatically paraclosed (see [2], Section I.3 for some details).

3 Quotient morphisms

Let $\mathcal{X}$ be a vector space over the field $\mathbb{K}$ and let $\mathrm{Lat}(\mathcal{X})$ denote the lattice (with respect to the inclusion) consisting of all vector subspaces of $\mathcal{X}$. Let also $\mathcal{Q}(\mathcal{X})$ be the family of all (quotient) vector spaces of the form X/X_0, with $X_0, X \in \mathrm{Lat}(\mathcal{X})$, $X_0 \subset X$. We note that in $\mathcal{Q}(\mathcal{X})$, the equality $X_1/Y_1 = X_2/Y_2$ holds if and only if $X_1 = X_2$ and $Y_1 = Y_2$, for all $X_1, X_2, Y_1, Y_2 \in \mathrm{Lat}(\mathcal{X})$ with $Y_1 \subset X_1$ and $Y_2 \subset X_2$.

Remark 3.1 (1) There is a natural partial order in $\mathcal{Q}(\mathcal{X})$, defined in the following way. We write $X/X_0 \prec Y/Y_0$ if $X \subset Y$ and $X_0 \subset Y_0$. In this case, there exists a natural map $X/X_0 \ni x + X_0 \mapsto x + Y_0 \in Y/Y_0$ called the *q-inclusion* of X/X_0 into Y/Y_0. This map is injective iff $X \cap Y_0 = X_0$, surjective iff $X + Y_0 = Y$, and therefore bijective iff $X \cap Y_0 = X_0$ and $Y = X + Y_0$. In fact, we have the following exact sequence:

$$0 \to (X \cap Y_0)/X_0 \to X/X_0 \to Y/Y_0 \to Y/(X + Y_0) \to 0,$$

where the arrows are the natural q-inclusions.

Note also that we have $X/(X \cap Y) \prec (X + Y)/Y$ but the q-inclusion is, in this case, a classical isomorphism. Nevertheless, if $X/X_0 \prec Y/Y_0$ and $Y/Y_0 \prec X/X_0$, then $X/X_0 = Y/Y_0$.

(2) In the set $\mathcal{Q}(X)$ we may define the *q-intersection* and the *q-sum* of two (or several) spaces, denoted by $\Cap$ and $\uplus$ respectively, via the formulas

$$X/X_0 \Cap Y/Y_0 = (X \cap Y)/(X_0 \cap Y_0), \quad X/X_0 \uplus Y/Y_0 = (X + Y)/(X_0 + Y_0),$$

for any pair of spaces $X/X_0, Y/Y_0 \in \mathcal{Q}(X)$. When $X_0 = Y_0$, the q-intersection is actually intersection and the q-sum is actually the sum of the corresponding vector spaces.

Let $\mathcal{Y}$ be another vector space over $\mathbb{K}$.

Definition 3.2 A *quotient morphism* (or, simply, a *q-morphism*) from $\mathcal{X}$ into $\mathcal{Y}$ is any linear map $T : X/X_0 \mapsto Y/Y_0$, where $X/X_0 \in \mathcal{Q}(\mathcal{X})$ and $Y/Y_0 \in \mathcal{Q}(\mathcal{Y})$.

When there exists a linear map $T_0 : X \mapsto Y$ with $T_0(X_0) \subset Y_0$ such that $T(x + X_0) = T_0 x + Y_0$, $x \in X$, the q-morphism $T : X/X_0 \mapsto Y/Y_0$ is said to be *induced* (by T_0).

This concept is similar to that of *morphism* defined by Waelbroeck (see [14]; see also [1, 11, 12], etc.). As noticed in [14], there exist q-morphisms which are not induced by any linear map.

The family of all quotient morphisms from $\mathcal{X}$ into $\mathcal{Y}$ will be denoted by $\mathcal{QM}(\mathcal{X},\mathcal{Y})$. When $\mathcal{X}=\mathcal{Y}$, the family $\mathcal{QM}(\mathcal{X},\mathcal{Y})$ will be denoted by $\mathcal{QM}(\mathcal{X})$.

For the study of q-morphisms, we adapt the same notation and terminology from [1].

Let $T : X/X_0 \mapsto Y/Y_0$ be a given q-morphism in $\mathcal{QM}(\mathcal{X},\mathcal{Y})$. The space X/X_0, also denoted by $D(T)$, is the *domain* (*of definition*) of T. It can be written as $D_0(T)/X_0$, where $D_0(T) = X$ is called the *lifted domain* of T.

The *range* $T(X/X_0)$ of T is also denoted by $R(T)$ and can be represented as $R_0(T)/Y_0$, where $R_0(T) \in \mathrm{Lat}(\mathcal{Y})$ is called the *lifted range* of T.

The *graph* $G(T)$ of T in $X/X_0 \times Y/Y_0$ is isomorphic to the space $G_0(T)/(X_0 \times Y_0)$, where

$$G_0(T) = \{(x, y) \in X \times Y; T(x + X_0) = y + Y_0\} \in \mathrm{Lat}(\mathcal{X} \times \mathcal{Y})$$

is called the *lifted graph* of T.

Definition 3.3 Let $T_j : X_j/X_{0j} \mapsto Y_j/Y_{0j}$, $j = 1, 2, \ldots, n$, be quotient morphisms from $\mathcal{QM}(\mathcal{X},\mathcal{Y})$. We define the *q-sum* of these morphisms, and denote it by $T_1 \uplus T_2 \uplus \cdots \uplus T_n$ or by $\uplus_{j=1}^n T_j$, as the q-morphism

$$\uplus_{j=1}^n T_j : \Cap_{j=1}^n X_j/X_{0j} \mapsto \uplus_{j=1}^n Y_j/Y_{0j}$$

given by the formula

$$\uplus_{j=1}^n T_j = \sum_{j=1}^n B_j T_j A_j,$$

where

$$A_j : \Cap_{k=1}^n X_k/X_{0k} \mapsto X_j/X_{0j}$$

and

$$B_j : Y_j/Y_{0j} \mapsto \uplus_{k=1}^n Y_k/Y_{0k}$$

are the q-inclusions ($j = 1, 2, \ldots, n$).

Remark 3.4 Let $X/X_0 \in \mathcal{Q}(\mathcal{X})$, $Y/Y_0, Z/Z_0 \in \mathcal{Q}(\mathcal{Y})$ and $W/W_0 \in \mathcal{Q}(\mathcal{W})$. Let also $T : X/X_0 \mapsto Y/Y_0$ and $S : Z/Z_0 \mapsto W/W_0$ be q-morphisms. Motivated by further applications, we shall define a "composition" of the maps S and T in the following way.

We first consider the subspace

$$(Y \cap Z + Y_0)/Y_0 = Y/Y_0 \Cap (Z + Y_0)/Y_0,$$

and the map

$$T_0 : D(T_0) \mapsto (Y \cap Z + Y_0)/Y_0,$$

where $D(T_0) = T^{-1}((Y \cap Z + Y_0)/Y_0)$ and $T_0 = T|D(T_0)$. Clearly, $N(T_0) = N(T)$, $R(T_0) = (R_0(T) \cap Z + Y_0)/Y_0$, and so $R_0(T_0) = R_0(T) \cap Z + Y_0$.

Secondly, we note that there exists a natural map

$$U : (Y \cap Z + Y_0)/Y_0 \mapsto Z/(Y_0 \cap Z + Z_0).$$

The map U is the composition of the isomorphism of $(Y \cap Z + Y_0)/Y_0$ onto the space $(Y \cap Z)/(Y_0 \cap Z)$ and the q-inclusion of $(Y \cap Z)/(Y_0 \cap Z)$ into $Z/(Y_0 \cap Z + Z_0)$.

Thirdly, let S° be the restriction $S|(Y_0 \cap Z + Z_0)/Z_0$ and $\widehat{S}_0 : (Z/Z_0)/((Y_0 \cap Z + Z_0)/Z_0) \mapsto W/R_0(S^\circ)$ be the linear map defined by $\widehat{S}_0(\xi + (Y_0 \cap Z + Z_0)/Z_0) = w + R_0(S^\circ)$, where $w \in S(\xi)$. Then, we define the linear map

$$S_0 : Z/(Y_0 \cap Z + Z_0) \mapsto W/R_0(S^\circ),$$

via the composition of $\widehat{S}_0$ and the natural isomorphism between $(Z/Z_0)/((Y_0 \cap Z + Z_0)/Z_0)$ and $Z/(Y_0 \cap Z + Z_0)$. Moreover, the space $(W/R_0(S^\circ))/R(S_0)$ is isomorphic to $W/R_0(S)$.

Clearly, the composition S_0UT_0 is well defined. The map S_0UT_0 will be designated by $S \circ_q T$. We therefore have

$$S \circ_q T : D(T_0) \mapsto W/R_0(S^\circ),$$

with $D(T_0) \subset X/X_0$ and $R(S \circ_q T) \subset R_0(S_0)/R_0(S^\circ)$.

Note also that

$$D_0(S \circ_q T) = \{x \in X; \exists z \in Z, (x, z) \in G_0(T)\}$$

and, if $x \in X, z \in Z, w \in W$ are such that $(x, z) \in G_0(T), (z, w) \in G_0(S)$, then $(S \circ_q T)(x + X_0) = w + R_0(S^\circ)$.

The map $S \circ_q T$ will be called the *q-composition* of the maps S and T.

An important particular case of the construction from above is obtained when $Z/Z_0 \prec Y/Y_0$. In this case, $T_0 : D(T_0) \mapsto (Y_0 + Z)/Y_0$, the map U is the natural isomorphism $U : (Y_0 + Z)/Y_0 \mapsto Z/(Y_0 \cap Z)$ and $S_0 : Z/(Y_0 \cap Z) \mapsto W/R_0(S^\circ)$.

Note that for two linear maps $T : D(T) \subset X \mapsto Y$ and $S : D(S) \subset Y \mapsto Z$, we have the equality $S \circ_q T = S \circ T$.

Remark 3.5 The q-composition defined in Remark 3.4 occurs in various situations. For instance, it occurs in relation with a concept of generalized inverse, defined for every q-morphism.

Let $T : X/X_0 \mapsto Y/Y_0$ be a q-morphism with $R(T) = Y/Y_0$. We have a natural isomorphism from $R(T)$ into $(X/X_0)/N(T)$, associating to each $\eta \in R(T)$ the class $\xi + N(T) \in (X/X_0)/N(T)$, whenever $T\xi = \eta$. Identifying the space $(X/X_0)/N(T)$ with $X/N_0(T)$, we therefore have an isomorphism from $R_0(T)/Y_0$ into $X/N_0(T)$, given by the assignment $y + Y_0 \mapsto x + N_0(T)$, whenever $(x, y) \in G_0(T)$. This q-morphism will be denoted by T^{-1} and called the *q-inverse* of T. It coincides with the usual inverse when T is bijective. Moreover,

$$T^{-1} \circ T = J_{X/X_0}^{X/N_0(T)}, \tag{16}$$

$$T \circ_q T^{-1} = I_{R(T)}, \tag{17}$$

where $J_{X/X_0}^{X/N_0(T)}$ is the q-inclusion $X/X_0 \prec X/N_0(T)$ and $I_{R(T)}$ is the identity on $R(T)$.

The usefulness of the q-compositions of quotient morphisms also follows from the fact that it is an associative operation.

Theorem 3.6 *Let $X/X_0 \in \mathcal{Q}(\mathcal{X})$, $Y/Y_0, Z/Z_0 \in \mathcal{Q}(\mathcal{Y})$, $W/W_0, U/U_0 \in \mathcal{Q}(\mathcal{W})$ and $V/V_0 \in \mathcal{Q}(\mathcal{V})$. Let also $T : X/X_0 \mapsto Y/Y_0$, $S : Z/Z_0 \mapsto W/W_0$ and $P : U/U_0 \mapsto V/V_0$ be q-morphisms. Then*

$$P \circ_q (S \circ_q T) = (P \circ_q S) \circ_q T.$$

Proof. Let us make the notation

$$S \circ_q T = L \quad \text{and} \quad P \circ_q S = K.$$

Hence,

$$L : D_0(L)/X_0 \mapsto W/R_0(S^\circ), \quad K : D_0(K)/Z_0 \mapsto V/R_0(\widetilde{P}^\circ),$$

where

$$D_0(L) = \{x \in X; \exists z \in Z, (x,z) \in G_0(T)\}, \quad S^\circ = S|(Y_0 \cap Z + Z_0)/Z_0,$$

$$D_0(K) = \{z \in Z; \exists u \in U, (z,u) \in G_0(S)\}, \quad \widetilde{P}^\circ = P|(W_0 \cap U + U_0)/U_0.$$

It follows that

$$P \circ_q L : D_0(P \circ_q L)/X_0 \mapsto V/R_0(P^\circ),$$

and $D_0(P \circ_q L)/X_0 \subset D_0(L)/X_0$. Moreover, it is easy to see that

$$\begin{aligned} D_0(P \circ_q L) &= \{x \in D_0(L); \exists u \in U, (x,u) \in G_0(L)\} \\ &= \{x \in X; \exists z \in Z, \exists u \in U, (x,z) \in G_0(T), (x,u) \in G_0(L)\}. \end{aligned}$$

Let us prove that actually

$$D_0(P \circ_q L) = \{x \in X; \exists \widetilde{z} \in Z, \exists u \in U, (x, \widetilde{z}) \in G_0(T), (\widetilde{z}, u) \in G_0(S)\}. \tag{18}$$

Consider $x \in X, z \in Z, u \in U$ such that $(x,z) \in G_0(T), (x,u) \in G_0(L)$. Let also $w \in W$ be fixed such that $(z,w) \in G_0(S)$. Then $L(x + X_0) = w + R_0(S^\circ)$, and because $L(x + X_0) = u + R_0(S^\circ)$, we infer that $w - u \in R_0(S^\circ)$, so we can find a $\overline{z} \in Y_0 \cap Z$ such that $(\overline{z}, w - u) \in G_0(S)$. It follows that $(z - \overline{z}, u) \in G_0(S)$. Consider $\tilde{z} = z - \overline{z} \in Z$ and note that $(\tilde{z}, u) \in G_0(S)$. On the other hand, because $\bar{z} \in Y_0$ and $(x,z) \in G_0(T)$, it follows that $(x, \tilde{z}) \in G_0(T)$.

Conversely, if $x \in X, \widetilde{z} \in Z, u \in U$ are such that $(x, \tilde{z}) \in G_0(T), (\tilde{z}, u) \in G_0(S)$ then $(x,u) \in G_0(L)$.

Note that, in the definition of $P \circ_q L$, we have that

$$P^\circ = P|(R_0(S^\circ) \cap U + U_0)/U_0.$$

On the other hand,

$$K \circ_q T : D_0(K \circ_q T)/X_0 \to V/R_0(K^\circ),$$

where

$$\begin{aligned} D_0(K \circ_q T) &= \{x \in X; \exists z \in D_0(K), (x,z) \in G_0(T))\} \\ &= \{x \in X; \exists z \in Z, \exists u \in U, (x,z) \in G_0(T), (z,u) \in G_0(S)\}, \end{aligned} \tag{19}$$

and

$$K^\circ = K|(Y_0 \cap D_0(K) + Z_0)/Z_0.$$

From (18) and (19) we deduce that

$$D_0(P \circ_q L) = D_0(K \circ_q T).$$

We show now that

$$R_0(P^\circ) = R_0(K^\circ).$$

It is easy to see the inclusion $R_0(P^\circ) \subset R_0(K^\circ)$.

Reciprocally, let $v \in R_0(K^\circ)$. It follows that there exists a $z \in Y_0 \cap D_0(K)$ such that $(z,v) \in G_0(K)$. Because $z \in D_0(K)$, we infer that there exists a $u \in U$ such that $(z,u) \in G_0(S)$. In particular, $u \in R_0(S^\circ) \cap U$. Let us fix $\bar{v} \in V$ satisfying $(u,\bar{v}) \in G_0(P)$. Hence $(z,\bar{v}) \in G_0(K)$ and $v - \bar{v} \in R_0(\widetilde{P}^\circ)$, implying that there exists $w \in W_0 \cap U$ such that $(w, v - \bar{v}) \in G_0(P)$. This together with $(u,\bar{v}) \in G_0(P)$ imply that $(w+u, v) \in G_0(P)$. Taking $\widetilde{u} = w + u$ and noticing that $\widetilde{u} \in R_0(S^\circ) \cap U$ we infer that $v \in R_0(P^\circ)$.

Finally, for $x \in X, z \in Z, u \in U$ such that $(x,z) \in G_0(T), (z,u) \in G_0(S)$, and $v \in V$ such that $(u,v) \in G_0(P)$, one obtains $(x,u) \in G_0(L)$ and $(z,v) \in G_0(K)$. Hence

$$P \circ_q L(x + X_0) = v + R_0(P^\circ) = v + R_0(K^\circ) = K \circ_q T(x + X_0),$$

which completes the proof of the theorem.

Remark 3.7 (1) Let $X/X_0 \in \mathcal{Q}(\mathcal{X})$, $Y/Y_0, Z/Z_0 \in \mathcal{Q}(\mathcal{Y})$ and $W/W_0 \in \mathcal{Q}(\mathcal{W})$. Let also $T : X/X_0 \mapsto Y/Y_0$ and $S : Z/Z_0 \mapsto W/W_0$ be q-morphisms. The previous result show that the q-composition have a behavior similar to that of the usual composition. For this reason, from now on we write the q-composition $S \circ_q T$ simply as $S \circ T$.

(2) Let $T : X/X_0 \mapsto Y/Y_0$ be a q-morphism with $R(T) = Y/Y_0$ and let $T^{-1} : R_0(T)/Y_0 \mapsto X/N_0(T)$ be defined as in Remark 3.5. Using Proposition 3.6, we can now show that T^{-1} is the only surjective q-morphism $S : R_0(T)/Y_0 \mapsto X/N_0(T)$ satisfying (17), that is, $T \circ S = I_{R(T)}$. Indeed, choosing an S with this property, we infer from (16) that

$$T^{-1} \circ (T \circ S) = J_{X/X_0}^{X/N_0(T)} \circ S = T^{-1} \circ I_{R(T)}.$$

According to Remark 3.4, the composition $J_{X/X_0}^{X/N_0(T)} \circ S$ is the composition of three maps. The map T_0 from Remark 3.4 is precisely S, because S is surjective.The map U from Remark 3.4 is the identity on $X/N_0(T)$. The map S° from Remark 3.4 is actually the map $0 : N_0(T)/X_0 \mapsto X/N_0(T)$, and so $R_0(0) = N_0(T)$, implying that the map S_0 from Remark 3.4 is just the identity on $X/N_0(T)$. Therefore, $J_{X/X_0}^{X/N_0(T)} \circ S = S$. Clearly, $T^{-1} \circ I_{R(T)} = T^{-1}$, and thus $S = T^{-1}$.

(3) As in the previous section, we say that a q-morphism $T : X/X_0 \mapsto Y/Y_0$ is *Fredholm* if both $\dim N(T)$ and $\dim(Y/Y_0)/R(T) = \dim Y/R_0(T)$ are finite. In that case, the *index* of T, denoted by $\text{ind}(T)$, is given by

$$\text{ind}(T) = \dim N(T) - \dim Y/R_0(T).$$

In particular, the *q-inclusion* of X/X_0 into Y/Y_0, say Q, is Fredholm if and only if $\dim N(Q) = \dim(X \cap Y_0)/X_0$ and $\text{codim} R(Q) = \dim Y/(X + Y_0)$ are both finite, where "codim" stands for *codimension*.

It follows directly from Theorem 2.1 that if $T : X/X_0 \mapsto Y/Y_0$ and $S : D(S) \subset Y/Y_0 \mapsto Z/Z_0$ are Fredholm quotient morphisms, then $S \circ T$ is Fredholm and

$$\text{ind}(S \circ T) = \text{ind}(T) + \text{ind}(S) + \dim Y/(R_0(T) + D_0(S)).$$

The next result is a more general assertion.

Theorem 3.8 *Let $X/X_0 \in \mathcal{Q}(\mathcal{X})$, $Y/Y_0, Z/Z_0 \in \mathcal{Q}(\mathcal{Y})$ and $W/W_0 \in \mathcal{Q}(\mathcal{W})$. Let also $T : X/X_0 \mapsto Y/Y_0$ and $S : Z/Z_0 \mapsto W/W_0$ be q-morphisms.*

Assume that the maps S, T are Fredholm and that the dimensions

$$\dim(Y \cap Z_0)/(Y_0 \cap Z_0) \quad \text{and} \quad \dim Z/(Y \cap Z + Z_0)$$

are finite. Then the dimensions

$$\dim Y/(R_0(T) + Y \cap Z) \quad \text{and} \quad \dim(Y_0 \cap N_0(S) + Z_0)/Z_0$$

are finite, the map $S \circ T$ is Fredholm and we have

$$\text{ind}(S \circ T) = \text{ind}(T) + \text{ind}(S) + \dim(Y \cap Z_0)/(Y_0 \cap Z_0) +$$

$$\dim Y/(R_0(T) + Y \cap Z) - \dim(N_0(S) \cap Y_0 + Z_0)/Z_0 - \dim Z/(Y \cap Z + Z_0).$$

Proof. We consider the maps T_0, U, S_0, as defined in Remark 3.4. The map T being Fredholm, the subspaces $R_0(T)$ and $R_0(T) + Y \cap Z$ are of finite codimension in Y. Therefore

$$\dim(R_0(T) + Y \cap Z)/R_0(T) = \dim(Y \cap Z)/(R_0(T) \cap Z) < +\infty. \tag{20}$$

The following q-inclusion

$$(Y \cap Z)/(R_0(T) \cap Z) \prec (Y \cap Z + Y_0)/(R_0(T) \cap Z + Y_0) \tag{21}$$

is an isomorphism, because

$$Y \cap Z + R_0(T) \cap Z + Y_0 = Y \cap Z + Y_0$$

and

$$Y \cap Z \cap (R_0(T) \cap Z + Y_0) = R_0(T) \cap Z,$$

as one can easily see.

We prove first that the map T_0 is Fredholm. Note that $N(T_0) = N(T)$, as observed in Remark 3.4, and the latter is finite dimensional. Note also that the codimension of the range of T_0 is finite, and we have

$$\mathrm{codim} R(T_0) = \dim(Y \cap Z + Y_0)/(R_0(T) \cap Z + Y_0) < +\infty,$$

via (20) and (21). Therefore,

$$\mathrm{ind}(T_0) = \dim N(T) - \dim(Y \cap Z)/(R_0(T) \cap Z),$$

using again the isomorphism (21). The exactness of the sequence

$$0 \to (Y \cap Z)/(R_0(T) \cap Z) \to Y/R_0(T) \to Y/(R_0(T) + Y \cap Z) \to 0,$$

where the arrows are the natural q-morphisms, shows that

$$\dim(Y \cap Z)/(R_0(T) \cap Z) = \mathrm{codim} R(T) - \dim Y/(R_0(T) + Y \cap Z).$$

Hence

$$\mathrm{ind}(T_0) = \mathrm{ind}(T) + \dim Y/(R_0(T) + Y \cap Z). \tag{22}$$

The map

$$U : (Y \cap Z + Y_0)/Y_0 \mapsto Z/(Y_0 \cap Z + Z_0)$$

is the composition of the isomorphism between the spaces $(Y \cap Z + Y_0)/Y_0$ and $(Y \cap Z)/(Y_0 \cap Z)$, and the q-inclusion

$$Q : (Y \cap Z)/(Y_0 \cap Z) \mapsto Z/(Y_0 \cap Z + Z_0).$$

Thus $N(U)$ is isomorphic to

$$N(Q) = (Y \cap (Y_0 \cap Z + Z_0))/(Y_0 \cap Z) = (Y_0 \cap Z + Y \cap Z_0)/(Y_0 \cap Z)$$

and so $N(U)$ is isomorphic to $(Y \cap Z_0)/(Y_0 \cap Z_0)$.

Note also that $R(Q) = (Y \cap Z + Z_0)/(Y_0 \cap Z + Z_0)$. Consequently, it follows from the hypothesis that U is Fredholm and

$$\mathrm{ind}(U) = \dim(Y \cap Z_0)/(Y_0 \cap Z_0) - \dim Z/(Y \cap Z + Z_0). \tag{23}$$

Finally, let us prove that the map S_0 is also Fredholm. Indeed, $N(S)$ is finite dimensional by hypothesis, $N(S_0)$ is isomorphic to $N(S)/N(S^o)$, and so $N(S_0)$ must be finite dimensional too. Since

$$N(S^o) = N(S) \cap ((Y_0 \cap Z + Z_0)/Z_0) = (Y_0 \cap N_0(S) + Z_0)/Z_0,$$

we have

$$\dim N(S_0) = \dim N(S) - \dim(Y_0 \cap N_0(S) + Z_0)/Z_0.$$

Note also that $\mathrm{codim} R(S_0) = \dim W/R_0(S) < \infty$, showing that S_0 is also Fredholm. Moreover,

$$\mathrm{ind}(S_0) = \mathrm{ind}(S) - \dim(Y_0 \cap N_0(S) + Z_0)/Z_0. \tag{24}$$

Consequently, $S \circ T = S_0 U T_0$ is a Fredholm map. In addition,

$$\mathrm{ind}(S \circ T) = \mathrm{ind}(T_0) + \mathrm{ind}(U) + \mathrm{ind}(S_0). \tag{25}$$

Formula from the statement is a direct consequence of (22), (23), (24) and (25).

Theorem 3.8 has the following important consequence.

Corollary 3.9 *Let $X/X_0 \in \mathcal{Q}(\mathcal{X})$, $Y/Y_0, Z/Z_0 \in \mathcal{Q}(\mathcal{Y})$ and $W/W_0 \in \mathcal{Q}(\mathcal{W})$ be such that $Z/Z_0 \prec Y/Y_0$. Let also $T : X/X_0 \mapsto Y/Y_0$ and $S : Z/Z_0 \mapsto W/W_0$ be q-morphisms.*

If the maps S, T are Fredholm, then the dimensions

$$\dim Y/(R_0(T) + Z) \quad \text{and} \quad \dim(Y_0 \cap N_0(S))/Z_0$$

are finite, the map $S \circ T$ is Fredholm and we have

$$\mathrm{ind}(S \circ T) = \mathrm{ind}(T) + \mathrm{ind}(S) +$$

$$\dim Y/(R_0(T) + Z) - \dim(N_0(S) \cap Y_0)/Z_0.$$

Proof. As noticed in Remark 3.4, if $Z/Z_0 \prec Y/Y_0$, the map U is an isomorphism, and so the index of U is null.

Remark 3.10 The map $U : (Y \cap Z + Y_0)/Y_0 \mapsto Z/(Y_0 \cap Z + Z_0)$ from Remark 3.4 is an isomorphism if and only if $Y \cap Z_0 = Y_0 \cap Z_0$ and $Y \cap Z + Z_0 = Z$ (see also the map Q from the proof of Theorem 3.8). For this reason, assuming T, S Fredholm and replacing the condition $Z/Z_0 \prec Y/Y_0$ by the more general conditions $Y \cap Z_0 = Y_0 \cap Z_0$ and $Y \cap Z + Z_0 = Z$ in the statement of Corollary 3.9, we get the formula

$$\mathrm{ind}(S \circ T) = \mathrm{ind}(T) + \mathrm{ind}(S) +$$

$$\dim Y/(R_0(T) + Y \cap Z) - \dim(N_0(S) \cap Y_0 + Z_0)/Z_0,$$

via a similar argument.

Let $T : X/X_0 \mapsto Y/Y_0$ be a q-morphism with $X/X_0 \prec Y/Y_0$. We may consider the iterates $T \circ T$, $T \circ T \circ T$ etc., which are unambiguously defined. In fact, defining T^0 as the q-inclusion $X/X_0 \prec Y/Y_0$, for every integer $n \geq 1$ we may define by induction $T^n = T \circ T^{n-1}$. Note that we may actually consider polynomials of T, as in [3]. We think that a study of such polynomials would be of some interest.

4 Linear relations as quotient morphisms

In this section we discuss linear relations (see [3, 5, 6, 9]), which will be regarded as particular cases of quotient morphisms.

Giving two linear spaces $\mathcal{X}$ and $\mathcal{Y}$, following Arens [3], a *linear relation* is any subspace $Z \in \mathrm{Lat}(\mathcal{X} \times \mathcal{Y})$. As we work only with linear relations, we shall often call them simply *relations*.

In this text, for a given relation $Z \in \mathrm{Lat}(\mathcal{X} \times \mathcal{Y})$, we use the following notation. Set

$$D(Z) = \{x \in \mathcal{X}; \exists y \in \mathcal{Y} : (x, y) \in Z\},$$

and

$$R(Z) = \{y \in \mathcal{Y}; \exists x \in \mathcal{X} : (x, y) \in Z\},$$

called the *domain*, respectively the *range* of Z. We shall also use the spaces

$$N(Z) = \{x \in D(Z); (x, 0) \in Z\}$$

and

$$M(Z) = \{y \in R(Z); (0, y) \in Z\},$$

called the *kernel*, respectively the *multivalued part* of Z.

The inverse $Z^{-1} \in \mathrm{Lat}(\mathcal{Y}, \mathcal{X})$ of the relation $Z \in \mathrm{Lat}(\mathcal{X}, \mathcal{Y})$ is given by $\{(y, x); (x, y) \in Z\}$. Clearly, $D(Z^{-1}) = R(Z)$, $R(Z^{-1}) = D(Z)$, $N(Z^{-1}) = M(Z)$ and $M(Z^{-1}) = N(Z)$.

If $V \subset \mathcal{X}$, we write $Z(V)$ to designate the set $\{y \in \mathcal{Y}; \exists x \in V : (x, y) \in Z\}$, which is empty if $V \cap D(Z) = \emptyset$. Clearly, $Z(V) = Z(V \cap D(Z))$. If $V = \{x\}$, $x \in \mathcal{X}$, we write $Z(x)$ for $Z(V)$. Note also that $R(Z) = Z(D(Z))$ and $M(Z) = Z(0)$.

The linear relation Z can be associated with the map $Q_Z : D(Z) \mapsto R(Z)/M(Z)$, defined by the formula $Q_Z(x) = y + M(Z)$ whenever $(x, y) \in Z$. The map Q_Z is correctly defined and linear. Moreover, it is surjective and its kernel is precisely $N(Z)$. In fact, Q_Z is a quotient morphism of a particular form.

Proposition 4.1 *The map*

$$\mathrm{Lat}(\mathcal{X} \times \mathcal{Y}) \ni Z \mapsto Q_Z \in \mathcal{QM}(\mathcal{X}, \mathcal{Y})$$

is injective, and its range consists of all quotient morphisms $Q \in \mathcal{QM}(\mathcal{X}, \mathcal{Y})$ of the form $Q : X \mapsto Y/Y_0$, which are surjective.

Proof. Recall that $X_1/Y_1 = X_2/Y_2$ is true if and only if $X_1 = X_2$ and $Y_1 = Y_2$.

Let us show that the map $Z \mapsto Q_Z$ is injective. If $Q_{Z_1} = Q_{Z_2}$, we obtain, via the previous remark, that $D(Z_1) = D(Z_2), R(Z_1) = R(Z_2)$, and $M(Z_1) = M(Z_2)$. Fix a vector $(x, y_1) \in Z_1$. Then $Q_{Z_1}(x) = y_1 + M = Q_{Z_2}(x) = y_2 + M$, for some $y_2 \in R(Z_2)$, where $M = M(Z_1) = M(Z_2)$. Therefore $y_2 - y_1 \in M$, and so $(x, y_1) = (x, y_2) - (0, y_2 - y_1) \in Z_2$. Similarly, $Z_2 \subset Z_1$, showing that $Z_1 = Z_2$.

Now, let $Q : X \mapsto Y/Y_0$ be surjective. Set

$$Z = G_0(Q) = \{(x, y) \in X \times Y; y + Y_0 = Q(x)\}.$$

We clearly have $D(Z) = X$ and $R(Z) \subset Y$. In fact, $R(Z) = Y$ because the map Q is surjective. Moreover, as we have $(0, y) \in G_0(Q)$ if and only if $y \in Y_0$, it follows that $M(Z) = Y_0$. Consequently, $Q_Z(x) = y+M(Z)$ whenever $(x, y) \in Z$, showing that $Q_Z = Q$.

The previous proposition allows us to designate the uniquely determined quotient morphism Q_Z as the *morphism of* Z, for any relation $Z \in \mathrm{Lat}(\mathcal{X} \times \mathcal{Y})$.

If $Z \subset X \times Y$ for some $X \in \mathrm{Lat}(\mathcal{X}), Y \in \mathrm{Lat}(\mathcal{Y})$, the q-morphism from $D(Z)$ into $Y/M(Z)$ induced by Q_Z will be designated by Q_Z^Y.

Remark 4.2 Given two relations $Z'' \in \mathrm{Lat}(\mathcal{X} \times \mathcal{Y})$ and $Z' \in \mathrm{Lat}(\mathcal{Y} \times \mathcal{Z})$, their product (or composition) is the relation $Z' \circ Z'' \in \mathrm{Lat}(\mathcal{X} \times \mathcal{Z})$ defined as

$$Z' \circ Z'' = \{(x, z) \in \mathcal{X} \times \mathcal{Z}; \exists y \in \mathcal{Y} : (x, y) \in Z'', (y, z) \in Z'\}.$$

It is well known, and easily seen, that

$$D(Z' \circ Z'') = Z''^{-1}(D(Z')),$$

$$R(Z' \circ Z'') = Z'(R(Z'')),$$

$$M(Z' \circ Z'') = Z'(M(Z'')).$$

Proposition 4.3 *Given two relations* $Z'' \in \mathrm{Lat}(\mathcal{X} \times \mathcal{Y})$ *and* $Z' \in \mathrm{Lat}(\mathcal{Y} \times \mathcal{Z})$*, we have the equality* $Q_{Z' \circ Z''} = Q_{Z'} \circ Q_{Z''}$*.*

Proof. Let $Q' = Q_{Z'}$ and $Q'' = Q_{Z''}$. Therefore, $D(Q'') = D(Z''), R_0(Q'') = R(Z'')$, and similar relations hold for Q'.

We follow the construction of $Q' \circ Q''$ as in Remark 3.4.

Let

$$x \in D(Z' \circ Z'') = Z''^{-1}(D(Z')).$$

We can find $y \in D(Z')$ and $z \in \mathcal{Z}$ such that that $(x, y) \in Z''$ and $(y, z) \in Z'$. Hence, $y \in D(Z') \cap R(Z'')$ and

$$Q_{Z' \circ Z''}(x) = z + M(Z' \circ Z'') \in R(Z' \circ Z'')/M(Z' \circ Z'').$$

We want to show that $Q' \circ Q''(x) = z + M(Z' \circ Z'')$. Note that

$$y + M(Z'') \in (R_0(Q'') \cap D(Q') + M(Z''))/M(Z'').$$

If

$$U : (D(Q') + M(Z''))/M(Z'') \mapsto D(Q')/(D(Q') \cap M(Z''))$$

is the natural isomorphism, then we have $U(y+M(Z''))=y+D(Q')\cap M(Z'')$. Let

$$Q'_0 : D(Q')/(D(Q')\cap M(Z''))\mapsto R_0(Q')/R_0(Q'^\circ),$$

which corresponds to the map S_0 in Remark 3.4, where Q'° is the restriction of Q' to $M(Z'')\cap D(Q')$. Then we have

$$Q'_0(y+D(Q')\cap M(Z''))=z+R_0(Q'^\circ).$$

We have only to observe that

$$Q'(M(Z'')\cap D(Q'))=Z'(M(Z''))/M(Z')=M(Z'\circ Z'')/M(Z'),$$

whence we infer that $R_0(Q'^\circ)=M(Z'\circ Z'')$. Consequently, $Q'\circ Q''(x)=Q_{Z'\circ Z''}(x)$, which is the assertion of the lemma.

Remark 4.4 Let $X\in \mathrm{Lat}(\mathcal{X}), Y\in \mathrm{Lat}(\mathcal{Y})$ and let $Z\subset X\times Y$ be a relation. Recall that the relation Z is said to be *Fredholm* [5] if both $\mathrm{dim}N(Z)$ and $\mathrm{dim}Y/R(Z)$ are finite. One sets $\mathrm{ind}(Z)=\mathrm{dim}N(Z)-\mathrm{dim}Y/R(Z)$, which is the *index* of Z. Clearly, the index of Z depends strongly on the space Y.

Note that $Z\subset X\times Y$ is Fredholm if and only if the map $Q_Z^Y : D(Z)\mapsto Y/M(Z)$ induced by the morphism Q_Z of Z is Fredholm, and in this case $\mathrm{ind}(Z)=\mathrm{ind}(Q_Z^Y)$.

The next result is a version of the index theorem for the product of Fredholm relations, extending the corresponding result from [5], Corollary I.6.12 (see also [4]).

Theorem 4.5 *Let X, Y, W be linear spaces and let $Z''\subset X\times Y$, $Z'\subset Y\times W$ be Fredholm relations. Then the dimensions* $\mathrm{dim}(Y/(D(Z')+R(Z''))$ *and* $\mathrm{dim}N(Z')\cap M(Z'')$ *are finite, $Z'\circ Z''$ is Fredholm and*

$$\mathrm{ind}(Z'\circ Z'')=\mathrm{ind}(Z')+\mathrm{ind}(Z'')+$$

$$\mathrm{dim}(Y/(D(Z')+R(Z''))-\mathrm{dim}N(Z')\cap M(Z'').$$

Proof. The proof of Proposition 4.3 shows that, in fact, we have $Q^W_{Z'\circ Z''}=Q^W_{Z'}\circ Q^Y_{Z''}$. Therefore,

$$\mathrm{ind}(Z'\circ Z'')=\mathrm{ind}(Q^W_{Z'\circ Z''})=\mathrm{ind}(Q^W_{Z'}\circ Q^Y_{Z''}).$$

The assertion is obtained directly from Corollary 3.9, applied to $Q^W_{Z'}, Q^Y_{Z''}$.

Corollary 4.6 *Let X, Y, W be linear spaces and let $Z''\subset X\times Y$, $Z'\subset Y\times W$ be Fredholm relations. Assume that $D(Z')=Y$. Then $Z'\circ Z''$ is Fredholm and*

$$\mathrm{ind}(Z'\circ Z'')=\mathrm{ind}(Z')+\mathrm{ind}(Z'')-\mathrm{dim}N(Z')\cap M(Z'').$$

This statement is precisely Corollary I.6.12 from [5].

Proof. Since $D(Z')=Y$, we have $\mathrm{dim}(Y/(D(Z')+M(Z''))=0$. The desired formula is then given by Theorem 4.5.

Let us finally remark that a version of Theorem 4.5 has been independently obtained in [10].

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