Dimensional Stability in Truncated Moment Problems

F.-H. Vasilescu

Abstract

An approach to truncated moment problems is developed, via the Riesz functional and an assumed dimensional stability of its associated Hilbert spaces. Although equivalent to a concept of "flatness" introduced by R. Curto and L. Fialkow, the dimensional stability discussed in this paper has a different geometric aspect and leads to statements parallel to those of Curto and Fialkow, as well as to some newer ones, obtained by simpler arguments. A stability equation, giving a local characterization of the dimensional stability, is also presented.

1 Introduction

The study of truncated moment problems is a subject of predilection in many works by R. E. Curto and L. A. Fialkow (see [4]–[7] and their references). Roughly speaking, this means that giving a finite multi-sequence of real numbers $\gamma = (\gamma_\alpha)_{|\alpha|\le 2m}$ with $\gamma_0 > 0$, where α's are multi-indices of a given length $n \ge 1$ and $m \ge 0$ is an integer, one looks for a positive measure μ on $\mathbb{R}^n$ such that $\gamma_\alpha = \int t^\alpha d\mu$ for all monomials t^α with $|\alpha| \le 2m$. Associating the sequence γ with the Hankel matrix $M_\gamma = (\gamma_{\alpha+\beta})_{|\alpha|,|\beta|\le m}$, which is supposed to be nonnegative when acting on a corresponding Euclidean space, Curto and Fialkow show that the existence of the measure μ, having a number of atoms equal to the rank of M_γ, is characterized by the existence of a rank preserving nonnegative extension $M_{\gamma'}$ of the matrix M_γ, associated to a larger finite multi-sequence $\gamma' = (\gamma_\alpha)_{|\alpha|\le 2m+2}$, which is said to be *a flat extension* of M_γ (see [4], Definition 5.1 and Theorem 7.10; see also our Remark 2.16). The flatness introduced by Curto and Fialkow can be extended and used in a more general framework, as done for instance in [3].

Unlike in the works by Curto and Fialkow quoted above, where the centrel object is the matrix M_γ, in this paper we mainly deal with the linear functional, often called the Riesz functional, induced by the assignment $t^\alpha \mapsto \gamma_\alpha$ on the space of polynomials of total degree less or equal to $2m$, which, in addition, is supposed to be nonnegative on the cone of sums of squares of real-valued polynomials. Our approach, based on different techniques (consisting mainly of the Cauchy-Schwarz inequality and elements of spectral theory for commuting tuples of self-adjoint operators), leads to important shortcuts of the proofs of several basic results in [4], [5] and [8], restated in our terms (see for instance Theorem 2.6), as well as to some more general results (see Theorem 2.10).

The use of the Riesz functional to solve various moment problems and related topics (as for instance the cubature formulas) appears in several works. In this respect, we cite the papers [9], [12], [13], [15]-[17] etc.

As mentioned before, the flatness introduced by Curto and Fialkow can be interpreted as a property of stability of the dimension of some associated Hilbert spaces, and therefore it has another geometric description, which will be exploited in the present work. This stability will be thoroughly analysed in the third section, via a quadratic equation, whose resolution leads to quite explicit solutions to truncated moment problems (see Theorem 3.3). Finally, in the last chapter, we study some connections between the finite moment sequences and their atomic representing measures.

In the rest of this section, we introduce the terminology used in the paper and recall some elementary facts, most of them well known, presented here in a more general context than the usual one (see also [23]).

Throughout this paper, $n \geq 1$ will be a fixed integer. Let $\mathcal{S}$ be a vector space consisting of complex-valued Borel functions, defined on $\mathbb{R}^n$ (we restrict ourselves to $\mathbb{R}^n$ but other joint domains of definitions may be considered). We assume that $1 \in \mathcal{S}$ and if $f \in \mathcal{S}$, then $\bar{f} \in \mathcal{S}$. For convenience, let us say that $\mathcal{S}$, having these properties, is a *function space.*

Let also $\mathcal{S}^{(1)}$ be the vector space spanned by all products of the form fg with $f, g \in \mathcal{S}$, which is itself a function space. We have $\mathcal{S} \subset \mathcal{S}^{(1)}$, and $\mathcal{S} = \mathcal{S}^{(1)}$ when $\mathcal{S}$ is an algebra.

Let $\mathcal{S}$ be a function space and let $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ be a linear map with the following properties:

(1) $\Lambda(\bar{f}) = \overline{\Lambda(f)}$ for all $f \in \mathcal{S}^{(1)}$;

(2) $\Lambda(|f|^2) \geq 0$ for all $f \in \mathcal{S}$.

(3) $\Lambda(1) = 1$.

Adapting some terminology from [13] to our context, a linear map Λ with the properties (1)-(3) is said to be a *unital square positive functional*, briefly a *uspf*.

When $\mathcal{S}$ is an algebra, conditions (2) and (3) imply condition (1). In this case, a map Λ with the property (2) is usually said to be *positive (semi)definite*.

Condition (3) may be replaced by $\Lambda(1) > 1$ but (looking for probability measures representing such a functional) we always assume (3) in the stated form, without loss of generality.

If $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ is a uspf, we have the *Cauchy-Schwarz inequality*

$$|\Lambda(fg)|^2 \le \Lambda(|f|^2)\Lambda(|g|^2),\ p, q \in \mathcal{S}. \tag{1.1}$$

Putting $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; \Lambda(|f|^2) = 0\}$, the Cauchy-Schwarz inequality shows that $\mathcal{I}_\Lambda$ is a vector subspace of $\mathcal{S}$ and that $\mathcal{S} \ni f \mapsto \Lambda(|f|^2)^{1/2} \in \mathbb{R}_+$ is a seminorm. Moreover, the quotient $\mathcal{S}/\mathcal{I}_\Lambda$ is an inner product space, with the inner product given by

$$\langle f + \mathcal{I}_\Lambda, g + \mathcal{I}_\Lambda\rangle = \Lambda(f\bar{g}). \tag{1.2}$$

Note that, in fact, $\mathcal{I}_\Lambda = \{f \in \mathcal{S}; \Lambda(fg) = 0\, \forall g \in \mathcal{S}\}$ and $\mathcal{I}_\Lambda \cdot \mathcal{S} \subset \ker(\Lambda)$.

If $\mathcal{S}$ is finite dimensional, then $\mathcal{S}/\mathcal{I}_\Lambda$ is actually a Hilbert space.

Now, let $\mathcal{T} \subset \mathcal{S}$ be a function subspace. If $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ is a uspf, then $\Lambda|\mathcal{T}^{(1)}$ is also a uspf, and setting $\mathcal{I}_{\Lambda,\mathcal{T}} = \{f \in \mathcal{T}; \Lambda(|f|^2) = 0\} = \mathcal{I}_\Lambda \cap \mathcal{T}$, there is a natural map

$$J_{\mathcal{T},\mathcal{S}} : \mathcal{T}/\mathcal{I}_{\Lambda,\mathcal{T}} \mapsto \mathcal{S}/\mathcal{I}_\Lambda,\ \ J_{\mathcal{T},\mathcal{S}}(f + \mathcal{I}_{\Lambda,\mathcal{T}}) = f + \mathcal{I}_\Lambda,\ f \in \mathcal{T}. \tag{1.3}$$

The equality

$$\langle f + \mathcal{I}_{\Lambda,\mathcal{T}}, f + \mathcal{I}_{\Lambda,\mathcal{T}}\rangle = \Lambda(|f|^2) = \langle f + \mathcal{I}_\Lambda, f + \mathcal{I}_\Lambda\rangle$$

shows that the map $J_{\mathcal{T},\mathcal{S}}$ is an isometry, in particular it is injective.

The dimensional stability suggested by the title of this work is related to the equality $J_{\mathcal{T},\mathcal{S}}(\mathcal{Q}/\mathcal{I}_{\Lambda,\mathcal{T}}) = \mathcal{S}/\mathcal{I}_\Lambda$. Formally, we have the following:

Definition We say that the uspf $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ it *stable* at $\mathcal{T}$, where $\mathcal{T} \subset \mathcal{S}$ is a function subspace, if we have the equality $J_{\mathcal{T},\mathcal{S}}(\mathcal{T}/\mathcal{I}_{\Lambda,\mathcal{T}}) = \mathcal{S}/\mathcal{I}_\Lambda$.

The equality $J_{\mathcal{T},\mathcal{S}}(\mathcal{T}/\mathcal{I}_{\Lambda,\mathcal{S}}) = \mathcal{S}/\mathcal{I}_\Lambda$ is equivalent to the property $\mathcal{T}+\mathcal{I}_\Lambda = \mathcal{S}$; in other words, for every $f \in \mathcal{S}$ we can find a $g \in \mathcal{T}$ such that $f - g \in \mathcal{I}_\Lambda$. In particular, the spaces $\mathcal{T}/\mathcal{I}_{\Lambda,\mathcal{T}}$ and $\mathcal{S}/\mathcal{I}_\Lambda$ have the same dimension.

This concept is an version of that of *flatness*, defined in [4], Definition 5.1 (see also Remark 2.16).

An important problem in this framework is to find *representing measures* for a uspf $\Lambda : \mathcal{S}^{(1)} \mapsto \mathbb{C}$ on the whole space $\mathcal{S}^{(1)}$ or only on a part of it. If $\mathcal{T}$ is a function subspace of $\mathcal{S}^{(1)}$, a representing measure of $\Lambda|\mathcal{T}$ is a probability measure μ with support in $\mathbb{R}^n$, such that $\Lambda(f) = \int f d\mu$ for all $f \in \mathcal{T}$. When such a measure μ exists, we say that $\Lambda|\mathcal{T}$ has an *integral representation.*

To present the most significant examples (from our point of view) of function spaces, we freely use multi-indices from $\mathbb{Z}_+^n$ and the standard notation related to them.

The symbol $\mathcal{P}$ will designate the algebra of all polynomials in $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, with complex coefficients. (Although the polynomials with real coefficients seem to be more appropriate for these problems, we prefer polynomials with complex coefficients because of the systematic use of some associated complex Hilbert spaces.)

For every integer $m \geq 1$, let $\mathcal{P}_m$ be the subspace of $\mathcal{P}$ consisting of all polynomials p with $\deg(p) \leq m$, where $\deg(p)$ is the total degree of p. Note that $\mathcal{P}_m^{(1)} = \mathcal{P}_{2m}$ and $\mathcal{P}^{(1)} = \mathcal{P}$, the latter being an algebra.

Giving a finite multi-sequence of real numbers $\gamma = (\gamma_\alpha)_{|\alpha|\leq 2m}$, $\gamma_0 = 1$, we associate it with a map $\Lambda_\gamma : \mathcal{P}_{2m} \mapsto \mathbb{C}$ given by $\Lambda_\gamma(t^\alpha) = \gamma_\alpha$, extended to $\mathcal{P}_{2m}$ by linearity. The map Λ_γ is called the *Riesz functional associated to* γ.

We clearly have $\Lambda_\gamma(1) = 1$ and $\Lambda_\gamma(\bar{p}) = \overline{\Lambda_\gamma(p)}$ for all $p \in \mathcal{P}_{2m}$. If, moreover, $\Lambda_\gamma(|p|^2) \geq 0$ for all $p \in \mathcal{P}_m$, then Λ_γ is a uspf. In this case, we say that γ itself is *square positive.*

Conversely, if $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ is a uspf, setting $\gamma_\alpha = \Lambda(t^\alpha)$, $|\alpha| \leq 2m$, we have $\Lambda = \Lambda_\gamma$, as above. The multi-sequence γ is said to be the *multi-sequence associated to the uspf* Λ.

To find an integral representation for the map Λ_γ means to solve a *truncated moment problem* (see [4]–[8] for other details).

To solve the *full (or the multidimensional Hamburger) moment problem* means to find an integral representation for the map $\Lambda_\gamma : \mathcal{P} \mapsto \mathbb{C}$, similarly defined for a multi-sequence $\gamma = (\gamma_\alpha)_{\alpha \geq 0}$, $\gamma_0 = 1$ (see [2] for other details). Various results concerning the integral representations for truncated and full moment problems will be given throughout this text.

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2 Dimensional Stability and Consequences

In this section, we present an extension theorem for some unital square positive functionals $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ to unital square positive functionals on whole algebra $\mathcal{P}$, and exhibit some of its consequences.

Remark 2.1 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, and let $0 \leq k \leq m$. As in the Introduction (see (1.2) and (1.3)), if we put $\mathcal{I}_k = \mathcal{I}_{\Lambda,\mathcal{P}_k} = \{p \in \mathcal{P}_k; \Lambda(|p|^2) = 0\} = \{p \in \mathcal{P}_k; \Lambda(pq) = 0\, \forall q \in \mathcal{P}_k\}$, then

$$\mathcal{H}_k = \mathcal{P}_k/\mathcal{I}_k, \tag{2.1}$$

is a finite dimensional Hilbert space, with the scalar product given by

$$\langle p + \mathcal{I}_k, q + \mathcal{I}_k \rangle = \Lambda(p\bar{q}),\ p, q \in \mathcal{P}_k. \tag{2.2}$$

Recall also that the map $\mathcal{P}_k \ni p \mapsto \Lambda(|p|^2)^{1/2}$ is a semi-norm.

Now, if $l \leq m$ is another integer with $k \leq l$, since $\mathcal{I}_k \subset \mathcal{I}_l$, we have a natural map $J_{k,l} : \mathcal{H}_k \mapsto \mathcal{H}_l$ given by $J_{k,l}(p + \mathcal{I}_k) = p + \mathcal{I}_l$, $p \in \mathcal{P}_{n,k}$, which is an isometry because $\|p + \mathcal{I}_k\|^2 = \Lambda(|p|^2) = \|p + \mathcal{I}_l\|^2$, whenever $p \in \mathcal{P}_k$. In particular, $J_{k,k}$ is the identity on $\mathcal{H}_k$.

For a given uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$, each of the spaces $\mathcal{H}_k = \mathcal{P}_k/\mathcal{I}_k, 0 \leq k \leq m$, will be referred to as a *Hilbert space built via the uspf* Λ, while every map $J_{k,l} : \mathcal{H}_k \mapsto \mathcal{H}_l$, $0 \leq k \leq l \leq m$, is designated as an *associated isometry.*

Similar constructions and a similar terminology will be used for a uspf $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$

As mentioned in the Introduction, equalities of the form $J_{k,l}(\mathcal{H}_k) = \mathcal{H}_l\ (k < l)$ play an important role in this paper. We note that $J_{k,l}(\mathcal{H}_k) = \mathcal{H}_l$ if and only if $\mathcal{P}_l = \mathcal{P}_k + \mathcal{I}_l$. In this case, $J_{k,l}$ is a unitary transformation.

When $l = k + 1$, we write sometimes J_k instead of $J_{k,k+1}$.

Definition 2.2 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a uspf, let $(\mathcal{H}_l)_{0 \leq l \leq m}$ be the Hilbert spaces built via Λ, and let $J_l : \mathcal{H}_l \mapsto \mathcal{H}_{l+1}\,(0 \leq l \leq m-1)$ be the associated isometries. If for some $k \in \{0, \ldots, m-1\}$ one has $J_k(\mathcal{H}_k) = \mathcal{H}_{k+1}$, we say that Λ is *dimensionally stable* (or simply *stable*) *at* k.

The uspf $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ is said to be *dimensionally stable* if there exist integers m, k, with $m > k \geq 0$, such that $\Lambda_\infty|\mathcal{P}_{2m}$ is stable at k. The number $\mathrm{sd}(\Lambda_\infty) = \dim \mathcal{H}_k$ will be called the *stable dimension* of Λ_∞.

We shall see later (see Corollary 2.8) that the stable dimension is unambiguously defined.

Lemma 2.3 *Let* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ *be a uspf. If* Λ *is stable at* $m-1$*, then* $(\sum_{j=1}^m t_j \mathcal{I}_m) \cap \mathcal{P}_m \subset \mathcal{I}_m$*. In particular,* $t_j \mathcal{I}_{m-1} \subset \mathcal{I}_m$ *for all* $j = 1, \ldots, m$*.*

Proof. Let $p = \sum_{j=1}^m t_j p_j \in \mathcal{P}_m$ with $p_j \in \mathcal{I}_m$ for all $j = 1, \ldots, m$, and let $q \in \mathcal{P}_{m-1}$. Then

$$|\Lambda(pq)| \leq \sum_{j=1}^m |\Lambda(t_j p_j q)| \leq \sum_{j=1}^m \Lambda(|p_j|^2)^{1/2} \Lambda(t_j^2 |q|^2)^{1/2} = 0$$

by the Cauchy-Schwarz inequality.

Now, let $q \in \mathcal{P}_{m-1}$ be such that $\bar{p} - q \in \mathcal{I}_m$, via the hypothesis on Λ. Then

$$\Lambda(p\bar{p}) = \Lambda(pq) + \Lambda(p(\bar{p} - q)) = 0,$$

by the previous computation and the inclusion $\mathcal{I}_m \cdot \mathcal{P}_m \subset \ker(\Lambda)$. Therefore $p \in \mathcal{I}_m$.

The last assertion is obvious.

Remark 2.4 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a uspf, stable at $m-1$. Lemma 2.3 allows us to define correctly the map $M_j : \mathcal{H}_{m-1} \mapsto \mathcal{H}_m$ by the equality $M_j(p + \mathcal{I}_{m-1}) = t_j p + \mathcal{I}_m$ for all $j = 1, \ldots, m$. Setting $J = J_{m-1}$ (see Remark 2.2), we may consider on the Hilbert space $\mathcal{H}_m$ the linear operators $A_j = M_j J^{-1}$ for all $j = 1, \ldots, m$. With this notation, we have the following.

Proposition 2.5 *The linear maps* A_j*,* $j = 1, \ldots, m$*, are self-adjoint operators, and* $A = (A_1, \ldots, A_n)$ *is a commuting tuple on* $\mathcal{H}_m$*.*

Proof. Let $p_k \in \mathcal{P}_m$ and $q_k \in \mathcal{P}_{m-1}$ be such that $p_k - q_k \in \mathcal{I}_m \, (k = 1, 2)$. Then

$$\langle A_j^*(p_1 + \mathcal{I}_m, p_2 + \mathcal{I}_m\rangle = \langle p_1 + \mathcal{I}_m, t_j q_2 + \mathcal{I}_m\rangle = \Lambda(p_1 t_j q_2)$$

$$= \Lambda(q_1 t_j p_2) = \langle A_j(p_1 + \mathcal{I}_m), p_2 + \mathcal{I}_m\rangle,$$

because $A_j(p_k + \mathcal{I}_m) = t_j q_k + \mathcal{I}_m \, (k = 1, 2; j = 1, \ldots, m)$, as one can easily see. Hence $A_1, \ldots, A_n$ are self-adjoint.

We prove now that $A_1, \ldots, A_n$ mutually commute. It suffices to show that $M_j J^{-1} M_k = M_k J^{-1} M_j$ for all $j, k = 1, \ldots, n$. To show this, fix a polynomial $p \in \mathcal{P}_{m-1}$. Thus $M_j(p + \mathcal{I}_{m-1}) = t_j p + \mathcal{I}_m$. We can choose $q_j \in \mathcal{P}_{m-1}$ such that $t_j p - q_j \in \mathcal{I}_m$. Therefore, $J^{-1}(t_j p + \mathcal{I}_m) = q_j + \mathcal{I}_{m-1}$, and $M_k(q_j + \mathcal{I}_{m-1}) = t_k q_j + \mathcal{I}_m$.

Similarly, $M_k(p + \mathcal{I}_{m-1}) = t_k p + \mathcal{I}_m$, and we can choose a $q_k \in \mathcal{P}_{m-1}$ such that $t_k p - q_k \in \mathcal{I}_m$, and $M_j(q_k + \mathcal{I}_{m-1}) = t_j q_k + \mathcal{I}_m$.

Let us show that $t_k q_j - t_j q_k \in \mathcal{I}_m$. We note that $t_j t_k p - t_j q_k \in t_j \mathcal{I}_m$ and $t_k t_j p - t_k q_j \in t_k \mathcal{I}_m$. Consequently,

$$t_k q_j - t_j q_k \in (t_k \mathcal{I}_m + t_j \mathcal{I}_m) \cap \mathcal{P}_m \subset \mathcal{I}_m,$$

via Lemma 2.4. This shows that $A_1, \ldots, A_n$ mutually commute.

The next result is a substitute for Theorem 7.8 and Corollary 7.9 from [4].

Theorem 2.6 *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C} \, (m \geq 1)$ be a uspf, stable at $m - 1$. Then there exists a unique extension $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ of Λ, which is a uspf.*

Proof. We keep the notation from Remark 2.4 and Proposition 2.5. Let us define the map $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ by the equality

$$\Lambda_\infty(p) = \langle p(A)(1 + \mathcal{I}_m), 1 + \mathcal{I}_m\rangle, \; p \in \mathcal{P}. \tag{2.3}$$

Using elementary properties of the (polynomial) functional calculus for tuples of self-adjoint operators (see for instance [22]), it follows easily that Λ_∞ is a uspf on $\mathcal{P}$.

Next we remark that if α, β are multi-indices with $\alpha \neq 0$ and $|\alpha + \beta| \leq m$, then $A^\alpha(t^\beta + \mathcal{I}_m) = t^{\alpha+\beta} + \mathcal{I}_m$, obtained from formula (2.4) (whose proof is given in Remark 2.9).

Taking an arbitrary polynomial $p \in \mathcal{P}_{2m}$, and considering a (finite) representation of p of the form $p = \sum_{j,k\geq 0} c_{jk} t^{\alpha_j} t^{\beta_k}$, with $|\alpha_j|, |\beta_k| \leq m$, we have:

$$\Lambda_\infty(p) = \sum_{j,k\geq 0} c_{jk} \langle A^{\alpha_j} A^{\beta_k}(1+\mathcal{I}_m), 1+\mathcal{I}_m\rangle =$$

$$\sum_{j,k\geq 0} c_{jk}\langle t^{\alpha_j}+\mathcal{I}_m, t^{\beta_k}+\mathcal{I}_m\rangle = \sum_{j,k\geq 0} c_{jk}\Lambda(t^{\alpha_j} t^{\beta_k}) = \Lambda(p),$$

showing that Λ_∞ is an extension of Λ.

In the last part of the proof, we show the uniqueness of the uspf Λ_∞.

Let $\Lambda', \Lambda'' : \mathcal{P} \mapsto \mathbb{C}$ be extensions of Λ, which are uspf. Let $\mathcal{H}'_k = \mathcal{P}_k/\mathcal{I}'_k$, $\mathcal{H}''_k = \mathcal{P}_k/\mathcal{I}''_k$ be the Hilbert spaces built via Λ', Λ'', respectively.

Fixing an integer $l \geq 1$, we prove by recurrence that for every multi-index α with $|\alpha| \leq 2m+2l$ there exists a polynomial $p_\alpha \in \mathcal{P}_{m-1}$, such that $t^\alpha - p_\alpha \in \mathcal{I}'_{|\alpha|} \cap \mathcal{I}''_{|\alpha|}$.

The assertion is obvious for $|\alpha| = m$. Assume the property true for all multi-indices of length $m+k-1$ and let us prove it for multi-indices of length $m+k$, where $1 \leq k \leq m+2l$. If $|\alpha| = m+k$, then there exists a $j \in \{1, \ldots, n\}$ and a multi-index β with $|\beta| = m+k-1$ such that $t^\alpha = t_j t^\beta$. By the induction hypothesis, we can find a polynomial $p_\beta \in \mathcal{P}_{m-1}$ such that $t^\beta - p_\beta \in \mathcal{I}'_{m+k-1} \cap \mathcal{I}''_{m+k-1}$. Therefore, $t^\alpha - t_j p_\beta \in \mathcal{I}'_{m+k} \cap \mathcal{I}''_{m+k}$, by applying the Cauchy-Schwarz inequality succesively to Λ' and Λ''. Further, $t_j p_\beta \in \mathcal{P}_m$ and so we can find a polynomial $p_{j,\beta} \in \mathcal{P}_{m-1}$ such that $t_j p_\beta - p_{j,\beta} \in \mathcal{I}_m$, via the equality $J_{m-1}(\mathcal{H}_{m-1}) = \mathcal{H}_m$. Consequently,

$$t^\alpha - p_\alpha = t^\alpha - t_j p_\beta + t_j p_\beta - p_{j,\beta} \in \mathcal{I}'_{m+k} \cap \mathcal{I}''_{m+k} + \mathcal{I}_m = \mathcal{I}'_{m+k} \cap \mathcal{I}''_{m+k},$$

where $p_\alpha = p_{j,\beta}$.

Extending the property from above to arbitrary polynomials, we deduce that for every polynomial $p \in \mathcal{P}_{2m+2l}$ we can find a polynomial $q \in \mathcal{P}_{m-1}$ such that $p - q \in \mathcal{I}'_{2m+2l} \cap \mathcal{I}''_{2m+2l}$. Moreover,

$$\Lambda'(p) = \langle p + \mathcal{I}'_{2m+2l}, 1 + \mathcal{I}'_{2m+2l}\rangle = \langle q + \mathcal{I}_{m-1}, 1 + \mathcal{I}_{m-1}\rangle =$$

$$\Lambda(q) = \langle p + \mathcal{I}''_{2m+2l}, 1 + \mathcal{I}''_{2m+2l}\rangle = \Lambda''(p),$$

showing that $\Lambda'|\mathcal{P}_{2m+2l} = \Lambda''|\mathcal{P}_{2m+2l}$.

As the integer $l \geq 1$ is arbitrary, we have, in fact, that $\Lambda' = \Lambda''$.

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a uspf, stable at $m-1$. The unique extension $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ of Λ which is a uspf (existing by Theorem 2.6) will be designated as the *sp-extension* of Λ.

Proposition 2.7 *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a uspf, stable at $m-1$, and let $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ be the sp-extension of Λ. Then Λ_∞ is stable at any $l \geq m-1$.*

Proof. Let $(\mathcal{H}_l)_{l\geq 0}$ be the Hilbert spaces built via Λ_∞, and let $J_l : \mathcal{H}_l \mapsto \mathcal{H}_{l+1}\,(l \geq 0)$ be the associated isometries. We prove, by induction, that $J_l(\mathcal{H}_l) = \mathcal{H}_{l+1}$ for all $l \geq m-1$.

The assertion is obvious for $l = m-1$. Assume the assertion true for some $l \geq m-1$ and let us get it for $l+1$.

Fix a monomial $t^\alpha \in \mathcal{P}_{l+2}$ with $|\alpha| = l+2$. Then there exists a $j \in \{1, \ldots, n\}$ and a multi-index β with $|\beta| = l+1$ such that $t^\alpha = t_j t^\beta$. Moreover, we can find a polynomial $p_\beta \in \mathcal{P}_l$ such that $\Lambda_\infty(|t^\beta - p_\beta|^2) = 0$, via the equality $\mathcal{P}_l + \mathcal{I}_{l+1} = \mathcal{P}_{l+1}$. Therefore

$$\Lambda_\infty(|t^\alpha - t_j p_\beta|^2)^2 \leq \Lambda_\infty(|t^\beta - p_\beta|^2)\Lambda_\infty(t_j^4|t^\beta - p_\beta|^2) = 0$$

by the Cauchy-Schwarz inequality, showing that $\mathcal{P}_{l+1} + \mathcal{I}_{l+2} = \mathcal{P}_{l+2}$.

Corollary 2.8 *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a uspf, and let $k \in \{0, \ldots, m-1\}$. If Λ is stable at k, then Λ is stable at any $l \in \{k, \ldots, m-1\}$.*

Let $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ be a uspf, stable at $k \geq 0$. Then Λ_∞ is stable at any $l \geq k$.

Proof. It suffices to apply the previous proposition to the uspf $\Lambda_{k+1} = \Lambda|\mathcal{P}_{2k+2}$, using the uniqueness of its sp-extension.

The second part follows easily from the first one.

Remark 2.9 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a uspf, stable at $m-1$, and let $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ be the sp-extension of Λ. Let also $(\mathcal{H}_l)_{l\geq 0}$ be the Hilbert spaces built via Λ_∞, and let $J_{l,j} : \mathcal{H}_l \mapsto \mathcal{H}_j\ (j \geq l)$ be the associated isometries (with $J_{l,l+1} = J_l$) . As Λ_∞ is stable at any $l \geq m-1$, by Corollary 2.8, we may construct the n-tuple of commuting self-adjoint operators $A_l = (A_{l,1}, \ldots, A_{l,n})$ on the space $\mathcal{H}_l$, for all $l \geq m$, as in Proposition 2.5. Defining, as in Remark 2.4, $M_{l-1,j} : \mathcal{H}_{l-1} \mapsto \mathcal{H}_l$ by the equality $M_{l-1,j}(p + \mathcal{I}_{l-1}) = t_j p + \mathcal{I}_l$, we have $A_{l,j} = M_{l-1,j} J_{l-1}^{-1}$ for all $j = 1, \ldots, n$ and $l \geq m$. It is also clear that $J_l M_{l-1,j} = M_{l,j} J_{l-1}$ for all $j = 1, \ldots, n$ and $l \geq m$. Therefore

$$M_{l,k} M_{l-1,j}(p + \mathcal{I}_{l-1}) = t_k t_j p + \mathcal{I}_{l+1}$$

for all $k, j = 1, \ldots, n$, $l \geq m$, and $p \in \mathcal{P}_{l-1}$. Using these remarks, we infer that

$$A_{l,k} A_{l,j}(p + \mathcal{I}_l) = M_{l-1,k} J_{l-1}^{-1} M_{l-1,j} J_{l-1}^{-1}(p + \mathcal{I}_l) =$$
$$J_l^{-1} J_{l+1}^{-1} M_{l+1,k} M_{l,j}(p + \mathcal{I}_l) = J_{l,l+2}^{-1}(t_k t_j p + \mathcal{I}_{l+2}),$$

for all $k, j = 1, \ldots, n$, $l \geq m$, and $p \in \mathcal{P}_l$.

A recurrence argument leads to the formula

$$A_l^\alpha(p + \mathcal{I}_l) = J_{l,l+k}^{-1}(t^\alpha p + \mathcal{I}_{l+k}), \tag{2.4}$$

for all $l \geq m$, $p \in \mathcal{P}_l$ and $|\alpha| \leq k$.

Theorem 2.10 *Let $m \geq 1$ be an integer, and let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf. If Λ is stable at $m-1$, then, endowed with an equivalent norm, the space $\mathcal{H}_m$ has the structure of a unital commutative C^*-algebra.*

Proof. Let $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ be the sp-extension of Λ given by Theorem 2.6. In particular, considering the Hilbert spaces $\mathcal{H}_k$ for all $k \geq m$ bilt via Λ_∞, the associated isometries $J_{k,l} : \mathcal{H}_k \mapsto \mathcal{H}_l$ are unitary operators for all integres k, l with $m \leq k \leq l$.

We identify the space $\mathcal{H}_m$ with a commutative sub-C^*-algebra $\mathcal{A}$ of the C^*-algebra $B(\mathcal{H}_m)$ of all linear (automatically bounded) operators on $\mathcal{H}_m$. We use the notation from Proposition 2.5.

First of all we define a map $\pi : \mathcal{H}_m \mapsto B(\mathcal{H}_m)$ by the equation $\hat{p} \mapsto p(A)$, where $\hat{p} = p + \mathcal{I}_m$, $p \in \mathcal{P}_m$. To check the correctness of this definition, we note that

$$p(A)(q + \mathcal{I}_m) = J_{m,m+k}^{-1}(pq + \mathcal{I}_{m+k}),$$

for all integers $k \geq 1$ and polynomials $p \in \mathcal{P}_k$, $q \in \mathcal{P}_m$, which is a direct consequence of Remark 2.9. If $p \in \mathcal{I}_m$, then $pq \in \mathcal{I}_{2m}$ for all $q \in \mathcal{P}_m$, and so $p(A) = 0$.

The map π is injective since $p(A) = 0$ for some $p \in \mathcal{P}_m$ implies $p(A)(1 + \mathcal{I}_m) = p + \mathcal{I}_m = 0$, and hence $p \in \mathcal{I}_m$.

Now, let $p \in \mathcal{P}$ be arbitrary, so $p \in \mathcal{P}_k$ for some integer $k \geq 0$. If $k \geq m+1$, we can find a polynomial $r \in \mathcal{P}_m$ such that $p - r \in \mathcal{I}_k$. Thus

$$p(A)(q + \mathcal{I}_m) = J_{m,k}^{-1}(pq + \mathcal{I}_k) = J_{m,k}^{-1}(rq + \mathcal{I}_k) = r(A)(q + \mathcal{I}_m)$$

for all $q \in \mathcal{P}_m$. This shows that the map $\pi : \mathcal{H}_m \mapsto \{p(A); p \in \mathcal{P}\}$ is surjective.

Let $\mathcal{A} = \{p(A); p \in \mathcal{P}\}$, which is a commutative sub-C^*-algebra $\mathcal{A}$ of $B(\mathcal{H}_m)$. The previous discussion shows that the map $\pi : \mathcal{H}_m \mapsto \mathcal{A}$ is a linear isomorphism. Identifying the algebra $\mathcal{A}$ with the space $\mathcal{H}_m$, we obtain the desired structure of the latter.

The next result is a substitute for Corollary 7.11 from [4]

Theorem 2.11 *Let $m \geq 1$ be an integer, and let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf.*

If Λ is stable at $m-1$, then there exists a d-atomic measure μ on $\mathbb{R}^n$, where $d = \dim \mathcal{H}_m$, such that

$$\Lambda(p) = \int p(t) d\mu(t),\ p \in \mathcal{P}_{2m}.$$

Proof. We use the n-tuple $A = (A_1, \ldots, A_n)$ consisting of commuting self-adjoint operators, generating the commutative unital C^*-subalgebra $\mathcal{A}$ (in the C^*-algebra $\mathcal{L}(\mathcal{H}_m)$), considered in the previous theorem. The C^*-algebra $\mathcal{A}$ must have precisely d characters, say $\phi_1, \ldots, \phi_d$. Therefore, the joint spectrum (for the necessary information concerning the multi-parameter spectral theory, we refer to [22]) of the n-tuple $A = (A_1, \ldots, A_n)$ equals the set

$$\Sigma_A = \{(\phi_k(A_1), \ldots, \phi_k(A_n)) \in \mathbb{R}^n;\ k = 1, \ldots, d\}.$$

We use now an idea which goes back to [10]. Let E be the joint spectral measure of A, which is concentrated on its joint spectrum Σ_A. If $\mu(*) = \langle E(*)(1 + \mathcal{I}_m), 1 + \mathcal{I}_m \rangle$, which is clearly a positive atomic measure with support in Σ_A, we have:

$$\Lambda(p\bar{q}) = \langle p + \mathcal{I}_m, q + \mathcal{I}_m \rangle = \langle p(A)(1 + \mathcal{I}_m), q(A)(1 + \mathcal{I}_m) \rangle =$$

$$\langle (p\bar{q})(A)(1 + \mathcal{I}_m), 1 + \mathcal{I}_m \rangle 1 + \mathcal{I}_m \rangle = \int_{\Sigma_A} p(t)\overline{q(t)} d\mu(t),$$

for all polynomials $p, q \in \mathcal{P}_m$, leading to the stated formula, via the equality $\mathcal{P}_m^{(1)} = \mathcal{P}_{2m}$.

Finally, the support of the measure μ is precisely the set Σ_A. Indeed, denoting by χ_ξ the characteristic function on Σ_A of an arbitrary point $\xi \in \Sigma_A$, the element $\chi_\xi(A)$, built via the functional calculus of A, is a nontrivial idempotent of $\mathcal{A}$ (in fact, a self-adjoint projection on $\mathcal{H}_m$), satisfying

$$\langle \chi_\xi(A)(1+\mathcal{I}_m), 1+\mathcal{I}_m\rangle = \int_{\Sigma_A} \chi_\xi(t)\mu(t) = \mu(\{\xi\}).$$

Assuming $\mu(\{\xi\}) = 0$, we derive that $\chi_\xi(A)(p+\mathcal{I}_m) = p(A)\chi_\xi(A)(1+\mathcal{I}_m) = 0$ for all $p \in \mathcal{P}_m$, and so $\chi_\xi(A) = 0$, which is a contradiction.

The next result is an assertion in the spirit of Corollary 2.6 of [6].

Theorem 2.12 *Let $\Lambda_\infty : \mathcal{P} \mapsto \mathbb{C}$ be a uspf.*

If Λ_∞ is dimensionally stable, then Λ_∞ has a unique representing measure, which is d-atomic, where $d = \mathrm{sd}(\Lambda_\infty)$.

Conversely, if Λ_∞ has a d-atomic representing measure, then Λ_∞ is dimensionally stable and $d = \mathrm{sd}(\Lambda_\infty)$.

Proof. Let $m \geq 1$ be an integer with the property that $\Lambda_\infty|\mathcal{P}_{2m}$ is stable at $m-1$. Therefore, $J_{m-1}(\mathcal{H}_{m-1}) = \mathcal{H}_m$, and there exists a d-atomic measure μ on $\mathbb{R}^n$ such that $\Lambda(p) = \int p(t)d\mu(t)$, $p \in \mathcal{P}_{2m}$, by Theorem 2.11, where $d = \mathrm{sd}(\Lambda_\infty)$. Setting $\Lambda'_\infty(p) = \int p(t)d\mu(t)$ for all $p \in \mathcal{P}$, we get a sp-extension of $\Lambda_m = \Lambda_\infty|\mathcal{P}_{2m}$. As Λ_∞ is also a sp-extension of Λ_m, we must have $\Lambda'_\infty = \Lambda_\infty$, by Theorem 2.6, and so Λ_∞ has a representing measure.

The uniqueness of μ follows from Theorem 3.4 in [24]. For the sake of completeness, we sketch the argument from [24] (see also [18] or [22] for the background). Let ν be another representing measure for Λ_∞, and let $B_j f(t) = t_j f(t)$, $f \in \{g \in L^2(\nu); t_j g \in L^2(\nu)\}$, $j = 1, \ldots, n$. Then B_j are (not necessarily bounded) commuting self-adjoint operators. Since $\int |p|^2 d\mu = \int |p|^2 d\nu$ for all polynomials $p \in \mathcal{P}$, the space $\mathcal{H}_m$ may be regarded as a closed subspace of $L^2(\nu)$, and B_j extends A_j for all j, where A_j is defined as in Theorem 2.11. Therefore, if E_B is the spectral measure of $B = (B_1, ..., B_n)$, then the spectral measure E of $A = (A_1, \ldots, A_n)$ equals $E_B|\mathcal{H}_m$, and therefore

$$\nu(*) = \langle E_B(*)1, 1\rangle = \langle E(*)(1+\mathcal{I}_m), 1+\mathcal{I}_m\rangle = \mu(*).$$

Conversely, assume that Λ_∞ has a d-atomic representing measure. If $d = 1$, then there exists a point $\xi \in \mathbb{R}^n$ such that $\Lambda_\infty(p) = p(\xi)$ for all $p \in \mathcal{P}$.

Then, for all $k \geq 1$, $\mathcal{I}_k = \{p \in \mathcal{P}_k; p(\xi) = 0\}$, the space $\mathcal{H}_k$ is isomorphic to $\mathbb{C}$, and so Λ_∞ is dimensionally stable with $\mathrm{sd}(\Lambda_\infty) = 1$.

Assume now that $d \geq 2$. Let $\Xi = \{\xi^{(1)}, \dots, \xi^{(d)}\} \subset \mathbb{R}^n$ be distinct points and let μ be an atomic measure concentrated on Ξ, such that $\Lambda_\infty(p) = \int p d\mu$ for all $p \in \mathcal{P}$.

Consider the polynomials

$$\chi_k(t) = \frac{\prod_{j \neq k} \|t - \xi^{(j)}\|^2}{\prod_{j \neq k} \|\xi^{(k)} - \xi^{(j)}\|^2},\ t \in \mathbb{R}^n,\ k = 1, \dots, d \tag{2.5}$$

(see also [4], (7.7)). Clearly, $\chi_k \in \mathcal{P}_{2d-2}$, $k = 1, \dots, d$, and $\chi_k(\xi^{(l)}) = \delta_{kl}$ (the Kronecker symbol) for all $k, l = 1, \dots, d$. In fact, the set $(\chi_k)_{1 \leq k \leq d}$ is an orthonormal basis of $L^2(\mu)$.

Since each polynomial $p \in \mathcal{P}_l$ can be written on the set Ξ as $p(t) = \sum_{j=1}^d p(\xi^{(j)})\chi_j(t)$, and so

$$\int |p(t) - \sum_{j=1}^d p(\xi^{(j)})\chi_j(t)|^2 d\mu(t) = 0,$$

it follows that, for every $l \geq 2d-2$, we have $\mathcal{I}_l = \{p \in \mathcal{P}_l; p|\Xi = 0\}$, and so $(\chi_k + \mathcal{I}_l)_{1 \leq k \leq d}$ is an orthonormal basis of $\mathcal{H}_l$. Therefore, all spaces $\mathcal{H}_l$, $l \geq 2d-2$, have the same dimension equal to $\dim L^2(\mu) = d$. In particular, Λ_∞ is dimensionally stable and $\mathrm{sd}(\Lambda_\infty) = d$.

Corollary 2.13 *The uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ has a uniquely determined d-atomic representing measure, where $d = \dim \mathcal{H}_m$, if and only if Λ is stable at $m-1$.*

Corollary 2.13 a direct consequence of Theorem 2.12, via Theorem 2.6.

The next result is a subsitute of Theorem 7.10 from [4]

Corollary 2.14 *The uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 0)$ has a d-atomic representing measure, where $d = \dim \mathcal{H}_m$, if and only if Λ has a sp-extension $\Lambda' : \mathcal{P}_{2m+2} \mapsto \mathbb{C}$, which is stable at m.*

Proof. If Λ has a sp-extension Λ' which is stable at m, the assertion follows from Corollary 2.13.

Conversely, if Λ has a d-atomic representing measure μ, we define $\Lambda_\infty(p) = \int p d\mu$, $p \in \mathcal{P}$. Then Λ_∞ is dimensionally stable and $d = \mathrm{sd}(\Lambda_\infty)$, by Theorem 2.12. As dim $L^2(\mu) = d =$ dim $\mathcal{H}_m$, we must have that $\Lambda' = \Lambda_\infty | \mathcal{P}_{2m+2}$ is a sp-extension of Λ, stable at m.

With our terminology, a consequence for the full moment problem of the previous results is the following (see also Proposition 5.9 from [5]).

Corollary 2.15 *If $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ $(\gamma_0 = 1)$ is a positive definite multi-sequence, then γ has a representing measure if and only if there are a sequence $(m_j)_{j \geq 1}$ of integers with $m_j \geq j$ for all j, and a sequence of uspf $\Lambda_j : \mathcal{P}_{2m_j+2} \mapsto \mathbb{C}$ that are stable at m_j, such that $\Lambda_j | \mathcal{P}_j$ is the Riesz functional associated to the finite multi-sequence $(\gamma_\alpha)_{|\alpha| \leq j}$ for all $j \geq 1$.*

Proof. Assume that γ is a moment sequence and let Λ_∞ be the associated uspf (see the Introduction). According to Tchakaloff's theorem (see [1]; see also [21], [14] etc.), for each integer $j \geq 1$ we can find an atomic measure μ_j on $\mathbb{R}^n$ such that $\Lambda_\infty(p) = \int p(t) d\mu_j(t)$ for all $p \in \mathcal{P}_j$. Let $\Xi_j = \{\xi^{(j,1)}, \ldots, \xi^{(j,d_j)}\} \subset \mathbb{R}^n$ be the support of μ_j. We fix integers m_j with $m_j \geq \max\{j, 2d_j - 2\}$, and define the uspf

$$\Lambda_j(p) = \int_{\Xi_{m_j}} p(t) d\mu_j(t),\ p \in \mathcal{P}_{2m_j+2}.$$

As noticed in the proof of Theorem 2.12 , the uspf Λ_j is stable at m_j. Moreover, $\Lambda_j | \mathcal{P}_j$ is actually the Riesz functional associated to $(\gamma_\alpha)_{|\alpha| \leq j}$ for all $j \geq 1$.

Conversely, assuming that there are a sequence $(m_j)_{j \geq 1}$ of integers with $m_j \geq j$ for all j, and a sequence of uspf $\Lambda_j : \mathcal{P}_{2m_j+2} \mapsto \mathbb{C}$ that are stable at m_j, such that $\Lambda_j | \mathcal{P}_j$ is the Riesz functional associated to $(\gamma_\alpha)_{|\alpha| \leq j}$ for all $j \geq 1$, we derive the existence of a representing measure for the finite multi-sequence $(\gamma_\alpha)_{|\alpha| \leq j}$ for all $j \geq 1$, by Theorem 2.11. The existence of a representing measure for the sequence γ is then a consequence of the main result from [20] (see also [25]).

Remark 2.16 Let $m \geq 0$ be an integer. The finite dimensional space $\mathcal{P}_m$ may be given a Hilbert space structure with the scalar product defined by

$$(p|q) = \sum_{|\alpha| \leq m} c_\alpha \bar{d}_\alpha,$$

where $p = \sum_{|\alpha|\le m} c_\alpha t^\alpha, q = \sum_{|\alpha|\le m} d_\alpha t^\alpha$. In other words, the family of monomials $(t^\alpha)_{|\alpha|\le m}$ is an orthonormal basis of $\mathcal{P}_m$.

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, and let $\{\mathcal{H}_k = \mathcal{P}_k/\mathcal{I}_k,\ 0 \le k \le m\}$ be the Hilbert spaces built via Λ. The sesquilinear form $(p,q) \mapsto \Lambda(p\bar{q})$ implies the existence of a positive operator A_k on $\mathcal{P}_k$ such that $(A_k p|q) = \Lambda(p\bar{q})$ for all $p, q \in \mathcal{P}_k$, where $0 \le k \le m$. Note that $p \in \mathcal{I}_k$ if and only if $A_k p = 0$. This implies that $\dim \mathcal{H}_k$ equals the rank of A_k. The concept of *flatness* for the finite multi-sequence associated to Λ (see [4], Definition 5.1) means precisely that Λ is stable at $m-1$, and it is equivalent to the fact that the rank of A_{m-1} is equal to the rank of A_m. The latter property is directly used to 'extend' A_m ('extension' in the sense of [19]) to a rank preserving uniquely determined matrix A_{m+1} (associated to a uspf $\Lambda' : \mathcal{P}_{2m+2} \mapsto \mathbb{C}$ which extends Λ)), which is the main tool of Curto and Fialkow (see [4], Theorem 7.8.).

Although implying statements parallel to those of Curto and Fialkow, our approach, completely different from that of Curto and Fialkow, leads more quickly to the essential results of this theory.

3 The Stability Equation

As we have seen in the previous section, if $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ is a uspf, $(\mathcal{H}_k)_{0\le k\le m}$ are the Hilbert spaces bilt via Λ and $J_{k,l}(\mathcal{H}_k) = \mathcal{H}_l$ $(0 \le k < l \le m)$ are the associared isometries (see Remark 2.1), equalities of the form $J_{k,l}(\mathcal{H}_k) = \mathcal{H}_l$ are of particular interest for us. In fact, the equality $J_k(\mathcal{H}_k) = \mathcal{H}_{k+1}$ (that is, Λ is stable at k) implies all equalities $J_{k,l}(\mathcal{H}_k) = \mathcal{H}_l$ $(0 \le k < l \le m)$, as shown by Corollary 2.8. This is a dimensional stability which will be analysed in this section.

Remark 3.1 Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf with $m \ge 1$ and let k be an integer such that $0 \le k < m$. It is easily checked that the uspf Λ is stable at k if and only if for each multi-index δ with $|\delta| = k+1$ the equation

$$\sum_{|\xi|,|\eta|\le k} \gamma_{\xi+\eta} c_\xi c_\eta - 2 \sum_{|\xi|\le k} \gamma_{\xi+\delta} c_\xi + \gamma_{2\delta} = 0$$

has a solution $(c_\xi)_{|\xi|\le k}$ consisting of real numbers, where $\gamma = (\gamma_\xi)_{|\xi|\le 2m}$ is the finite multi–sequence associated to Λ.

To study the existence of solutions for such an equation, it is convenient to use an abstract framework.

Let $N \geq 1$ be an arbitrary integer, let $A = (a_{jk})_{j,k=1}^N$ be a matrix with real entries, that is positive on $\mathbb{C}^N$ (endowed with the standard scalar product denoted by $(*|*)$, and associated norm $\| * \|$), let $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$, and let $c \in \mathbb{R}$. We look for necessary and sufficient conditions insuring the existence of a solution $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ of the equation

$$(Ax|x) - 2(b|x) + c = 0. \tag{3.1}$$

This is a quadratic equation which will be solved in detail in the following. The range and the kernel of A, regarded as an operator on $\mathbb{C}^N$, will be denoted by $R(A)$, $N(A)$, respectively.

Proposition 3.2 *We have the following alternative:*

1) *If $b \notin R(A)$, equation* (3.1) *always has solutions.*

2) *If $b \in R(A)$, equation* (3.1) *has solutions if and only if for some (and therefore for all) $d \in A^{-1}(\{b\})$ we have $c \leq (d|b)$.*

In particular, if $N(A) = \{0\}$, then A is invertible and the equation (3.1) *has solutions if and only if $c \leq (A^{-1}b|b)$.*

Proof. Let $C : \mathbb{C}^N \to \mathbb{C}^N$ be the conjugation $z = (z_1, \ldots, z_N) \mapsto \bar{z} = (\bar{z}_1, \ldots, \bar{z}_N)$. The space $\mathbb{R}^N$ will be identified with the space $\{z \in \mathbb{C}^N; z = \bar{z}\}$. Moreover, $A(\mathbb{R}^N) \subset \mathbb{R}^N$, because the entries of A are real.

1) Assume first that $b \notin R(A)$. Then we must have $N(A) \neq \{0\}$, because otherwise $R(A)$ would be the whole space. Let P_0 be the orthogonal projection of $\mathbb{C}^N$ onto $N(A)$, and let P_1 be the orthogonal projection of $\mathbb{C}^N$ onto the space $N(A)^\perp$, which is precisely $R(A)$.

Assuming that $x \in \mathbb{R}^N$ is a solution of the equation (3.1), we set $x^{(j)} = P_j x$, $j = 0, 1$. Put also $b^{(j)} = P_j b$, $j = 0, 1$. Then from (3.1) we derive the equality

$$(A_1 x^{(1)}|x^{(1)}) - 2(b^{(1)}|x^{(1)}) - 2(b^{(0)}|x^{(0)}) + c = 0, \tag{3.2}$$

where $A_1 = A|R(A)$, which is positive and invertible.

As we clearly have $AC = CA$, implying $CN(A) = N(A)$ and $CR(A) = R(A)$, we must have $Cx^{(1)} = x^{(1)}$ and $Cb^{(1)} = b^{(1)}$. Therefore, $(b^{(1)}|x^{(1)}) = (x^{(1)}|b^{(1)})$ and equation (3.2) can be written as

$$\|B_1 x^{(1)} - B_1^{-1} b^{(1)}\|^2 = \|B_1^{-1} b^{(1)}\|^2 + 2(b^{(0)}|x^{(0)}) - c, \tag{3.3}$$

where $B_1 = A_1^{1/2}$, with $B_1^{-1}b^{(1)} \in R(A) \cap \mathbb{R}^N$. Indeed, if $C_1 = C|R(A)$, the equality $A_1C_1 = C_1A_1$ implies the equality $B_1C_1 = C_1B_1$, and so $B_1^{-1}b^{(1)} \in R(A) \cap \mathbb{R}^N$.

A necessary condition to have a solution $x^{(1)}$ of (3.3) is that

$$(b^{(0)}|x^{(0)}) \geq \frac{1}{2}(c - \|B_1^{-1}b^{(1)}\|^2). \tag{3.4}$$

This condition is also sufficient and valid for such a choice of b. Indeed, because $b \notin R(A)$, then $b^{(0)} \neq 0$ and we can fix a vector $x^{(0)} \in N(A) \cap \mathbb{R}^N$, $x^{(0)} \neq 0$, such that $2(b^{(0)}|x^{(0)}) \geq c - \|B_1^{-1}b^{(1)}\|^2$, which is possible because $N(A) \neq \{0\}$. Putting

$$r^2 = 2(b^{(0)}|x^{(0)}) - c + \|B_1^{-1}b^{(1)}\|^2,$$

we choose a vector $y^{(1)} \in R(A) \cap \mathbb{R}^N$ such that $\|y^{(1)} - B_1^{-1}b^{(1)}\|^2 = r^2$. If $x^{(1)} = B_1^{-1}y^{(1)}$, then $x = x^{(0)} + x^{(1)}$ is a solution of (3.1).

2) If $b \in R(A)$, then $b^{(0)} = 0$ and $b = b^{(1)}$. Setting $d_1 = B_1^{-1}b^{(1)}$, condition (3.4) becomes $c \leq (d_1|b) = (d|b)$, where d is an arbitrary solution of the equation $b = Ad$, which is a necessary and sufficient condition to determine a solution of (3.1).

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be a uspf and let $\gamma = (\gamma_\alpha)_{|\alpha| \leq 2m}$ the multi-sequence associated to Λ. Then $A_{m-1} = (\gamma_{\xi+\eta})_{|\xi|,|\eta| \leq m-1}$ is a positive matrix with real entries, acting as an operator on $\mathbb{C}^N$, where N is the cardinal of the set $\{\xi \in \mathbb{Z}_+^n; |\xi| \leq m-1\}$. In fact, by identifying the space $\mathcal{P}_{m-1}$ with $\mathbb{C}^N$, A_{m-1} is the operator with the property $(A_{m-1}p|q) = \Lambda(p\bar{q})$ for all $p, q \in \mathcal{P}_{m-1}$ (see Remark 2.16).

For each multi-index δ with $|\delta| = m$, we put $b_\delta = (\gamma_{\xi+\delta})_{|\xi| \leq m-1} \in \mathbb{R}^N$ and $c_\delta = \gamma_{2\delta}$. With this notation, equation (3.1) becomes

$$(A_{m-1}x|x) - 2(b_\delta|x) + c_\delta = 0, \tag{3.5}$$

which may be called the *stability equation* of the uspf Λ.

Theorem 3.3 *Let $\gamma = (\gamma_\alpha)_{|\alpha| \leq 2m}\,(\gamma_0 = 1, m \geq 1)$ be a square positive finite multi-sequence and let $A_{m-1} = (\gamma_{\xi+\eta})_{|\xi|,|\eta| \leq m-1}$, acting on $\mathbb{C}^N$, where N is the cardinal of the set $\{\xi \in \mathbb{Z}_+^n; |\xi| \leq m-1\}$. For each multi-index δ with $|\delta| = m$, set $b_\delta = (\gamma_{\xi+\delta})_{|\xi| \leq m-1} \in \mathbb{R}^N$ and $c_\delta = \gamma_{2\delta}$. The multi-sequence γ has a unique r-atomic representing measure if and anly if, whenever $b_\delta \in R(A_{m-1})$, we have $c_\delta = (d_\delta|b_\delta)$ for some (and therefore for all) $d_\delta \in A_{m-1}^{-1}(\{b_\delta\})$, where r is the rank of the matrix A_{m-1}.*

Proof. Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \geq 1)$ be the uspf associated to γ. Assuming the equality $c_\delta = (d_\delta|b_\delta)$ for some $d_\delta \in A_{m-1}^{-1}(\{b_\delta\})$ whenever $b_\delta \in R(A_{m-1})$ and $|\delta| = m$, we infer that Λ is stable at $m-1$, via Proposition 3.2 and Remark 3.1.

Conversely, if the multi-sequence γ has a unique r-atomic representing measure, in virtue of Corollary 2.13, the uspf Λ should be stable at $m-1$. Then the matrix A_m is a flat extension of the matrix A_{m-1} (see Remark 2.16). Writting $\mathcal{P}_m = \mathcal{P}_{m-1} \oplus \mathcal{R}_m$, there exists a linear map $W : \mathcal{R}_m \mapsto \mathcal{P}_{m-1}$ such that $A_m|\mathcal{R}_m = A_{m-1}W + W^*A_{m-1}W$ (see for instance [4] for details). Particularly, for a fixed index δ with $|\delta| = m$, as the monomial $t^\delta \in \mathcal{R}_m$, we have $A_m t^\delta = A_{m-1}Wt^\delta \oplus W^*A_{m-1}Wt^\delta$. Therefore, $(A_m t^\delta|t^\delta) = (A_{m-1}Wt^\delta|Wt^\delta)$. But $A_{m-1}Wt^\delta = P_m A_m t^\delta = \sum_{|\xi|\leq m-1} \gamma_{\xi+\delta} t^\xi = b_\delta$, where P_m is the orthogonal projection of $\mathcal{P}_m$ onto $\mathcal{P}_{m-1}$, and the vector $b_\delta \in \mathbb{R}^N$ is identified with the corresponding polynomial from $\mathcal{P}_{m-1}$. Putting $d_\delta = Wt^\delta$, the equation $c_\delta = (d_\delta|b_\delta)$ is fulfilled.

Corollary 3.4 *Assume the matrix A_{m-1} invertible. There exists a d-atomic representing measure on $\mathbb{R}^n$ for the uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ if and only if for each δ with $|\delta| = m$ we have $c_\delta = (A_{m-1}^{-1} b_\delta|b_\delta)$, where $d = \dim \mathcal{P}_{m-1}$.*

Remark 3.5 Related to the previous result, a natural question arises (see Conjecture 6.6 from [4]): Given a usps $\Lambda_{m-1} : \mathcal{P}_{2m-2} \mapsto \mathbb{C}$ such that A_{m-1} is invertible, is it true that Λ_{m-1} has an extension $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ which is stable at m? For $n = m = 2$, it is shown in [4], Section 6.1 (in the context of complex moment problems), that such extensions do exist. Note that, in general, a necessary condition is that

$$\dim\mathcal{H}_{m-1} = \dim\mathcal{P}_{m-1} = \dim\mathcal{P}_m - \dim\mathcal{I}_m.$$

As we have $\dim\mathcal{P}_m = \binom{n+m}{n}$ (see for instance [13]), the necessary condition mentioned above is equivalent to

$$\dim\mathcal{I}_m = \binom{n-1+m}{n-1}.$$

Corollary 3.4 offers, in particular, a numerical test to decide whether a uspf $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$, which extends Λ_{m-1}, is stable at $m-1$.

4 Representing Measures for Unital Square Positive Functionals

It is well known, and already discussed in the second section of this work, that the representing measures for truncated moment problems may always be supposed to be atomic, via Tchakaloff's theorem. In this section, we present some properties of square positive functionals having representing measure with prescribed atoms.

For every integer $m \ge 0$ we set

$$\mathcal{S}_{2m} = \{\sum_{j\in J} |p_j|^2;\ p_j \in \mathcal{P}_m, \operatorname{card} J < \infty\}, \tag{4.1}$$

which is a positive cone in $\mathcal{P}_{2m}$.

If $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$ is a finite set of distinct points in $\mathbb{R}^n$, we put

$$\mathcal{J}_{2m,\Xi} = \{p \in \mathcal{P}_{2m},\ p|\Xi = 0\}. \tag{4.2}$$

We start with a Riesz-Haviland type result (see [11]) for truncated moment problems. A more general result appears in [7], Proposition 3.6 (see also [21]), which addresses to arbitrary compact sets. For further use, we state the result, and give a different proof, in the context of finite sets.

Theorem 4.1 *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}\,(m \ge 1)$ be a uspf, and let Ξ be a finite set of distinct points in $\mathbb{R}^n$. The uspf Λ has a representing measure with support in Ξ if and only if $\Lambda(p) \ge 0$ whenever $p|\Xi \ge 0$.*

Proof. The condition is clearly necessary.

To prove the sufficiency, we fix an integer $k \ge \max\{m, 2d-2\}$, where d is the cardinal of Ξ. To continue the proof, we need the following.

Remark 4.2 Let $C(\Xi)$ be the space of all complex-valued (continuous) functions of Ξ, endowed with its natural norm $\|f\|_\Xi = \max_{1\le j\le d}|f(\xi^{(j)}|$, $f \in C(\Xi)$. Then the kernel of the restriction $\mathcal{P}_{2k} \ni p \mapsto p|\Xi \in C(\Xi)$ is precisely $\mathcal{J}_{2k,\Xi}$ (given by (4.2)). Therefore, there exists an injective maps, say J_Ξ, defined on $\mathcal{P}_{2k}/\mathcal{J}_{2k,\Xi}$, with values in $C(\Xi)$, induced by the restriction. The choice of k insure the surjectivity of this map J_Ξ. Indeed, using the (squares of) polynomials (2.5), and defining the linear map

$$\ell_\Xi(p) = \sum_{j=1}^{d} p(\xi^{(j)})\chi_j^2,\ p \in \mathcal{P}_{2k},$$

we clearly have $p - \ell_\Xi(p) \in \mathcal{J}_{2k,\Xi}$. Taking an arbitrary $f \in C(\Xi)$, the polynomial $p_f = \sum_{j=1}^d f(\xi^{(j)})\chi_j^2$ has the property $J_\Xi(p_f) = f$. In fact, $J_\Xi : \mathcal{P}_{2k} \mapsto C(\Xi)$ is an isomorphism.

We return to the proof of Theorem 4.1. We show first that $p|\Xi \geq 0$ if and only if $p \in \mathcal{P}_{2m} \cap (\mathcal{S}_{2k} + \mathcal{J}_{2k,\Xi})$. Indeed, if $p \in \mathcal{P}_{2m} \cap (\mathcal{S}_{2k} + \mathcal{J}_{2k,\Xi})$, the property $p|\Xi \geq 0$ is obvious. Reciprocally, if $p|\Xi \geq 0$, the polynomial $\ell_\Xi(p) = \sum_{j=1}^d p(\xi^{(j)})\chi_j^2$ (see Remark 4.2) belongs to $\mathcal{S}_{2k}$ and $p - \ell_\Xi(p) \in \mathcal{J}_{2k,\Xi}$.

Consequently, the hypothesis implies that for all $p \in \mathcal{P}_{2m} \cap (\mathcal{S}_{2k} + \mathcal{J}_{2k,\Xi})$, we have $\Lambda(p) \geq 0$.

For every integer $l \geq 0$, the symbol $\mathcal{RS}$ will stands for the subspace of $\mathcal{P}_l$, consisting of all real parts of functions from the subspace $\mathcal{S} \subset \mathcal{P}_l$.

Let the $\mathbb{R}$-linear functional $\Phi : \mathcal{RP}_{2m} \mapsto \mathbb{R}$ be given by $\Phi(p) = 2^{-1}(\Lambda(p) + \overline{\Lambda(p)})$ for all $p \in \mathcal{RP}_{2m}$. We want to extend the functional Φ to a functional $\Psi : \mathcal{RP}_{2k} \mapsto \mathbb{R}$, with $\Psi|\mathcal{D}_{2k} \geq 0$, where $\mathcal{D}_{2k} = \mathcal{S}_{2k} + \mathcal{RJ}_{2k,\Xi}$.

We show now that $\mathcal{RP}_{2k} = \mathcal{D}_{2k} + \mathcal{RP}_{2m}$. Indeed, if $p \in \mathcal{RP}_{2k}$ is a given polynomial, then the polynomial $q = p + \|p\|_\Xi$ is positive, when restricted to Ξ. Therefore, as above, $\ell_\Xi(q) \in \mathcal{S}_{2k}$, $q - \ell_\Xi(q) \in \mathcal{RJ}_{2k,\Xi}$, showing that $p \in \mathcal{D}_{2k} + \mathcal{RP}_{2m}$.

As we clearly have $\mathcal{RP}_{2m} \cap (\mathcal{S}_{2k} + \mathcal{RJ}_{2k,\Xi}) \subset \mathcal{P}_{2m} \cap (\mathcal{S}_{2k} + \mathcal{J}_{2k,\Xi})$, and so $\Phi(p) \geq 0$ if $p \in \mathcal{RP}_{2m} \cap (\mathcal{S}_{2k} + \mathcal{RP}_{2k,\Xi})$, we may apply Corollary 1.2.7 from [2], which asserts the existence of an extension Ψ of Φ such that Ψ is nonnegative on $\mathcal{S}_{2k} + \mathcal{RJ}_{2k,\Xi}$. As $\mathcal{RJ}_{2k,\Xi}$ is a nonnull vector space, the restriction of Ψ to $\mathcal{RJ}_{2k,\Xi}$ should be zero.

Let now $\Lambda_1(p) = \Psi(p_1) + i\Psi(p_2)$ for all $p = p_1 + ip_2 \in \mathcal{P}_{2k}$, with $p_1, p_2 \in \mathcal{RP}_{2k}$. Then Λ_1 is an extension of Λ to $\mathcal{P}_{2k}$, whose restriction to $\mathcal{S}_{2k}$ is nonnegative and the restriction to $\mathcal{J}_{2k,\Xi}$ is zero.

According to Remark 4.2, the map $J_\Xi : \mathcal{P}_{2k}/\mathcal{J}_{2k,\Xi} \mapsto C(\Xi)$ is an isomorphism. Moreover, as $\Lambda_1|\mathcal{J}_{2k,\Xi} = 0$, there exists a unique functional Λ_2 on $C(\Xi)$ such that $\Lambda_1 = \Lambda_2 \circ J_\Xi$. The functional Λ_2 is also positive. Indeed, as before, if $h \in C(\Xi)$ is a positive function, as $p_h = \sum_{j=1}^d h(\xi^{(j)})\chi_j^2 \in \mathcal{S}_{2k}$ and $J_\Xi(p_h) = h$, we have $\Lambda_2(h) = \Lambda_1(p_h) \geq 0$.

Taking into account the structure of a positive linear functional on $C(\Xi)$, we infer the existence of nonnegative real numbers $\alpha_1, \ldots, \alpha_d$ such that $\Lambda_2(f) = \sum_{j=1}^d \alpha_j f(\xi^{(j)})$, $f \in \mathcal{C}(\Xi)$. Consequently, $\Lambda(p) = \Lambda_2(J_\Xi(p)) = \sum_{j=1}^d \alpha_j p(\xi^{(j)})$, $p \in \mathcal{P}_{2m}$.

The next result is an approach to the existence of representing measures for truncated moment problems in the spirit of [25].

Corollary 4.3 *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf. Let also $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$ be a subset of $\mathbb{R}^n$. The uspf Λ has a representing measure with support in Ξ if and only if*

$$\sup\{|\Lambda(p)|; p \in \mathcal{P}_{2m}, \|p\|_\Xi \leq 1\} = 1. \tag{4.3}$$

Proof. Assume first that $\Lambda(p) = \sum_{j=1}^d \alpha_j p(\xi^{(j)})$, $p \in \mathcal{P}_{2m}$ for some non-negative scalars $\alpha_1, \ldots, \alpha_d$. Putting $\Theta(f) = \sum_{j=1}^d \alpha_j f(\xi^{(j)})$, $f \in C(\Xi)$, we have the equality $\Theta(p|\Xi) = \Lambda(p)$ for all $p \in \mathcal{P}_{2m}$. Therefore,

$$\sup\{|\Lambda(p)|; p \in \mathcal{P}_{2m}, \|p\|_\Xi \leq 1\} =$$

$$\sup\{|\Theta(f)|; f \in C(\Xi), \|f\|_\Xi \leq 1\} = \|\Theta\| = 1,$$

because Θ is positive on $C(\Xi)$ and $\Lambda(1) = \Theta(1) = 1$.

Conversely, assume that (4.3) holds. This implies that $\Lambda|\mathcal{J}_{2m,\Xi} = 0$. Hence Λ induces a functional Λ_Ξ on $\mathcal{P}_{2m}/\mathcal{J}_{2m,\Xi}$, which is identified with a subspace of $C(\Xi)$. Moreover, we clearly have $\Lambda_\Xi(1+\mathcal{J}_{2m,\Xi}) = 1$ and $\|\Lambda_\Xi\| = 1$. The Hahn-Banach theorem implies the existence of an extension Θ of Λ_Ξ to $C(\Xi)$, with the properties $\Theta(1) = 1$ and $\|\Theta\| = 1$, insuring the positivity of Θ. The existence of a representing measure for Λ follows as in the last part of the previous theorem.

Remark With Λ and Ξ as in Corollary 4.3, the uspf Λ has a representing measure with support in Ξ if and only if there exists a polynomial $p_\Xi \in \mathcal{RP}_{2m}$ such that $\|p_\Xi\|_\Xi = 1$ and $1 = \Lambda(p_\Xi) \geq \Lambda(p)$ for all $p \in \mathcal{RP}_{2m}$ with $\|p\|_\Xi \leq 1$.

The proof is based on the fact that the upper bound in (4.3) may be computed only on $\mathcal{RP}_{2m}$, and it is attained.

Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf. Following [5], we define the algebraic variety of Λ by $\mathcal{V}_\Lambda = \cap_{p \in \mathcal{I}_m} \mathcal{Z}(p)$, where $\mathcal{Z}(p)$ is the set of zeros of p, and $\mathcal{I}_m$ has the meaning from (2.1).

An important assertion related to the extremal truncated moment problem (see [8], Lemma 2.5) can be also obtained in our context.

Corollary 4.4 *Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbb{C}$ be a uspf, and let $\Xi = \{\xi^{(1)}, \ldots, \xi^{(d)}\}$ be a finite subset of $\mathcal{V}_\Lambda$. Assume that there are complex numbers $\alpha_1, \ldots, \alpha_d$ such that $\Lambda(p) = \sum_{j=1}^d \alpha_j p(\xi^{(j)})$ for every $p \in \mathcal{P}_{2m}$. If dim $\mathcal{H}_m = d$, then $\alpha_j > 0$ for all $j = 1, \ldots, d$.*

Proof. Note the equality $\mathcal{J}_{m,\Xi} = \mathcal{I}_m$ (for notation, see (2.1) and (4.2)). Indeed, the inclusion $\mathcal{I}_m \subset \mathcal{J}_{m,\Xi}$ follows from the definition of $\mathcal{V}_\Lambda$, while the inclusion $\mathcal{J}_{m,\Xi} \subset \mathcal{I}_m$ follows from the representation of Λ. This implies that the restriction $\mathcal{H}_m = \mathcal{P}_m/\mathcal{I}_m \ni p + \mathcal{I}_m \mapsto p|\Xi \in C(\Xi)$ is well defined and injective. It is also surjective because of the equality $\dim \mathcal{H}_m = d$.

Let $\chi_j \in C(\Xi)$ be the function equal to 1 in $\xi^{(j)}$ and equal to 0 in $\xi^{(k)}$ if $k \neq j$, $j, k = 1\ldots, d$. Then we can find polynomials $p_j \in \mathcal{P}_m$ such that $p_j|\Xi = \chi_j$ for all $j = 1\ldots, d$. Consequently, $\Lambda(\bar{p}_k p_k) = \alpha_k \geq 0$ for all $k = 1\ldots, d$. If $\alpha_k = 0$ for some k, then $p_k \in \mathcal{I}_m$, which is impossible.

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Author's address:
Department of Mathematics
University of Lille 1
59655 Villeneuve d'Ascq Cedex
France