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Regularity of functions: Genericity and multifractal analysis

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Céline Esser
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Françoise Bastin, Université de Liège

*“Cada libro, cada tomo que ves, tiene alma.
El alma de quien lo escribió,
y el alma de quienes lo leyeron y vivieron y soñaron con él.”*

Carlos Ruiz Zafón, *La sombra del viento*

Abstract

As surprising as it may seem, there exist functions of $\mathcal{C}^\infty(\mathbb{R})$ which are nowhere analytic. When such an unexpected object is found, a natural question is to ask whether many similar ones may exist. A classical technique is to use the Baire category theorem and the notion of residuality. This notion is purely topological and does not give any information about the measure of the set of objects satisfying such a property. In this purpose, the notion of prevalence has been introduced. Moreover, one could also wonder whether large algebraic structures of such objects can be constructed. This question is formalized by the notion of lineability.

The first objective of this thesis is to go further into the study of nowhere analytic functions. It is known that the set of nowhere analytic functions is residual and lineable in $\mathcal{C}^\infty([0, 1])$. We prove that the set of nowhere analytic functions is also prevalent in $\mathcal{C}^\infty([0, 1])$. Those results of genericity are then generalized using Gevrey classes, which can be seen as intermediate between the space of analytic functions and the space of infinitely differentiable functions. We also study how far such results of genericity could be extended to spaces of ultradifferentiable functions, defined using weight sequences or using weight functions.

Our second main objective is to study the pointwise regularity of functions via their multifractal spectrum. Computing the multifractal spectrum of a function using directly its definition is an unattainable goal in most of the practical cases, but there exist heuristic methods, called multifractal formalisms, which allow to estimate this spectrum and which give satisfactory results in many situations. The Frisch-Parisi conjecture, classically used and based on Besov spaces, presents two disadvantages: it can only hold for spectra that are concave and it can only yield the increasing part of spectra. Concerning the first problem, the use of \mathcal{S}^ν spaces allows to deal with non-concave increasing spectra. Concerning the second problem, a generalization of the Frisch-Parisi conjecture obtained by replacing the role played by wavelet coefficients by wavelet leaders allows to recover the decreasing part of concave spectra.

Our purpose in this thesis is to combine both approaches and define a new formalism derived from large deviations based on statistics of wavelet leaders. As expected, we show that this method yields non-concave spectra and is not limited to their increasing part. From the theoretical point of view, we prove that this formalism is more efficient than the previous wavelet-based multifractal formalisms. We present the underlying function space and endow it with a topology.

Résumé

Aussi surprenant que cela puisse paraître, il existe des fonctions de $\mathcal{C}^\infty(\mathbb{R})$ qui sont nulle part analytiques. Lorsqu'un tel objet est trouvé, il est naturel de se demander s'il peut en exister beaucoup d'autres. Une technique classique consiste à utiliser le théorème de Baire et la notion de résidualité. Cette notion est purement topologique et ne donne aucune information sur la mesure de l'ensemble formé de tels objets. C'est pourquoi la notion de prévalence a été introduite. En outre, on peut aussi se demander s'il est possible de construire de larges structures algébriques formées de ces objets. Cette question est formalisée par la notion de linéabilité.

Le premier objectif de cette thèse est de poursuivre l'étude de l'ensemble des fonctions qui sont nulle part analytiques. Il a été prouvé que cet ensemble est résiduel et linéable dans $\mathcal{C}^\infty([0, 1])$. Nous montrons qu'il est également prévalent dans $\mathcal{C}^\infty([0, 1])$. Ces résultats de généricité sont ensuite généralisés en utilisant les classes de Gevrey. Elles peuvent être considérées comme des espaces intermédiaires entre l'espace des fonctions analytiques et l'espace des fonctions infiniment continûment différentiables. Nous étudions ensuite dans quelle mesure ces résultats peuvent s'étendre aux espaces de fonctions ultradifférentiables, définis en utilisant des suites ou des fonctions de poids.

Notre deuxième objectif consiste en l'étude de la régularité ponctuelle de fonctions via leur spectre multifractal. Dans la plupart des cas pratiques, il est impossible de calculer le spectre multifractal d'une fonction en se basant uniquement sur sa définition. Néanmoins, il existe des méthodes heuristiques, appelées formalismes multifractals, qui permettent d'estimer ce spectre et qui donnent des résultats satisfaisants dans de nombreuses situations. La conjecture de Frisch-Parisi, classiquement utilisée et basée sur les espaces de Besov, présente deux inconvénients: elle ne permet de détecter que des spectres concaves et ne donne une indication que sur leur partie croissante. Concernant le premier problème, l'utilisation des espaces \mathcal{S}^ν mène à des résultats pour la détection de spectres non-concaves croissants. Pour le deuxième problème, une généralisation de la conjecture de Frisch-Parisi, obtenue en remplaçant le rôle joué par les coefficients d'ondelettes par les coefficients dominants, permet de récupérer la partie décroissante de spectres concaves.

Dans cette thèse, nous combinons les deux approches et définissons un nouveau formalisme basé sur les coefficients dominants. Comme attendu, nous montrons que cette méthode permet d'estimer des spectres non-concaves et n'est pas limitée à leur partie croissante. D'un point de vue théorique, nous montrons que ce formalisme est plus efficace que deux précédents. Nous présentons l'espace fonctionnel sous-jacent et le munissons d'une topologie.

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Introduction

In the early nineteenth century, most of the mathematicians believed that a continuous function had derivative at a significant set of points. A.M. Ampère even attempted to give a theoretical justification for this. It was therefore a veritable shock among the mathematical community, when, during a presentation at the Berlin Academy in 1872, K. Weierstraß proved that this conjecture was false. He presented the nowadays classical example [140] of a function which is continuous on \mathbb{R} but nowhere differentiable. This function is given by

$$W(x) = \sum_{n=0}^{+\infty} a^n \cos(b^n \pi x)$$

where $a \in (0, 1)$, b is any odd integer and $ab > 1 + \frac{3\pi}{2}$.

When such a surprising object is found, a natural question is to ask whether many similar ones can exist, or if this example is atypical. After the publication of the result concerning the Weierstraß function, many other mathematicians made their own contributions and constructed variants of this function. In particular, in 1931, as a nice application of the Baire category theorem, Banach [18] and Mazurkiewicz [110] proved that most of the continuous functions are nowhere differentiable. More precisely, they proved that the set of nowhere differentiable functions contains a countable union of dense open sets of $\mathcal{C}([0, 1])$; we say that such a set is *residual* in $\mathcal{C}([0, 1])$.

The notion of residuality gives the dominant behavior of the functions of the considered space from a topological point of view. Nevertheless, it does not provide any information about the “measure” of such a set. In finite dimensional spaces, we say that a property is verified almost everywhere if the set of points where it is not satisfied has a Lebesgue measure zero. The particular role played by the Lebesgue measure is justified by the fact that it is the only σ -finite measure which is invariant by translation. The notion of *prevalence* was introduced by Christensen [53] in 1972, and rediscovered in 1992 by Hunt et al. [82], in order to generalize the notion of almost everywhere to infinite dimensional spaces. The prevalence of the set of nowhere differentiable functions in $\mathcal{C}(\mathbb{R})$ was obtained by Hunt [81] in 1994.

Independently of the notions of residuality and prevalence, one could also wonder if it is possible to find large algebraic structures in the set of nowhere differentiable functions. In this context, V.I. Gurariy introduced the concept of *lineability* that first appeared in [6]. Basically, we say that a set M is lineable if $M \cup \{0\}$ contains a infinite dimensional vector space. The research of algebraic structures in the class of nowhere differentiable functions has been undertaken by many authors, but the first result in this direction is due to Gurariy [74] in 1966 who proved that the set of continuous nowhere differentiable functions on $[0, 1]$ is lineable. A constructive proof of this result, with

a maximal dimension for the existing subspace, was also given recently by Jiménez-Rodríguez et al. [96].

When looking at the Weierstraß function, a second question that can arise naturally is to wonder whether one can characterize its local behavior. For example, given two continuous functions which are nowhere differentiable, is it possible to see if the first one is, at a given point, more regular than the second one? The information concerning the local regularity of a function f at a given point x_0 can be obtained via its Hölder exponent $h_f(x_0)$. In the case of the Weierstraß function, Hardy [76] proved that its regularity is the same at every point, and more precisely, that its Hölder exponent is equal to $-\frac{\log a}{\log b}$ at every point. Nevertheless, for a highly irregular function, the function h_f can be itself very irregular. In order to get a concrete idea of the distribution of the singularities of f and their importance, one tries instead to estimate the “size” of the iso-Hölder sets $E^f(h)$ defined by

$$E^f(h) = \{x_0 \in \mathbb{R}^n : h_f(x_0) = h\}.$$

Such sets can be fractal sets, therefore by “size” one usually means Hausdorff dimension. Roughly speaking, the information about the Hölder-regularity of a function is summarized by its multifractal spectrum, defined by the function

$$d_f : [0, \infty] \rightarrow \{-\infty\} \cup [0, n] : h \mapsto \dim_{\mathcal{H}}(E_h).$$

The example of the Weierstraß function illustrates the two problems we address in the present thesis. In Part I, given a particular “strange” property, we study how large the set of functions enjoying this property is. For this purpose, we will use the notions of residuality, prevalence and lineability. These three concepts are recalled with more details in Chapter 1. In Part II, we introduce a new method which allows to get information about the regularity of irregular functions, and more precisely, to estimate their multifractal spectrum. Let us be more precise about these two parts.

The starting point of Part I is the existence of functions which are infinitely continuously differentiable on an interval of the real line but which are nowhere analytic on this interval. The existence of nowhere analytic functions can be surprising but is known since the construction of du Bois Reymond [62], in 1876. A nice example is due to Cellérier [51] in 1890, with the function defined for all $x \in \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{+\infty} \frac{\sin(a^n x)}{n!}$$

where a is a positive integer larger than 1. Actually, a generic function in the space $\mathcal{C}^\infty([0, 1])$ of infinitely continuously differentiable functions on $[0, 1]$ is nowhere analytic. Indeed, in 1954, Morgenstern [115] proved that the set of nowhere analytic functions is residual in $\mathcal{C}^\infty([0, 1])$. More recent results using the notion of lineability have been obtained. The first result is due to Bernal-González [33] in 2008. Besides, in Chapter 2, we prove that the set of nowhere analytic functions is also prevalent in $\mathcal{C}^\infty([0, 1])$. Those results of genericity are then generalized using Gevrey classes. These classes can be seen as intermediate between the space of analytic functions and the space of infinitely differentiable functions and are defined imposing growth conditions on the derivatives of the functions. We give two explicit constructions of functions of $\mathcal{C}^\infty([0, 1])$ which are nowhere Gevrey differentiable, that is to say which are not in a Gevrey class of any

order at any point. We prove the residuality, the prevalence and the lineability of the set of nowhere Gevrey differentiable functions. We go further with our main result which consists in the construction of a dense algebra of $C^\infty([0, 1])$ whose elements are nowhere Gevrey differentiable.

Chapter 3 aims at studying how far such results of genericity could extend to the non-quasianalytic classes. These classes are spaces of smooth functions which contain not identically zero elements with compact support. There are essentially two ways to introduce them: using weight sequences M and imposing weight conditions on the derivatives of the function [100], or using weight functions ω and imposing conditions on the Fourier Laplace transform of the function [38]. In both cases, we distinguish the classes $\mathcal{E}_{\{M\}}$ and $\mathcal{E}_{\{\omega\}}$ of ultradifferentiable functions of Roumieu type and the classes $\mathcal{E}_{(M)}$ and $\mathcal{E}_{(\omega)}$ of ultradifferentiable functions of Beurling type. Then, given a class E of ultradifferentiable functions of Beurling type on the real line that strictly contains another non-quasianalytic class F of Roumieu type, we handle the question of knowing how large the set of functions in E that are nowhere in the class F is. In particular, we obtain that F is a rather small subspace of E and in this way, we complement a work of Schmets and Valdivia [128]. Consequences for the Gevrey classes are also given. More recently, Rainer and Schindl [120] extended the definition of ultradifferentiable classes of functions using weight matrices. Similar results of genericity are studied in this context.

The main objective of Part II is to study the pointwise regularity of functions using their multifractal spectrum. Although the multifractal spectrum of many mathematical functions can be directly determined from its definition, for real-life signals, it is clearly impossible to estimate this spectrum numerically since it involves the successive determination of several intricate limits. Therefore one tries instead to estimate this spectrum from quantities which are numerically computable. Mathematically, these quantities are interpreted as indicating that the signal belongs to a certain family of function spaces. Such a method is called a *multifractal formalism*. It never holds in complete generality, but a first step in the justification of its use consists in showing that this method yields an upper bound for the multifractal spectrum of the functions in the underlying function space. This is the best that can be expected: usually, there are no non-trivial minorations for the multifractal spectrum of all functions in the space. Nevertheless, one can hope that for most of the functions in the space, that is to say for a generic subset of the space, the inequality becomes an equality.

Several multifractal formalisms based on the wavelet coefficients of a function have been proposed to estimate its multifractal spectrum [3, 86, 88, 91]. The starting point of all these methods is a wavelet characterization of the Hölder exponent [91]. They share the advantage of being easy to compute and relatively stable from a numerical point of view. The Frisch-Parisi conjecture, classically used, gives such an estimation based on the characterization of Besov spaces in terms of wavelet coefficients [88, 117]. Nevertheless, it appeared that this use of Besov spaces is not sufficient to handle all the information concerning the pointwise regularity contained in the distribution of the wavelet coefficients [90]. In particular, it can only lead to recover the increasing and concave hull of spectra.

In order to get a suitable context to obtain multifractal results in the non-concave case, spaces based on large deviation estimates of the repartition of wavelet coefficients, called \mathcal{S}^ν spaces, have then been introduced by Jaffard [90]. Although the formalism based on these spaces allows to effectively recover non-concave spectra, the problem

met with the Frish-Parisi approach reappears: one cannot access the decreasing part of spectra through the \mathcal{S}^ν spaces.

Concerning the estimation of the decreasing part of spectra, it appeared that more accurate information can be obtained when relying on wavelet leaders, which are local suprema of wavelet coefficients [94]. In this context, Oscillation spaces have been introduced as generalization of Besov spaces using wavelet leaders [92]. They lead to the so-called *wavelet leaders method*. In particular, this method allows to recover the increasing and decreasing parts of spectra. Nevertheless, it is still limited to concave spectra. So, a natural idea is to combine both approaches in order to derive a multifractal formalism which allows to recover the decreasing and non-concave parts of spectra.

In Chapter 4, we recall the definitions of pointwise regularity and of Hausdorff dimension. We also present the notion of wavelets and the characterization of the Hölder exponents in terms of decay rates of wavelet coefficients. The formalisms based on the Frisch-Parisi conjecture and on \mathcal{S}^ν spaces are recalled. Finally, we define the wavelet leaders of a function and we present the associated wavelet leaders method.

Chapter 5 consists in the presentation of a new multifractal formalism, the *leaders profile method*, based on the distribution of the wavelet leaders of the function. We show that this method yields an upper bound for the spectrum. Since it is defined through wavelet coefficients of the function, the independence from the sufficiently smooth wavelet basis which is chosen is a natural requirement. We prove that it is indeed the case for the leaders profile method. We illustrate then this formalism on classical models and in particular, for thresholded wavelet cascades whose spectra have a non-concave decreasing part. We end this chapter with a theoretical comparison of the leaders profile method with both the wavelet leaders method and the method based on \mathcal{S}^ν spaces. In particular, we prove that this new method gives a sharper upper bound than the previous approaches.

In Chapter 6, we present the function spaces, denoted \mathcal{L}^ν , which underlie the leaders profile method. These new spaces encapsulate the information supplied by the distribution of the wavelet leaders. We endow these spaces with a topology and obtain generic results about the form of the wavelet leaders profile of the functions in \mathcal{L}^ν : we show that the subset of functions in \mathcal{L}^ν whose wavelet leaders profile is equal to ν is large in \mathcal{L}^ν . While comparing with \mathcal{S}^ν spaces, the main difference is that now the profile includes an increasing and a decreasing part, and is therefore much more realistic for most multifractal models, but it implies that the \mathcal{L}^ν spaces are not vector spaces.

Finally, in Chapter 7, we construct functions with prescribed multifractal spectra which satisfy the leaders profile method. This construction is the first step toward the proof of the generic validity of this new method.

Part I

Generic results in classes of ultradifferentiable functions

Chapter 1

Notions of genericity

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1.1 Introduction

Historically, mathematicians have been confronted to objects which have properties that contradict their intuition. When such an object is found, a natural question is to ask whether many more similar ones may exist, i.e. whether the set of objects enjoying such a property is “large” in the considered space. In general, we will say that such a set is *generic*. Coming up with a concrete example of a special object can be difficult. Therefore, it may seem that there cannot be many functions of that kind. Actually, this statement is in general not true and a classical technique to prove that many such objects exist is to use the Baire category theorem. This gives a definition of genericity from a topological point of view.

Definition 1.1.1. If X is a Baire space, a subset $A \subseteq X$ is of *first category* (or *meager*) if it is included in a countable union of closed sets of X with empty interior. The complement of a set of first category is called *residual* (or *comeager*). Therefore, a set is residual if it contains a countable intersection of dense open sets of X .

The notion of sets of first category satisfies natural properties that one could expect for “small sets”, as presented in the following proposition.

Proposition 1.1.2.

- If A is a set of first category and if $B \subseteq A$, then B is of first category.
- Any countable union of sets of first category is of first category.
- If A is a set of first category, then A has empty interior.

In \mathbb{R}^n , the notion of sets with Lebesgue measure zero is a natural way to obtain “small sets”. However, the notion of sets of first category is purely topological and

in particular, it does not give any information about the measure of such a set. For example, the set of Liouville numbers is residual but has Lebesgue measure zero. In contrast, the set of Diophantine numbers is of the first category but has full Lebesgue measure in every interval [82].

In order to get a generalization of the notion of sets with Lebesgue measure zero, Christensen [53] introduced in 1972 the notion of prevalence, rediscovered in 1992 by Hunt et al. [82]. It is presented in the Section 1.2.

Finally, given an object which satisfies a special or unexpected property, one could also wonder if large algebraic structures of such objects can be constructed. This question is formalized in Section 1.3 where the notions of lineability and algebraability are presented.

1.2 Prevalence

In \mathbb{R}^n we say that a property holds almost everywhere if the set of points where it does not hold is included in a set with Lebesgue measure zero. The Lebesgue measure plays a particular role, due to the fact that this is the only σ -finite measure which is invariant by translation. However, in infinite dimensional spaces, such a measure does not exist as illustrated in the following proposition.

Proposition 1.2.1. [136] *In an infinite dimensional locally convex vector space E , there does not exist any non-zero σ -finite measure μ defined on the Borel sets which is quasi-invariant, i.e. such that for every Borel subset A ,*

$$\mu(A) = 0 \Rightarrow \mu(A + x) = 0 \quad \forall x \in E.$$

Since it is not possible to find an analogous to the Lebesgue measure in infinite dimensional spaces, one has to find another characterization of the Borel sets having a Lebesgue measure zero.

Proposition 1.2.2. *Let A be a Borel subset of \mathbb{R}^n . Then A has Lebesgue measure zero if and only if there exists a Borel probability measure μ on \mathbb{R}^n with compact support such that $\mu(A + x) = 0$ for every $x \in \mathbb{R}^n$.*

The proof of this result is straightforward. One can take μ equal to the Lebesgue measure on the unit cube. The converse implication is an application of Fubini's theorem.

This last characterization does not refer explicitly to the Lebesgue measure and can therefore easily be transposed in infinite dimensional spaces. This is the idea of the definition of prevalence.

Definition 1.2.3. [53, 82] A Borel subset A in a complete metrizable topological vector space E is *shy* (or *Haar-null*) if there exists a Borel probability measure μ on E with compact support such that $\mu(A + x) = 0$ for every $x \in E$. Such a measure μ is called *transverse* to A . A subset of E is *shy* if it is included in a shy Borel subset. The complement of a shy subset is called *prevalent*.

Remark 1.2.4. If the space E is a Polish space, it is known that the condition on the compact support is automatically satisfied (see [82] for example).

The following proposition gives basic properties of prevalence. It shows in particular that the notion of shyness satisfies properties that one could expect for “small sets”.

Proposition 1.2.5. [82] *Let E be a complete metrizable topological vector space.*

- *If $A \subseteq E$ is shy and if $B \subseteq A$, then B is shy.*
- *Any countable union of shy sets of E is shy.*
- *If $A \subseteq E$ is shy, then A has empty interior.*
- *If $A \subseteq E$ is shy, then $x + A$ is shy for every $x \in E$.*

There are two important techniques to prove that a set is prevalent. The first one is based on the construction of a probe. In \mathbb{R}^n , the Lebesgue measure is the best possible candidate to be transverse to a given Borel set. Therefore, when looking for a transverse measure in an infinite-dimensional space, a natural type of measure to try is the Lebesgue measure supported by some finite-dimensional subspace. This idea leads to the following definition.

Definition 1.2.6. Let P be a finite-dimensional subspace of a complete metrizable topological vector space E and let us denote by \mathcal{L}_P the Lebesgue measure supported by P . Then P is called a *probe* for a Borel subset B of E if $\mathcal{L}_P(x + B^c) = 0$ for every $x \in E$.

Therefore, a sufficient condition for a Borel subset to be prevalent is to have a probe (it suffices to consider the Lebesgue measure on the unit ball supported by P). As a particular case, we get the following useful result. It simply means that a proper vector subspace which is a Borel set is always shy.

Lemma 1.2.7. *If A is a non-empty Borel subset of E such that the complement of A is a vector subspace of E , then A is prevalent.*

Proof. A probe is given by the linear span of any element a of A . Indeed, since $B = E \setminus A$ is a vector subspace, for every $e \in E$, the set

$$\{\alpha \in \mathbb{R} : \alpha a + e \in B\}$$

contains only one element, so has Lebesgue measure zero. □

The second classical technique to prove that a set is prevalent is to use a stochastic process. We assume that E is a Polish space of complex-valued functions defined on \mathbb{R}^n . Let \mathcal{P} be a property satisfied by some elements of E and let A denote the set of those elements. Assume that there exists a stochastic process X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with values in E such that for every $f \in E$, $f + X$ has almost surely property \mathcal{P} . Then, if A is a Borel subset of E , we get that A is prevalent. Indeed, the measure law of the process is transverse to the complement of A .

Remark 1.2.8. In general, this technique is used in Polish spaces of functions. Indeed, the measure law of the stochastic process is not necessarily compactly supported.

Let us end this section by mentioning that in general, there are no correspondances between the notions of residuality and of prevalence.

Proposition 1.2.9. [118] *Every separable Banach space X can be decomposed into two sets U and V such that U is shy in X and V is of first category in X . In particular, U is shy and residual in X and V is of first category and prevalent in X .*

1.3 Algebraic genericity

For the last decade there has been an increasing interest toward the search for large algebraic structures of special objects. Given a property, we say that the subset M of functions which satisfy it is *lineable* if $M \cup \{0\}$ contains an infinite dimensional vector space (not necessarily closed). The concept of lineability was coined by V. I. Gurariy and it first appeared in [6]. In a more general framework we have the following.

Definition 1.3.1. [6] Let X be a vector space, M a subset of X , and κ a cardinal number.

- (1) The subset M is said to be κ -*lineable* if $M \cup \{0\}$ contains a vector space of dimension κ . At times, we shall be referring to the set M as simply *lineable* if the existing subspace is infinite dimensional.
- (2) We also let $\lambda(M)$ be the maximum cardinality (if it exists) of such a vector space.
- (3) When X is a topological vector space and when the above vector space can be chosen to be dense in X , we shall say that M is κ -*dense-lineable* (or, simply, *dense-lineable* if κ is infinite).

Remark 1.3.2.

- Let us recall that the $\lambda(M)$ from Definition 1.3.1 might actually not exist. It is not difficult to provide natural examples of sets which are n -lineable for every $n \in \mathbb{N}$ but which are not lineable. For instance [27], let $j_1 \leq k_1 < j_2 \leq \dots \leq k_m < j_{m+1} \leq \dots$ be positive integers and let $M = \cup_m \{\sum_{i=j_m}^{k_m} a_i x^i : a_i \in \mathbb{R}\}$. Since the sets $\{\sum_{i=j_m}^{k_m} a_i x^i : a_i \in \mathbb{R}\}$ ($m \in \mathbb{N}$) are pairwise disjoint, M is finitely (but not infinitely) lineable in $\mathcal{C}([0, 1])$, the set of continuous functions in $[0, 1]$.
- Let us mention that, in [22], the authors introduced the lineability number of a set M as follows

$$L(M) = \min\{\kappa : M \text{ is not } \kappa - \text{lineable}\}.$$

This number always exists and, whenever $\lambda(M)$ exists, one has $L(M) = \lambda(M)^+$ (the successor cardinal of $\lambda(M)$).

Since this concept appeared, it has attracted the attention of many authors who became interested in the study of subsets of functions enjoying certain special or, as they sometimes are called, “*pathological*” properties. For more details and several examples, we refer the reader to the nice review of Bernal-González et al. [35].

Let us also recall that, recently, Bernal-González [34] introduced the notion of *maximal lineable* (and that of *maximal dense-lineable*) meaning that, when keeping the above notation, the dimension of the existing vector space is equal to $\dim(X)$.

Let us now state a necessary condition on a lineable set to be dense-lineable. We will also present the proof of this result in order to show that it can be easily slightly generalized. Following Aron et al. [8], we introduce the notion of strong set.

Definition 1.3.3. Let A and B be subsets of a vector space X . We say that A is *stronger* than B if $A + B \subseteq A$.

Proposition 1.3.4. [8] Let E be a separable metrizable topological vector space and consider two subsets A and B of E such that A is lineable and B is dense-lineable in E . If A is stronger than B , then A is dense-lineable.

Proof. If $0 \in A$, then $B \subseteq A$ and the result is direct. So we can assume that $0 \notin A$. Let $(U_n)_{n \in \mathbb{N}}$ be a countable basis of convex balanced absorbing 0-neighborhoods in E . We may assume that $U_{n+1} + U_{n+1} \subseteq U_n$ for every $n \in \mathbb{N}$. We know that there exist infinite dimensional subspaces Y and Z of E satisfying $Y \subseteq A \cup \{0\}$, $Z \subseteq B \cup \{0\}$ and such that Z is dense in E . Since E is separable, we can find a countable subset $\{z_n : n \in \mathbb{N}\}$ of Z dense in E . Let also $\{y_n : n \in \mathbb{N}\}$ be a countable linearly independent family of Y . For every $n \in \mathbb{N}$, there is $\lambda_n > 0$ such that $\lambda_n y_n \in U_n$. We consider

$$W = \text{span}\{\lambda_n y_n + z_n : n \in \mathbb{N}\}.$$

From the construction, the set $\{\lambda_n y_n + z_n : n \in \mathbb{N}\}$ is dense in E , and therefore the same holds for W . Moreover, let $w \in W \setminus \{0\}$. Then, there are c_1, \dots, c_N not all equal to 0 such that

$$w = \sum_{n=1}^N c_n (\lambda_n y_n + z_n) = \sum_{n=1}^N c_n \lambda_n y_n + \sum_{n=1}^N c_n z_n \in (Y \setminus \{0\}) + Z.$$

It follows that $w \in (Y \setminus \{0\}) + Z \subseteq A + B \cup \{0\} \subseteq A$ and $W \subseteq A \cup \{0\}$. Finally, W is infinite dimensional. Indeed, assume that there are c_1, \dots, c_N not all equal to 0 such that

$$\sum_{n=1}^N c_n (\lambda_n y_n + z_n) = 0.$$

As done before, we get that $0 \in A$ hence a contradiction. □

Remark 1.3.5. If A is κ -lineable with $\kappa > \dim(\mathbb{N})$, the dense subspace can also be chosen with dimension κ . Indeed, keeping the notation of the previous proof, it suffices to consider

$$W = \text{span}\{\lambda(y)y + z(y) : y \in Y\},$$

where $\lambda(y) = \lambda_n$ and $z(y) = z_n$ if $y = y_n$, and $\lambda(y) = 1$ and $z(y) = 0$ if $y \notin \{y_n : n \in \mathbb{N}\}$. As previously, W is dense in E and $W \subseteq A \cup \{0\}$. Moreover, $\dim W = \kappa$ since W contains $Y \setminus \{y_n : n \in \mathbb{N}\}$.

Besides asking for vector subspaces one could also study other structures, such as algebras, which motivated the following concept.

Definition 1.3.6. [5, 7] Let \mathcal{A} be an algebra and \mathcal{B} be a subset of \mathcal{A} .

- (1) We say that \mathcal{B} is *algebrable* if there is a subalgebra \mathcal{C} of \mathcal{A} such that $\mathcal{C} \subseteq \mathcal{B} \cup \{0\}$ and the cardinality of any system of generators of \mathcal{C} is infinite.
- (2) When having \mathcal{A} endowed with a topology, we would say that \mathcal{B} is *dense-algebrable* if in addition \mathcal{C} can be taken dense in \mathcal{A} .
- (3) At times we shall say that \mathcal{B} is κ (dense)-algebrable if there exists a κ -generated (dense) subalgebra \mathcal{C} of \mathcal{A} with $\mathcal{C} \subseteq \mathcal{B} \cup \{0\}$ (where κ is some cardinal number).

Remark 1.3.7. We say that $X = \{x_\alpha : \alpha \in \Gamma\}$ is a minimal system of generators of \mathcal{C} if \mathcal{C} is the algebra generated by X and if for every $\alpha_0 \in \Gamma$, x_{α_0} does not belong to the algebra generated by $X \setminus \{x_{\alpha_0}\}$.

Some of the first examples of algebrable sets appeared in [5, 28]. Of course, any algebrable set is, automatically, lineable as well. In general, the converse is false. An example of this [27] can be the set of improper Riemann integrable functions on \mathbb{R} (see for example [141]) that are not Lebesgue integrable, denoted $\mathcal{R}(\mathbb{R}) \setminus \mathcal{L}(\mathbb{R})$. This set is lineable (see [71]) but it is also clearly not algebrable. Indeed, for every $f \in \mathcal{R}(\mathbb{R})$, either $f^2 \notin \mathcal{R}(\mathbb{R})$ or $f^2 = |f^2| \in \mathcal{R}(\mathbb{R})$ and, therefore, $f^2 \in \mathcal{L}(\mathbb{R})$.

Remark 1.3.8. As we did in Remark 1.3.2, one could also define the algebrability number by

$$\min\{\kappa : M \text{ is not } \kappa\text{-algebrable}\}.$$

A strengthened notion of algebrability was introduced by Bartoszewicz and Głab [21]. Let us recall that a subset X of a commutative algebra generates a *free subalgebra* if for each polynomial P without a constant term and any $x_1, \dots, x_n \in X$, we have $P(x_1, \dots, x_n) = 0$ if and only if $P = 0$ (that is, the set of all elements of the form $x_1^{k_1} \dots x_n^{k_n}$ where $x_1, \dots, x_n \in X$ and where $k_1, \dots, k_n \in \mathbb{N}_0$ are not all equal to 0, is linearly independent).

Remark 1.3.9. If $Z = \{z_\alpha : \alpha \in \Gamma\}$ is a minimal system of generators of \mathcal{C} , the set of generators does not necessary generate a free algebra. For example, let us consider the functions $f(x) = x^2$ and $g(x) = x^3$. They constitute a minimal system of generators for the algebra formed by the polynomials of degree greater than 2, but the non-zero polynomial $P(s, t) = s^3 - t^2$ is such that $P(f(x), g(x)) = 0$ for every x .

Definition 1.3.10. [21] Given a commutative algebra \mathcal{A} and a cardinal number κ , a subset $\mathcal{B} \subseteq \mathcal{A}$ is *strongly κ -algebrable* if there exists a κ -generated free algebra \mathcal{C} contained in $\mathcal{B} \cup \{0\}$. A subset $\mathcal{B} \subseteq \mathcal{A}$ is *strongly algebrable* if it is strongly κ -algebrable for an infinite κ and it is *densely strongly κ -algebrable* if it is strongly κ -algebrable and the respective free subalgebra is dense in \mathcal{A} , provided that \mathcal{A} is endowed with a topology.

Remark that there are subsets of algebra which are algebrable but not strongly algebrable. An example is given by the subset c_{00} of c_0 consisting of all sequences with real terms equal to 0 from some index. This set is algebrable in c_0 but not strongly 1-algebrable [21].

Let us present now a technique, the so-called exponential-like function method, which allows to get strong algebrability of some sets of functions defined on $[0, 1]$. This method was used in [70], rediscovered in [17] and very recently, studied in depth in [23].

Definition 1.3.11. We say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *exponential-like (of range m)* whenever f is given by

$$f(x) = \sum_{i=1}^m a_i e^{\beta_i x}, \quad x \in [0, 1]$$

for some distinct non-zero real numbers β_1, \dots, β_m and some non-zero real numbers a_1, \dots, a_m .

In [17], the authors proved a very useful property of exponential-like functions. Let us recall it here.

Lemma 1.3.12. [17] *For every $m \in \mathbb{N}$, every exponential-like function f of range m and every $c \in \mathbb{R}$, the preimage $f^{-1}(\{c\})$ has at most m elements. Consequently, f is not constant in every subinterval of \mathbb{R} . In particular, there exists a finite decomposition of \mathbb{R} into intervals such that f is strictly monotone in each of them.*

Proof. We proceed by induction. If $m = 1$, then there are $a \neq 0$ and $\beta \neq 0$ such that the function f is of the form $f(x) = ae^{\beta x}$, $x \in [0, 1]$. So f is strictly monotone and the property is obvious. Let us now assume that the property holds for all exponential-like functions of range m . Let $f(x) = \sum_{i=1}^{m+1} a_i e^{\beta_i x}$, $x \in [0, 1]$, for some distinct non-zero real numbers $\beta_1, \dots, \beta_{m+1}$ and some non-zero real numbers a_1, \dots, a_{m+1} . Remark that the derivative of f can be written as

$$f'(x) = e^{\beta_1 x} (\beta_1 a_1 + g(x))$$

where $g(x) = \sum_{i=2}^{m+1} \beta_i a_i e^{(\beta_i - \beta_1)x}$, $x \in [0, 1]$. Note that g is an exponential-like function of range m since $\beta_2 - \beta_1, \dots, \beta_{m+1} - \beta_1$ are distinct non-zero numbers. The induction hypothesis gives that $g^{-1}(\{-\beta_1 a_1\}) = (f')^{-1}(\{0\})$ has at most m elements and hence f has at most m local extrema on \mathbb{R} . This implies that for every $c \in \mathbb{R}$, the preimage $f^{-1}(\{c\})$ has at most $m + 1$ elements. \square

The technique recently developed in [17] is presented in the following proposition. Let us recall that a Hamel basis of \mathbb{R} is a basis of \mathbb{R} while considered as a \mathbb{Q} -vector space.

Proposition 1.3.13. [17] *Let $\mathcal{F} \subseteq \mathbb{R}^{[0,1]}$ and assume that there exists $F \in \mathcal{F}$ such that $f \circ F \in \mathcal{F} \setminus \{0\}$ for every exponential-like function f . Then \mathcal{F} is strongly \mathfrak{c} -algebrable. More precisely, if \mathcal{H} is a Hamel basis of \mathbb{R} , then the functions $\exp \circ (rF)$, $r \in \mathcal{H}$, are free generators of an algebra contained in $\mathcal{F} \cup \{0\}$.*

Proof. By assumption, we have that

$$\{\exp \circ (rF) : r \in \mathcal{H}\} \subseteq \mathcal{F}.$$

Let us show first that the subalgebra generated by this set is contained in \mathcal{F} . Consider $n \in \mathbb{N}$ and a polynomial P of n variables without any constant term. The function g given by

$$g(x) = P(e^{r_1 F(x)}, e^{r_2 F(x)}, \dots, e^{r_n F(x)}), \quad x \in [0, 1],$$

is of the form

$$\sum_{i=1}^m a_i (e^{r_1 F(x)})^{k_{i,1}} (e^{r_2 F(x)})^{k_{i,2}} \dots (e^{r_n F(x)})^{k_{i,n}} = \sum_{i=1}^m a_i \exp \left(F(x) \sum_{j=1}^n r_j k_{i,j} \right),$$

where we can assume that $r_1, \dots, r_n \in \mathcal{H}$ are distinct, a_1, \dots, a_m are non-zero real numbers and the matrix $(k_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$ has distinct non-zero rows with $k_{j,k} \in \mathbb{N}_0$. Since \mathcal{H} is a Hamel basis, the non-zero real numbers $\sum_{j=1}^n r_j k_{i,j}$, $i \in \{1, \dots, m\}$, are distinct. Then, the function

$$x \in [0, 1] \mapsto \sum_{i=1}^m a_i \exp \left(x \sum_{j=1}^n r_j k_{i,j} \right)$$

is exponential-like and the assumption gives that the function g belongs to $\mathcal{F} \setminus \{0\}$. Finally, the generated subalgebra is free since $g \neq 0$ if a_1, \dots, a_m are real numbers not all equal to 0. \square

Of course, the result is still valid if we consider a subset of $\mathbb{R}^{\mathbb{R}}$.

A set that is not lineable or algebrable can be viewed as “small” since it does not contain a large algebraic structure. It means that being non-lineable or non-algebrable is an algebraic notion of smallness. Those notions of smallness are different from the topological and probabilistic ones, as illustrated in the two following examples.

Example 1.3.14. Let us consider the subset A of $L^1([0, 1])$ composed of functions f such that $\int_0^1 f(x)dx \neq 0$. A probe for A is given by the one-dimensional space of all constant functions and therefore, A is prevalent in $L^1([0, 1])$ [82]. Moreover, from its definition, it is an open dense subset of $L^1([0, 1])$ and is therefore residual in this space. Nevertheless, A is not lineable. Indeed, assume that there exist $f, g \in L^1([0, 1])$ linearly independent such that $\text{span}\{f, g\} \subseteq A \cup \{0\}$. If we set

$$\alpha := -\frac{\int_0^1 f(x)dx}{\int_0^1 g(x)dx},$$

then $f + \alpha g$ is not identically 0 and does not belong to A , which is a contradiction.

Example 1.3.15. Let us consider a Hamel basis \mathcal{H} and a sequence $(\alpha_m)_{m \in \mathbb{N}}$ of different elements of \mathcal{H} . We define the functions e_α , $\alpha \in \mathcal{H}$, by setting $e_\alpha(x) := \exp(\alpha x)$ for every $x \in \mathbb{R}$, and the function f by $f(x) = \exp(x^2)$ for every $x \in \mathbb{R}$. For every $m \in \mathbb{N}$, we take $k_m > 0$ such that $\sup_{x \in [0, 1]} |k_m e_{\alpha_m}(x) f(x)| < \frac{1}{m}$. Let also $(P_m)_{m \in \mathbb{N}}$ be a sequence of polynomials whose elements form a dense subset of $\mathcal{C}([0, 1])$. Finally, we define $k_\alpha = 1$ and $P_\alpha = 0$ for $\alpha \in \mathcal{H} \setminus \{\alpha_m : m \in \mathbb{N}\}$. We consider the algebra \mathcal{A} generated by the set $\{P_\alpha + k_\alpha e_\alpha f : \alpha \in \mathcal{H}\}$. Then, \mathcal{A} is a \mathfrak{c} -generated free algebra which is dense in $\mathcal{C}([0, 1])$. To prove this, it suffices to use similar arguments that those which will be developed in Lemma 2.3.21 and Proposition 2.3.20, as well as the linear independence of the functions e_α , see Lemma 2.3.11. Therefore, \mathcal{A} is densely strongly \mathfrak{c} -algebrable. On the other hand, the set \mathcal{A} is included in the set of functions of $\mathcal{C}([0, 1])$ which are differentiable at some points of $[0, 1]$. It has long been known that the set of nowhere differentiable functions is residual in $\mathcal{C}([0, 1])$. This result was proved originally by Banach [18] and Mazurkiewicz [110]. Moreover, Hunt [81] proved that this set is also prevalent in $\mathcal{C}([0, 1])$. Consequently, \mathcal{A} is also shy and of first category.

Chapter 2

Genericity and Gevrey classes

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2.1 Introduction

In this chapter, we are interested in the class of functions which are infinitely continuously differentiable on an interval of the real line but which are nowhere analytic on this interval. Let us first recall that if f is a C^∞ function on an open interval containing x_0 , its Taylor series at x_0 is denoted by

$$T(f, x_0)(x) = \sum_{n=0}^{+\infty} \frac{D^n f(x_0)}{n!} (x - x_0)^n.$$

We say that f is *analytic* at x_0 if $T(f, x_0)$ converges to f on an open neighborhood of x_0 ; if this is not the case, we say that f has a *singularity* at x_0 . It is well known that the set of singularities of a function is closed, since a function which is analytic at a point is analytic in a neighborhood of this point (see [124] for example). A standard example of a C^∞ function which has a singularity at a point is given by the function defined on \mathbb{R} by

$$f(x) = \begin{cases} \exp\left(\frac{-1}{x^2}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

This function is infinitely continuously differentiable on \mathbb{R} and its derivative of any order at 0 is 0. Therefore, the Taylor series of this function at 0 converges at every point x , but not to $f(x)$ except for $x = 0$. Another example was given by du Bois Reymond [62]

in 1876 with the function

$$f(x) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^{2n}}{(2n)!(x^2 + a_n^2)},$$

where $(a_n)_{n \in \mathbb{N}}$ is any sequence that converges to 0. It is the first construction of a function whose Taylor series at 0 does not converge for any $x \neq 0$. The proof given by du Bois Reymond was not correct but the result has been shown to be right [37].

Those two functions illustrate the two only ways a function can fail to be analytic at a point: if f has a singularity at x_0 , either the radius of convergence of the series is 0 (i.e. the series only converges at x_0), or the series converges in some neighborhood of x_0 but the limit does not represent f , as small as you take the neighborhood of x_0 . Following Boas [40] and Ramsamujh [121], we say that x_0 is a *Pringsheim singularity* if the radius of convergence at x_0 is 0 and a *Cauchy singularity* in the other case. Let $PS(f)$ denote the set of Pringsheim singularities of f and $CS(f)$ denote the set of its Cauchy singularities. In 1893, Pringsheim [119] proved that $CS(f)$ is never very large; specifically, it cannot contain an interval. The exact structure of $CS(f)$ and $PS(f)$ was given by Zahorski [143] in 1947.

Proposition 2.1.1. [143] *Let A and B be two subsets of an interval (open or closed) I of \mathbb{R} . There exists $f \in C^\infty(I)$ with $PS(f) = A$ and $CS(f) = B$ if and only if*

- A is a G_δ subset;
- B is a F_σ subset of first category;
- $A \cup B$ is closed and $A \cap B = \emptyset$.

As a consequence, there exist no functions with a Cauchy singularity at each point of a given interval. This result had already been obtained by Boas [39] in 1935. Nevertheless, functions with a Pringsheim singularity at each point exist and in particular, functions with a singularity at each point of an interval exist. Such a function is called *nowhere analytic* on the interval. Note that in case of a closed interval $[a, b]$, the convergence of the Taylor series $T(f, a)$ and $T(f, b)$ is only considered on the restriction to $[a, b]$. The existence of nowhere analytic functions can be surprising but is known since the construction of du Bois Reymond [62]. Many other examples can be found in [125]. A very nice example was given by Cell erier [51] in 1890 with the function defined for all $x \in \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{+\infty} \frac{\sin(a^n x)}{n!}$$

where a is a positive integer larger than 1. This function has Pringsheim singularities at points of the form $x = (2\pi m)/a^k$ where m is an integer and $k \in \mathbb{N}$. These points are dense in \mathbb{R} and since the set of singularities of a function is a closed set, we get that f is nowhere analytic on \mathbb{R} . A similar example was given by Lerch [103]. Let us present in more details a last example, given by Kim and Kwon [98], which will be a nice source of inspiration in Section 2.3, see Proposition 2.3.18.

Proposition 2.1.2. [80, 98] *Consider the function Φ defined on \mathbb{R} by*

$$\Phi(x) = \begin{cases} \exp(-x^{-2}) \exp(-(x-1)^{-2}) & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the function Ψ defined on \mathbb{R} by

$$\Psi(x) = \sum_{j=1}^{+\infty} \frac{1}{j!} \Phi(2^j x - \lfloor 2^j x \rfloor)$$

is in $C^\infty(\mathbb{R})$ but is nowhere analytic.

Proof. The function Φ is in $C^\infty(\mathbb{R})$ and analytic at all points except 0 and 1. Moreover, it is flat at those two points, that is $D^k \Phi(0) = D^k \Phi(1) = 0$ for every $k \in \mathbb{N}_0$. Let us set $\Phi_j(x) := \Phi(2^j x - \lfloor 2^j x \rfloor)$ for every $x \in \mathbb{R}$, $j \in \mathbb{N}$. The behaviour of Φ over the interval $[0, 1]$ is replicated by Φ_j on any interval of the form $[\frac{m}{2^j}, \frac{m+1}{2^j}]$ for all $m \in \mathbb{Z}$. Because of the flatness of Φ at 0 and 1, we get that Φ_j is in $C^\infty(\mathbb{R})$. Moreover, $\sum_{j=1}^{+\infty} \frac{1}{j!} D^k \Phi_j(x)$ is uniformly convergent on \mathbb{R} for all $k \in \mathbb{N}$ and by Weierstraß theorem, it follows that the function Ψ belongs to $C^\infty(\mathbb{R})$.

Let us now prove that Ψ is nowhere analytic. Assume that Ψ is analytic at a point. Then it is analytic in a neighborhood of this point. Since the dyadic numbers form a dense subset in \mathbb{R} , Ψ is analytic at some $x_0 = \frac{m}{2^n}$, with m an odd integer and $n \in \mathbb{N}$. Since Φ_j is analytic at x_0 for $j \in \{1, \dots, n-1\}$, we get that the function

$$\tilde{\Psi} := \sum_{j=n}^{+\infty} \frac{1}{j!} \Phi_j$$

is also analytic at x_0 . However, $D^k \tilde{\Psi}(x_0) = 0$ for all $k \in \mathbb{N}_0$, which is a contradiction with the fact that $\tilde{\Psi}$ is positive in some punctured neighborhood of x_0 . \square

With those multiple examples in hands, a natural question is to ask whether the set of nowhere analytic functions is not only a non-empty family but even generic (using different notions). We will work in the classical context of functions defined on the interval $[0, 1]$. Nevertheless, the results can easily be adapted to the case of the real line \mathbb{R} .

In what follows, $\mathcal{C}^\infty([0, 1])$ denotes the vector space of all complex-valued¹ functions which are infinitely continuously differentiable on $(0, 1)$ such that the derivative of any order can be continuously extended to $[0, 1]$. We endow the space $\mathcal{C}^\infty([0, 1])$ with the sequence $(p_k)_{k \in \mathbb{N}_0}$ of semi-norms defined by

$$p_k(f) := \sup_{j \leq k} \sup_{x \in [0, 1]} |D^j f(x)|$$

or equivalently with the distance d defined by

$$d(f, g) := \sum_{k=0}^{+\infty} 2^{-k} \frac{p_k(f - g)}{1 + p_k(f - g)}.$$

This space is a Fréchet space.

In Section 2.2 we present a brief history of the results of genericity obtained in the context of nowhere analytic functions. Then, we show that the set of nowhere analytic functions is prevalent. This result is already mentioned in [134] but one of the arguments was the fact that

$$A(I, x_I) = \{f \in \mathcal{C}^\infty([0, 1]) : T(f, x_I) \text{ converges to } f \text{ on } I\}$$

¹Certain results only work in the real context. This will be clearly explicated.

(where I is a closed interval of $[0, 1]$ with x_I as center point) is closed in $\mathcal{C}^\infty([0, 1])$. However, this is not possible since the set of polynomials is included in $A(I, x_I)$ and also dense in $\mathcal{C}^\infty([0, 1])$.

In the Section 2.3, we define the notion of *nowhere Gevrey differentiability* and we examine the set of functions which are nowhere Gevrey differentiable. In this case, we also obtain generic results, from the three different points of view. Since analytic functions are a particular class of Gevrey type functions, these results generalize those obtained in the analytic case. However, we kept separated sections since analytic functions are somehow more classical than Gevrey-type ones and since the result about nowhere analytic functions directly brings a complement to an already mentioned one in literature. The results presented in this chapter are mainly from the articles [26] and [27].

2.2 Genericity of nowhere analytic functions

In this section, we present different results concerning the genericity of the set of nowhere analytic functions (or even about the set of functions with a Pringsheim singularity at each point) in $\mathcal{C}^\infty([0, 1])$. The first result is due to Morgenstern [115], in 1954. Several authors gave other proofs of this result, namely Darst [59] and Cater [49].

Proposition 2.2.1. [115] *The set of nowhere analytic functions is residual in $\mathcal{C}^\infty([0, 1])$.*

This implies in particular that this set is dense in $\mathcal{C}^\infty([0, 1])$. A deeper result concerning the set of functions with a Pringsheim singularity at each point was given by Salzmann and Zeller [125] in 1955. Another proof of this result was given by Bernal-González [32] (this result was also proved by Ramsamujh [121], but there is a gap in the proof, as mentioned in [33]).

Proposition 2.2.2. [125] *The set of functions with a Pringsheim singularity at each point is residual in $\mathcal{C}^\infty([0, 1])$.*

More recent results using the notion of genericity from the algebraic point of view have been proved. Those results can differ depending on whether the space $\mathcal{C}^\infty([0, 1])$ is the space of \mathcal{C}^∞ real-valued functions or complex-valued functions on $[0, 1]$. The first result is due to Bernal-González [33] in 2008.

Proposition 2.2.3. [33] *The set of functions with a Pringsheim singularity at each point is dense-lineable in $\mathcal{C}^\infty([0, 1])$. In the complex case, the dense-lineability is maximal. In the real case, the set of nowhere analytic functions is maximal dense-lineable.*

In 2012, Conejero et al. [57] constructed an algebra \mathcal{A} of real-valued functions enjoying simultaneously each of the following properties:

- \mathcal{A} is infinitely generated,
- every non-zero element of \mathcal{A} is nowhere analytic,
- $\mathcal{A} \subseteq \mathcal{C}^\infty(\mathbb{R})$,
- every non-zero element of \mathcal{A} has infinitely many zeros in \mathbb{R} ,
- for every $f \in \mathcal{A}$ and $n \in \mathbb{N}$, $D^n f$ is also in $\mathcal{C}^\infty(\mathbb{R})$, nowhere analytic and possesses infinitely many zeros in \mathbb{R} .

Let us mention here the construction of such an algebra. It follows the construction of Kim and Kwon [98] presented in Proposition 2.1.2. Let \mathcal{H} denote a Hamel basis of \mathbb{R} . We can assume that all the elements of \mathcal{H} are positive. As in Proposition 2.1.2, the function Φ is defined by

$$\Phi(x) = \begin{cases} \exp(-x^{-2}) \exp(-(x-1)^{-2}) & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

For every $\alpha \in \mathcal{H}$, consider the function ρ_α defined on \mathbb{R} by

$$\rho_\alpha(x) = \sum_{j=1}^{+\infty} \frac{\lambda_j(x)}{\mu_j} \Phi(2^j x - \lfloor 2^j x \rfloor) \alpha^j.$$

Here, λ_j is defined for every $j \in \mathbb{N}$ by

$$\lambda_j(x) = \begin{cases} 1 & \text{if } |x| \geq 2^{-j}, \\ 0 & \text{otherwise,} \end{cases}$$

and $\mu_j = (s_k)!$ if $s_{k-1} < j \leq s_k$, where s_k is the sum of the first k positive integers. The minimum algebra that contains the family of functions ρ_α , $\alpha \in \mathcal{H}$ has the desired properties.

In particular, the following result was obtained.

Proposition 2.2.4. [57] *The set of nowhere analytic functions is algebraic.*

Let us finish by mentioning this last very recent result of Bartoszewicz et al. [23] which use the exponential-like function method presented in Chapter 1. Let us recall first that if a function f is analytic and invertible, then its inverse is also analytic.

Proposition 2.2.5. [23] *The set of nowhere analytic functions is strongly-c-algebraic in $\mathcal{C}^\infty([0, 1])$.*

Proof. Let F be a nowhere analytic function and let f be an exponential-like function. Assume that $g = f \circ F$ is analytic at $x_0 \in [0, 1]$. Then, there is a neighborhood V of x_0 such that g is analytic in V . By Lemma 1.3.12, there is an open subset $V \subseteq F(V)$ on which f is invertible. Hence, $F = f^{-1} \circ g$ is analytic on $F^{-1}(V) \cap V$ as composition of analytic functions. This is a contradiction. \square

As announced, let us now prove the result concerning the prevalence of the set of nowhere analytic functions in $\mathcal{C}^\infty([0, 1])$.

Proposition 2.2.6. [26] *The set of nowhere analytic functions on $[0, 1]$ is a prevalent subset of $\mathcal{C}^\infty([0, 1])$.*

Proof. For any closed subinterval I of $[0, 1]$ and x_I the center point of I , let

$$A(I, x_I) = \{f \in \mathcal{C}^\infty([0, 1]) : T(f, x_I) \text{ converges to } f \text{ on } I\}.$$

Since a function which is analytic at a point is analytic in a neighborhood of this point, the set of nowhere analytic functions is the complement of the union of all $A(I, x_I)$ over rational subintervals $I \subseteq [0, 1]$. From Proposition 1.2.5, we know that any countable union of shy sets is shy and therefore, it is enough to prove that every $A(I, x_I)$ is shy.

Since $A(I, x_I)$ is a proper vector subspace of $\mathcal{C}^\infty([0, 1])$, using Lemma 1.2.7, this will be done if we show that it is a Borel set.

For any $j, n \in \mathbb{N}$, let

$$F_{n,j} = \bigcap_{x \in I} \left\{ f \in \mathcal{C}^\infty([0, 1]) : |T_j(f, x_I)(x) - f(x)| \leq \frac{1}{n} \right\},$$

where

$$T_j(f, x_I)(x) = \sum_{k=0}^j \frac{f^{(k)}(x_I)}{k!} (x - x_I)^k.$$

The definition of the topology of $\mathcal{C}^\infty([0, 1])$ and the fact that only a finite number of derivatives are involved directly imply that $F_{n,j}$ is closed in $\mathcal{C}^\infty([0, 1])$.

Using typical properties of power series, the convergence of $T(f, x_I)$ on I is equivalent to uniform convergence on I . Hence

$$A(I, x_I) = \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \bigcap_{j \geq k} F_{n,j},$$

and we conclude. \square

To conclude this section, let us emphasise that, as far as we know, the problem concerning the prevalence or the (dense-)algebraicity of the set of functions with a Pringsheim singularity at each point is still open.

2.3 Nowhere Gevrey differentiable functions

The Gevrey classes play an important role in the theory of the linear partial differential equations. They were introduced in order to classify the smoothness of \mathcal{C}^∞ functions according to how close they are to analytic functions. Gevrey classes are widely used as intermediates between the space of analytic functions and the space of infinitely differentiable functions. We refer to Rodino [122] for an introduction to this topic. Let us just mention the basic example which was the source of investigation of Gevrey [73], in 1918. It is given by the heat operator in \mathbb{R}^n , $n \geq 2$, defined by

$$L = \frac{\partial}{\partial x_n} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}.$$

A fundamental solution of this operator is given by a function E which belongs to $\mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$, but which is not analytic on $\{x \in \mathbb{R}^n : x_n = 0\}$. In order to characterize the regularity of this \mathcal{C}^∞ but not analytic function, he observed that for every compact subset K of \mathbb{R}^n , there exist two constants $C, h > 0$ such that

$$\sup_{x \in K} |D^\alpha E(x)| \leq Ch^{|\alpha|} (\alpha!)^2, \quad \forall \alpha \in \mathbb{N}_0^n.$$

A generalization of this observation leads to the following definition (see for example [54, 122]).

Definition 2.3.1. For a real number $s \geq 1$ and an open subset Ω of \mathbb{R}^n , an infinitely differentiable function f in Ω is said to be *Gevrey differentiable of order s at $x_0 \in \Omega$* if there exist a compact neighborhood K of x_0 and two constants $C, h > 0$ such that

$$\sup_{x \in K} |D^\alpha f(x)| \leq Ch^{|\alpha|} (\alpha!)^s, \quad \forall \alpha \in \mathbb{N}_0^n.$$

Remark 2.3.2. Let us mention that this definition extends naturally to the case $s < 1$. Nevertheless, in this case, the properties of quasianalyticity differ, see Chapter 3.1, Remark 3.2.11.

Clearly, if a function is Gevrey differentiable of order s at x_0 , it is also Gevrey differentiable of any order $s' > s$ at x_0 . Remark also that the case $s = 1$ corresponds to analyticity (by the Cauchy's estimate). We denote by $G^s(\Omega)$ the set of functions which are Gevrey differentiable of order s at every point of Ω . Let us mention some direct properties of this space.

Proposition 2.3.3. [122] *For every $s \geq 1$ and every open subset Ω of \mathbb{R} , $G^s(\Omega)$ is an algebra for the pointwise multiplication of functions. Moreover, it is closed under differentiation.*

In order to generalize the results about nowhere analyticity of the previous section, we introduce the following notion.

Definition 2.3.4. Given an interval I of \mathbb{R} , a function $f \in C^\infty(I)$ is *nowhere Gevrey differentiable* on I if f is not Gevrey differentiable of order s at x_0 , for any $x_0 \in I$ and $s \geq 1$, where the compact neighborhoods K are considered in I .

The existence of a nowhere Gevrey differentiable function is provided by the following lemma, where an explicit construction of such a function is proposed. Let us mention that a similar approach has been proposed by Bernal-González [32].

Lemma 2.3.5. [26] *Let $(\lambda_k)_{k \in \mathbb{N}}$ be a sequence of strictly positive numbers such that*

$$\lambda_k \geq (k+1)^{(k+1)^2} \quad \text{and} \quad \lambda_{k+1} \geq 2 \sum_{j=1}^k \lambda_j^{2+k-j}, \quad \forall k \in \mathbb{N}$$

and let f be the function defined on \mathbb{R} by

$$f(x) = \sum_{k=1}^{+\infty} c_k e^{i\lambda_k x} \quad \text{with} \quad c_k = \lambda_k^{1-k}, \quad k \in \mathbb{N}.$$

This function belongs to the class $C^\infty(\mathbb{R})$ and is nowhere Gevrey differentiable on \mathbb{R} .

Proof. For every $n, k \in \mathbb{N}$, we have $c_k \lambda_k^n = \lambda_k^{1+n-k}$. Hence the series

$$\sum_{k=1}^{+\infty} c_k \lambda_k^n e^{i\lambda_k x}$$

is uniformly convergent on \mathbb{R} . Using the Weierstraß theorem, f belongs to $C^\infty(\mathbb{R})$.

Moreover, for every $n \in \mathbb{N}$, $n \geq 2$ and $x \in \mathbb{R}$, we have

$$\begin{aligned}
 |D^n f(x)| &= \left| \sum_{k=1}^{n-1} \lambda_k^{n+1-k} e^{i\lambda_k x} + \lambda_n e^{i\lambda_n x} + \sum_{k>n} \lambda_k^{n+1-k} e^{i\lambda_k x} \right| \\
 &\geq \lambda_n - \sum_{k=1}^{n-1} \lambda_k^{n+1-k} - \sum_{k>n} \lambda_k^{n+1-k} \\
 &\geq \sum_{k=1}^{n-1} \lambda_k^{n+1-k} - \sum_{k>n} \lambda_k^{n+1-k} \\
 &\geq \lambda_{n-1}^2 - \sum_{j=0}^{+\infty} \frac{1}{\lambda_j} \\
 &\geq n^{2n^2} - e \geq \frac{1}{2} n^{2n^2}.
 \end{aligned}$$

Then, given strictly positive s, C, h , we have

$$n^{2n^2} = n^{n^2} (n^n)^n \geq Ch^n (n^n)^s \geq Ch^n (n!)^s$$

for n large enough. This proves that f is nowhere Gevrey differentiable on \mathbb{R} . \square

In this section, we will denote by NG the set of nowhere Gevrey differentiable functions on $[0, 1]$. We shall settle the question of how large the set NG is. First of all, we give a direct proof of the prevalence and of the maximal dense-lineability of NG in $\mathcal{C}^\infty([0, 1])$. To achieve this result we use *any* nowhere Gevrey differentiable function. However, to tackle the problem of algebraicity, a more precise knowledge of a very particular “key” function in NG is needed. Following some ideas from Chung and Chung [54], Kim and Kwon [98], Conejero et al. [57] (see Lemma 2.3.17 and Propositions 2.1.2 and 2.2.4), we construct a real-valued infinitely differentiable nowhere Gevrey differentiable function. This construction allows us to prove the maximal dense-algebraicity of the set of nowhere Gevrey differentiable functions in $\mathcal{C}^\infty([0, 1])$ (even in the real setting).

2.3.1 Prevalence and residuality of NG

Let us start by giving a characterization of NG that follows directly from its definition.

Lemma 2.3.6. [26] *The set NG is the complement of*

$$\bigcup_{s \in \mathbb{N}} \bigcup_{I \subseteq [0, 1]} B(s, I)$$

where I denotes a rational subinterval of $[0, 1]$ and $B(s, I)$ is the set of functions f of $\mathcal{C}^\infty([0, 1])$ for which there exist $C, h > 0$ such that

$$\sup_{x \in I} |D^n f(x)| \leq Ch^n (n!)^s, \quad \forall n \in \mathbb{N}_0.$$

The proof of the prevalence of NG in $\mathcal{C}^\infty([0, 1])$ uses the same arguments as in the analytic case. We use the notations introduced in the previous lemma.

Proposition 2.3.7. [26] *The set NG is a prevalent subset of $\mathcal{C}^\infty([0, 1])$.*

Proof. In a complete metric space, we know from Proposition 1.2.5 any countable union of shy sets is shy; so the result will be proved if we show that every $B(s, I)$ is shy. To get this, using Lemma 1.2.7, it suffices to prove that $B(s, I)$ is a proper vector subspace of $\mathcal{C}^\infty([0, 1])$ which is also a Borel set.

It is direct to see that $B(s, I)$ is a vector subspace of $\mathcal{C}^\infty([0, 1])$. Moreover, using Lemma 2.3.5, it is strictly included in $\mathcal{C}^\infty([0, 1])$. We also have

$$B(s, I) = \bigcup_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}_0} \left\{ f \in \mathcal{C}^\infty([0, 1]) : \sup_{x \in I} |D^n f(x)| \leq m^{n+1} (n!)^s \right\},$$

where

$$\left\{ f \in \mathcal{C}^\infty([0, 1]) : \sup_{x \in I} |D^n f(x)| \leq m^{n+1} (n!)^s \right\}$$

is closed in $\mathcal{C}^\infty([0, 1])$. Hence $B(s, I)$ is a Borel subset of $\mathcal{C}^\infty([0, 1])$. □

Now, let us show that the generic result also holds in the topological sense.

Proposition 2.3.8. [26] *The set NG is a residual subset of $\mathcal{C}^\infty([0, 1])$.*

Proof. We use the same definition as before for the set $B(s, I)$. So, as we have done previously, the set of nowhere Gevrey differentiable functions of $\mathcal{C}^\infty([0, 1])$ is the complement of

$$\bigcup_{s \in \mathbb{N}} \bigcup_{I \subseteq [0, 1]} B(s, I)$$

where I denotes a rational subinterval of $[0, 1]$. We also have

$$B(s, I) = \bigcup_{m \in \mathbb{N}} A(s, I, m)$$

where

$$A(s, I, m) = \left\{ f \in \mathcal{C}^\infty([0, 1]) : \sup_{x \in I} |D^n f(x)| \leq m^{n+1} (n!)^s, \forall n \in \mathbb{N}_0 \right\}.$$

To conclude, it suffices then to notice that the closed set $A(s, I, m)$ is a set with empty interior since it is included in $B(s, I)$ which is a proper vector subspace of the locally convex space $\mathcal{C}^\infty([0, 1])$. □

Let us remark that this last proposition has already been proved by Cater [50]. It can also be obtained as a special case of the following result of Bernal-González [33] (Lemma 2.1 and Remark 2.2).

Proposition 2.3.9. [33] *For each infinite set $M \subseteq \mathbb{N}_0$ and each sequence $(c_n)_{n \in \mathbb{N}_0}$ of strictly positive numbers, the family of functions $f \in \mathcal{C}^\infty([0, 1])$ for which there exist infinitely many $n \in M$ such that*

$$\max \{ |D^n f(x)|, |D^{n+1} f(x)| \} > c_n, \quad \forall x \in [0, 1]$$

is a residual subset of $\mathcal{C}^\infty([0, 1])$.

Indeed, for $c_n = (n!)^n$ and $M = \mathbb{N}_0$, this last family is contained in the set of nowhere Gevrey differentiable functions, since for any $s \in \mathbb{N}$ and $h, C > 0$, one has $(n!)^n > Ch^n (n!)^s$ for n sufficiently large.

Similarly, a generalization of Proposition 2.3.7 can be obtained.

Proposition 2.3.10. [26] For each sequence $(c_n)_{n \in \mathbb{N}_0}$ of strictly positive numbers, the set

$$\{f \in \mathcal{C}^\infty([0, 1]) : \forall I \subseteq [0, 1], \sup_{n \in \mathbb{N}_0} \frac{\sup_{x \in I} |D^n f(x)|}{c_n} = +\infty\}$$

is a prevalent subset of $\mathcal{C}^\infty([0, 1])$.

Proof. The complement of this set can be written as the countable union over rational subintervals I of $[0, 1]$ of the subset D_I defined by

$$D_I = \{f \in \mathcal{C}^\infty([0, 1]) : \sup_{n \in \mathbb{N}_0} \frac{\sup_{x \in I} |D^n f(x)|}{c_n} < +\infty\}.$$

Again, in a complete metric space, we know from Proposition 1.2.5 that any countable union of shy sets is shy. Therefore, it suffices to show that D_I is shy for each I . This is obtained as before, using Lemma 1.2.7. First, it is clear that D_I is a vector space. Moreover, using Proposition 2.3.9 with the sequence $(nc_n)_{n \in \mathbb{N}_0}$, D_I is strictly included in $\mathcal{C}^\infty([0, 1])$. Finally, it is a Borel subset of $\mathcal{C}^\infty([0, 1])$ since it can be written as

$$D_I = \bigcup_{k \in \mathbb{N}} \bigcap_{n \in \mathbb{N}_0} \{f \in \mathcal{C}^\infty([0, 1]) : \sup_{x \in I} |D^n f(x)| \leq kc_n\}$$

which is a countable union of closed sets of $\mathcal{C}^\infty([0, 1])$. \square

Taking again $c_n = (n!)^n$, we see that the set mentioned in the Proposition above is contained in the set of nowhere Gevrey differentiable functions and this proposition generalizes Proposition 2.3.7.

2.3.2 Maximal dense-lineability of NG

The aim of this section is to prove that the set NG is dense-lineable in $\mathcal{C}^\infty([0, 1])$ and that $\lambda(\text{NG}) = \mathfrak{c}$. The dense-lineability is, of course, a consequence of the dense-algebrability of NG in $\mathcal{C}^\infty([0, 1])$ (next subsection). Nevertheless, the dense-lineability is here directly obtained, using *any* function belonging to NG; this is the reason why we show it here as well, to illustrate the differences that one might encounter when dealing with dense-lineability and dense-algebrability.

In order to simplify notations, for every $\alpha \in \mathbb{R}$, we introduce the function e_α defined by $e_\alpha(x) := \exp(\alpha x)$, $x \in [0, 1]$. Let us start by mentioning this result of linear independence.

Lemma 2.3.11. [33] For each non empty subset $A \subseteq \mathbb{R}$, the functions e_α , $\alpha \in A$, are linearly independant.

Proof. Assume that it is not the case. Then there exist $N \geq 2$, $c_1, \dots, c_N \in \mathbb{C}$ with $c_N \neq 0$ and $\alpha_1 < \dots < \alpha_N$ in A such that $c_1 e_{\alpha_1} + \dots + c_N e_{\alpha_N} = 0$ on $[0, 1]$. From the analytic continuation principle, we obtain that the last equality holds on the whole line \mathbb{R} . We get that

$$c_N = -(c_1 e_{\alpha_1 - \alpha_N} + \dots + c_{N-1} e_{\alpha_{N-1} - \alpha_N})$$

which converges to 0 as x tends to $+\infty$. This is a contradiction and the conclusion follows. \square

Proposition 2.3.12. [27] *If f is nowhere Gevrey differentiable on \mathbb{R} , if $a_1, \dots, a_N \in \mathbb{C}$ are not all equal to 0 and if $\alpha_1 < \dots < \alpha_N$ are real numbers, then the function*

$$g = \sum_{j=1}^N a_j f e_{\alpha_j}$$

is nowhere Gevrey differentiable on \mathbb{R} .

Proof. Let us first remark that

$$g = \sum_{j=1}^N a_j f e_{\alpha_j} = f h$$

where $h := \sum_{j=1}^N a_j e_{\alpha_j}$. We proceed by contradiction. Let us assume there are $x_0 \in [0, 1]$ and $s > 1$ such that g is Gevrey differentiable of order s at x_0 . This implies that g is Gevrey differentiable of order s in some neighborhood V of x_0 . From Lemma 2.3.11, the functions $e_{\alpha_1}, \dots, e_{\alpha_N}$ are linearly independent and consequently, h is not identically equal to 0. Moreover, h is analytic on \mathbb{R} . It follows that there is $x_1 \in V$ such that $h(x_1) \neq 0$. Then, $\frac{1}{h}$ is analytic at x_1 and consequently, $f = \frac{g}{h}$ is Gevrey differentiable of order s at x_1 . This is a contradiction. \square

Next, let us show that the set NG is lineable and that its lineability dimension is the largest possible one.

Proposition 2.3.13. [27] *The set NG is lineable and $\lambda(\text{NG}) = \mathfrak{c}$.*

Proof. Let us fix a function $f \in \text{NG}$. We consider the subspace \mathcal{D} defined by

$$\mathcal{D} = \text{span}\{f e_{\alpha} : \alpha \in \mathbb{R}\}.$$

From Proposition 2.3.12, we just have to show that $\dim \mathcal{D} = \mathfrak{c}$. For this, it suffices to show that the functions $f e_{\alpha}$, $\alpha \in \mathbb{R}$, are linearly independent. Let us assume that it is not the case. Then there exist $c_1, \dots, c_N \in \mathbb{C}$ not all zero, and $\alpha_1 < \dots < \alpha_N$ in \mathbb{R} such that $c_1 f e_{\alpha_1} + \dots + c_N f e_{\alpha_N} = 0$ on $[0, 1]$, i.e. $f(c_1 e_{\alpha_1} + \dots + c_N e_{\alpha_N}) = 0$ on $[0, 1]$. Since the functions $e_{\alpha_1}, \dots, e_{\alpha_N}$ are linearly independent from Lemma 2.3.11, there exists $x \in [0, 1]$ such that $c_1 e_{\alpha_1}(x) + \dots + c_N e_{\alpha_N}(x) \neq 0$. By continuity, there exists a subinterval $J \subseteq [0, 1]$ such that $c_1 e_{\alpha_1} + \dots + c_N e_{\alpha_N} \neq 0$ on J . It follows that $f = 0$ on J , which is impossible since f is nowhere Gevrey differentiable. \square

In order to get the dense-lineability of NG, we use the condition presented in Chapter 1, Proposition 1.3.4.

Lemma 2.3.14. [27] *If \mathcal{P} denotes the set of polynomials, then NG is stronger than \mathcal{P} .*

Proof. Let us consider $g \in \text{NG}$ and a polynomial P . We proceed by contradiction. Assume that $g + P$ is Gevrey differentiable of order $s > 1$ at $x_0 \in [0, 1]$. Since P is analytic at x_0 , P is also Gevrey differentiable of order s at x_0 and the same holds for $g = (g + P) - P$ hence a contradiction. \square

With this result in hand, we can now infer the following.

Proposition 2.3.15. [27] *The set NG is \mathfrak{c} -dense-lineable in $\mathcal{C}^\infty([0, 1])$.*

Proof. It follows directly from Proposition 2.3.13, Lemma 2.3.14, Proposition 1.3.4 and Remark 1.3.5. \square

Let us end this section by showing that we can also get the strong \mathfrak{c} -algebrability of NG using any nowhere Gevrey differentiable function. The proof of this result employs the exponential-like function method presented in Chapter 1 and follows the lines of the proof of Proposition 2.2.5.

Proposition 2.3.16. *The set NG is strongly \mathfrak{c} -algebrable.*

Proof. Let F be a function of NG and let f be an exponential-like function. Assume that $g = f \circ F$ is Gevrey differentiable of order $s > 1$ at $x_0 \in [0, 1]$. Then, there is a neighborhood V of x_0 such that g is Gevrey differentiable of order s in V . By Lemma 1.3.12, there is an open subset $V \subseteq F(V)$ on which f is invertible. Hence, $F = f^{-1} \circ g$ is Gevrey differentiable of order s on $F^{-1}(V) \cap V$ as composition of Gevrey differentiable functions [142]. This is a contradiction. \square

2.3.3 Dense-algebrability of NG

The strategy to tackle the dense-algebrability problem will be different from that of the previous section. Here, we shall need a very particular NG function. We achieve this by means of a function defined as a series, in which the n^{th} term is built via a special function which is Gevrey differentiable of order n on \mathbb{R} .

For any $s > 1$, let f_s denote the function defined on \mathbb{R} by

$$f_s(x) = \begin{cases} \exp\left(-x^{-\frac{1}{s-1}}\right) & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.3.17. [54] *For any $s > 1$, the function f_s is Gevrey differentiable of order s on \mathbb{R} .*

Let us consider the function ψ_s defined on \mathbb{R} by

$$\psi_s(x) = f_s(x)f_s(1-x).$$

Since the space of Gevrey differentiable functions of order s is an algebra (see Proposition 2.3.3), the function ψ_s is Gevrey differentiable of order s on \mathbb{R} . Moreover, it is analytic on $(0, 1)$ since f_s is analytic on $(0, +\infty)$. It is clear that the support of ψ_s is $[0, 1]$ and we have $D^p\psi_s(0) = D^p\psi_s(1) = 0$ for every $p \in \mathbb{N}_0$ (i.e. ψ is flat at 0 and 1). Consequently, for every $n \geq 2$, there exist $D_n > 0$ and $h_n > 0$ such that

$$\sup_{x \in \mathbb{R}} |D^p\psi_n(x)| \leq D_n(h_n)^p(p!)^n, \quad \forall p \in \mathbb{N}_0.$$

Keeping the previous notation, we have:

Proposition 2.3.18. [27] *The function ρ defined by*

$$\rho(x) = \sum_{n=2}^{+\infty} C_n \psi_n(2^n x - \lfloor 2^n x \rfloor)$$

for every $x \in \mathbb{R}$, where $C_n = (D_n(h_n 2^n n!)^n)^{-1}$, is nowhere Gevrey differentiable on \mathbb{R} .

Proof. Because of the flatness of ψ_n at 0 and 1, as done in the proof of Proposition 2.1.2, the function $x \mapsto \psi_n(2^n x - \lfloor 2^n x \rfloor)$ belongs to $\mathcal{C}^\infty(\mathbb{R})$ for every $n \geq 2$. Moreover, for every p , from the choice of the coefficients C_n , the series

$$\sum_{n=2}^{+\infty} C_n 2^{np} \sup_{x \in \mathbb{R}} |D^p \psi_n(x)|$$

converges. Therefore, we obtain that the function ρ belongs to $\mathcal{C}^\infty(\mathbb{R})$.

Let us show that ρ is nowhere Gevrey differentiable. The set Q of all points of the form $2^{-m}k$, where $m \geq 3$ is a natural number and k is an odd number, is dense in \mathbb{R} . Therefore, it suffices to show that ρ is not Gevrey differentiable of any order at each point of Q . On the contrary, assume that ρ is Gevrey differentiable of order $s > 1$ at some point $x_0 \in Q$. Let $x_0 = 2^{-m_0}k_0$. Then for $n \in \{2, \dots, m_0 - 1\}$, the function $\psi_n(2^n x - \lfloor 2^n x \rfloor)$ is analytic at x_0 and hence Gevrey differentiable of order s at x_0 . Consequently, the function

$$\Theta_{m_0}(x) := \sum_{n=m_0}^{+\infty} C_n \psi_n(2^n x - \lfloor 2^n x \rfloor) = \rho(x) - \sum_{n=2}^{m_0-1} C_n \psi_n(2^n x - \lfloor 2^n x \rfloor)$$

is also Gevrey differentiable of order s at x_0 . Since Θ_{m_0} is periodic of period 2^{-m_0} , we can assume that $x_0 = 0$. Then, there exist $\varepsilon > 0$, $C > 0$ and $h > 0$ such that

$$\sup_{|x| \leq \varepsilon} |D^p \Theta_{m_0}(x)| \leq Ch^p (p!)^s, \quad \forall p \in \mathbb{N}_0.$$

Since each derivative of Θ_{m_0} at 0 is equal to 0, Taylor's formula gives that for every $x \in \mathbb{R}$ and every $p \in \mathbb{N}$, there exists a real number ξ between 0 and x such that

$$\Theta_{m_0}(x) = \frac{D^p \Theta_{m_0}(\xi)}{p!} x^p.$$

Then, we have

$$0 \leq \Theta_{m_0}(x) \leq Cx^p h^p (p!)^{s-1} \quad \forall p \in \mathbb{N}, \quad \forall 0 < x \leq \varepsilon,$$

and it follows that

$$0 \leq C_n \psi_n(2^n x - \lfloor 2^n x \rfloor) \leq Cx^p h^p (p!)^{s-1}$$

for every $p \in \mathbb{N}$, $n \geq m_0$ and $0 < x \leq \varepsilon$. Let us fix n large enough such that $n \geq s$, $n \geq m_0$ and $h2^{-n}e < 1$. For every $p \in \mathbb{N}$, we define then $x_p := 2^{-n}p^{-(n-1)}$. For p sufficiently large, we have $0 < x_p < \varepsilon$ and we obtain then

$$0 \leq C_n \psi_n(p^{-(n-1)}) \leq Ch^p 2^{-np} p^{-p(n-1)} (p!)^{s-1},$$

where $\psi_n(p^{-(n-1)}) = e^{-p} f_n(1 - p^{-(n-1)})$. Consequently, we have

$$C_n f_n(1 - p^{-(n-1)}) \leq Ch^p 2^{-np} e^p (p^{-p} p!)^{s-1}$$

for every p large enough. The left-hand side converges to $C_n f_n(1) = C_n e^{-1} > 0$ and the right-hand side converges to 0 when $p \rightarrow +\infty$. This leads to a contradiction. \square

The following proposition improves Proposition 2.3.12. It is the second key of the main result in this section.

Proposition 2.3.19. [27] *If F_1, \dots, F_N are analytic on \mathbb{R} and not all identically equal to 0, and if ρ is the function from Proposition 2.3.18, then the function*

$$g = \sum_{i=1}^N F_i \rho^i$$

is nowhere Gevrey differentiable on \mathbb{R} .

Proof. As previously, consider the set Q of all points of the form $2^{-m}k$, where $m \geq 3$ is a natural number and k is an odd number. Since Q is dense in \mathbb{R} , we just have to show that g is not Gevrey differentiable of any order at each point of Q . On the contrary, assume that g is Gevrey differentiable of order $s > 1$ at some point $x_0 = 2^{-m_0}k_0$.

Recall that we do not necessarily have flatness of ρ at x_0 . This is the reason why we set

$$A_{m_0}(x) := \sum_{n=2}^{m_0-1} C_n \psi_n(2^n x - \lfloor 2^n x \rfloor) \quad \text{and} \quad \Theta_{m_0}(x) := \sum_{n=m_0}^{+\infty} C_n \psi_n(2^n x - \lfloor 2^n x \rfloor)$$

for every $x \in \mathbb{R}$. Then, A_{m_0} is analytic at x_0 and Θ_{m_0} is flat at x_0 . Of course, we also have

$$\rho = A_{m_0} + \Theta_{m_0}$$

and it follows that

$$\begin{aligned} g(x) &= \sum_{i=1}^N F_i(x) (A_{m_0}(x) + \Theta_{m_0}(x))^i = \sum_{i=1}^N F_i(x) \sum_{j=0}^i \binom{i}{j} (A_{m_0}(x))^{i-j} (\Theta_{m_0}(x))^j \\ &= \sum_{i=1}^N F_i(x) (A_{m_0}(x))^i + \sum_{i=1}^N F_i(x) \sum_{j=1}^i \binom{i}{j} (A_{m_0}(x))^{i-j} (\Theta_{m_0}(x))^j \\ &= \sum_{i=1}^N F_i(x) (A_{m_0}(x))^i + \sum_{j=1}^N \left(\sum_{i=j}^N F_i(x) \binom{i}{j} (A_{m_0}(x))^{i-j} \right) (\Theta_{m_0}(x))^j \\ &= \sum_{i=1}^N F_i(x) (A_{m_0}(x))^i + \sum_{j=1}^N c_j(x) (\Theta_{m_0}(x))^j, \end{aligned}$$

where for every $j \in \{1, \dots, N\}$

$$c_j(x) := \sum_{i=j}^N F_i(x) \binom{i}{j} (A_{m_0}(x))^{i-j}.$$

Let us fix a neighborhood V of x_0 and let us show that there exists $j \in \{1, \dots, N\}$ such that c_j is not identically 0 in V . We proceed by contradiction. Assume that $c_j(x) = 0$

for every $j \in \{1, \dots, N\}$ and $x \in V$. This would mean that

$$\begin{pmatrix} 1 & \binom{2}{1}A_{m_0}(x) & \cdots & \binom{N}{1}(A_{m_0}(x))^{N-1} \\ 0 & 1 & \cdots & \binom{N}{2}(A_{m_0}(x))^{N-2} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \binom{N}{N-1}A_{m_0}(x) \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_N(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

for every $x \in V$. Since F_1, \dots, F_N are not all identically equal to 0, there is $x \in V$ and $j \in \{1, \dots, N\}$ such that $F_j(x) \neq 0$, which gives a contradiction since the matrix is invertible. Let k be the smallest element of $\{1, \dots, N\}$ for which c_k is not identically equal to 0 on V . Then, in this neighborhood, we have

$$g(x) = \sum_{i=1}^N F_i(x)(A_{m_0}(x))^i + \sum_{j=k}^N c_j(x)(\Theta_{m_0}(x))^j.$$

Since $\sum_{i=1}^N F_i(x)(A_{m_0}(x))^i$ is analytic at x_0 and since g is Gevrey differentiable of order s at x_0 , we have that the function

$$\Phi_{m_0}(x) := \sum_{j=k}^N c_j(x)(\Theta_{m_0}(x))^j$$

is also Gevrey differentiable of order s at x_0 . Then, there exist $\varepsilon > 0$, $C > 0$ and $h > 0$ such that

$$\sup_{|x-x_0| \leq \varepsilon} |D^p \Phi_{m_0}(x)| \leq Ch^p(p!)^s, \quad \forall p \in \mathbb{N}_0.$$

From the flatness of Θ_{m_0} at x_0 , we also get that Φ_{m_0} is flat at x_0 . Then, by Taylor's formula, for every $x \in \mathbb{R}$ and every $p \in \mathbb{N}$, there is ξ between x and x_0 such that

$$\Phi_{m_0}(x) = \frac{D^p \Phi_{m_0}(\xi)}{p!} (x - x_0)^p.$$

Consequently, we have

$$|\Phi_{m_0}(x)| \leq Ch^p(p!)^{s-1} |x - x_0|^p$$

for every x such that $|x - x_0| \leq \varepsilon$ and for every $p \in \mathbb{N}$.

Recall that the function c_k is analytic at x_0 and not identically equal to 0 in a neighborhood of x_0 . Thus, there exist $J \in \mathbb{N}_0$ and d_k analytic at x_0 with $d_k(x_0) \neq 0$ and such that

$$c_k(x) = (x - x_0)^J d_k(x)$$

in a neighborhood of x_0 . Let us fix $n \in \mathbb{N}$ such that $n > s$, $n \geq m_0$ and $he^k 2^{-n} < 1$.

As before, we consider $x_p := x_0 + 2^{-n} p^{-(n-1)}$ for every $p \in \mathbb{N}$. Then, on one hand, we have

$$\frac{\Phi_{m_0}(x_p)}{(\Theta_{m_0}(x_p))^k (x_p - x_0)^J} = d_k(x_p) + \sum_{j=k+1}^N c_j(x_p) \frac{(\Theta_{m_0}(x_p))^{j-k}}{(x_p - x_0)^J}$$

which converges to $d_k(x_0) \neq 0$ as p goes to infinity (the second term of the sum converges to 0 since Θ_{m_0} is flat at x_0). On the other hand, for p large enough, we have $|x_p - x_0| \leq \varepsilon$ and it follows that

$$|\Phi_{m_0}(x_p)| \leq Ch^p(p!)^{s-1}|x_p - x_0|^p.$$

Moreover, for p large enough, we have $2^n x_p - \lfloor 2^n x_p \rfloor = p^{-(n-1)}$ and $f_n(1 - p^{-(n-1)})$ converges to $f_n(1) = e^{-1} > 0$ as p goes to infinity. Therefore, we obtain that

$$\begin{aligned} \left| \frac{\Phi_{m_0}(x_p)}{(\Theta_{m_0}(x_p))^k (x_p - x_0)^J} \right| &\leq \frac{Ch^p(p!)^{s-1}|x_p - x_0|^p}{\left(C_n \psi_n(2^n x_p - \lfloor 2^n x_p \rfloor)\right)^k |x_p - x_0|^J} \\ &= \frac{Ch^p(p!)^{s-1} 2^{-n(p-J)} p^{-(p-J)(n-1)}}{\left(C_n e^{-p} f_n(1 - p^{-(n-1)})\right)^k} \\ &\leq \frac{C 2^{nJ}}{\left(C_n f_n(1 - p^{-(n-1)})\right)^k} \left(\frac{p!}{p^p}\right)^{n-1} p^{J(n-1)} (he^k 2^{-n})^p, \end{aligned}$$

which converges to 0 as p goes to infinity. This contradiction gives the conclusion. \square

Let \mathcal{H} denote a Hamel basis of \mathbb{R} . The ‘‘potential’’ candidate to obtain maximal algebraicity of NG is the minimum algebra \mathcal{A} which contains the family of the nowhere Gevrey functions ρe_α , with $\alpha \in \mathcal{H}$ and e_α defined as previously by $e_\alpha(x) = \exp(\alpha x)$, $x \in \mathbb{R}$. First of all, let us describe the structure of the elements in this algebra.

An element f of \mathcal{A} can be written as

$$f = \sum_{l=1}^L a_l \prod_{j=1}^J (\rho e_{\gamma_j})^{n(l,j)}$$

where $J, L \in \mathbb{N}$, $a_l \in \mathbb{R}$ for all $l \in \{1, \dots, L\}$, $\gamma_j \in \mathcal{H}$ for all $j \in \{1, \dots, J\}$ (with $\gamma_j \neq \gamma_{j'}$ if $j \neq j'$) and where $n(l, j) \in \mathbb{N}_0$ are such that $n(l, j) \neq n(l', j)$ for at least one j in case $l \neq l'$. For every l , we have

$$\prod_{j=1}^J (\rho e_{\gamma_j})^{n(l,j)} = \rho^{n_l} e_{\beta_l}, \quad \beta_l := \sum_{j=1}^J n(l, j) \gamma_j$$

where $n_l := \sum_{j=1}^J n(l, j) \in \mathbb{N}$ and where $\beta_l \neq \beta_{l'}$ if $l \neq l'$ because of the properties of the data. So we have

$$f = \sum_{l=1}^L a_l \rho^{n_l} e_{\beta_l} \quad \text{with} \quad \beta_l \neq \beta_{l'} \text{ if } l \neq l'.$$

We are now ready to state and prove the following result.

Proposition 2.3.20. [27] *The algebra \mathcal{A} is a \mathfrak{c} -generated free algebra contained in $\text{NG} \cup \{0\}$.*

Proof. Using Proposition 2.3.19 and the description of any element of \mathcal{A} here above, we directly get that $\mathcal{A} \subseteq \text{NG} \cup \{0\}$. Using the periodicity of ρ and the properties of Vandermonde determinants, let us show that the functions $\rho^{n_l} e_{\beta_l}$ are linearly independent.

We use the same notations as above. Let us now show that, if

$$f = \sum_{l=1}^L a_l \prod_{j=1}^J (\rho e_{\gamma_j})^{n(l,j)} = \sum_{l=1}^L a_l \rho^{n_l} e_{\beta_l} \quad \text{with} \quad \beta_l \neq \beta_{l'} \text{ if } l \neq l'$$

is identically 0, then $a_l = 0$ for every l .

Take $x_0 \in \mathbb{R}$ such that $\rho(x_0) \neq 0$ and consider the system obtained from the conditions $f(x_0 + l) = 0$ for $l \in \{0, \dots, L-1\}$. Using the periodicity of the function ρ , we have

$$A \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where

$$A = \begin{pmatrix} \rho^{n_1}(x_0)e^{\beta_1 x_0} & \rho^{n_2}(x_0)e^{\beta_2 x_0} & \dots & \rho^{n_L}(x_0)e^{\beta_L x_0} \\ \rho^{n_1}(x_0)e^{\beta_1(x_0+1)} & \rho^{n_2}(x_0)e^{\beta_2(x_0+1)} & \dots & \rho^{n_L}(x_0)e^{\beta_L(x_0+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \rho^{n_1}(x_0)e^{\beta_1(x_0+L-1)} & \rho^{n_2}(x_0)e^{\beta_2(x_0+L-1)} & \dots & \rho^{n_L}(x_0)e^{\beta_L(x_0+L-1)} \end{pmatrix}.$$

To conclude, it suffices to prove that the matrix A is non-singular. Up to the non-zero factor $\rho^{n_1+\dots+n_L}(x_0) e^{(\beta_1+\dots+\beta_L)x_0}$, the determinant of A is equal to the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ e^{\beta_1} & e^{\beta_2} & \dots & e^{\beta_L} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\beta_1(L-1)} & e^{\beta_2(L-1)} & \dots & e^{\beta_L(L-1)} \end{pmatrix}$$

which is a Vandermonde-type matrix. Since the e^{β_l} , $l \in \{1, \dots, L\}$ are different, we conclude. \square

In order to obtain the strong dense-algebrability of NG, we are now going to modify a little bit the definition of the previous algebra as explained in what follows. First we need some additional notations and a lemma.

Let $(\alpha_m)_{m \in \mathbb{N}}$ be a sequence of real numbers. Using the continuity of the multiplication by scalars, for every m , we take $k_m > 0$ such that $d(0, k_m e_{\alpha_m} \rho) < \frac{1}{m}$. Let also $(P_m)_{m \in \mathbb{N}}$ be a sequence of polynomials whose elements form a dense subset of $\mathcal{C}^\infty([0, 1])$.

Lemma 2.3.21. [27] *The family $\mathcal{G}_0 := \{P_m + k_m \rho e_{\alpha_m} : m \in \mathbb{N}\}$ is dense in $\mathcal{C}^\infty([0, 1])$.*

Proof. For every $f \in \mathcal{C}^\infty([0, 1])$ and for every m , we have

$$d(f, P_m + k_m e_{\alpha_m} \rho) \leq d(f, P_m) + d(0, k_m e_{\alpha_m} \rho) \leq d(f, P_m) + \frac{1}{m}.$$

Since there is a subsequence $M(k) \in \mathbb{N}$ ($k \in \mathbb{N}$) such that $\lim_k d(f, P_{M(k)}) = 0$, we conclude. \square

Now, take a sequence $(\alpha_m)_{m \in \mathbb{N}}$ of different elements of \mathcal{H} and define $k_\alpha = 1$, $P_\alpha = 0$ for $\alpha \in \mathcal{H} \setminus \{\alpha_m : m \in \mathbb{N}\}$. The ‘‘candidate’’ we are looking for is the algebra \mathcal{A}_d generated by

$$\mathcal{G} := \{P_\alpha + k_\alpha \rho e_\alpha : \alpha \in \mathcal{H}\}.$$

Theorem 2.3.22. [27] *The algebra \mathcal{A}_d is a \mathfrak{c} -generated free dense-algebra in $\mathcal{C}^\infty([0, 1])$ which is contained in $\text{NG} \cup \{0\}$. It follows that NG is densely strongly \mathfrak{c} -algebraable.*

Proof. Since the set of generators \mathcal{G} contains \mathcal{G}_0 , Lemma 2.3.21 provides the density. Let us show that $\mathcal{A}_d \subseteq \text{NG} \cup \{0\}$. An element $f \neq 0$ of \mathcal{A}_d can be written as

$$f = \sum_{l=1}^L a_l \prod_{j=1}^J \left(P_{\gamma_j} + k_{\gamma_j} e_{\gamma_j} \rho \right)^{n(l,j)}$$

where $J, L \in \mathbb{N}$, $a_l \in \mathbb{R} \setminus \{0\}$ for all $l \in \{1, \dots, L\}$, $\gamma_j \in \mathcal{H}$ for all $j \in \{1, \dots, J\}$ (with $\gamma_j \neq \gamma_{j'}$ if $j \neq j'$) and where $n(l, j) \in \mathbb{N}_0$ are such that $n(l, j) \neq n(l', j)$ for at least one j in case $l \neq l'$. As before, we set $\beta_l := \sum_{j=1}^J n(l, j) \gamma_j$ ($l \in \{1, \dots, L\}$) and we have $\beta_l \neq \beta_{l'}$ if $l \neq l'$.

For each $l \in \{1, \dots, L\}$, the term

$$\prod_{j=1}^J \left(P_{\gamma_j} + k_{\gamma_j} e_{\gamma_j} \rho \right)^{n(l,j)}$$

is a “polynomial” (with coefficients which are analytic functions) in the “variable” ρ ; the “degree” of this polynomial is $n_l = \sum_{j=1}^J n(l, j) \in \mathbb{N}$ and the coefficient of ρ^{n_l} is

$$c_l = \left(\prod_{j=1}^J k_{\gamma_j}^{n(l,j)} \right) e_{\beta_l}.$$

Let $N = \sup\{n_1, \dots, n_L\}$. The function f also appears as a “polynomial” (with coefficients which are analytic functions) in the “variable” ρ and the coefficient of the term with the highest power N is

$$F_N := \sum_{1 \leq l \leq L, n_l = N} a_l c_l = \sum_{1 \leq l \leq L, n_l = N} a_l \left(\prod_{j=1}^J k_{\gamma_j}^{n(l,j)} \right) e_{\beta_l}.$$

Since the coefficients a_l are not zero and since the β_l are different, F_N is not identically 0. We get that $f \in \text{NG}$ using Proposition 2.3.19 and the fact that the sum of a polynomial and a NG function is still a NG function. In particular, we have also obtained that \mathcal{A}_d is a \mathfrak{c} -generated free algebra. Indeed, if

$$\sum_{l=1}^L a_l \prod_{j=1}^J \left(P_{\gamma_j} + k_{\gamma_j} e_{\gamma_j} \rho \right)^{n(l,j)} = 0,$$

then $a_l = 0$ for all $l \in \{1, \dots, L\}$. Otherwise, as done previously, we would get $0 \in \text{NG}$, which is impossible. \square

Chapter 3

Classes of ultradifferentiable functions

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3.1 Introduction

Classes of ultradifferentiable functions are spaces of smooth functions which satisfy growth conditions on their derivatives. They are usually defined using weight sequences M or weight functions ω . We distinguish the classes $\mathcal{E}_{\{M\}}$ and $\mathcal{E}_{\{\omega\}}$ of ultradifferentiable functions of Roumieu type and the classes $\mathcal{E}_{(M)}$ and $\mathcal{E}_{(\omega)}$ of ultradifferentiable functions of Beurling type.

We will first work with the notion of ultradifferentiable classes defined using weight sequences. Such spaces are called Denjoy-Carleman classes. Let E be a Denjoy-Carleman class of ultradifferentiable functions of Beurling type on the real line \mathbb{R} that strictly contains another class F of Roumieu type. In this chapter, we investigate how large is the set of functions in the class E that are nowhere in the class F , i.e. such that the restriction of the function to any open subset of \mathbb{R} does not belong to this class. Then, we handle the same question but in the context of classes of ultradifferentiable

functions defined using weight functions, or equivalently imposing conditions on the Fourier-Laplace transform of the function.

An arbitrary sequence of positive real numbers $M = (M_k)_{k \in \mathbb{N}_0}$ is called a *weight sequence*. For every weight sequence M , every compact subset K of \mathbb{R}^n and every $h > 0$, we define the space $\mathcal{E}_{M,h}(K)$ as the space of functions $f \in \mathcal{C}^\infty(K)$ such that

$$\|f\|_{K,h}^M := \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}} < +\infty.$$

Endowed with the norm $\|\cdot\|_{K,h}^M$, the space $\mathcal{E}_{M,h}(K)$ is a Banach space.

Definition 3.1.1. Let Ω be an open subset of \mathbb{R}^n and M be a weight sequence. The space $\mathcal{E}_{\{M\}}(\Omega)$ is defined by

$$\mathcal{E}_{\{M\}}(\Omega) := \{f \in \mathcal{C}^\infty(\Omega) : \forall K \subseteq \Omega \text{ compact } \exists h > 0 \text{ such that } \|f\|_{K,h}^M < +\infty\}.$$

If $f \in \mathcal{E}_{\{M\}}(\Omega)$, we say that f is *M-ultradifferentiable of Roumieu type* on Ω . We obtain a locally convex topology on these spaces via the representation

$$\mathcal{E}_{\{M\}}(\Omega) = \underset{K \subseteq \Omega}{\text{proj}} \underset{h > 0}{\text{ind}} \mathcal{E}_{M,h}(K).$$

Fundamental examples of Roumieu spaces are given by the weight sequences $(k!)_{k \in \mathbb{N}_0}$ and $((k!)^s)_{k \in \mathbb{N}_0}$ with $s > 1$. They correspond respectively to the space of real analytic functions on Ω and the space of Gevrey differentiable functions of order s on Ω .

Definition 3.1.2. We say that a function is *nowhere in $\mathcal{E}_{\{M\}}$* if its restriction to any open and non-empty subset Ω of \mathbb{R} never belongs to $\mathcal{E}_{\{M\}}(\Omega)$.

Let us remark that similar results to those presented in Chapter 2 can be obtained with the class of functions of $\mathcal{C}^\infty([0, 1])$ which are nowhere in $\mathcal{E}_{\{M\}}$. Following Lemma 2.3.5, we consider a sequence $(\lambda_k)_{k \in \mathbb{N}}$ of strictly positive numbers such that

$$\lambda_k \geq \sqrt{M_{k+1}}(k+1)^{(k+1)^2} \quad \text{and} \quad \lambda_{k+1} \geq 2 \sum_{j=1}^k \lambda_j^{2+k-j}, \quad \forall k \in \mathbb{N}.$$

We construct the function f by setting

$$f(x) = \sum_{k=1}^{+\infty} c_k e^{i\lambda_k x} \quad \text{with } c_k = \lambda_k^{1-k}, \quad k \in \mathbb{N}$$

for every $x \in \mathbb{R}$. This function belongs to $\mathcal{C}^\infty(\mathbb{R})$ and is nowhere in $\mathcal{E}_{\{M\}}$. Using the same arguments as those presented in Propositions 2.3.7 and 2.3.8, we directly get that the set of functions of $\mathcal{C}^\infty([0, 1])$ which are nowhere in $\mathcal{E}_{\{M\}}$ is prevalent and residual in $\mathcal{C}^\infty([0, 1])$. This result can also be seen as a consequence of Proposition 2.3.9 and 2.3.10 with $c_n = M_n n^n$.

Let us now introduce the second type of Denjoy-Carleman classes.

Definition 3.1.3. Let Ω be an open subset of \mathbb{R}^n and M be a weight sequence. The space $\mathcal{E}_{(M)}(\Omega)$ is defined by

$$\mathcal{E}_{(M)}(\Omega) := \{f \in \mathcal{C}^\infty(\Omega) : \forall K \subseteq \Omega \text{ compact}, \forall h > 0, \|f\|_{K,h}^M < +\infty\}.$$

If $f \in \mathcal{E}_{(M)}(\Omega)$, we say that f is M -ultradifferentiable of Beurling type on Ω and we use the representation

$$\mathcal{E}_{(M)}(\Omega) = \text{proj}_{\overline{K} \subseteq \Omega} \text{proj}_{\overline{h} > 0} \mathcal{E}_{M,h}(K)$$

to endow $\mathcal{E}_{(M)}(\Omega)$ with a structure of Fréchet space.

This chapter is based on the article [63] and is structured as follows. In Section 3.2, we present basic properties of Denjoy-Carleman classes and we study the existing inclusions between these spaces. We also introduce the notion of quasianalyticity. In Section 3.3, given two weight sequences N and M such that $\mathcal{E}_{\{M\}}(\mathbb{R})$ is strictly included in $\mathcal{E}_{(N)}(\mathbb{R})$ and such that M is non-quasianalytic, we construct a function of $\mathcal{E}_{(N)}(\mathbb{R})$ which is nowhere in $\mathcal{E}_{\{M\}}$. We obtain then generic results about the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$. We extend this result using any countable union of Roumieu classes included in $\mathcal{E}_{(N)}(\mathbb{R})$. An application to the classes of Gevrey differentiable functions is given. In Section 3.4, the same question is handled but working with ultradifferentiable functions defined imposing conditions on the Fourier-Laplace transform of the function. Finally, in Section 3.5, we present new spaces of ultradifferentiable functions defined with weight matrices and we generalize the results presented in the previous sections.

3.2 Properties of Denjoy-Carleman classes

In this section, we consider that the dimension is $n = 1$. This section is divided in four parts. In the first and second parts, we present classical conditions on weight sequences and we see what it implies on the corresponding Denjoy-Carleman spaces. It can be resumed as follows:

- If the weight sequence M is logarithmically convex, then the space $\mathcal{E}_{\{M\}}(\Omega)$ is an algebra.
- If the weight sequence M is non-quasianalytic, then given an open subset Ω of \mathbb{R} and a compact $K \subseteq \Omega$, there exists a function of $\mathcal{E}_{\{M\}}(\mathbb{R})$ having a compact support included in Ω and being identically equal to 1 in K .

In the third part of this section, we study inclusions between Denjoy-Carleman spaces. These inclusions can be characterized by relations on weight sequences, defined as follows:

$$\begin{cases} M \preceq N & \iff \sup_{k \in \mathbb{N}} \left(\frac{M_k}{N_k} \right)^{\frac{1}{k}} < +\infty, \\ M \triangleleft N & \iff \lim_{k \rightarrow +\infty} \left(\frac{M_k}{N_k} \right)^{\frac{1}{k}} < +\infty. \end{cases}$$

Of course, for any open subset Ω of \mathbb{R} , if $M \preceq N$, then $\mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{\{N\}}(\Omega)$ and $\mathcal{E}_{(M)}(\Omega) \subseteq \mathcal{E}_{(N)}(\Omega)$. Moreover, if $M \triangleleft N$, then $\mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{(N)}(\Omega)$. Actually, all converse implications are true, using the assumption that the weight sequence M is log-convex. We also show that in the case $M \triangleleft N$, the inclusion is even strict.

In the last part, we present a result of separability of the Denjoy-Carleman classes. In order to simplify notations, if a statement is true for both the Roumieu space $\mathcal{E}_{\{M\}}(\Omega)$ and the Beurling space $\mathcal{E}_{(M)}(\Omega)$, we write $\mathcal{E}_{[M]}(\Omega)$. Since we have $\mathcal{E}_{[M]}(\Omega) = \mathcal{E}_{\{M'\}}(\Omega)$ where $M'_k = \frac{M_k}{M_0}$ for every k , we will always assume that any weight sequence M is such that $M_0 = 1$.

3.2.1 Log-convex weight sequences

An important condition usually imposed on weight sequences is the logarithm convexity.

Definition 3.2.1. A weight sequence M is *logarithmically convex* (or shortly *log-convex*) if $M_k^2 \leq M_{k-1}M_{k+1}$ for every $k \in \mathbb{N}_0$.

This condition means that the sequence $(\log(M_k))_{k \in \mathbb{N}_0}$ is convex. Let us remark that an equivalent condition is that the sequence $\left(\frac{M_k}{M_{k-1}}\right)_{k \in \mathbb{N}_0}$ is increasing. It follows that if M is log-convex, one has $M_k M_l \leq M_{k+l}$ for every $k, l \in \mathbb{N}_0$. Indeed, first we have $M_0 M_l \leq M_{0+l}$ for every $l \in \mathbb{N}_0$ since $M_0 = 1$. Using a simple induction on k , we get

$$M_k M_l = \frac{M_k}{M_{k-1}} M_{k-1} M_l \leq \frac{M_k}{M_{k-1}} M_{k-1+l} \leq \frac{M_{k+l}}{M_{k+l-1}} M_{k-1+l} = M_{k+l}$$

for every $l \in \mathbb{N}_0$. The following result follows directly (see [100] for example).

Proposition 3.2.2. *If the weight sequence M is log-convex, the space $\mathcal{E}_{[M]}(\Omega)$ is an algebra for the pointwise multiplication of functions.*

Proof. Let f, g be two functions of $\mathcal{E}_{[M]}(\Omega)$ and let K be a compact of Ω . Then, there exist $h_1, h_2 > 0$ and $C_1, C_2 > 0$ such that

$$\sup_{x \in K} |D^j f(x)| \leq C_1 h_1^j M_j \quad \text{and} \quad \sup_{x \in K} |D^j g(x)| \leq C_2 h_2^j M_j$$

for every $j \in \mathbb{N}$. By Leibnitz' rule, we have

$$\begin{aligned} |D^j(fg)(x)| &\leq \sum_{k \leq j} \binom{j}{k} |D^k f(x)| |D^{j-k} g(x)| \\ &\leq \sum_{k \leq j} \binom{j}{k} C_1 h_1^k M_k C_2 h_2^{j-k} M_{j-k} \\ &= C_1 C_2 (h_1 + h_2)^j M_j \end{aligned}$$

for every $x \in K$ and every $j \in \mathbb{N}$. The proof of the Beurling case is similar. \square

Let us state this other simple result about log-convex weight sequences (see [126] for example).

Proposition 3.2.3. *If the weight sequence M is log-convex, the sequence $((M_k)^{\frac{1}{k}})_{k \in \mathbb{N}}$ is increasing.*

Proof. It suffices to prove that $\log(M_{k+1}) \geq \frac{k+1}{k} \log(M_k)$ for every $k \geq 1$. Let us remark that since M is log-convex, we have

$$2 \log(M_k) \leq \log(M_{k+1}) + \log(M_{k-1}), \quad \forall k \geq 1.$$

We will prove the result by induction on k . For $k = 1$, since $M_0 = 1$, we have directly $2 \log(M_1) \leq \log(M_2) + \log(M_0) = \log(M_2)$. Moreover, if the result is true for $k - 1$, we get

$$2 \log(M_k) \leq \log(M_{k+1}) + \log(M_{k-1}) \leq \log(M_{k+1}) + \frac{k-1}{k} \log(M_k)$$

and it follows that $\frac{k+1}{k} \log(M_k) \leq \log(M_{k+1})$. \square

Let us end this section by the construction of the largest log-convex minorant of a given weight sequence [79]. See [126] for a proof.

Proposition 3.2.4. *The sequence M^c defined by*

$$\begin{cases} M_0^c & := M_0 = 1 \\ M_j^c & := \inf \left\{ M_k^{\frac{l-j}{l-k}} M_l^{\frac{j-k}{l-k}} : k \leq j \leq l, k \neq l \right\} \end{cases}$$

is the largest log-convex minorant (for \leq) of the sequence M .

Remark 3.2.5. Another construction is given by the so-called method of regularization of a sequence [107]. For every weight sequence M such that $(M_k)^{\frac{1}{k}}$ tends to infinity as k tends to infinity, we set

$$T_M(t) = \sup_{k \in \mathbb{N}_0} \frac{t^k}{M_k}, \quad t > 0, \quad \text{and} \quad M_k^{b(c)} = \sup_{t > 0} \frac{t^k}{T_M(t)}, \quad k \in \mathbb{N}_0.$$

The sequence $M^{b(c)}$ is the largest log-convex minorant of M .

3.2.2 Quasianalyticity

If f is an analytic function on an open interval I of \mathbb{R} and if $x_0 \in I$, the values of f in a neighborhood of x_0 are completely determined by the derivatives $D^n f(x_0)$ ($n \in \mathbb{N}_0$) of the function at x_0 . In particular, if there is $x_0 \in I$ such that all derivatives of f at x_0 are equal to 0, the function is identically equal to 0 on I . On the opposite, there exist non-zero infinitely continuously differentiable functions which are identically equal to 0 on an interval. So, there is a unicity property on the class of analytic functions which is not true in general for infinitely differentiable functions. For analytic functions, Cauchy's estimates gives growth conditions on the derivatives. A natural question is to ask whether this property of unicity is due to a good control of the growth of the derivatives. The Denjoy-Carleman theorem presented in this section gives an answer to this question. Let us start by giving some definitions.

Definition 3.2.6. Let I be an open interval of \mathbb{R} . A class $\mathcal{E}_{[M]}(I)$ is *quasianalytic* if 0 is its unique function f for which there is a point $x \in I$ such that $D^n f(x) = 0$ for every $n \in \mathbb{N}_0$. If this is not the case, we say that the class $\mathcal{E}_{[M]}(I)$ is *non-quasianalytic*.

Quasianalytic classes are classes of functions for which the statement of unicity mentioned above is true. One has the following result (see for example [124]).

Proposition 3.2.7. *A class $\mathcal{E}_{[M]}(\mathbb{R})$ is quasianalytic if and only if $\mathcal{E}_{[M]}(\mathbb{R})$ does not contain any function not identically zero with compact support.*

Proof. Of course, if $\mathcal{E}_{[M]}(\mathbb{R})$ is quasianalytic, it does not contain any function not identically zero with compact support. Assume that $\mathcal{E}_{[M]}(\mathbb{R})$ is non-quasianalytic, we can find $f \in \mathcal{E}_{[M]}(\mathbb{R})$ such that there are $x_0, x_1 \in \mathbb{R}$ with $D^n f(x_0) = 0$ for every $n \in \mathbb{N}_0$ and $f(x_1) \neq 0$. Let us consider the case where $x_1 > x_0$. We consider the function g defined on \mathbb{R} by

$$g(x) = \begin{cases} f(x) & \text{if } x \geq x_0, \\ 0 & \text{if } x < x_0. \end{cases}$$

Then g is infinitely continuously differentiable in \mathbb{R} and one directly checks that g belongs to $\mathcal{E}_{[M]}(\mathbb{R})$. Let us set

$$h(x) = g(x)g(2x_1 - x), \quad \forall x \in \mathbb{R}.$$

By Proposition 3.2.2, $h \in \mathcal{E}_{[M]}(\mathbb{R})$. Moreover, h is compactly supported since $h(x) = 0$ if $x < x_0$ or if $x > 2x_1 - x_0$. Finally, $h(x_1) = f(x_1)^2 \neq 0$ and this concludes the proof. \square

Corollary 3.2.8. *If the class $\mathcal{E}_{[M]}(\mathbb{R})$ is non-quasianalytic, given an open interval I of \mathbb{R} and a compact $K \subseteq I$, there exists a function of $\mathcal{E}_{\{M\}}(\mathbb{R})$ having a compact support included in I and being identically equal to 1 in K .*

Proof. By Proposition 3.2.7, there is $f \in \mathcal{E}_{[M]}(\mathbb{R})$ not identically zero with compact support. Consider a constant $a > 0$ such that the support of f is included in $[-a, a]$. Up to a multiplication by a constant, we can suppose that $\int_{\mathbb{R}} f(x)dx = 1$. For every $\varepsilon > 0$, let us set $f_\varepsilon(x) = \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}\right)$, $x \in \mathbb{R}$. Then, the support of f_ε is included in $[-a\varepsilon, a\varepsilon]$. For every $\varepsilon > 0$, consider the compact

$$K_\varepsilon := K + [-a\varepsilon, a\varepsilon]$$

and its indicator function χ_{K_ε} . Using typical properties of the convolution product, the function $f_\varepsilon * \chi_{K_\varepsilon}$ has its support included in $K + [-2a\varepsilon, 2a\varepsilon] \subseteq I$ if $\varepsilon > 0$ is small enough. Moreover, $f_\varepsilon * \chi_{K_\varepsilon}$ belongs to $\mathcal{C}^\infty(\mathbb{R})$ and $D^n(f_\varepsilon * \chi_{K_\varepsilon})(x) = (D^n f_\varepsilon * \chi_{K_\varepsilon})(x)$ for every $n \in \mathbb{N}_0$. Therefore, for every $n \in \mathbb{N}_0$ and every $x \in \mathbb{R}$,

$$\begin{aligned} |D^n(f_\varepsilon * \chi_{K_\varepsilon})(x)| &= \left| \int_{\mathbb{R}} D^n f_\varepsilon(x-y) \chi_{K_\varepsilon}(y) dy \right| \\ &\leq \sup_{\mathbb{R}} |D^n f_\varepsilon| \int_{K_\varepsilon} dy \\ &= \frac{1}{\varepsilon^{n+1}} \sup_{\mathbb{R}} |D^n f| \int_{K_\varepsilon} dy \end{aligned}$$

and since $f \in \mathcal{E}_{[M]}(\mathbb{R})$ it follows easily that $f_\varepsilon * \chi_{K_\varepsilon} \in \mathcal{E}_{[M]}(\mathbb{R})$. It remains to show that $f_\varepsilon * \chi_{K_\varepsilon}$ is identically equal to 1 on K . Let us fix $x \in K$. Then the support of the function $f_\varepsilon(x - \cdot)$ is included in K_ε and it follows that

$$f_\varepsilon * \chi_{K_\varepsilon}(x) = \int_{\mathbb{R}} f_\varepsilon(x-y) \chi_{K_\varepsilon}(y) dy = \int_{\mathbb{R}} f_\varepsilon(x-y) dy = 1,$$

hence the conclusion. \square

Let us now state the announced result of Denjoy [61] and Carleman [48]. Let us recall that M^c denotes the largest log-convex minorant of M (see Proposition 3.2.4). We also set $L_0 = 1$ and $L_k = \inf_{j \geq k} M_j^{1/j}$ for every $k \in \mathbb{N}$. The weight sequence L is the largest increasing minorant of $(M_k^{1/k})_{k \in \mathbb{N}}$.

Theorem 3.2.9 (Denjoy-Carleman). *Let M be a weight sequence and let I be an open interval of \mathbb{R} . The following conditions are equivalent:*

1. $\mathcal{E}_{\{M\}}(I)$ is quasianalytic,

2. $\sum_{k=0}^{+\infty} \frac{1}{L_k} = +\infty,$

3. $\sum_{k=1}^{+\infty} \frac{M_{k-1}^c}{M_k^c} = +\infty,$

$$4. \sum_{k=1}^{+\infty} (M_k^c)^{-1/k} = +\infty.$$

A contemporary proof of this result can be found in [79]. In view of this theorem, the following definition is natural.

Definition 3.2.10. If one of the equivalent conditions of Theorem 3.2.9 is satisfied, we say that the weight sequence M is *quasianalytic*. If this is not the case, we say that the sequence is *non-quasianalytic*.

Example 3.2.11. Consider the weight sequences $\mathbf{k}^s = ((k!)^s)_{k \in \mathbb{N}_0}$ with $s > 0$. The Roumieu space $\mathcal{E}_{\{\mathbf{k}^s\}}(\Omega)$ corresponds to the class of Gevrey differentiable functions of order s on Ω . We get that the class $\mathcal{E}_{\{\mathbf{k}^s\}}(\Omega)$ is quasianalytic if and only if $s \leq 1$.

3.2.3 Inclusions between Denjoy-Carleman classes

In this section, we study inclusions that exist between Denjoy-Carleman classes defined on an open subset Ω of \mathbb{R} . Of course, for every weight sequence M , we have the inclusion $\mathcal{E}_{(M)}(\Omega) \subseteq \mathcal{E}_{\{M\}}(\Omega)$. Moreover, conditions on two weight sequences M and N to have the inclusion $\mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{\{N\}}(\Omega)$ and $\mathcal{E}_{[M]}(\Omega) \subseteq \mathcal{E}_{[N]}(\Omega)$ are known and presented in this subsection. Let us start by defining some relations on weight sequences, using the notations of Rainer and Schindl [120].

Definition 3.2.12. Given two weight sequences M and N , we write $M \preceq N$ if there exist $C > 0$ and $\rho > 0$ such that $M_k \leq C\rho^k N_k$ for every $k \in \mathbb{N}_0$. Therefore, we have

$$M \preceq N \iff \sup_{k \in \mathbb{N}} \left(\frac{M_k}{N_k} \right)^{\frac{1}{k}} < +\infty.$$

If $M \preceq N$ and $N \preceq M$, we write $M \approx N$.

From the definition, it is clear that the relation \preceq is reflexive and transitive. Therefore, the relation \approx is an equivalence relation on the set of weight sequences.

It is clear that if $M \preceq N$, then $\mathcal{E}_{[M]}(\Omega) \subseteq \mathcal{E}_{[N]}(\Omega)$. We will see that, up to an additional assumption on M , the converse implication is also true. Let us first present a construction of Thilliez [138].

Lemma 3.2.13. [138] *Let M be a log-convex weight sequence and θ be the function defined on \mathbb{R} by*

$$\theta(x) = \sum_{k=1}^{+\infty} \frac{M_k}{2^k} \left(\frac{M_{k-1}}{M_k} \right)^k \exp \left(2i \frac{M_k}{M_{k-1}} x \right).$$

Then $\theta \in \mathcal{E}_{\{M\}}(\mathbb{R})$ and $|D^j \theta(0)| \geq M_j$ for all $j \in \mathbb{N}_0$. In particular, this function belongs to $\mathcal{E}_{\{M\}}(\mathbb{R}) \setminus \mathcal{E}_{(M)}(\mathbb{R})$.

Proof. First, using the log-convexity of the sequence M , let us show that

$$\left(\frac{M_{k-1}}{M_k} \right)^{k-j} \leq \frac{M_j}{M_k}$$

for every $k \in \mathbb{N}$, $j \in \mathbb{N}_0$. If $j > k$, we have

$$\frac{M_j}{M_k} = \frac{M_j}{M_{j-1}} \frac{M_{j-1}}{M_{j-2}} \cdots \frac{M_{k+1}}{M_k} \geq \left(\frac{M_k}{M_{k-1}} \right)^{j-k}$$

since the sequence $\left(\frac{M_{l+1}}{M_l}\right)_{l \in \mathbb{N}_0}$ is increasing. Similarly, if $j < k$, we have

$$\frac{M_j}{M_k} = \frac{M_j}{M_{j+1}} \frac{M_{j+1}}{M_{j+2}} \cdots \frac{M_{k-1}}{M_k} \leq \left(\frac{M_{k-1}}{M_k}\right)^{k-j}.$$

If $j = k$, the result is obvious. Then, for every $j \in \mathbb{N}_0$, we have

$$\sum_{k=1}^{+\infty} \frac{M_k}{2^{k-j}} \left(\frac{M_{k-1}}{M_k}\right)^{k-j} \leq M_j \sum_{k=0}^{+\infty} \frac{1}{2^{k-j}}$$

and the Weierstraß theorem implies that θ is well defined and belongs to $\mathcal{C}^\infty(\mathbb{R})$. Moreover, we get that

$$\sup_{x \in \mathbb{R}} |D^j \theta(x)| \leq M_j \sum_{k=1}^{+\infty} \frac{1}{2^{k-j}} = 2^j M_j$$

for every $j \in \mathbb{N}_0$ and consequently, $\theta \in \mathcal{E}_{\{M\}}(\mathbb{R})$. Finally, we have

$$|D^j \theta(0)| = \sum_{k=1}^{+\infty} \frac{M_k}{2^{k-j}} \left(\frac{M_{k-1}}{M_k}\right)^{k-j} \geq M_j.$$

□

Proposition 3.2.14. [138] *Let M and N be two weight sequences. If $M \preceq N$, then $\mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{\{N\}}(\Omega)$. If moreover M is log-convex, the converse implication is also true.*

Proof. It suffices to prove that if M is log-convex and if $\mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{\{N\}}(\Omega)$, then $M \preceq N$. Up to a translation, we can assume that $0 \in \Omega$. From Lemma 3.2.13, there is $\theta \in \mathcal{E}_{\{M\}}(\mathbb{R})$ such that $|D^j \theta(0)| \geq M_j$ for all $j \in \mathbb{N}_0$. By assumption, $\theta \in \mathcal{E}_{\{N\}}(\Omega)$ and there exists $C, h > 0$ such that

$$M_j \leq |D^j \theta(0)| \leq Ch^j N_j$$

for all $j \in \mathbb{N}_0$, which gives the conclusion. □

As a consequence, on the set of all log-convex weight sequences, the equivalence relation \approx characterizes entirely the equivalence of two function spaces of Roumieu type. The same holds for Beurling classes, as presented in the following result of Bruna [44].

Remark 3.2.15. If M is a log-convex non-quasianalytic weight sequence, then by Denjoy-Carleman's theorem, we obtain that $(k!)_{k \in \mathbb{N}_0} \preceq M$. Therefore, the set of analytic functions on an open set Ω is included in $\mathcal{E}_{\{M\}}(\Omega)$.

Proposition 3.2.16. [44] *Let M and N be two weight sequences. Assume that the sequence M is log-convex and that the sequence $((M_k)^{\frac{1}{k}})_{k \in \mathbb{N}}$ tends to infinity. We have $\mathcal{E}_{(M)}(\Omega) \subseteq \mathcal{E}_{(N)}(\Omega)$ if and only if $M \preceq N$.*

Proof. Again, if $M \preceq N$, it is clear that $\mathcal{E}_{(M)}(\Omega) \subseteq \mathcal{E}_{(N)}(\Omega)$. Let us assume that $\mathcal{E}_{(M)}(\Omega) \subseteq \mathcal{E}_{(N)}(\Omega)$. We know that those spaces are Fréchet spaces whose topologies are stronger than the pointwise topology. Then, using the closed graph theorem, we get that the inclusion map $\mathcal{E}_{(M)}(\Omega) \hookrightarrow \mathcal{E}_{(N)}(\Omega)$ is continuous. In particular, for every compact $K \subseteq \Omega$ and every $h > 0$, there is a compact $K' \subseteq \Omega$, $h' > 0$ and $C > 0$ such that

$$\|f\|_{K,h}^N \leq C \|f\|_{K',h'}^M, \quad \forall f \in \mathcal{E}_{(M)}(\Omega). \quad (3.1)$$

From the assumption that the sequence $((M_k)^{\frac{1}{k}})_{k \in \mathbb{N}}$ tends to infinity, we have that for every $t > 0$, the function $f_t(x) = \exp(itx)$, $x \in \mathbb{R}$ belongs to $\mathcal{E}_{(M)}(\mathbb{R})$. The inequality (3.1) with f_t and $h = 1$ gives $h', C > 0$ such that

$$\sup_{k \in \mathbb{N}_0} \frac{t^k}{N_k} \leq C \sup_{k \in \mathbb{N}_0} \frac{t^k}{(h')^k M_k}.$$

Since M is log-convex and using the Remark 3.2.5, we get

$$M_k = \sup_{t > 0} \frac{t^k}{T_M(t)} \leq C \sup_{t > 0} \frac{t^k}{T_N(th)} \leq C \left(\frac{1}{h}\right)^k N_k$$

for every $k \in \mathbb{N}_0$ and the conclusion follows. \square

Let us now study inclusions existing between Denjoy-Carleman classes of different types. First, let us introduce another relation on weight sequences, as done by Rainer and Schindl [120].

Definition 3.2.17. Given two weight sequences M and N , we write $M \triangleleft N$ if for every $\rho > 0$ there exists $C > 0$ such that $M_k \leq C\rho^k N_k$ for every $k \in \mathbb{N}_0$. Consequently, we have

$$M \triangleleft N \iff \lim_{k \rightarrow +\infty} \left(\frac{M_k}{N_k}\right)^{\frac{1}{k}} = 0.$$

If $M \triangleleft N$, it is direct that $\mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{\{N\}}(\Omega)$. Let us now study the converse implication. First, let us recall the following lemma of Rainer and Schindl [120] which directly imply that in the case $M \triangleleft N$, the inclusion is even strict.

Lemma 3.2.18. [120] *Let M and N be two weight sequences satisfying $M \triangleleft N$ and such that $(k!N_k)^{\frac{1}{k}}$ tends to infinity as k tends to infinity. There exist two weight sequences L^1, L^2 such that $(k!L_k^i)^{\frac{1}{k}}$ tends to infinity as k tends to infinity for $i = 1, 2$ and satisfying*

$$M \leq L^1 \triangleleft L^2 \triangleleft N.$$

Proof. First, we set $L_k^1 = \max \left\{ \sqrt{\frac{N_k}{k!}}, M_k \right\}$ for every $k \in \mathbb{N}_0$. Then $L_k^1 \geq M_k$ for every $k \in \mathbb{N}_0$ and $L^1 \triangleleft N$ since

$$\left(\frac{L_k^1}{N_k}\right)^{\frac{1}{k}} = \max \left\{ \left(\frac{1}{k!N_k}\right)^{\frac{1}{2k}}, \left(\frac{M_k}{N_k}\right)^{\frac{1}{k}} \right\}$$

which tends to 0 as k tends to infinity. Moreover, $L_k^1 k! \geq \sqrt{N_k k!}$ so that $(k!L_k^1)^{\frac{1}{k}}$ tends to 0 as k tends to infinity. To conclude, it suffices to set $L_k^2 = \sqrt{L_k^1 N_k}$ for every $k \in \mathbb{N}_0$. \square

Remark 3.2.19.

- The assumption $(k!N_k)^{\frac{1}{k}} \rightarrow +\infty$ as $k \rightarrow +\infty$ is automatically satisfied if the weight sequence N is log-convex. Indeed, in this case the sequence $(N_k^{\frac{1}{k}})_{k \in \mathbb{N}}$ is increasing as proved in Lemma 3.2.3 and the sequence $(k!)^{\frac{1}{k}}$ tends to infinity as k tends to infinity.
- If the weight sequence M is log-convex, we can assume that the weight sequence L^2 is also log-convex. Indeed, using Proposition 3.2.4, a simple computation shows that if M and L are two positive sequences such that $M \triangleleft L$, then $M^c \triangleleft L^c$.

The next result follows directly.

Proposition 3.2.20. *Let M, N be two weight sequences and let Ω be an open subset of \mathbb{R} . If M is log-convex, then*

$$M \triangleleft N \iff \mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{\{N\}}(\Omega)$$

and in this case, the inclusion is strict.

Proof. It suffices to show that if $\mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{\{N\}}(\Omega)$, then $M \triangleleft N$. Up to a translation, we can consider that $0 \in \Omega$. Since M is log-convex, Lemma 3.2.13 gives a function $\theta \in \mathcal{E}_{\{M\}}(\Omega)$ such that $|D^k \theta(0)| \geq M_k$ for every $k \in \mathbb{N}_0$. Then, $\theta \in \mathcal{E}_{\{N\}}(\Omega)$ and for every $\rho > 0$, there is $C > 0$ such that

$$M_k \leq |D^k \theta(0)| \leq C \rho^k N_k, \quad \forall k \in \mathbb{N}_0.$$

Let us now prove that in this case, the inclusion is strict. By Lemma 3.2.18, there exists a log-convex weight sequence L such that $M \triangleleft L \triangleleft N$. Again, Lemma 3.2.13 gives a function f which belongs to $\mathcal{E}_{\{L\}}(\Omega)$ but not to $\mathcal{E}_{\{N\}}(\Omega)$. Since $L \triangleleft N$, we have that $\mathcal{E}_{\{L\}}(\Omega) \subseteq \mathcal{E}_{\{N\}}(\Omega)$ and since $M \triangleleft L$, we have $\mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{\{L\}}(\Omega)$. The conclusion follows. \square

3.2.4 Separability

In order to end this section about properties of Denjoy-Carleman classes, let us mention this last result of density, due to Komatsu [100].

Proposition 3.2.21. [100] *Let M be a log-convex non-quasianalytic weight sequence. The polynomials form a dense subset of $\mathcal{E}_{\{M\}}(\Omega)$ for any open subset Ω of \mathbb{R}^n .*

The idea of the proof is to show that the space of analytic functions on Ω is dense in $\mathcal{E}_{\{M\}}(\Omega)$ and that the inclusion mapping is continuous. The density of the polynomials in the space of analytic functions gives the result.

3.3 Generic results in Denjoy-Carleman classes

Agreement. In this section, we will always assume that any weight sequence is log-convex.

Let us consider two weight sequences M and N such that $M \triangleleft N$. In this section, we study the set of functions of $\mathcal{E}_{\{N\}}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$. More precisely, we show that such a function exists and is generic in $\mathcal{E}_{\{N\}}(\mathbb{R})$, from three different points of view. Let us first start by an explicit construction of such a function.

Proposition 3.3.1. [63] *Assume that M and N are two weight sequences such that $M \triangleleft N$. If M is non-quasianalytic, there exists a function of $\mathcal{E}_{\{N\}}(\mathbb{R})$ which is nowhere in $\mathcal{E}_{\{M\}}$.*

Proof. From Lemma 3.2.18, we know that there is a log-convex weight sequence N^* such that $M \triangleleft N^* \triangleleft N$. Applying recursively this lemma, we get a sequence $(L^{(p)})_{p \in \mathbb{N}}$ of log-convex weight sequences such that

$$M \triangleleft L^{(1)} \triangleleft L^{(2)} \triangleleft \dots \triangleleft L^{(p)} \triangleleft \dots \triangleleft N^* \triangleleft N.$$

For every $p \in \mathbb{N}$, Lemma 3.2.13 allows us to consider a function f_p that belongs to the class $\mathcal{E}_{\{L^{(p)}\}}(\mathbb{R})$ and such that $|D^j f_p(0)| \geq L_j^{(p)}$ for every $j \in \mathbb{N}_0$. Since M is non-quasianalytic, Corollary 3.2.8 gives $\phi \in \mathcal{E}_{\{M\}}(\mathbb{R})$ with compact support and identically equal to 1 in a neighborhood of the origin. If we consider a countable dense subset $\{x_p : p \in \mathbb{N}_0\}$ of \mathbb{R} with $x_0 = 0$, then for every $p \in \mathbb{N}$, we can find $k_p > 0$ such that the function

$$\phi_p := \phi(k_p(\cdot - x_p))$$

has its support disjoint from $\{x_0, \dots, x_{p-1}\}$. We introduce the function g_p defined on \mathbb{R} by

$$g_p(x) := f_p(x - x_p)\phi_p(x).$$

Since $f_p \in \mathcal{E}_{\{L^{(p)}\}}(\mathbb{R}) \subseteq \mathcal{E}_{\{N^*\}}(\mathbb{R})$ and $\phi_p \in \mathcal{E}_{\{M\}}(\mathbb{R}) \subseteq \mathcal{E}_{\{N^*\}}(\mathbb{R})$, we obtain that g_p is a function with compact support that belongs to $\mathcal{E}_{\{N^*\}}(\mathbb{R})$. Then, there exists $\gamma_p > 0$ such that

$$\sup_{x \in \mathbb{R}} |D^j g_p(x)| \leq \gamma_p N_j^*, \quad \forall j \in \mathbb{N}_0.$$

Finally, we define the function g by

$$g := \sum_{p=1}^{+\infty} \frac{1}{\gamma_p 2^p} g_p.$$

Let us show that this function has the desired properties.

First, let us remark that for every $j \in \mathbb{N}_0$ and every $x \in \mathbb{R}$, we have

$$\sum_{p=1}^{+\infty} \frac{1}{\gamma_p 2^p} |D^j g_p(x)| \leq \sum_{p=1}^{+\infty} \frac{1}{2^p} N_j^* \leq N_j^*$$

which implies that g belongs to $\mathcal{E}_{\{N^*\}}(\mathbb{R})$. Since $N^* \triangleleft N$, we get that $g \in \mathcal{E}_{\{N\}}(\mathbb{R})$.

Let us now prove that the function g is nowhere in $\mathcal{E}_{\{M\}}$. We proceed by contradiction and we assume that there exists an open subset Ω of \mathbb{R} such that $g \in \mathcal{E}_{\{M\}}(\Omega)$. Since the subset $\{x_p : p \in \mathbb{N}\}$ is dense in \mathbb{R} , there is $p_0 \in \mathbb{N}$ such that $x_{p_0} \in \Omega$. Remark that the function $\sum_{p=1}^{p_0-1} \frac{1}{\gamma_p 2^p} g_p$ belongs to $\mathcal{E}_{\{L^{(p_0)}\}}(\mathbb{R})$ and that g belongs to $\mathcal{E}_{\{M\}}(\Omega)$ which is included in $\mathcal{E}_{\{L^{(p_0)}\}}(\Omega)$. Consequently, the function

$$\sum_{p=p_0}^{+\infty} \frac{1}{\gamma_p 2^p} g_p = g - \sum_{p=1}^{p_0-1} \frac{1}{\gamma_p 2^p} g_p$$

also belongs to $\mathcal{E}_{\{L^{(p_0)}\}}(\Omega)$. But, since the support of g_p is disjoint of x_{p_0} for every $p > p_0$, we also have

$$\begin{aligned} \left| \sum_{p=p_0}^{+\infty} \frac{1}{\gamma_p 2^p} D^j g_p(x_{p_0}) \right| &= \frac{1}{\gamma_{p_0} 2^{p_0}} |D^j g_{p_0}(x_{p_0})| \\ &= \frac{1}{\gamma_{p_0} 2^{p_0}} |D^j f_{p_0}(0)| \\ &\geq \frac{1}{\gamma_{p_0} 2^{p_0}} L_j^{p_0} \end{aligned}$$

for every $j \in \mathbb{N}_0$, hence a contradiction with the definition of the space $\mathcal{E}_{\{L^{(p_0)}\}}(\Omega)$. \square

Proposition 3.3.2. [63] *Assume that M and N are two weight sequences such that $M \triangleleft N$. If M is non-quasianalytic, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is prevalent in $\mathcal{E}_{(N)}(\mathbb{R})$.*

Proof. The set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are somewhere in $\mathcal{E}_{\{M\}}$ is given by

$$\bigcup_{I \subseteq \mathbb{R}} \bigcup_{m \in \mathbb{N}} E(I, m),$$

where I denotes rational subintervals of \mathbb{R} and

$$E(I, m) := \left\{ f \in \mathcal{E}_{(N)}(\mathbb{R}) : \exists C > 0 \text{ such that } \sup_{x \in I} |D^j f(x)| \leq C m^j M_j, \forall j \in \mathbb{N}_0 \right\}.$$

We know from Proposition 1.2.5 that any countable union of shy sets is shy. So we just have to prove that $E(I, m)$ is shy for every I and every m . It is clear that $E(I, m)$ is a vector subspace of $\mathcal{E}_{(N)}(\mathbb{R})$ which is proper using Proposition 3.3.1. Moreover, it is a Borel subset of $\mathcal{E}_{(N)}(\mathbb{R})$. Indeed, we have

$$E(I, m) = \bigcup_{s \in \mathbb{N}} \left\{ f \in \mathcal{E}_{(N)}(\mathbb{R}) : \sup_{x \in I} |D^j f(x)| \leq s m^j M_j, \forall j \in \mathbb{N}_0 \right\}$$

which is a countable union of closed sets in $\mathcal{E}_{(N)}(\mathbb{R})$. Lemma 1.2.7 gives the conclusion. \square

Proposition 3.3.3. [63] *Assume that M and N are two weight sequences such that $M \triangleleft N$. If M is non-quasianalytic, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is residual in $\mathcal{E}_{(N)}(\mathbb{R})$.*

Proof. As done in the previous proof, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are somewhere in $\mathcal{E}_{\{M\}}$ can be written as

$$\bigcup_{I \subseteq \mathbb{R}} \bigcup_{m \in \mathbb{N}} \bigcup_{s \in \mathbb{N}} \left\{ f \in \mathcal{E}_{(N)}(\mathbb{R}) : \sup_{x \in I} |D^j f(x)| \leq s m^j M_j, \forall j \in \mathbb{N}_0 \right\}.$$

Each closed set $\{f \in \mathcal{E}_{(N)}(\mathbb{R}) : \sup_{x \in I} |D^j f(x)| \leq s m^j M_j, \forall j \in \mathbb{N}_0\}$ has empty interior since it is included in $E(I, m)$ which is a proper vector subspace of the locally convex space $\mathcal{E}_{(N)}(\mathbb{R})$. The conclusion follows. \square

The next construction used to prove the lineability follows an idea of Schmets and Valdivia [128]. Fix two weight sequences M and N such that M is non-quasianalytic and $M \triangleleft N$. For every $t \in (0, 1)$, we define a weight sequence $L^{(t)}$ by

$$L_k^{(t)} := (M_k)^{1-t} (N_k)^t, \quad \forall k \in \mathbb{N}_0.$$

Since N, M are log-convex, it is straightforward to see that $L^{(t)}$ is also log-convex. Moreover, the assumption $M \triangleleft N$ leads directly to the relations

$$M \triangleleft L^{(t)} \triangleleft N \text{ if } t \in (0, 1) \text{ and } L^{(t)} \triangleleft L^{(s)} \text{ if } t < s.$$

For every $p \in \mathbb{N} \setminus \{1\}$ and for every $t \in (0, 1)$, using Lemma 3.2.13, we consider a function $f_{p,t} \in \mathcal{E}_{\{L^{((1-\frac{1}{p})t)}\}}(\mathbb{R})$ such that $|D^j f_{p,t}(0)| \geq L_j^{((1-\frac{1}{p})t)}$ for every $j \in \mathbb{N}_0$.

Since M is non-quasianalytic, using Corollary 3.2.8, we can choose a function ϕ in $\mathcal{E}_{\{M\}}(\mathbb{R})$ with compact support and identically equal to 1 in a neighborhood of 0. Let us consider a countable dense subset $\{x_p : p \in \mathbb{N}\}$ of \mathbb{R} with $x_1 = 0$. For every $p \geq 2$, we fix $k_p > 0$ such that the function

$$\phi_p := \phi(k_p(\cdot - x_p))$$

has its support disjoint from $\{x_1, \dots, x_{p-1}\}$. We introduce then for every $t \in (0, 1)$ the function $g_{p,t}$ defined by

$$g_{p,t} := f_{p,t}(\cdot - x_p)\phi_p.$$

It is clear that $f_{p,t}(\cdot - x_p) \in \mathcal{E}_{\{L^{((1-\frac{1}{p})t)}\}}(\mathbb{R}) \subseteq \mathcal{E}_{\{L^{(t)}\}}(\mathbb{R})$. Moreover ϕ belongs to $\mathcal{E}_{\{M\}}(\mathbb{R}) \subseteq \mathcal{E}_{\{L^{(t)}\}}(\mathbb{R})$. Since the support of $g_{p,t}$ is compact and using Proposition 3.2.2, there exists $\gamma_{p,t} > 0$ such that

$$\sup_{x \in \mathbb{R}} |D^j g_{p,t}(x)| \leq \gamma_{p,t} L_j^{(t)}, \quad \forall j \in \mathbb{N}_0.$$

For every $t \in (0, 1)$, we define the function g_t by setting

$$g_t := \sum_{p=2}^{+\infty} \frac{1}{\gamma_{p,t} 2^p} g_{p,t}.$$

Remark that we are in the same situation than in the proof of Proposition 3.3.1 since

$$M \triangleleft L^{(\frac{t}{2})} \triangleleft L^{(\frac{2t}{3})} \triangleleft L^{(\frac{3t}{4})} \triangleleft \dots \triangleleft L^{(t)} \triangleleft N, \quad \forall t \in (0, 1).$$

Therefore, as done previously, we get that the function g_t belongs to $\mathcal{E}_{\{L^{(t)}\}}(\mathbb{R})$ and is not in $\mathcal{E}_{\{L^{((1-\frac{1}{p_0})t)}\}}(\Omega)$, for any open neighborhood Ω of x_{p_0} and for any $p_0 \geq 2$. This construction leads to the following lemma.

Lemma 3.3.4. [63] *If \mathcal{D} denotes the subspace of $\mathcal{E}_{\{N\}}(\mathbb{R})$ spanned by the functions g_t , $t \in (0, 1)$, then $\dim \mathcal{D} = \mathfrak{c}$ and every non-zero function of \mathcal{D} is nowhere in $\mathcal{E}_{\{M\}}$.*

Proof. First, assume there exist $\alpha_1, \dots, \alpha_N \in \mathbb{C}$ with $\alpha_N \neq 0$ and $t_1 < \dots < t_N$ in $(0, 1)$ such that $\sum_{n=1}^N \alpha_n g_{t_n} = 0$. Then

$$g_{t_N} = -\frac{1}{\alpha_N} \sum_{n=1}^{N-1} \alpha_n g_{t_n}$$

and since $g_{t_n} \in \mathcal{E}_{\{L^{(t_n)}\}}(\mathbb{R}) \subseteq \mathcal{E}_{\{L^{(t_{N-1})}\}}(\mathbb{R})$ for every $n \leq N-1$, we get that

$$g_{t_N} \in \mathcal{E}_{\{L^{(t_{N-1})}\}}(\mathbb{R}) \subseteq \mathcal{E}_{\{L^{((1-\frac{1}{p_0})t_N)}\}}(\mathbb{R})$$

if p_0 is such that $(1 - \frac{1}{p_0})t_N > t_{N-1}$. This is a contradiction and it follows that the functions g_t , $t \in (0, 1)$, are linearly independent.

To conclude, it remains to show that every non-zero linear combination of the functions g_t , $t \in (0, 1)$, is nowhere in $\mathcal{E}_{\{M\}}$. Let us fix $\alpha_1, \dots, \alpha_N \in \mathbb{C}$ with $\alpha_N \neq 0$ and $t_1 < \dots < t_N$ in $(0, 1)$, and let us consider the function

$$G = \sum_{n=1}^N \alpha_n g_{t_n}.$$

Assume that there exists an open subset Ω of \mathbb{R} such that $G \in \mathcal{E}_{\{M\}}(\Omega)$. We fix $p_0 \in \mathbb{N}$ such that $x_{p_0} \in \Omega$ and $t_{N-1} < \left(1 - \frac{1}{p_0}\right) t_N$. Again, the function g_{t_n} belongs to $\mathcal{E}_{\{L^{(t_{N-1})}\}}(\mathbb{R})$ for every $n \leq N-1$ and it follows that the function

$$g_{t_N} = \frac{1}{\alpha_N} \left(G - \sum_{n=1}^{N-1} \alpha_n g_{t_n} \right)$$

belongs to $\mathcal{E}_{\{L^{(t_{N-1})}\}}(\Omega)$. From the choice of p_0 , we have

$$\mathcal{E}_{\{L^{(t_{N-1})}\}}(\Omega) \subseteq \mathcal{E}_{\left\{L^{\left(1 - \frac{1}{p_0}\right)t_N}\right\}}(\Omega)$$

and this leads to a contradiction with the construction of g_{t_N} . \square

The dense-lineability in $\mathcal{E}_{(N)}(\mathbb{R})$ of the set of functions which are nowhere in $\mathcal{E}_{\{M\}}$ will be obtained using the condition presented in Chapter 1, Proposition 1.3.4.

Lemma 3.3.5. *If \mathcal{P} denotes the set of polynomials, then the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is stronger than \mathcal{P} .*

Proof. Let us fix a function g of $\mathcal{E}_{(N)}(\mathbb{R})$ which is nowhere in $\mathcal{E}_{\{M\}}$ and P a polynomial. We proceed by contradiction. Assume that there is an open subset Ω such that $g + P$ belongs to the class $\mathcal{E}_{\{M\}}(\Omega)$. Of course, P belongs also to this class and it follows that $g = (g + P) - P \in \mathcal{E}_{\{M\}}(\Omega)$, hence a contradiction. \square

Consequently, we directly obtain the following.

Theorem 3.3.6. [63] *Assume that M and N are two weight sequences such that M is non-quasianalytic and $M \triangleleft N$. Then the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is \mathfrak{c} -dense-lineable in $\mathcal{E}_{(N)}(\mathbb{R})$.*

Proof. From Proposition 3.2.21, we know that the set of polynomials is dense in $\mathcal{E}_{(N)}(\mathbb{R})$. The result is then a direct consequence of Lemmas 3.3.4 and 3.3.5, Proposition 1.3.4 and Remark 1.3.5. \square

Let us end this section by the study of the case of countable unions. As a consequence, we will get results about Gevrey classes. If $(M^{(n)})_{n \in \mathbb{N}}$ is a sequence of weight sequences, we say that a function is *nowhere in* $\bigcup_{n \in \mathbb{N}} \mathcal{E}_{\{M^{(n)}\}}$ if its restriction to any open and non-empty subset Ω of \mathbb{R} does not belongs to $\mathcal{E}_{\{M^{(n)}\}}(\Omega)$ for any $n \in \mathbb{N}$.

Lemma 3.3.7. [63] *Let N be a weight sequence and let $(M^{(n)})_{n \in \mathbb{N}}$ be a sequence of weight sequences such that $M^{(n)} \triangleleft N$ for every $n \in \mathbb{N}$. Then, there exists a weight sequence P such that*

$$M^{(n)} \preceq P, \quad \forall n \in \mathbb{N} \text{ and } P \triangleleft N.$$

Proof. By assumption, we know that there exists a sequence $(C_n)_{n \in \mathbb{N}}$ of positive numbers such that

$$M_k^{(n)} \leq C_n n^{-k} N_k, \quad \forall k \in \mathbb{N}_0, n \in \mathbb{N}.$$

Then, for every $k \in \mathbb{N}_0$, $\sup \left\{ \frac{M_k^{(n)}}{C_n} : n \in \mathbb{N} \right\} < +\infty$ and we define a weight sequence P by setting

$$P_k := \sup \left\{ \frac{M_k^{(n)}}{C_n} : n \in \mathbb{N} \right\}, \quad k \in \mathbb{N}_0.$$

It is clear that $M^{(n)} \preceq P$ for every $n \in \mathbb{N}$. Let us now show that $P \triangleleft N$. If we consider $\rho > 0$, there exists $N \in \mathbb{N}$ such that $\rho \geq \frac{1}{n}$ for every $n \geq N$. We get that

$$M_k^{(n)} \leq C_n n^{-k} N_k \leq C_n \rho^k N_k, \quad \forall k \in \mathbb{N}_0$$

if $n \geq N$. Moreover, if $n < N$, the assumption $M^{(n)} \triangleleft N$ gives a constant $D > 0$ such that

$$M_k^{(n)} \leq D \rho^k N_k, \quad \forall k \in \mathbb{N}_0, \forall n < N.$$

It follows that the constant $C := \max\{1, \max\{\frac{D}{C_n} : n < N\}\} > 0$ is such that

$$P_k \leq C \rho^k N_k, \quad \forall k \in \mathbb{N}_0.$$

This means that $P \triangleleft N$. Finally, it is straightforward to see that the sequence P is log-convex. This leads to the conclusion. \square

This Lemma and the results obtained previously lead directly to the following proposition.

Proposition 3.3.8. [63] *Let N be a log-convex weight sequence and let $(M^{(n)})_{n \in \mathbb{N}}$ be a sequence of weight sequences such that $M^{(n)} \triangleleft N$ for every $n \in \mathbb{N}$. If there is $n_0 \in \mathbb{N}$ such that the weight sequence $M^{(n_0)}$ is non-quasianalytic, then the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\bigcup_{n \in \mathbb{N}} \mathcal{E}_{\{M^{(n)}\}}$ is prevalent, residual and \mathfrak{c} -dense-lineable in $\mathcal{E}_{(N)}(\mathbb{R})$.*

Proof. From Lemma 3.3.7, there is a log-convex weight sequence P such that

$$\bigcup_{n \in \mathbb{N}} \mathcal{E}_{\{M^{(n)}\}}(\Omega) \subseteq \mathcal{E}_{\{P\}}(\Omega) \subsetneq \mathcal{E}_{(N)}(\Omega)$$

for every open subset Ω of \mathbb{R} . Moreover, since the weight sequence $M^{(n_0)}$ is non-quasianalytic and $M^{(n_0)} \preceq P$, the weight sequence P is also non-quasianalytic. The result follows then directly from Propositions 3.3.2, 3.3.3 and Theorem 3.3.6. \square

As mentioned before, an important example of ultradifferentiable classes of Roumieu type is given by the classes of Gevrey differentiable functions of order $\alpha > 1$. They correspond to the weight sequences

$$M_k := (k!)^\alpha, \quad k \in \mathbb{N}_0.$$

Remark that for every $\alpha > 1$, the class $\mathcal{E}_{\{(k!)^\alpha\}}(\mathbb{R})$ is non-quasianalytic. Moreover, for every α, β such that $1 < \beta < \alpha$, we have

$$\mathcal{E}_{\{(k!)^\beta\}}(\mathbb{R}) \subseteq \mathcal{E}_{\{(k!)^\alpha\}}(\mathbb{R}).$$

In 1999, Schmets and Valdivia [128] proved the following result.

Proposition 3.3.9. [128] *Let $\alpha > 1$. The set of functions of $\mathcal{E}_{\{(k!)^\alpha\}}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{(k!)^\beta\}}$ for every $\beta \in (1, \alpha)$ is residual in $\mathcal{E}_{\{(k!)^\alpha\}}(\mathbb{R})$.*

This result can be seen as a consequence of Proposition 3.3.8 applied to the weight sequences $M^{(n)}$ ($n \in \mathbb{N}$) given by

$$M_k^{(n)} := (k!)^{\beta_n}, \quad k \in \mathbb{N}_0,$$

where $(\beta_n)_{n \in \mathbb{N}}$ is an increasing sequence of $(1, \alpha)$ that converges to α .

Here is a direct consequence of Proposition 3.3.8 which completes the result of Schmets and Valdivia given in Proposition 3.3.9.

Proposition 3.3.10. [63] Let $\alpha > 1$. The set of functions of $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{(k!)^\beta\}}$ for every $\beta \in (1, \alpha)$ is prevalent and \mathfrak{c} -dense-lineable in $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$.

3.4 Braun, Meise and Taylor classes

In the present section, we handle the same kind of questions than previously but in the context of non-quasianalytic classes of ultradifferentiable functions which have been introduced by Beurling [36], see Björck [38] for more details. They pointed out that the smoothness of a \mathcal{C}^∞ compactly supported function can also be measured using decay properties of its Fourier-Laplace transform and weight functions ω . This method was modified by Braun et al. [43] who showed that these classes can also be defined by the decay properties of their derivatives through the Legendre(-Fenchel-Young) transform of the function $t \mapsto \omega(e^t)$. It is in this context that we will work in this section.

In this section, we will first define classes of ultradifferentiable functions as introduced by Braun et al. [43] and give their first properties. We will then present their dual space. This characterization will be used while studying inclusions between classes of Roumieu and Beurling type. Finally, we present generic results about functions which are in a given Beurling class but nowhere in a given Roumieu class.

3.4.1 Definition and first properties

Definition 3.4.1. [43] A function $\omega : [0, +\infty) \rightarrow [0, +\infty)$ is called a *weight function* if it is continuous, increasing and satisfies $\omega(0) = 0$ as well as the following conditions:

- (α) there exists $L \geq 1$ such that $\omega(2t) \leq L\omega(t) + L$, $t \geq 0$,
- (β) $\int_1^{+\infty} \frac{\omega(t)}{t^2} dt < +\infty$,
- (γ) $\log(t) = o(\omega(t))$ as t tends to infinity,
- (δ) $\varphi_\omega : t \mapsto \omega(e^t)$ is convex on $[0, +\infty)$.

The Legendre(-Fenchel-Young) transform of φ_ω is defined by

$$\varphi_\omega^*(x) := \sup\{xy - \varphi_\omega(y) : y > 0\}, \quad x \geq 0.$$

Let us now introduce function spaces of Beurling and Roumieu type associated with a weight function ω . For a compact subset K of \mathbb{R}^n and every $m \in \mathbb{N}$, we define the space $\mathcal{E}_\omega^m(K)$ as the space of functions $f \in \mathcal{C}^\infty(K)$ such that

$$\|f\|_{K,m}^\omega := \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} |D^\alpha f(x)| \exp\left(-\frac{1}{m} \varphi_\omega^*(m|\alpha|)\right) < +\infty.$$

Clearly, it is a Banach space.

Definition 3.4.2. If ω is a weight function and if Ω is an open subset of \mathbb{R}^n , we define the space $\mathcal{E}_{\{\omega\}}(\Omega)$ of ω -ultradifferentiable functions of Roumieu type on Ω by

$$\mathcal{E}_{\{\omega\}}(\Omega) := \{f \in \mathcal{C}^\infty(\Omega) : \forall K \subseteq \Omega \text{ compact } \exists m \in \mathbb{N} \text{ such that } \|f\|_{K,m}^\omega < +\infty\}.$$

It is endowed with the topology given by the representation

$$\mathcal{E}_{\{\omega\}}(\Omega) = \text{proj}_{K \subseteq \Omega} \text{ind}_{m \in \mathbb{N}} \mathcal{E}_\omega^m(K),$$

where K runs over all compact subsets of Ω .

Definition 3.4.3. If ω is a weight function and if Ω is an open subset of \mathbb{R}^n , the space $\mathcal{E}_{(\omega)}(\Omega)$ of ω -ultradifferentiable functions of Beurling type on Ω is defined by

$$\mathcal{E}_{(\omega)}(\Omega) := \left\{ f \in \mathcal{C}^\infty(\Omega) : \forall K \subseteq \Omega \text{ compact}, \forall m \in \mathbb{N}, p_{K,m}^\omega(f) < +\infty \right\},$$

where for every compact subset K of \mathbb{R}^n and every $m \in \mathbb{N}$

$$p_{K,m}^\omega(f) := \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} |D^\alpha f(x)| \exp \left(-m \varphi_\omega^* \left(\frac{|\alpha|}{m} \right) \right).$$

We endow the space $\mathcal{E}_{(\omega)}(\Omega)$ with its natural Fréchet space topology.

As done in the case of Denjoy-Carleman classes, when a statement holds both for the space $\mathcal{E}_{(\omega)}(\Omega)$ and the space $\mathcal{E}_{\{\omega\}}(\Omega)$, we will write $\mathcal{E}_{[\omega]}(\Omega)$.

Remark 3.4.4. Fix $\alpha > 1$ and consider the weight function $\omega(t) = t^{1/\alpha}$ and the weight sequence $M = ((k!)^\alpha)_{k \in \mathbb{N}_0}$. It is well known that for every open subset $\Omega \subseteq \mathbb{R}^n$, the equality

$$\mathcal{E}_{[\omega]}(\Omega) = \mathcal{E}_{[M]}(\Omega)$$

holds as locally convex spaces. In particular, the space $\mathcal{E}_{\{\omega\}}(\Omega)$ corresponds to the space of Gevrey differentiable functions of order α on Ω . However, in general, the definitions of ultradifferentiable functions using weight sequences or weight functions lead to different classes. We refer to Bonet et al. [42] for a complete study of the comparison of the two approaches.

Given a weight function ω , the property (α) and the convexity of φ_ω^* lead to the following result.

Proposition 3.4.5. [43] *Let Ω be an open subset of \mathbb{R}^n and ω be a weight function. The space $\mathcal{E}_{[\omega]}(\Omega)$ is an algebra for the pointwise multiplication of functions.*

Proof. Let $f, g \in \mathcal{E}_{\{\omega\}}(\Omega)$ and let K be a compact of Ω . Then, there exist $C_1, C_2 > 0$ and $m_1, m_2 \in \mathbb{N}$ such that

$$\sup_{x \in K} |D^\alpha f(x)| \leq C_1 \exp \left(\frac{1}{m_1} \varphi_\omega^*(m_1 |\alpha|) \right)$$

and

$$\sup_{x \in K} |D^\alpha g(x)| \leq C_2 \exp \left(\frac{1}{m_2} \varphi_\omega^*(m_2 |\alpha|) \right)$$

for every $\alpha \in \mathbb{N}_0^n$. Leibnitz' rule gives

$$\begin{aligned} |D^\alpha(fg)(x)| &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |D^\beta f(x)| |D^{\alpha-\beta} g(x)| \\ &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} C_1 C_2 \exp \left(\frac{1}{m} (\varphi_\omega^*(m|\beta|) + \varphi_\omega^*(m|\alpha - \beta|)) \right) \end{aligned}$$

for every $x \in K$ and every $\alpha \in \mathbb{N}_0^n$, where $m = \max\{m_1, m_2\}$. Moreover, since φ_ω^* is convex, we have $\varphi_\omega^*(\lambda t) \leq \lambda \varphi_\omega^*(t)$ for every $\lambda \in [0, 1]$, hence

$$\varphi_\omega^*(m|\beta|) \leq \frac{|\beta|}{|\alpha|} \varphi_\omega^*(m|\alpha|) \text{ and } \varphi_\omega^*(m|\alpha - \beta|) \leq \frac{|\alpha - \beta|}{|\alpha|} \varphi_\omega^*(m|\alpha|).$$

Therefore,

$$\begin{aligned} \sup_{x \in K} |D^\alpha(fg)(x)| &\leq C_1 C_2 2^{|\alpha|} \exp\left(\frac{1}{m} \varphi_\omega^*(m|\alpha|)\right) \\ &\leq C_1 C_2 \exp\left(|\alpha| + \frac{1}{m} \varphi_\omega^*(m|\alpha|)\right). \end{aligned}$$

To conclude, let us remark that since ω is continuous and increasing, the condition (α) provides a constant $L > 0$ such that

$$\varphi_\omega(x+1) \leq L(1 + \varphi_\omega(x))$$

for every $x \geq 0$. Then, an easy computation shows that

$$\varphi_\omega^*(y) - y \geq L \varphi_\omega^*\left(\frac{y}{L}\right) - L$$

for every $y \geq 0$. By taking $y = mL|\alpha|$, we get

$$\begin{aligned} \sup_{x \in K} |D^\alpha(fg)(x)| &\leq C_1 C_2 \exp\left(\frac{1}{mL}(mL|\alpha| + L\varphi_\omega^*(m|\alpha|))\right) \\ &\leq C_1 C_2 \exp\left(\frac{1}{m} + \frac{1}{mL} \varphi_\omega^*(mL|\alpha|)\right). \end{aligned}$$

The proof of the Beurling case is similar. □

The next result of Braun et al. [43] gives the non-quasianalyticity of the space $\mathcal{E}_{[\omega]}(\Omega)$.

Proposition 3.4.6. [43] *Let Ω be an open subset of \mathbb{R}^n and ω be a weight function. The space $\mathcal{E}_{[\omega]}(\Omega)$ contains non-zero functions with compact support.*

Therefore given an open subset Ω of \mathbb{R}^n and a compact $K \subseteq \Omega$, it is possible to find a function in $\mathcal{E}_{[\omega]}(\mathbb{R}^n)$ with compact support included in Ω and identically equal to 1 on K [43]. Such a function can be obtained with a technique similar to the one used in Corollary 3.2.8.

3.4.2 Dual space

When dealing with ultradifferentiable classes defined using weight functions, it is generally difficult to construct an explicit function with some expected properties. That is the reason why, given a weight function ω , we will need the characterization of the strong dual spaces of $\mathcal{E}_{\{\omega\}}(\Omega)$ and $\mathcal{E}_{(\omega)}(\Omega)$, respectively denoted $\mathcal{E}'_{\{\omega\}}(\Omega)$ and $\mathcal{E}'_{(\omega)}(\Omega)$. Let us first introduce weighted spaces of entire functions, where we denote the space of entire functions on \mathbb{C}^n by $\mathcal{H}(\mathbb{C}^n)$.

Definition 3.4.7. For each compact set K of \mathbb{R}^n , the *support functional* of K is defined as

$$h_K : \mathbb{R}^n \rightarrow \mathbb{R} : x \mapsto h_K(x) := \sup_{y \in K} \langle x, y \rangle .$$

For $\lambda > 0$, let

$$A(K, \lambda) := \left\{ f \in \mathcal{H}(\mathbb{C}^n) : |f|_{K, \lambda}^\omega := \sup_{z \in \mathbb{C}^n} |f(z)| \exp(-h_K(\Im z) - \lambda \omega(|z|)) < +\infty \right\}$$

endowed with its natural topology. We define

$$\mathcal{A}_{\{\omega\}}(\Omega) := \underset{K \subseteq \Omega}{\text{ind}} \underset{n \in \mathbb{N}}{\text{ind}} A(K, n)$$

and

$$\mathcal{A}_{\{\omega\}}(\Omega) := \underset{K \subseteq \Omega}{\text{ind}} \underset{n \in \mathbb{N}}{\text{proj}} A\left(K, \frac{1}{n}\right).$$

It is easy to check that $A(K, \lambda)$ is a Banach space, $\mathcal{A}_{\{\omega\}}(\Omega)$ is an (LB)-space and $\mathcal{A}_{\{\omega\}}(\Omega)$ is a (LF)-space.

The following result was proved by Heinrich and Meise [77] (Theorems 3.6 and 3.7). Let us mention that the Roumieu case was already proved by Rösner [123] (Theorem 2.19).

Proposition 3.4.8. *For each weight function ω and each convex open set Ω in \mathbb{R}^n , the Fourier-Laplace transform*

$$\mathcal{F} : \mathcal{E}'_{\{\omega\}}(\Omega) \rightarrow \mathcal{A}_{\{\omega\}}(\Omega), \mathcal{F}(u) : z \mapsto u \left(\exp(-i \langle x, z \rangle) \right)$$

is a linear topological isomorphism. The same holds for the Beurling type provided that $\omega(t) = o(t)$ as t tends to infinity.

Remark 3.4.9. If ω and σ are two weight functions such that $\sigma(t) = o(\omega(t))$ as t tends to infinity, then the condition $\sigma(t) = o(t)$ as t tends to infinity is automatically satisfied.

3.4.3 Inclusions between Braun, Meise and Taylor classes

Following [127], we define the following relations on weight sequences:

$$\begin{aligned} \omega \preceq \sigma &\iff \sigma(t) = O(\omega(t)) \text{ as } t \rightarrow +\infty, \\ \omega \sim \sigma &\iff \omega \preceq \sigma \text{ and } \sigma \preceq \omega, \\ \omega \triangleleft \sigma &\iff \sigma(t) = o(\omega(t)) \text{ as } t \rightarrow +\infty. \end{aligned}$$

Of course, if $\omega \preceq \sigma$, we have $\mathcal{E}_{[\omega]}(\Omega) \subseteq \mathcal{E}_{[\sigma]}(\Omega)$ for any open subset Ω of \mathbb{R}^n . Inclusion between Roumieu and Beurling classes follows also directly from the definitions and is given by the following result of Braun et al. [43].

Proposition 3.4.10. [43] *Let Ω be an open set of \mathbb{R}^n . If $\omega \triangleleft \sigma$, then the space $\mathcal{E}_{\{\omega\}}(\Omega)$ is continuously included in $\mathcal{E}_{\{\sigma\}}(\Omega)$.*

Proof. Since $\omega \triangleleft \sigma$, for every $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\sigma(t) \leq \varepsilon \omega(t) + C_\varepsilon \quad \forall t \geq 0.$$

Consequently, we have

$$\varphi_\omega^*(x) \leq \varepsilon \varphi_\sigma^*\left(\frac{x}{\varepsilon}\right) + \frac{C_\varepsilon}{\varepsilon}$$

for every $x \geq 0$. This gives directly the conclusion. □

In this section, we will show that if $\omega \triangleleft \sigma$, the inclusion given by the previous proposition is even strict. We will use the characterization of the dual spaces. Let us also recall the following proposition that follows from Hörmander [78] (Theorem 4.4.2). See Bonet and Meise [41] (Proposition 12).

Proposition 3.4.11. *For every $n \in \mathbb{N}$, there exist $C_1, C_2 > 0$ such that for every plurisubharmonic function $u : \mathbb{C}^n \rightarrow \mathbb{R}$ and every $a \in \mathbb{C}^n$, there exists $f \in \mathcal{H}(\mathbb{C}^n)$ that satisfies*

$$f(a) = \exp \left(\inf_{|v-a| \leq 1} u(v) - n \log(1 + |a|^2) \right)$$

and

$$|f(z)| \leq C_1 \exp \left(\sup_{|v-z| \leq 1} u(v) + C_2 \log(1 + |z|^2) \right), \quad \forall z \in \mathbb{C}^n.$$

Remark 3.4.12. Let $\omega : [0, +\infty) \rightarrow [0, +\infty)$ be an increasing continuous function such that the function $\varphi_\omega : t \mapsto \omega(e^t)$ is convex on $[0, +\infty)$. Then its radial extension $\tilde{\omega}$ defined on \mathbb{C}^n by $\tilde{\omega}(z) := \omega(|z|)$ is continuous and plurisubharmonic on \mathbb{C}^n . Indeed, it suffices to note that $\tilde{\omega}(z) = \varphi_\omega(\log(|z|))$ for every $z \in \mathbb{C}^n$. In particular, the radial extension of any weight function is plurisubharmonic on \mathbb{C}^n . We refer the reader to Hörmander [78] for more information about theory of plurisubharmonic functions.

In what follows, we will also use the following results of Braun et al. [43] (Lemma 1.7).

Lemma 3.4.13. *Let ω be a weight function and assume that $g : [0, +\infty) \rightarrow [0, +\infty)$ satisfies $g(t) = o(\omega(t))$ as t tends to infinity. Then, there exists a weight function τ such that*

$$g(t) = o(\tau(t)) \quad \text{and} \quad \tau(t) = o(\omega(t))$$

as t tends to infinity.

Let us finally recall the localization theorem of De Wilde (see for example [95], Corollary 5.6.4).

Theorem 3.4.14 (De Wilde). *Let E be a Baire topological vector space and let the Hausdorff topological vector space F be the reduced inductive limit of a sequence of strictly webbed topological vector spaces F_n , $n \in \mathbb{N}$. Let $T : E \rightarrow F$ be a closed linear map. Then there exists $n \in \mathbb{N}$ such that $T(E) \subseteq F_n$ and the map $E \rightarrow F_n$ induced by T is continuous.*

The proof of our next result is inspired by the proofs of Propositions 13 and 18 in Bonet and Meise [41].

Proposition 3.4.15. [63] *Let ω and σ be two weight functions such that $\omega \triangleleft \sigma$. If Ω is a convex open subset of \mathbb{R}^n , then $\mathcal{E}_{\{\omega\}}(\Omega)$ is strictly included in $\mathcal{E}_{\{\sigma\}}(\Omega)$.*

Proof. Up to a translation, we can assume that $0 \in \Omega$. Suppose that $\mathcal{E}_{\{\omega\}}(\Omega) = \mathcal{E}_{\{\sigma\}}(\Omega)$. Then, the continuity of the inclusion $\mathcal{E}_{\{\omega\}}(\Omega) \subseteq \mathcal{E}_{\{\sigma\}}(\Omega)$ and the closed graph theorem imply that $\mathcal{E}_{\{\omega\}}(\Omega) = \mathcal{E}_{\{\sigma\}}(\Omega)$ as locally convex spaces. Consequently they have the same dual spaces, i.e. by Proposition 3.4.8, the spaces $\mathcal{A}_{\{\omega\}}(\Omega)$ and $\mathcal{A}_{\{\sigma\}}(\Omega)$ coincide as locally convex spaces. In particular, the inclusion

$$\mathcal{A}_{\{\omega\}}(\Omega) \rightarrow \mathcal{A}_{\{\sigma\}}(\Omega)$$

is continuous. It follows that for every compact $K \subseteq \Omega$, the inclusion

$$\text{proj}_{\substack{\leftarrow \\ m \in \mathbb{N}}} \mathcal{A}^\omega \left(K, \frac{1}{m} \right) \rightarrow \mathcal{A}_{\{\sigma\}}(\Omega)$$

is also continuous. Let us fix a compact subset K of Ω such that $0 \in K$. Now, we apply Theorem 3.4.14 to get a compact K' of Ω and a natural number m'_0 such that

$$\underset{m \in \mathbb{N}}{\text{proj}} \mathcal{A}^\omega \left(K, \frac{1}{m} \right) \subseteq \mathcal{A}^\sigma(K', m'_0)$$

continuously. Therefore, there are $m_0 \in \mathbb{N}$ and $C > 0$ such that

$$|f|_{K', m'_0}^{\sigma} \leq C |f|_{K, \frac{1}{m_0}}^{\omega}, \quad \forall f \in \underset{m \in \mathbb{N}}{\text{proj}} \mathcal{A}^\omega \left(K, \frac{1}{m} \right). \quad (3.2)$$

Since $\sigma(t) = o(\omega(t))$ as t tends to infinity, Lemma 3.4.13 gives a weight function τ such that $\sigma(t) = o(\tau(t))$ and $\tau(t) = o(\omega(t))$ as t tends to infinity. Next, we consider the radial extension $\tilde{\tau}$ of τ to \mathbb{C}^n defined by

$$\tilde{\tau}(z) := \tau(|z|), \quad z \in \mathbb{C}^n.$$

As mentioned in Remark 3.4.12, this function is plurisubharmonic on \mathbb{C}^n . For every $j \in \mathbb{N}$, we apply Proposition 3.4.11 with $a_j = (j, 0, \dots, 0)$ to get a function $f_j \in \mathcal{H}(\mathbb{C}^n)$ such that

$$f_j(a_j) = \exp \left(\inf_{|v - a_j| \leq 1} \tilde{\tau}(v) - n \log(1 + j^2) \right) \quad (3.3)$$

and

$$|f_j(z)| \leq C_1 \exp \left(\sup_{|v - z| \leq 1} \tilde{\tau}(v) + C_2 \log(1 + |z|^2) \right), \quad \forall z \in \mathbb{C}^n. \quad (3.4)$$

Let us first show that for every $j \in \mathbb{N}$, the function f_j belongs to $\underset{m \in \mathbb{N}}{\text{proj}} \mathcal{A}^\omega \left(K, \frac{1}{m} \right)$.

We know from condition (α) that there is $L > 0$ such that

$$\tau(1 + |z|) \leq \tau(2|z|) \leq L\tau(|z|) + L$$

for every $|z| > 1$ since τ is an increasing function. Moreover, using the continuity of τ , there is $D_1 > 0$ such that $\tau(1 + |z|) \leq D_1$ if $|z| \leq 1$. So, we have

$$\tau(1 + |z|) \leq L\tau(|z|) + L + D_1, \quad \forall z \in \mathbb{C}^n. \quad (3.5)$$

Consequently, using condition (γ) , there exists $D_2 \geq 0$ such that

$$2C_2 \log(1 + |z|) \leq L\tau(|z|) + L + D_2, \quad \forall z \in \mathbb{C}^n. \quad (3.6)$$

If we use (3.4), (3.5) and (3.6), we get

$$\begin{aligned} |f_j(z)| &\leq C_1 \exp \left(\sup_{|v - z| \leq 1} \tilde{\tau}(v) + C_2 \log(1 + |z|^2) \right) \\ &\leq C_1 \exp(\tau(1 + |z|) + 2C_2 \log(1 + |z|)) \\ &\leq C_1 \exp(L\tau(|z|) + L + D_1 + L\tau(|z|) + L + D_2) \\ &\leq D_3 \exp(2L\tau(|z|)) \end{aligned}$$

for every $z \in \mathbb{C}^n$, where we have set $D_3 := C_1 \exp(2L + D_1 + D_2)$.

Moreover, since $0 \in K$, we have $h_K(x) \geq 0$ for every $x \in \mathbb{R}^n$. Therefore, for every $m \in \mathbb{N}$ fixed, we get

$$\begin{aligned} |f_j|_{K, \frac{1}{m}}^\omega &= \sup_{z \in \mathbb{C}^n} |f_j(z)| \exp\left(-h_K(\Im z) - \frac{\omega(|z|)}{m}\right) \\ &\leq D_3 \sup_{z \in \mathbb{C}^n} \exp\left(2L\tau(|z|) - h_K(\Im z) - \frac{\omega(|z|)}{m}\right) \\ &\leq D_3 \sup_{z \in \mathbb{C}^n} \exp\left(2L\tau(|z|) - \frac{\omega(|z|)}{m}\right) \end{aligned}$$

for every $j \in \mathbb{N}$. We know that $\tau(t) = o(\omega(t))$ as t tends to infinity and consequently, the function $x \in [0, +\infty) \mapsto 2L\tau(x) - \frac{\omega(x)}{m}$ is bounded from above. This implies that

$$f_j \in \underset{m \in \mathbb{N}}{\text{proj}} \mathcal{A}^\omega \left(K, \frac{1}{m} \right), \quad \forall j \in \mathbb{N}.$$

In particular, we have also got the existence of a constant $D > 0$ such that

$$|f_j|_{K, \frac{1}{m_0}}^\omega \leq D \quad \forall j \in \mathbb{N}. \quad (3.7)$$

On the other hand, τ is increasing and consequently we have

$$\inf_{|v-a_j| \leq 1} \tilde{\tau}(v) \geq \tau(j-1)$$

for every $j \in \mathbb{N}$. Moreover, we have that $\Im a_j = 0$ for every $j \in \mathbb{N}$. Using (3.3), the condition (α) and the assumption that τ is increasing, we get then

$$\begin{aligned} \|f_j\|_{K', m'_0}^\sigma &\geq |f_j(a_j)| \exp\left(-h_{K'}(\Im a_j) - m'_0 \sigma(j)\right) \\ &\geq \exp\left(\tau(j-1) - n \log(1+j^2) - m'_0 \sigma(j)\right) \\ &\geq \exp\left(\tau\left(\frac{j}{2}\right) - 2n \log(1+j) - m'_0 \sigma(j)\right) \\ &\geq \exp\left(\frac{\tau(j)}{L} - 1 - 2n \log(1+j) - m'_0 \sigma(j)\right) \\ &= \exp\left(\frac{\tau(j)}{L} \left(1 - \frac{L}{\tau(j)} - 2Ln \frac{\log(1+j)}{\tau(j)} - m'_0 L \frac{\sigma(j)}{\tau(j)}\right)\right) \end{aligned}$$

for every $j \geq 2$. Moreover, from the condition (γ) and the assumption $\sigma(t) = o(\tau(t))$, the term

$$\frac{L}{\tau(j)} + 2Ln \frac{\log(1+j)}{\tau(j)} + m'_0 L \frac{\sigma(j)}{\tau(j)}$$

converges to 0 as j tends to infinity and therefore, there is $J \in \mathbb{N}$ such that

$$|f_j|_{K', m'}^\sigma \geq \exp\left(\frac{\tau(j)}{2L}\right)$$

for every $j \geq J$. Combining this with the relations (3.2) and (3.7), we finally get

$$\exp\left(\frac{\tau(j)}{2L}\right) \leq CD$$

for $j \geq J$. Taking $j \rightarrow +\infty$, we obtain a contradiction. \square

3.4.4 Generic results

In this subsection, we handle the same kind of questions of genericity than those presented in the case of Denjoy-Carleman classes.

Definition 3.4.16. Given a weight sequence ω , we say that a function is *nowhere in $\mathcal{E}_{\{\omega\}}$* if its restriction to any open and non-empty subset Ω of \mathbb{R}^n never belongs to $\mathcal{E}_{\{\omega\}}(\Omega)$.

We will obtain generic results about the class of functions which are in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ but nowhere in $\mathcal{E}_{\{\omega\}}$.

Unlike the case of weight sequences, if $\omega \triangleleft \sigma$, we have obtained the strict inclusion of $\mathcal{E}_{\{\omega\}}(\Omega)$ into $\mathcal{E}_{(\sigma)}(\Omega)$ without exhibiting a particular function. The construction of a function of $\mathcal{E}_{(\sigma)}(\Omega)$ which is nowhere in $\mathcal{E}_{\{\omega\}}$ is therefore more complicated, but it will be obtained thanks to the following results.

Lemma 3.4.17. [63] *Let ω and σ be two weight functions such that $\omega \triangleleft \sigma$. Fix $x \in \mathbb{R}^n$, $r, m \in \mathbb{N}$ and define $b_r := B(x, \frac{1}{r})$. Then the set*

$$E(x, r, m) = \left\{ f \in \mathcal{E}_{(\sigma)}(\mathbb{R}^n) : \sup_{\alpha \in \mathbb{N}_0^n} \sup_{y \in \overline{b_r}} |D^\alpha f(y)| \exp\left(-\frac{1}{m} \varphi_\omega^*(m|\alpha|)\right) < +\infty \right\}$$

is a proper vector subspace of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$.

Proof. It is clear that the set $E(x, r, m)$ is a vector subspace of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$. Moreover, Proposition 3.4.15 provides a function $f \in \mathcal{E}_{(\sigma)}(b_r) \setminus \mathcal{E}_{\{\omega\}}(b_r)$ so that there is a compact K included in b_r such that

$$\sup_{\alpha \in \mathbb{N}_0^n} \sup_{y \in K} |D^\alpha f(y)| \exp\left(-\frac{1}{m} \varphi_\omega^*(m|\alpha|)\right) = +\infty$$

for every $m \in \mathbb{N}$. Multiplying f by any function of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ with compact support and identically equal to 1 on K , we get a function of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which does not belong to $E(x, r, m)$. This gives the conclusion. \square

Proposition 3.4.18. [63] *Let ω and σ be two weight functions such that $\omega \triangleleft \sigma$. The set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which are nowhere in $\mathcal{E}_{\{\omega\}}$ is prevalent in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$.*

Proof. Consider a countable dense subset $\{x_p : p \in \mathbb{N}\}$ in \mathbb{R}^n . The set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which are somewhere in $\mathcal{E}_{\{\omega\}}$ is given by

$$\bigcup_{p \in \mathbb{N}} \bigcup_{r \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} E(x_p, r, m),$$

using the notation of Lemma 3.4.17. As done previously, since any countable union of shy sets is shy ([82]), it is enough to prove that $E(x_p, r, m)$ is shy for every $p, r, m \in \mathbb{N}$. Remark that $E(x_p, r, m)$ is a Borel subset of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$. Indeed, we have that the set $E(x_p, r, m)$ is the union

$$\bigcup_{s \in \mathbb{N}} \left\{ f \in \mathcal{E}_{(\sigma)}(\mathbb{R}^n) : \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in \overline{b_{p,r}}} |D^\alpha f(x)| \exp\left(-\frac{1}{m} \varphi_\omega^*(m|\alpha|)\right) \leq s \right\},$$

where $b_{p,r}$ denotes the open ball $B(x_p, r)$, and an easy computation shows that every set of the countable union is closed in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$. We get the conclusion using Lemma 1.2.7 and Lemma 3.4.17. \square

A prevalent subset is not empty (it is even dense in the considered space, see Proposition 1.2.5) and therefore, we get the following corollary.

Corollary 3.4.19. [63] *For every weight functions ω and σ such that $\omega \triangleleft \sigma$, there exists a function of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which is nowhere in $\mathcal{E}_{\{\omega\}}$.*

Proposition 3.4.20. [63] *Let ω and σ be two weight functions such that $\omega \triangleleft \sigma$. The set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which are nowhere in $\mathcal{E}_{\{\omega\}}$ is residual in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$.*

Proof. From the previous proof, we know that the set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which are somewhere in $\mathcal{E}_{\{\omega\}}$ is a countable union of sets closed in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$. Moreover, each closed set has empty interior since it is included in $E(x_p, r, m)$ which is a proper vector subspace of the locally convex space $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$. \square

Proposition 3.4.21. [63] *Let ω and σ be two weight functions such that $\omega \triangleleft \sigma$. The set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which are nowhere in $\mathcal{E}_{\{\omega\}}$ is lineable.*

Proof. Since $\omega \triangleleft \sigma$, using Lemma 3.4.13, we get a weight function $\omega^{(1)}$ which satisfies $\omega \triangleleft \omega^{(1)} \triangleleft \sigma$. Repeating this procedure, we construct recursively a sequence $(\omega^{(p)})_{p \in \mathbb{N}}$ of weight functions such that

$$\omega \triangleleft \omega^{(1)} \triangleleft \dots \triangleleft \omega^{(p)} \triangleleft \omega^{(p+1)} \triangleleft \sigma$$

for every $p \in \mathbb{N}$. For every $p \in \mathbb{N}$, Corollary 3.4.19 gives a function $g_p \in \mathcal{E}_{(\omega^{(2p+1)})}(\mathbb{R}^n)$ which is nowhere in $\mathcal{E}_{\{\omega^{(2p)}\}}$. In particular, every g_p is in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$. Moreover, the functions $g_p, p \in \mathbb{N}$, are linearly independent. Indeed, assume there exist $\alpha_1, \dots, \alpha_N \in \mathbb{C}$ with $\alpha_N \neq 0$ and $p_1 < \dots < p_N$ such that $\sum_{j=1}^N \alpha_j g_{p_j} = 0$. Then

$$g_{p_N} = \frac{-1}{\alpha_N} \sum_{j=1}^{N-1} \alpha_j g_{p_j}$$

so that $g_{p_N} \in \mathcal{E}_{\{\omega^{(2p_N)}\}}(\mathbb{R}^n)$ since $\mathcal{E}_{(\omega^{(2p_j+1)})}(\mathbb{R}^n) \subseteq \mathcal{E}_{\{\omega^{(2p_N)}\}}(\mathbb{R}^n)$ for every $j \leq N-1$, which is impossible.

With the same technique, let us also show that every non-zero linear combination of the functions $g_p, p \in \mathbb{N}$, is nowhere in $\mathcal{E}_{\{\omega\}}$. Let $\alpha_1, \dots, \alpha_N \in \mathbb{C}$ with $\alpha_N \neq 0, p_1 < \dots < p_N$ and

$$G = \sum_{j=1}^N \alpha_j g_{p_j}.$$

If there is an open set Ω such that G belongs to $\mathcal{E}_{\{\omega\}}(\Omega) \subseteq \mathcal{E}_{\{\omega^{(2p_N)}\}}(\Omega)$, then the function

$$g_{p_N} = \frac{1}{\alpha_N} \left(G - \sum_{j=1}^{N-1} \alpha_j g_{p_j} \right)$$

belongs to $\mathcal{E}_{\{\omega^{(2p_N)}\}}(\Omega)$, which is impossible. This concludes the proof. \square

As for the case of classes of ultradifferentiable functions defined using weight sequences, we have the following result of density.

Lemma 3.4.22. [77] *For each weight function ω such that $\omega(t) = o(t)$ as t tends to infinity and each open subset Ω of \mathbb{R}^n , the polynomials form a dense subset of $\mathcal{E}_{(\omega)}(\Omega)$.*

Consider two weight functions ω and σ such that $\omega \triangleleft \sigma$. As done previously, it is easy to see that the set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which are nowhere in $\mathcal{E}_{\{\omega\}}$ is stronger than the set of polynomials. From the last lemma, using Remark 3.4.9, the set of polynomials is dense-lineable in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$. Therefore, we can apply Proposition 1.3.4 and Remark 1.3.5 to obtain the following result.

Proposition 3.4.23. [63] *Let ω and σ be two weight functions such that $\omega \triangleleft \sigma$. The set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which are nowhere in $\mathcal{E}_{\{\omega\}}$ is dense-lineable in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$.*

3.5 More with weight matrices

Recently, Rainer and Schindl [120] proved that $\mathcal{E}_{[\omega]}(\Omega)$ can be represented as locally convex space through intersections and unions of ultradifferentiable classes defined by means of weight sequences. More precisely, given a weight function ω , we define a family of associated weight sequences $M^{(l)}$, $l > 0$, by setting

$$M_j^{(l)} = \exp\left(\frac{1}{l}\varphi_\omega^*(lj)\right), \quad j \in \mathbb{N}_0.$$

Then, as it will be stated in Proposition 3.5.4, we have

$$\mathcal{E}_{\{\omega\}}(\Omega) = \bigcap_{K \subseteq \Omega} \bigcup_{l > 0} \mathcal{E}_{\{M^{(l)}\}}(K) \quad \text{and} \quad \mathcal{E}_{(\omega)}(\Omega) = \bigcap_{l > 0} \mathcal{E}_{(M^{(l)})}(\Omega).$$

Following this representation, Rainer and Schindl [120] have introduced new spaces of ultradifferentiable functions using weight matrices.

Definition 3.5.1. A *weight matrix* $\mathcal{M} = \{M^{(l)} : l > 0\}$ is a family of log-convex weight sequences $M^{(l)} = (M_j^{(l)})_{j \in \mathbb{N}_0}$ satisfying $M_0^{(l)} = 1$ and $M^{(l)} \leq M^{(l')}$ if $0 < l \leq l'$, i.e. $M_j^{(l)} \leq M_j^{(l')}$ for every $j \in \mathbb{N}_0$.

If ω is a weight function and if $M_j^{(l)} = \exp\left(\frac{1}{l}\varphi_\omega^*(lj)\right)$ for every $j \in \mathbb{N}_0$, $l > 0$, then $\mathcal{M} = \{M^{(l)} : l > 0\}$ is a weight matrix. We say that it is the *weight matrix associated to ω* .

Before introducing classes of ultradifferentiable functions defined with weight matrices, let us recall that, given a weight sequence M , $\mathcal{E}_{M,h}(K)$ denotes the space of functions $f \in \mathcal{C}^\infty(K)$ such that

$$\|f\|_{K,h}^M := \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}} < +\infty.$$

Definition 3.5.2. Let \mathcal{M} be a weight matrix and let Ω be an open subset of \mathbb{R}^n . The space $\mathcal{E}_{\{\mathcal{M}\}}(\Omega)$ of \mathcal{M} -ultradifferentiable functions of Roumieu type on Ω is defined by

$$\mathcal{E}_{\{\mathcal{M}\}}(\Omega) := \bigcap_{K \subseteq \Omega} \bigcup_{l > 0} \bigcup_{h > 0} \mathcal{E}_{M^{(l)},h}(K).$$

Similarly, the space $\mathcal{E}_{(\mathcal{M})}(\Omega)$ of \mathcal{M} -ultradifferentiable functions of Beurling type on Ω is defined by

$$\mathcal{E}_{(\mathcal{M})}(\Omega) := \bigcap_{l > 0} \mathcal{E}_{(M^{(l)})}(\Omega).$$

Those spaces are endowed with their natural topology through the representations

$$\mathcal{E}_{\{\mathcal{M}\}}(\Omega) = \text{proj}_{\overline{K \subseteq \Omega}} \text{ind}_{l>0} \text{ind}_{h>0} \mathcal{E}_{M^{(l)},h}(K) \quad \text{and} \quad \mathcal{E}_{(\mathcal{M})}(\Omega) = \text{proj}_{\overline{K \subseteq \Omega}} \text{proj}_{l>0} \text{proj}_{h>0} \mathcal{E}_{M^{(l)},h}(K).$$

Remark 3.5.3. Intersections of non-quasianalytic ultradifferentiable classes have already been studied by several authors. Among others, let us mention Chaumat and Chollet [52], Beugendre [29, 30] and Schmets and Valdivia [129, 130].

If \mathcal{M} is a weight matrix, we have assumed that $M^{(l)} \leq M^{(l')}$ if $l \leq l'$. Therefore, all the occurring limits are countable. In the Roumieu case, we can restrict ourselves to the inductive limits over $l \in \mathbb{N}$ and $h \in \mathbb{N}$, and we can take a countable covering of Ω by compact sets. In the Beurling case, we take $l = \frac{1}{n}$ for $n \in \mathbb{N}$, $h \in \mathbb{N}$ and we take again a countable covering of Ω by compact sets. Therefore, the space $\mathcal{E}_{(\mathcal{M})}(\Omega)$ is a Fréchet space.

As done previously, if a statement holds for both $\mathcal{E}_{\{\mathcal{M}\}}(\Omega)$ and $\mathcal{E}_{(\mathcal{M})}(\Omega)$, we will write $\mathcal{E}_{[\mathcal{M}]}\Omega$.

Among the spaces $\mathcal{E}_{[\mathcal{M}]}\Omega$, we recover the spaces $\mathcal{E}_{[M]}\Omega$ defined with weight sequences by taking $\mathcal{M} = \{M\}$. Moreover, the spaces $\mathcal{E}_{[\omega]}\Omega$ defined with weight functions are also recovered, as proved by Rainer and Schindl [120].

Proposition 3.5.4. [120] *Let ω be a weight function and denote by \mathcal{M} its associated weight matrix. For any open subset Ω of \mathbb{R}^n ,*

$$\mathcal{E}_{[\omega]}\Omega = \mathcal{E}_{[\mathcal{M}]}\Omega$$

as locally convex spaces.

Let us mention nevertheless that in [120], it is proved that there exist weight matrix spaces $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R})$ and $\mathcal{E}_{(\mathcal{M})}(\mathbb{R})$ which don't coincide with $\mathcal{E}_{\{M\}}(\mathbb{R})$, $\mathcal{E}_{\{\omega\}}(\mathbb{R})$, $\mathcal{E}_{(M)}(\mathbb{R})$, $\mathcal{E}_{(\omega)}(\mathbb{R})$ for any weight sequence M and any weight function ω .

As done in the case of weight sequences and weight functions, let us define a relation on weight matrices, following [120].

Definition 3.5.5. Let \mathcal{M} and \mathcal{N} be two weight matrices. We write $\mathcal{M} \triangleleft \mathcal{N}$ if $M \triangleleft N$ for every $M \in \mathcal{M}$ and every $N \in \mathcal{N}$.

Of course, $\mathcal{M} \triangleleft \mathcal{N}$ implies $\mathcal{E}_{\{\mathcal{M}\}}(\Omega) \subseteq \mathcal{E}_{\{\mathcal{N}\}}(\Omega)$.

Definition 3.5.6. We say that a function is *nowhere in $\mathcal{E}_{\{\mathcal{M}\}}$* if its restriction to any open and non-empty subset Ω of \mathbb{R} never belongs to $\mathcal{E}_{\{\mathcal{M}\}}(\Omega)$.

As in the previous sections, if $\mathcal{M} \triangleleft \mathcal{N}$, we wish to characterize the size of functions of $\mathcal{E}_{(\mathcal{N})}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{\mathcal{M}\}}$. Let us start by the following lemma, inspired by Schindl [127] (Lemma 9.4.1).

Lemma 3.5.7. *Let \mathcal{M} and \mathcal{N} be two weight matrices. If $\mathcal{M} \triangleleft \mathcal{N}$, then there exists a weight sequence L such that*

- $M \triangleleft L$ for every $M \in \mathcal{M}$,
- $L \triangleleft N$ for every $N \in \mathcal{N}$.

Proof. By assumption, for every $n \in \mathbb{N}$, $M^{(n)} \triangleleft N^{(1/n)}$. Therefore, there is $C_n > 0$ such that

$$M_j^{(n)} \leq C_n \frac{1}{n^j} N_j^{(1/n)}, \quad \forall j \in \mathbb{N}_0.$$

Let us fix $A > 1$. For every $n \in \mathbb{N}$, we choose $j_n \in \mathbb{N}$ such that $C_n \leq A^{j_n}$. Of course, we can assume that the sequence $(j_n)_{n \in \mathbb{N}}$ is strictly increasing. We define the sequence L by setting

$$L_j = \begin{cases} \sqrt{M_j^{(1)} N_j^{(1)}} & \text{if } 0 \leq j < j_1, \\ \sqrt{M_j^{(n)} N_j^{(1/n)}} & \text{if } j_n \leq j < j_{n+1}. \end{cases}$$

Let us fix $l > 0$. First, let us show that $M^{(l)} \triangleleft L$. Indeed, let us fix $\rho > 0$ and $n_0 \geq l$ such that $A \leq \rho^2 n_0$. If $j_n \leq j < j_{n+1}$ with $n \geq n_0$, we have

$$\frac{M_j^{(l)}}{L_j} \leq \frac{M_j^{(n)}}{\sqrt{M_j^{(n)} N_j^{(1/n)}}} = \sqrt{\frac{M_j^{(n)}}{N_j^{(1/n)}}} \leq \sqrt{C_n \frac{1}{n^j}} \leq \sqrt{A^{j_n} \frac{1}{n^j}} \leq \left(\frac{A}{n_0}\right)^{\frac{j}{2}} \leq \rho^j$$

and therefore $M_j^{(l)} \leq \rho^j L_j$ for every $j \geq j_{n_0}$.

Secondly, let us show that $L \triangleleft N^{(l)}$. Let us fix $\rho > 0$ and let us choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} \leq l$ and $A \leq \rho^2 n_0$. Then, if j is such that $j_n \leq j < j_{n+1}$ with $n \geq n_0$, we have

$$L_j = \sqrt{M_j^{(n)} N_j^{(1/n)}} \leq \sqrt{C_n \frac{1}{n^j} N_j^{(1/n)}} \leq \rho^j N_j^{(l)}.$$

□

Remark 3.5.8. Since the weight sequences of \mathcal{M} and \mathcal{N} are all log-convex, we can assume that L is log-convex by taking its largest log-convex minorant.

Lemma 3.5.9. *Let \mathcal{M} be a weight matrix. The polynomials form a dense subset of $\mathcal{E}_{(\mathcal{M})}(\Omega)$ for any open subset Ω of \mathbb{R}^n .*

Proof. This is clear since we know from Proposition 3.2.21 that the polynomials form a dense subspace of each one of the spaces $\mathcal{E}_{(M^{(v)})}(\Omega)$. □

Proposition 3.5.10. *Let \mathcal{M} and \mathcal{N} be two weight matrices such that $\mathcal{M} \triangleleft \mathcal{N}$. If there exists $M \in \mathcal{M}$ such that M is non-quasianalytic, the set of functions of $\mathcal{E}_{(\mathcal{N})}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is \mathfrak{c} -dense-lineable.*

Proof. Since $\mathcal{M} \triangleleft \mathcal{N}$, Lemma 3.5.7 gives a weight sequence L such that $M \triangleleft L$ for every $M \in \mathcal{M}$ and $L \triangleleft N$ for every $N \in \mathcal{N}$. Consider the weight matrix $\mathcal{L} = \{L\}$. Then $\mathcal{L} \triangleleft \mathcal{N}$ and applying again Lemma 3.5.7, we get a weight sequence P such that $L \triangleleft P$ and $P \triangleleft N$ for every $N \in \mathcal{N}$. Moreover, remark that since there exists $M \in \mathcal{M}$ such that M is non-quasianalytic, L is also non-quasianalytic.

If we use Lemma 3.3.4, we get a subspace \mathcal{D} of functions of $\mathcal{E}_{(P)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$. Remark that

$$\mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{\{L\}}(\Omega) \quad \text{and} \quad \mathcal{E}_{(P)}(\Omega) \subseteq \mathcal{E}_{(\mathcal{N})}(\Omega)$$

for every open subset Ω of \mathbb{R} . It follows that the set of functions of $\mathcal{E}_{(\mathcal{N})}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{L\}}$ is \mathfrak{c} -lineable. Lemma 3.5.9, Proposition 1.3.4 and Remark 1.3.5 give the conclusion. □

Before we state a corollary of this result, let us first recall this result of Schindl [127].

Lemma 3.5.11. [127] *Let ω and σ be two weight functions and denote by \mathcal{M} and \mathcal{N} their associated weight matrices. If $\omega \triangleleft \sigma$, then $\mathcal{M} \triangleleft \mathcal{N}$.*

Let us also observe that Denjoy-Carleman's theorem implies that $\mathcal{E}_{\{\mathcal{M}\}}(\Omega)$ contains non-zero functions with compact support if and only if there exists $M \in \mathcal{M}$ such that M is non-quasianalytic. Therefore, given a weight function ω and its associated weight matrix \mathcal{M} , since $\mathcal{E}_{\{\omega\}}(\Omega)$ contains non-zero functions with compact support, there is $M \in \mathcal{M}$ such that M is non quasi-analytic (it is even true for any $M \in \mathcal{M}$, see [127]).

This observation, Lemma 3.5.11 and of Proposition 3.5.10 directly give an improvement of the result 3.4.23 in the case $n = 1$. The dense-lineability obtained is now maximal.

Corollary 3.5.12. *Let ω and σ be two weight functions such that $\omega \triangleleft \sigma$. The set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{\omega\}}$ is \mathfrak{c} -dense-lineable in $\mathcal{E}_{(\sigma)}(\mathbb{R})$.*

We get also the following generalization of the results 3.3.2 and 3.3.3 presented for weight sequences, and the results 3.4.18 and 3.4.20 for weight functions.

Proposition 3.5.13. *Let \mathcal{M} and \mathcal{N} be two weight matrices such that $\mathcal{M} \triangleleft \mathcal{N}$. If there exists $M \in \mathcal{M}$ such that M is non-quasianalytic, the set of functions of $\mathcal{E}_{(\mathcal{N})}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{\mathcal{M}\}}$ is prevalent and residual in $\mathcal{E}_{(\mathcal{N})}(\mathbb{R})$.*

Proof. The set of functions of $\mathcal{E}_{(\mathcal{N})}(\mathbb{R})$ which are somewhere in $\mathcal{E}_{\{\mathcal{M}\}}$ is given by

$$\bigcup_{l \in \mathbb{N}} \bigcup_{I \subseteq \mathbb{R}} \bigcup_{m \in \mathbb{N}} E(l, I, m),$$

where I denotes rational subintervals of \mathbb{R} and $E(l, I, m)$ is the set defined by

$$\left\{ f \in \mathcal{E}_{(\mathcal{N})}(\mathbb{R}) : \exists C > 0 \text{ such that } \sup_{x \in I} |D^j f(x)| \leq C m^j M_j^{(l)}, \forall j \in \mathbb{N}_0 \right\}.$$

It is direct to check that $E(l, I, m)$ is a vector subspace of $\mathcal{E}_{(\mathcal{N})}(\mathbb{R})$ which is proper using Proposition 3.5.10. Moreover, it is a Borel subset of $\mathcal{E}_{(\mathcal{N})}(\mathbb{R})$ since

$$E(l, I, m) = \bigcup_{s \in \mathbb{N}} \left\{ f \in \mathcal{E}_{(\mathcal{N})}(\mathbb{R}) : \sup_{x \in I} |D^j f(x)| \leq s m^j M_j^{(l)}, \forall j \in \mathbb{N}_0 \right\}$$

which is a countable union of closed sets in $\mathcal{E}_{(\mathcal{N})}(\mathbb{R})$. Lemma 1.2.7 gives the prevalence. Each closed set $\left\{ f \in \mathcal{E}_{(\mathcal{N})}(\mathbb{R}) : \sup_{x \in I} |D^j f(x)| \leq s m^j M_j^{(l)}, \forall j \in \mathbb{N}_0 \right\}$ has empty interior since it is included in $E(l, I, m)$ which is a proper vector subspace of the locally convex space $\mathcal{E}_{(\mathcal{N})}(\mathbb{R})$. The residuality follows. \square

Let us end this chapter with a few words about algebrability. It can be obtained directly using the exponential-like method, up to an additional assumption on the weight matrix. Given a weight sequence m , we define the sequence m° by setting $m_0^\circ = 1$ and

$$m_k^\circ = \max\{m_j m_{\alpha_1} \dots m_{\alpha_j} : \alpha_i \in \mathbb{N}, \alpha_1 + \dots + \alpha_j = k\}, \quad \forall k \in \mathbb{N}.$$

Following Rainer and Schindl [120], we say that a weight matrix \mathcal{M} satisfies the property $(\mathfrak{M}_{\{FdB\}})$ if for every $l > 0$, there is $l' > 0$ such that $(m^{(l)})^\circ \preceq m^{(l')}$, where $m_k^{(l)} := \frac{M_k^{(l)}}{k!}$ and $m_k^{(l')} = \frac{M_k^{(l')}}{k!}$ for every $k \in \mathbb{N}_0$.

Proposition 3.5.14. [120] *Let \mathcal{M} be a weight matrix and let Ω be an open subset of \mathbb{R} . Then $\mathcal{E}_{\{\mathcal{M}\}}(\Omega)$ is stable under composition if and only if \mathcal{M} satisfies the property $(\mathfrak{M}_{\{FdB\}})$.*

Moreover, remark that if \mathcal{M} is a weight matrix for which there is a non-quasianalytic weight sequence $M \in \mathcal{M}$, then $\mathcal{E}_{\{M\}}(\Omega)$, and therefore $\mathcal{E}_{\{\mathcal{M}\}}(\Omega)$, contains the set of analytic functions on Ω (see Remark 3.2.15).

Proposition 3.5.15. *Let \mathcal{M} and \mathcal{N} be two weight matrices such that $\mathcal{M} \triangleleft \mathcal{N}$. If there exists $M \in \mathcal{M}$ such that M is non-quasianalytic and if \mathcal{M} satisfies the property $(\mathfrak{M}_{\{FdB\}})$, then the set of functions of $\mathcal{E}_{\{\mathcal{N}\}}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{\mathcal{M}\}}$ is strongly-c-algebrable.*

Proof. Using Proposition 3.5.10, we can consider a function F of $\mathcal{E}_{\{\mathcal{N}\}}(\mathbb{R})$ which is nowhere in $\mathcal{E}_{\{\mathcal{M}\}}$. Let f be an exponential-like function. Assume that there exists an open subset Ω of \mathbb{R} such that $g = f \circ F$ belongs to $\mathcal{E}_{\{\mathcal{M}\}}(\Omega)$. By Lemma 1.3.12, there is an open subset $V \subseteq F(V)$ on which f is invertible. Hence, $F = f^{-1} \circ g$ belongs to $\mathcal{E}_{\{\mathcal{M}\}}(F^{-1}(\Omega) \cap V)$ using Proposition 3.5.14. This is a contradiction and the conclusion follows from Proposition 1.3.13. \square

In the case of a weight matrix associated with a weight function ω , the property $(\mathfrak{M}_{\{FdB\}})$ can be written

$$\exists C > 0 \exists t_0 > 0 : \omega(\lambda t) \leq C\lambda\omega(t) \quad \forall t \geq t_0, \forall \lambda \geq 1. \quad (\alpha_0)$$

So, we get that $\mathcal{E}_{\{\omega\}}(\Omega)$ is stable under composition if and only if ω satisfies (α_0) [120]. This result had already been proved by Fernández and Galbis [68].

Corollary 3.5.16. *Let ω and σ be two weight functions such that $\omega \triangleleft \sigma$. If ω satisfies (α_0) , then the set of functions of $\mathcal{E}_{\{\sigma\}}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{\omega\}}$ is strongly-c-algebrable.*

Finally, in the case of weight sequences, we obtain this last result.

Corollary 3.5.17. *Assume that N and M are two weight sequences such that M is non-quasianalytic and $M \triangleleft N$. If $m^\circ \preceq m$, where $m_k := \frac{M_k}{k!}$ for every $k \in \mathbb{N}_0$, then the set of functions of $\mathcal{E}_{\{N\}}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is strongly-c-algebrable.*

Part II

Revisiting S^ν spaces with wavelet leaders

Chapter 4

Preliminaries

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4.1 Introduction

Multifractal analysis of functions started to be developed by physicists in the context of fully developed turbulence. More precisely, in the 1940's, Kolmogorov predicted that the scaling function $\eta(p)$ of the velocity field v of a turbulent fluid included in a domain U , defined by

$$\int_U |v(x+h) - v(x)|^p dx \sim |h|^{\eta(p)} \quad \text{when } |h| \rightarrow 0,$$

should be linear: $\eta(p) = \frac{p}{3}$. Subsequent experiences showed that η is actually a strictly concave function, which is believed to be independent of the considered fluid and central to the understanding of turbulence. This problem was addressed by Kolmogorov and Mandelbrot among others. Parisi and Frisch [117] proposed an explanation by interpreting the nonlinearity of η as the signature of the presence of several kinds of Hölder singularities (the regularity of the velocity of a turbulent fluid fluctuates widely from point to point). More precisely, they proposed a formula which is expected to connect together the scaling function and the singularities through a Legendre transform. This phenomenon of large variability in the local regularity received the name multifractality. Let us be more precise about this notion of regularity.

Definition 4.1.1. Let us fix $x_0 \in \mathbb{R}^n$ and $\alpha \geq 0$. A locally bounded function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ belongs to the Hölder space $C^\alpha(x_0)$ if there exist a constant $C > 0$, a polynomial P of degree strictly less than α and a neighborhood V of x_0 such that

$$|f(x) - P(x - x_0)| < C|x - x_0|^\alpha, \quad \forall x \in V.$$

Remark that one has $C^{\alpha+\varepsilon}(x_0) \subseteq C^\alpha(x_0)$ for any $\varepsilon \geq 0$. We consider then the following definition.

Definition 4.1.2. The Hölder exponent of f at x_0 is defined by

$$h_f(x_0) = \sup\{\alpha \geq 0 : f \in C^\alpha(x_0)\}.$$

The Hölder exponent $h_f(x_0)$ gives information about the local regularity of f at x_0 . In particular, the smaller its value is, the less regular the graph of f looks around x_0 .

Some functions have a Hölder exponent which is the same at every point. An example of such a function is given by the Weierstraß function, defined by

$$W : x \in \mathbb{R} \mapsto \sum_{n=0}^{+\infty} a^n \cos(b^n \pi x).$$

for $a \in (0, 1)$ and $b > 0$ such that $ab > 1$.

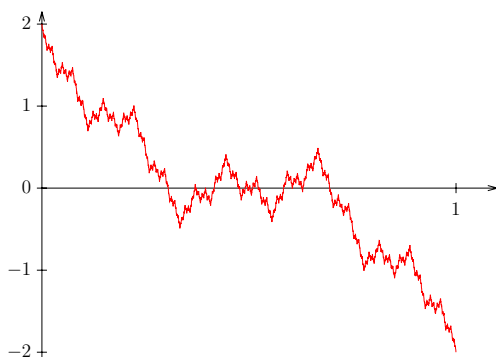


Figure 4.1: The Weierstraß function with parameters $a = 0.5$ and $b = 3$

This application is known as the first published example (1872) of a function that is continuous everywhere but nowhere differentiable [140]. It is among the first examples which contradicted the idea of the time that every continuous function is differentiable except on a set of isolated points. Hardy [76] proved the following result.

Theorem 4.1.3. [76] *The Weierstraß function is continuous but nowhere differentiable. Moreover, its Hölder exponent equals $-\frac{\log a}{\log b}$ at every point.*

Nevertheless, for an arbitrary locally bounded function, the behaviour of the function h_f can be very erratic. In particular, from a practical point of view, it is very difficult to estimate the Hölder exponents of a function obtained from real-life data. Moreover, the knowledge of the function h_f does not give a concrete idea of the distribution of the singularities of f and their importance. This is why iso-Hölder sets

$$E^f(h) = \{x_0 \in \mathbb{R}^n : h_f(x_0) = h\}$$

are usually considered. In order to give some precise meaning to their importance, one has to find a notion of “size”. The Lebesgue measure is not the appropriate notion in

this context: in general, one iso-Hölder set has full Lebesgue measure and the others have a vanishing one. One should therefore use a notion a “size” which allows to distinguish sets with Lebesgue measure zero. Such a tool is supplied by the different “fractal dimensions”: in general, those dimensions are the box dimension or the Hausdorff dimension. Nevertheless, it appears that most functions of interest or sample paths of stochastic processes have dense iso-Hölder sets. The box dimension gives dimension n to any dense set of \mathbb{R}^n and therefore, this notion does not allow to distinguish them. This explains why the Hausdorff dimension is considered. This last notion is recalled briefly in Section 4.2. This leads to the following definition.

Definition 4.1.4. The *multifractal spectrum* of a locally bounded function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the function

$$d_f : [0, +\infty] \rightarrow \{-\infty\} \cup [0, n] : h \mapsto \dim_{\mathcal{H}} E^f(h),$$

where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension.

The multifractal spectrum of a function f gives a geometrical idea about the distribution of the singularities of f . Note that d_f is defined on $[0, +\infty]$ since $h_f(x_0)$ can be infinite. Furthermore, we use the convention that $\dim_{\mathcal{H}}(\emptyset) = -\infty$ so that d_f takes values in $[0, n] \cup \{-\infty\}$. Let us mention that the multifractal spectrum is sometimes called the *Hölder spectrum* or the *spectrum of singularities*.

A very famous example of a function for which the Hölder exponent is not constant is given by the Riemann function, defined on \mathbb{R} by

$$R(x) = \sum_{n=1}^{+\infty} \frac{\sin(n^2\pi x)}{n^2}.$$

Many authors worked on the multifractal properties of this function, see [72, 76] for example, but the computation of the multifractal spectrum of the Riemann function was completed by Jaffard [85].

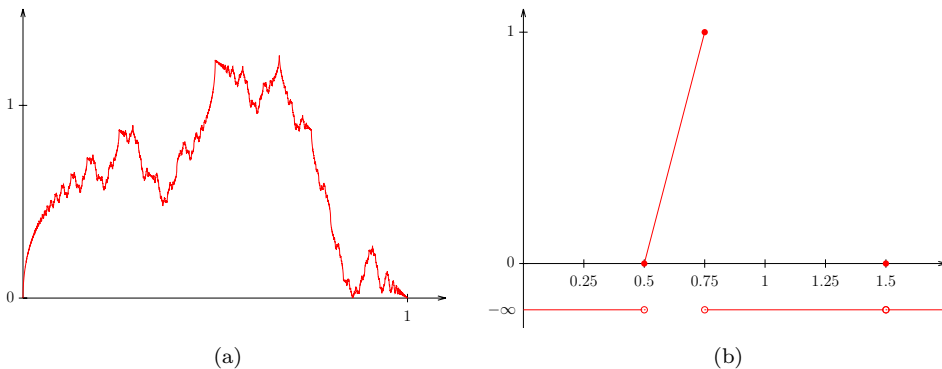


Figure 4.2: The Riemann function (a) and its multifractal spectrum (b)

Theorem 4.1.5. [85] *The multifractal spectrum of the Riemann function R is given by*

$$d_R(h) = \begin{cases} 4h - 2 & \text{if } h \in [\frac{1}{2}, \frac{3}{4}], \\ 0 & \text{if } h = \frac{3}{2}, \\ -\infty & \text{otherwise.} \end{cases}$$

Although the multifractal spectrum of many mathematical functions can be directly determined from their definition, for real-life signals, as mentioned, before the Hölder exponent is expected to be very erratic and the numerical determination of their Hölder regularity is not feasible. Therefore, one cannot expect to have direct access to their spectrum. Moreover, since the definition of the multifractal spectrum involves successive intricate limits, there is no algorithm to directly obtain the spectrum d_f associated with a signal f . In such cases, one has to find an indirect way to compute the spectrum. A *multifractal formalism* is a formula which is expected to yield the spectrum of a function from “global” quantities which are numerically computable. Mathematically, these quantities are interpreted as indicating that the signal belongs to a certain family of function spaces. The validity of such formulas never holds in complete generality. However, three types of verification can be performed [91]:

- The multifractal formalism is proved under additional assumptions on the signal or even for specific functions.
- The multifractal formalism is proved for a generic subset of the function space considered.
- The multifractal formalism is shown to yield an upper bound of the multifractal spectrum of any (uniformly Hölder, see Section 4.3) function.

Several multifractal formalisms based on the wavelet coefficients of a signal have been proposed to estimate its multifractal spectrum [3, 86, 88, 91]. The starting point of all these methods is a wavelet characterization of the Hölder exponent [91] (see Section 4.3). They share the advantage of being easy to compute and relatively stable from a numerical point of view; however, they require the assumption of a uniform Hölder regularity. The most widespread of these formulas is the so-called *thermodynamic multifractal formalism*, proposed by Parisi and Frisch (presented in Section 4.5), and which is based on the computation of scaling exponents derived from the L^p norm of increments of the data, followed by a Legendre transform. This formalism presents two disadvantages: firstly, by construction, it can only hold for spectra that are concave and secondly, it can only yield the increasing part of the spectrum.

The following possibilities have been proposed in order to meet these two problems. Regarding the first one, the use of function spaces, based on large deviating estimates of the repartition of wavelet coefficients (the so-called \mathcal{S}^ν spaces [14], see Section 4.6) allows to deal with non-concave spectra. Regarding the second problem, it has been proposed to replace the role played by wavelet coefficients in the analysis by wavelet leaders (which are local suprema of wavelet coefficients, see [92, 94] and Section 4.7). In particular, it has been shown that a Legendre transform formula based on such quantities allows to recover the decreasing part of the spectrum for many multifractal models (cascades, Lévy processes... see [2, 16, 91]).

Our purpose in this part of the thesis is to combine both approaches and define a new formalism derived from large deviations based on statistics of wavelet leaders. We show

that, as expected, this method allows to access to both the increasing and decreasing parts of non-concave multifractal spectra.

In Section 4.2, we recall the definition and the first properties of the Hausdorff dimension. In Section 4.3, the notion of wavelets is presented as well as the characterization of the Hölder exponent in terms of decay rates of wavelet coefficients. The formalism based on the Frisch-Parisi conjecture is recalled in Section 4.5. We also give the definition and the first topological properties of the \mathcal{S}^ν spaces, as well as the corresponding multifractal formalism in Section 4.6. Finally, in Section 4.7, we define the wavelet leaders of a signal and we present the associated formalism.

Let us end this section by mentioning that the Hölder exponent does not fully describe the local behavior of a function. For example, it does not take into account the oscillating comportment of a function in the neighborhood of a point: at 0, the function “cusp” defined by $x \mapsto |x|^\alpha$ and the oscillating function $x \mapsto |x|^\alpha \sin\left(\frac{1}{|x|}\right)$ have the same Hölder exponent. Other regularity exponents have been introduced to complete the information given by the Hölder exponents, such as the oscillating exponents [4, 93] and the local Hölder exponents [131] for example.

4.2 Hausdorff dimension

In this section, we present the notion of Hausdorff dimension (for more details, we refer the reader to [66]). A first step consists in the introduction of the Hausdorff measures.

Definition 4.2.1. Let $B \subseteq \mathbb{R}^n$ and for any $\varepsilon > 0$, let $\Lambda_\varepsilon(B)$ denote the collection of all countable coverings of B by sets with diameter less than ε . For every $s \geq 0$ and $\varepsilon > 0$, one sets

$$\mathcal{H}_\varepsilon^s(B) = \inf_{(B_j)_{j \in \mathbb{N}} \in \Lambda_\varepsilon(B)} \sum_{j \in \mathbb{N}} \text{diam}(B_j)^s.$$

Since $\mathcal{H}_\varepsilon^s$ is a decreasing function with respect to ε , one can define the *s-dimensional Hausdorff measure* of B by

$$\mathcal{H}^s(B) = \sup_{\varepsilon > 0} \mathcal{H}_\varepsilon^s(B) = \lim_{\varepsilon \rightarrow 0^+} \mathcal{H}_\varepsilon^s(B).$$

This limit exists for any subset B of \mathbb{R}^n , but the limit value can be (and is usually) 0 or $+\infty$. For every $s > 0$, the function \mathcal{H}^s defines an outer measure. Moreover, it can be shown that its restriction to the Borel subsets of \mathbb{R}^n is a measure.

For subsets of \mathbb{R}^n , the n -dimensional Hausdorff measure is related with the n -dimensional Lebesgue measure. More precisely, we have

$$\mathcal{H}^n(B) = c_n \mathcal{L}^n(B) \quad \text{where} \quad c_n = \frac{\pi^{n/2}}{2^n \Gamma(n/2)}.$$

Moreover, Hausdorff measures behave nicely under translations and dilations in \mathbb{R}^n . Given a subset B of \mathbb{R}^n , $\lambda > 0$ and $x \in \mathbb{R}^n$, we set

$$\lambda B = \{\lambda x : x \in B\} \quad \text{and} \quad B + x = \{b + x : b \in B\}.$$

As might be expected, we have

$$\mathcal{H}^s(\lambda B) = \lambda^s \mathcal{H}^s(B) \quad \text{and} \quad \mathcal{H}^s(B + x) = \mathcal{H}^s(B).$$

Such scaling properties are fundamental in the theory of fractals.

Let us remark that for any subset B of \mathbb{R}^n , any $\varepsilon > 0$ and any $0 < s < \gamma$, we have

$$\mathcal{H}_\varepsilon^s(B) \geq \frac{\mathcal{H}_\varepsilon^\gamma(B)}{\varepsilon^{\gamma-s}}.$$

Taking $\varepsilon \rightarrow 0$, we get the following important result.

Proposition 4.2.2. *Let $0 < s < \gamma$.*

1. *If $\mathcal{H}^s(B) < +\infty$, then $\mathcal{H}^\gamma(B) = 0$.*
2. *If $\mathcal{H}^\gamma(B) > 0$, then $\mathcal{H}^s(B) = +\infty$.*

Thus, a graph of $\mathcal{H}^s(B)$ with respect to s shows that there is a critical value of s for which $\mathcal{H}^s(B)$ jumps from $+\infty$ to 0. The critical value is called the Hausdorff dimension of B .

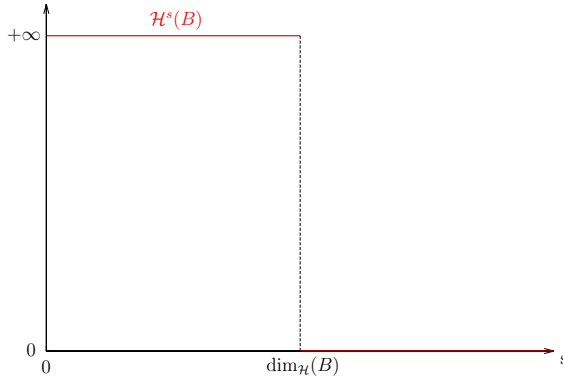


Figure 4.3: The Hausdorff dimension of B

More precisely, we have the following definition.

Definition 4.2.3. The *Hausdorff dimension* $\dim_{\mathcal{H}}(B)$ of a subset B of \mathbb{R}^n is defined by

$$\dim_{\mathcal{H}}(B) = \sup\{s \geq 0 : \mathcal{H}^s(B) = +\infty\}.$$

We use the convention that $\sup(\emptyset) = -\infty$ so that $\dim_{\mathcal{H}}(\emptyset) = -\infty$.

If $s = \dim_{\mathcal{H}}(B)$, then $\mathcal{H}^s(B)$ can be 0 or $+\infty$, or can satisfy $0 < \mathcal{H}^s(B) < +\infty$. Let us now give some properties of the Hausdorff dimension.

Proposition 4.2.4.

1. *If $A \subseteq B \subseteq \mathbb{R}^n$, then $\dim_{\mathcal{H}}(A) \leq \dim_{\mathcal{H}}(B)$.*
2. *If $B \subseteq \mathbb{R}^n$, then $\dim_{\mathcal{H}}(B) \leq n$ and if Ω is an open subset of \mathbb{R}^n , then $\dim_{\mathcal{H}}(\Omega) = n$.*
3. *If $(B_k)_{k \in \mathbb{N}}$ is a sequence of subsets of \mathbb{R}^n , then*

$$\dim_{\mathcal{H}} \left(\bigcap_{k \in \mathbb{N}} B_k \right) = \sup_{k \in \mathbb{N}} (\dim_{\mathcal{H}}(B_k)).$$

4. If B is a countable subset of \mathbb{R}^n , then $\dim_{\mathcal{H}}(B) = 0$.

5. If $B \subseteq \mathbb{R}^n$ satisfies $\dim_{\mathcal{H}}(B) < 1$, then B is totally disconnected.

It can also be shown that if $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ and if there are $c > 0$ and $\alpha > 0$ such that

$$|f(x) - f(y)| \leq c|x - y|^\alpha, \quad \forall x, y \in X,$$

then $\dim_{\mathcal{H}}(f(X)) \leq \frac{1}{\alpha} \dim_{\mathcal{H}}(X)$. The next result follows.

Proposition 4.2.5. *If $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz, then $\dim_{\mathcal{H}}(f(X)) \leq \dim_{\mathcal{H}}(X)$. If f is bi-Lipschitz, i.e. if there exist $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y|, \quad \forall x, y \in X,$$

then $\dim_{\mathcal{H}}(f(X)) = \dim_{\mathcal{H}}(X)$.

This proposition gives a fundamental property of the Hausdorff dimension: it is invariant under bi-Lipschitz transformations.

In order to get an upper bound for the Hausdorff dimension of a set, it is enough to consider a particular covering. In contrast, it is more difficult to get a lower bound directly from the definition. Indeed, all coverings have to be taken into account. The mass distribution principle replaces the consideration of all coverings by the construction of a particular measure.

Proposition 4.2.6 (Mass distribution principle). *Let μ be a probability measure with support included in $B \subseteq \mathbb{R}^n$. Assume that there exist $s > 0$, $C > 0$ and $\varepsilon > 0$ such that*

$$\mu(U) \leq C \operatorname{diam}(U)^s$$

for every set U such that $\operatorname{diam}(U) \leq \varepsilon$. Then $\mathcal{H}^s(B) \geq \frac{\mu(B)}{C}$ and $\dim_{\mathcal{H}}(B) \geq s$.

4.3 Wavelets

As mentioned previously, orthonormal wavelet bases appeared to be a useful tool to study multifractal properties of functions. A first reason is that classical function spaces, such as Sobolev or Besov spaces, can be characterized by conditions on the wavelet coefficients. Another reason is that the Hölder pointwise regularity can also be characterized by decay conditions on the wavelet coefficients. Besides, wavelet bases allow to construct easily functions which satisfy particular properties: for example, in [84], Jaffard constructed functions with prescribed Hölder exponents and more recently, Buczolich and Seuret [45] constructed functions with prescribed multifractal spectrum, see also Chapter 7.

An *orthonormal wavelet basis* of $L^2(\mathbb{R})$ is an orthonormal basis of $L^2(\mathbb{R})$ of the form

$$2^{j/2} \psi(2^j \cdot -k), \quad j, k \in \mathbb{Z},$$

where the function ψ is called the *mother wavelet*. The first construction of an orthonormal wavelet basis is due to Haar [75] in 1910 (the name “wavelet” was not already used). In 1981, Stomberg [135] constructed orthonormal wavelet bases with arbitrary regularities. Wavelets in the Schwartz class were introduced by Lemarié and Meyer [102] in 1986. The classical construction of wavelets using a multiresolution analysis of $L^2(\mathbb{R})$

was performed by Mallat [105] and Meyer [112] in 1989. In 1992, Daubechies [60] constructed wavelets with compact support. We refer the reader to [60, 105, 111, 112] for the construction and the main properties of wavelets. Note that the results exposed here also hold for higher dimensions.

The aim of this section is to state the characterization of the Hölder exponent using decay rates of wavelet coefficients. Given a mother wavelet ψ , we set

$$\psi_{j,k} := \psi(2^j \cdot -k), \quad j, k \in \mathbb{Z}.$$

Any $f \in L^2(\mathbb{R})$ can be decomposed as

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c_{j,k} \psi_{j,k}$$

where

$$c_{j,k} = 2^j \int_{\mathbb{R}} f(x) \psi_{j,k}(x) dx.$$

The values $c_{j,k}$ are called the *wavelet coefficients* of f . The index j is called the scale and k represents the position. Let us remark that we do not choose the L^2 normalization for the wavelets, but rather an L^∞ normalization, which is more appropriate to the study of the Hölder regularity.

We will also need decompositions on biorthogonal wavelet bases, which are a useful extension of orthogonal wavelet bases [55]. A *Riesz basis* of $L^2(\mathbb{R})$ is a collection of functions $(f_n)_{n \in \mathbb{N}}$ such that the vector space spanned by the set $\{f_n : n \in \mathbb{N}\}$ is dense in $L^2(\mathbb{R})$ and for which there are $C_1 > 0, C_2 > 0$ such that

$$C_1 \sum_{n \in \mathbb{N}} a_n^2 \leq \left\| \sum_{n \in \mathbb{N}} a_n f_n \right\|_{L^2(\mathbb{R})}^2 \leq C_2 \sum_{n \in \mathbb{N}} a_n^2$$

for every sequence $(a_n)_{n \in \mathbb{N}}$ of ℓ^2 . *Biorthogonal wavelet bases* are a couple of two Riesz wavelet bases generated respectively by ψ and $\tilde{\psi}$ and such that

$$2^{j/2} 2^{j'/2} \int_{\mathbb{R}} \psi(2^j x - k) \tilde{\psi}(2^{j'} x - k') dx = \delta_{j,j'} \delta_{k,k'}.$$

In that case, any function $f \in L^2(\mathbb{R})$ can be decomposed as

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c_{j,k} \psi_{j,k}$$

where

$$c_{j,k} = 2^j \int_{\mathbb{R}} f(x) \tilde{\psi}_{j,k}(x) dx.$$

Biorthogonal wavelet bases are particularly well adapted to the decomposition of the fractional Brownian motion (see Chapter 5): indeed, well chosen biorthogonal wavelet bases allow to decorrelate the wavelet coefficients of these processes (the wavelet coefficients become independent random variables), and therefore greatly simplify their analysis.

In what follows, since we are interested in local behavior of functions, we will work with periodic functions. We denote by \mathbb{T} the torus \mathbb{R} / \mathbb{Z} and we consider the space $L^2(\mathbb{T})$

of functions of period 1 which locally belong to $L^2(\mathbb{R})$. With the constant function $\phi(x) := 1$, the periodized wavelets

$$\psi_{j,k}^{per} := \sum_{l \in \mathbb{Z}} \psi(2^j(\cdot - l) - k), \quad j \in \mathbb{N}_0, k \in \{0, \dots, 2^j - 1\}$$

form an orthogonal wavelet basis of $L^2(\mathbb{T})$ [58]. The corresponding coefficients

$$c_{j,k}^{per} = 2^j \int_0^1 f(x) \psi_{j,k}^{per}(x) dx$$

are naturally called the *periodized wavelet coefficients*. In this thesis, we shall systematically use periodized wavelets and the corresponding periodized wavelet coefficients. For the sake of simplicity, we will again write them $\psi_{j,k}$ and $c_{j,k}$ and call them again wavelets and wavelet coefficients of the function. Moreover, we denote by Ω the set of complex sequences $\vec{c} = (c_{j,k})_{j \in \mathbb{N}_0, k \in \{0, \dots, 2^j - 1\}}$. It will usually be interpreted as the sequence of wavelet coefficients of a periodic function f .

We will also use the notation ψ_λ to denote the wavelet $\psi_{j,k}$, where λ is the dyadic interval

$$\lambda = \lambda(j, k) = \{x \in \mathbb{R} : 2^j x - k \in [0, 1)\} = \left[\frac{k}{2^j}, \frac{k+1}{2^j} \right).$$

The interval λ gives an indication concerning the localization of the corresponding wavelet ψ_λ [111]. We shall use both notations $c_{j,k}$ and c_λ for the wavelet coefficients. Finally, we denote by Λ the set of all dyadic intervals of $[0, 1)$ and at a given scale $j \in \mathbb{N}_0$, we denote by Λ_j the set of all dyadic intervals of $[0, 1)$ of size 2^{-j} .

Let us now present the important result which relates the pointwise regularity to decay conditions on the wavelet coefficients of a function. First, let us introduce a notion of uniform regularity.

For every $r \geq 0$, the space $C^r(\mathbb{T})$ is the space of Hölder continuous periodic functions of order r . If $r \notin \mathbb{N}$, the wavelet characterization of Hölder spaces, see [111], allows to identify this space with the subspace C^r of Ω composed of sequences satisfying

$$\|\vec{c}\|_{C^r} := \sup_{j \in \mathbb{N}_0} \sup_{k \in \{0, \dots, 2^j - 1\}} 2^{rj} |c_{j,k}| < +\infty.$$

In this thesis, when $r \in \mathbb{N}$, we will also denote by $C^r(\mathbb{T})$ the space of functions satisfying this condition. A function is *uniformly Hölder* if it belongs to a space $C^r(\mathbb{T})$, for an $r > 0$. Note that this is a stronger requirement than continuity.

The pointwise Hölder exponent $h_f(x_0)$ can be characterized by the decay rate of the wavelet coefficients around x_0 . This result is due to Jaffard [83].

Proposition 4.3.1. *If ψ has more than $\lfloor h_f(x_0) \rfloor + 1$ vanishing moments (that is to say $\int_{\mathbb{R}} x^n \psi(x) dx = 0$ for every $n \in \{0, \dots, \lfloor h_f(x_0) \rfloor\}$) and if f is uniformly Hölder, then*

$$h_f(x_0) = \liminf_{j \rightarrow +\infty} \inf_{k \in \{0, \dots, 2^j - 1\}} \frac{\log |c_{j,k}|}{\log(2^{-j} + |k2^{-j} - x_0|)}. \quad (4.1)$$

As mentioned before, the wavelet ψ can be chosen with compact support, see [60]. Nevertheless it introduces technical complications: indeed, if a compactly supported wavelet ψ is used, it has a finite number of vanishing moments so that (4.1) is in concurrence with the regularity of ψ . This justifies the following agreement.

Agreement. In this thesis, the considered mother wavelets ψ always belong to the Schwartz class $\mathcal{S}(\mathbb{R})$, as constructed in [102]. For such wavelets, all the moments of positive order are null (see [60] for example), so that (4.1) holds at every x_0 . Moreover, the assumption $\psi \in \mathcal{S}(\mathbb{R})$ is needed to get results of robustness, see next Section 4.4.

Remark 4.3.2. The condition $\psi \in \mathcal{S}(\mathbb{R})$ is clearly a drawback in applications. However, one can use the following heuristic: if the wavelet basis is r -smooth, then all results hold when dealing with exponent $h < r$.

4.4 Robustness criteria

In what follows, several quantities associated to a function will be defined through its wavelet coefficients. The independence from the sufficiently smooth wavelet basis which is chosen is a natural requirement. In practice, one often uses a stronger requirement but easier to handle which implies that the condition considered has some additional stability. This notion was introduced by Meyer in [111] (Chapter 8.9) as follows.

Definition 4.4.1. If γ is a positive number and if $\lambda = \lambda(j, k)$, $\lambda' = \lambda(j', k')$ are two dyadic intervals, let

$$\omega_\gamma(\lambda, \lambda') = \frac{2^{-(\gamma+2)|j-j'|}}{(1 + 2^{\inf\{j, j'\}} \text{dist}(\lambda, \lambda'))^{\gamma+2}},$$

where $\text{dist}(\lambda, \lambda') = |k2^{-j} - k'2^{-j'}|$. An infinite matrix $A = (A(\lambda, \lambda'))_{(\lambda, \lambda') \in \Lambda \times \Lambda}$ belongs to \mathcal{A}^γ if there exists $C \geq 0$ such that

$$|A(\lambda, \lambda')| \leq C\omega_\gamma(\lambda, \lambda'), \quad \forall \lambda, \lambda' \in \Lambda.$$

We denote by $\|A\|_\gamma$ the infimum of all possible such constants C . A matrix is *almost diagonal* if it belongs to \mathcal{A}^γ for every $\gamma > 0$. Moreover, we say that a matrix is *quasidiagonal* if it is almost diagonal, invertible on l^2 , and if its inverse is also almost diagonal.

Matrices of operators which map an orthonormal wavelet basis in the Schwartz class into another orthonormal wavelet basis in the Schwartz class are quasidiagonal [111]. Let us note that it also holds for biorthogonal wavelet bases. Therefore, in order to check that a condition defined on wavelet coefficients is independent of the chosen wavelet basis (in the Schwartz class), one can check the stronger property that it is invariant under the action of quasidiagonal matrices.

Definition 4.4.2. Let \mathcal{C} be a collection of coefficients indexed by dyadic intervals. A property \mathcal{P} is *linear robust* if the following conditions hold:

- The set of \mathcal{C} 's such that $\mathcal{P}(\mathcal{C})$ holds is a vector space;
- If $\mathcal{P}(\mathcal{C})$ holds, then for any almost diagonal operator \mathbf{M} , $\mathcal{P}(\mathbf{M}\mathcal{C})$ holds.

A property \mathcal{P} is *robust* if the following condition holds: if $\mathcal{P}(\mathcal{C})$ holds, then for any quasidiagonal operator \mathbf{M} , $\mathcal{P}(\mathbf{M}\mathcal{C})$ holds.

4.5 Besov Spaces and the Frisch-Parisi formalism

Let us recall the definition of the Besov spaces $b_{p,\infty}^s$ of sequences.

Definition 4.5.1. For $s \in \mathbb{R}$ and $p > 0$, a sequence $\vec{c} \in \Omega$ belongs to $b_{p,\infty}^s$ if

$$\|\vec{c}\|_{b_{p,\infty}^s} := \sup_{j \in \mathbb{N}_0} 2^{(s-\frac{1}{p})j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^p \right)^{\frac{1}{p}} < +\infty$$

The definition is extended to the case $p = \infty$ by setting $b_{\infty,\infty}^s = C^s$.

We endow naturally these spaces with the $(1 \wedge p)$ -norm $\|\cdot\|_{b_{p,\infty}^s}$ so that $b_{p,\infty}^s$ are complete topological vector spaces. Moreover, they are the discrete counterparts of the Besov spaces of functions $B_{p,\infty}^s$ (see [111]): $f \in B_{p,\infty}^s$ if and only if the sequence of its wavelet coefficients belongs to $b_{p,\infty}^s$. The information concerning the Besov spaces that contain \vec{c} can be stored through the *scaling function* $\eta_{\vec{c}}$ which is defined by

$$\eta_{\vec{c}}(p) = \sup \left\{ s \in \mathbb{R} : \vec{c} \in b_{p,\infty}^s \right\}, \quad \forall p > 0.$$

If \vec{c} denotes the sequence of wavelet coefficients of a function f , its *scaling function* η_f is defined by $\eta_f = \eta_{\vec{c}}$. From the previous characterization, this function does not depend on the chosen wavelet basis. Using the definition of the Besov spaces, it is direct to check that

$$\eta_f(p) = \eta_{\vec{c}}(p) = \liminf_{j \rightarrow +\infty} \frac{\log S_{\vec{c}}(j, p)}{\log 2^{-j}}$$

where

$$S_{\vec{c}}(j, p) = 2^{-j} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p.$$

Let us mention that this function was initially introduced in the context of fully developed turbulence by Parisi and Frisch [117]. They proposed a formula to estimate the multifractal spectrum of a function based on L^p norms of the increments of the function. This formula, referred generally as the *Frisch-Parisi conjecture* or the *thermodynamic multifractal formalism*, was generalized by Jaffard [86] where the connection with Besov spaces was made. This conjecture states that

$$d_f(h) = \inf_p (ph - \eta_f(p) + 1). \quad (4.2)$$

The heuristic argument that underlies this method is the following. If λ is a dyadic interval containing a point whose Hölder exponent is h , from the equality (4.1), one should have $|c_\lambda| \sim 2^{-hj}$ as j tends to infinity. If we cover each such singularity by dyadic intervals of size 2^{-j} , it follows from the definition of the Hausdorff dimension that there are about $2^{d_f(h)j}$ such intervals. The most important contribution in the sum $\sum |c_\lambda|^p$ is the one corresponding to the value of h associated with the biggest exponent in $2^{(d_f(h)-hq)j}$. Moreover, from the definition of the scaling function, one can expect to have $\sum_\lambda |c_\lambda|^q \sim 2^{(-\eta_f(q)+1)j}$. Consequently, we are led to the following heuristic formula: $-\eta_f(q) + 1 = \sup_h \{d_f(h) - hq\}$. Using an inverse Legendre transform, we obtain the estimation (4.2).

Although it is based on heuristic arguments and approximations, formula (4.2) holds for many mathematical objects. For example, Jaffard [86] proved that it allows to recover

the increasing part of spectra of self-similar functions. He also obtained the following upper bound: if f is a uniformly Hölder function, its multifractal spectrum satisfies

$$d_f(h) \leq \inf_{p \geq p_c} (ph - \eta_f(p) + 1)$$

where p_c is the solution of $\eta_f(p) = 1$.

Besides, Jaffard [88] and Fraysse [69] justified this formalism with the introduction of the Baire space

$$\mathcal{B}^n = \bigcap_{\varepsilon > 0} \bigcap_{p > 0} B_{p, \infty}^{(\eta(p)/p - \varepsilon)}$$

for every function η such that $s(q) = q\eta\left(\frac{1}{q}\right)$ is concave, $0 \leq s'(q) \leq 1$ for every $q \geq 0$ and $s(0) > 0$. These conditions follow naturally from the result of Jaffard [88]: any scaling function η_f associated with a uniformly Hölder function f satisfies these properties and conversely, any function η which satisfies these properties is the scaling function of a uniformly Hölder function.

Theorem 4.5.2. [69, 88] *If p_c denotes the unique solution of $\eta(p) = 1$, the set of functions of \mathcal{B}^n such that*

$$d_f(h) = \begin{cases} \inf_{p \geq p_c} (ph - \eta(p) + 1) & \text{if } h \in \left[s(0), \frac{1}{p_c} \right], \\ -\infty & \text{otherwise,} \end{cases}$$

and $\eta_f = \eta$ is residual and prevalent in \mathcal{B}^n .

Therefore, a generic function in \mathcal{B}^n satisfies the Frisch-Parisi conjecture. Let us remark that since this method is based on a Legendre transform, it can hold only for spectra that are concave. Moreover, since $\eta_f(p)$ is defined only for positive p , it can only yield the increasing part of the spectrum. However, this first problem can be avoided using \mathcal{S}^ν spaces. The second one can be avoided using wavelet leaders and a generalization of Besov spaces, the Oscillation spaces.

4.6 \mathcal{S}^ν spaces

In this subsection, we first recall some definitions and some basic topological results obtained for \mathcal{S}^ν spaces. Furthermore, we expose the multifractal formalism based on these spaces. We refer the reader to [13] for details about the topology and [9, 10, 11, 14] for more results. The spaces \mathcal{S}^ν are defined as function spaces through conditions on the wavelet coefficients. Let us first introduce the spaces \mathcal{S}^ν as sequence spaces.

Following [13], the *wavelet profile* of a sequence $\vec{c} \in \Omega$ is the function $\nu_{\vec{c}}$ defined by

$$\nu_{\vec{c}}(\alpha) := \lim_{\varepsilon \rightarrow 0^+} \limsup_{j \rightarrow +\infty} \frac{\log \#E_j(1, \alpha + \varepsilon)(\vec{c})}{\log 2^j}, \quad \alpha \in \mathbb{R},$$

where

$$E_j(C, \alpha)(\vec{c}) := \{ \lambda \in \Lambda_j : |c_\lambda| \geq C2^{-\alpha j} \}$$

for $j \in \mathbb{N}_0$, $C > 0$ and $\alpha \in \mathbb{R}$. Remark that the function $\nu_{\vec{c}}$ is non-decreasing and right-continuous. Moreover, it takes values in $\{-\infty\} \cup [0, 1]$. If $\vec{c} \in C^{\alpha_0}$, then $\nu_{\vec{c}}(\alpha) = -\infty$ for every $\alpha < \alpha_0$.

Following this remark, an *admissible profile* is a non-decreasing right-continuous function of a real variable, with values in $\{-\infty\} \cup [0, 1]$ such that there exists $\alpha_{\min} \in \mathbb{R}$ for which $\nu(\alpha) = -\infty$ for all $\alpha < \alpha_{\min}$ and $\nu(\alpha) \geq 0$ for all $\alpha \geq \alpha_{\min}$.

Definition 4.6.1. Given an admissible profile ν , a sequence \vec{c} belongs to S^ν if

$$\nu_{\vec{c}}(\alpha) \leq \nu(\alpha), \quad \forall \alpha \in \mathbb{R}.$$

Equivalently, \vec{c} belongs to S^ν if and only if for every $\alpha \in \mathbb{R}$, $\varepsilon > 0$ and $C > 0$, there exists $J \in \mathbb{N}_0$ such that

$$\#E_j(C, \alpha)(\vec{c}) \leq 2^{(\nu(\alpha)+\varepsilon)j}, \quad \forall j \geq J,$$

with the convention that $2^{-\infty} := 0$.

Heuristically, a sequence \vec{c} belongs to S^ν if at each large scale j , the number of k such that $|c_{j,k}| \geq 2^{-\alpha j}$ is of order smaller than $2^{\nu(\alpha)j}$. This space is a vector space.

Jaffard [90] proved that if $\alpha_{\min} > 0$, the definition of the S^ν spaces is robust, hence independent of the chosen wavelet basis. We will then consider them equivalently as sequence or function spaces (as for Besov spaces): we say that a function f belongs to S^ν if its sequence of wavelet coefficients belongs to the sequence space S^ν .

In order to define a complete metrizable topology on S^ν , auxiliary spaces were introduced. For any $\alpha \in \mathbb{R}$ and any $\beta \in \{-\infty\} \cup [0, +\infty)$, the space $A(\alpha, \beta)$ is defined by

$$A(\alpha, \beta) := \{\vec{c} \in \Omega : \exists C, C' \geq 0 \text{ such that } \#E_j(C, \alpha)(\vec{c}) \leq C' 2^{\beta j}, \forall j \in \mathbb{N}_0\}.$$

This space is endowed with the distance

$$\delta_{\alpha, \beta}(\vec{c}, \vec{c}') := \inf \{C + C' : C, C' \geq 0 \text{ and } \#E_j(C, \alpha)(\vec{c} - \vec{c}') \leq C' 2^{\beta j}, \forall j \in \mathbb{N}_0\}.$$

Remark that if $\beta = -\infty$, then $(A(\alpha, -\infty), \delta_{\alpha, -\infty})$ is the topological normed space C^α . If $\beta \geq 1$, then $A(\alpha, \beta) = \Omega$. Moreover, in the case $\beta > 1$, the topology defined by the distance $\delta_{\alpha, \beta}$ is equivalent to the topology of pointwise convergence.

The properties of auxiliary spaces are summarized in the following proposition.

Proposition 4.6.2. *Let $\alpha \in \mathbb{R}$ and $\beta \in \{-\infty\} \cup [0, +\infty)$.*

1. *The addition is continuous on $(A(\alpha, \beta), \delta_{\alpha, \beta})$ but the scalar multiplication is not continuous.*
2. *The space $(A(\alpha, \beta), \delta_{\alpha, \beta})$ has a stronger topology than the pointwise topology and every Cauchy sequence in $(A(\alpha, \beta), \delta_{\alpha, \beta})$ is also a pointwise Cauchy sequence.*
3. *If $\beta > 1$, the topology defined by the distance $\delta_{\alpha, \beta}$ is equivalent to the uniform topology.*
4. (a) *If B is a bounded set of $(A(\alpha, \beta), \delta_{\alpha, \beta})$, then there exists $r > 0$ such that*

$$\begin{aligned} B &\subseteq \{\vec{c} \in \Omega : \#\{\lambda \in \Lambda_j : |c_\lambda| \geq r 2^{-\alpha j}\} \leq r 2^{\beta j}, \forall j \in \mathbb{N}_0\} \\ &\subseteq \{\vec{c} \in \Omega : \#\{\lambda \in \Lambda_j : |c_\lambda| > r 2^{-\alpha j}\} \leq r 2^{\beta j}, \forall j \in \mathbb{N}_0\}. \end{aligned}$$

(b) Let $r, r' \geq 0$, $\alpha' \geq \alpha$ and $\beta' \leq \beta$. The set

$$B = \left\{ \vec{c} \in \Omega : \#\{\lambda \in \Lambda_j : |c_\lambda| > r 2^{-\alpha'j}\} \leq r' 2^{\beta'j}, \forall j \in \mathbb{N}_0 \right\}$$

is a bounded set of $(A(\alpha, \beta), \delta_{\alpha, \beta})$. Moreover, B is closed for the pointwise convergence.

5. The space $(A(\alpha, \beta), \delta_{\alpha, \beta})$ is a complete metric space.

The following proposition gives the connection between auxiliary spaces $A(\alpha, \beta)$ and the space \mathcal{S}^ν .

Proposition 4.6.3. *For any sequence $(\alpha_n)_{n \in \mathbb{N}}$ dense in \mathbb{R} and any sequence $(\varepsilon_m)_{m \in \mathbb{N}}$ of $(0, +\infty)$ which converges to 0, one has*

$$\mathcal{S}^\nu = \bigcap_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} A(\alpha_n, \nu(\alpha_n) + \varepsilon_m).$$

The topology of \mathcal{S}^ν is defined as the projective limit topology, i.e. the coarsest topology that makes each inclusion $\mathcal{S}^\nu \subseteq A(\alpha_n, \nu(\alpha_n) + \varepsilon_m)$ continuous. This topology is equivalent to the topology given by the distance

$$\delta = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} 2^{-(m+n)} \frac{\delta_{m,n}}{1 + \delta_{m,n}},$$

where $\delta_{m,n}$ denotes the distance $\delta_{\alpha_n, \nu(\alpha_n) + \varepsilon_m}$. The topology of \mathcal{S}^ν is independent of the sequences $(\alpha_n)_{n \in \mathbb{N}}$ and $(\varepsilon_m)_{m \in \mathbb{N}}$ chosen as above. Therefore, this distance is denoted δ , independently of the sequences chosen. The next result concerns the compact subsets of $(\mathcal{S}^\nu, \delta)$.

Proposition 4.6.4. *For $m, n \in \mathbb{N}$, let $C(m, n)$ and $C'(m, n)$ be positive constants and let us define*

$$K_{m,n} := \left\{ \vec{c} \in \Omega : \#\{\lambda \in \Lambda_j : |c_\lambda| > C(m, n) 2^{-\alpha_n j}\} \leq C'(m, n) 2^{(\nu(\alpha_n) + \varepsilon_m)j}, \forall j \in \mathbb{N}_0 \right\}$$

and

$$K := \bigcap_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} K_{m,n}.$$

Every sequence of K which converges pointwise converges also in $(\mathcal{S}^\nu, \delta)$ to an element of K . It follows that K is a compact of $(\mathcal{S}^\nu, \delta)$.

Let us now present some connections with Besov spaces. If we define the *concave conjugate* η of the admissible profile ν by

$$\eta(p) := \inf_{\alpha \geq \alpha_{\min}} (\alpha p - \nu(\alpha) + 1), \quad p > 0,$$

we get the following embedding of \mathcal{S}^ν spaces into Besov spaces.

Proposition 4.6.5. *[13] If $(p_n)_{n \in \mathbb{N}}$ is a dense sequence of $(0, +\infty)$ and if $(\varepsilon_m)_{m \in \mathbb{N}}$ is a sequence of $(0, +\infty)$ which converges to 0, then*

$$\mathcal{S}^\nu \subseteq \bigcap_{p>0} \bigcap_{\varepsilon>0} b_{p, \infty}^{\frac{\eta(p)}{p} - \varepsilon} = \bigcap_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} b_{p_n, \infty}^{\frac{\eta(p_n)}{p_n} - \varepsilon_m}$$

and this inclusion becomes an equality if and only if ν is concave.

This result justifies the introduction of the \mathcal{S}^ν spaces: the spaces \mathcal{B}^η do not contain more information about the multifractal spectra of their functions than their concave hull since for most of the functions in this intersection, the multifractal spectrum is given by a Legendre transform of η (see Theorem 4.5.2). By contrast, if ν is not concave, the space \mathcal{S}^ν gives an additional information and leads to estimation of spectra which are not concave, as presented below.

In order to state the multifractal formalism based on the \mathcal{S}^ν spaces and justify its validity, we assume now that $\alpha_{\min} > 0$ and we consider the space \mathcal{S}^ν as a function space. The multifractal formalism based on the \mathcal{S}^ν spaces, called also the *wavelet profile method*, consists in the estimation of the spectrum of a function f by the formula

$$d_f(h) = h \sup_{h' \in (0, h]} \frac{\nu_f(h')}{h'}, \quad \forall h \leq \inf_{\alpha \geq \alpha_{\min}} \frac{\alpha}{\nu_f(\alpha)}. \quad (4.3)$$

Aubry and Jaffard [12] proved that for any Random Wavelet Series (see Chapter 5), formula (4.3) holds. They also established that if f is a uniformly Hölder function, then

$$d_f(h) \leq h \sup_{h' \in (0, h]} \frac{\nu_f(h')}{h'}, \quad \forall h \leq \inf_{\alpha \geq \alpha_{\min}} \frac{\alpha}{\nu_f(\alpha)}.$$

Furthermore, an implementation of this formalism has been proposed by Kleynstssens et al. [99] where it is tested on several theoretical examples such as fractional Brownian motions, Lévy processes, sum of binomial cascades...

The results about the validity of this formalism are given by the following theorem.

Theorem 4.6.6. [11, 14] *If ν is an admissible profile, we denote*

$$\nu_I(h) = \begin{cases} -\infty & \text{if } h < \alpha_{\min}, \\ h \sup_{h' \in (0, h]} \frac{\nu(h')}{h'} & \text{if } \alpha_{\min} \leq h \leq h_{\max}, \\ 1 & \text{otherwise,} \end{cases}$$

where $h_{\max} = \inf_{h \geq \alpha_{\min}} \frac{h}{\nu(h)}$. If $\alpha_{\min} > 0$, the set of functions $f \in \mathcal{S}^\nu$ such that

$$d_f(h) = \begin{cases} \nu_I(h) & \text{if } h \leq h_{\max}, \\ -\infty & \text{otherwise,} \end{cases}$$

and $\nu = \nu_f$ is residual and prevalent in \mathcal{S}^ν .

It follows that for a generic function in \mathcal{S}^ν , formula (4.3) holds. Although it allows to estimate non-concave spectra, this formalism is still limited to the increasing part of spectra. Moreover, the conversion of ν into ν_I transforms the admissible profile into another admissible profile with an additional property, called the increasing-visibility (see [106] and the definition below), and is therefore limited to spectra enjoying this property.

Definition 4.6.7. Take $0 \leq a < b \leq +\infty$. A function $g : [a, b] \mapsto [0, +\infty)$ is with *increasing-visibility* on $[a, b]$ if g is continuous at a and if the function

$$x \mapsto \frac{g(x)}{x}$$

is increasing on $(a, b]$.

In other words, a function g is with increasing-visibility if for all $x \in (a, b]$, the segment $[(0, 0); (x, g(x))]$ lies above the graph of g on $(a, x]$.

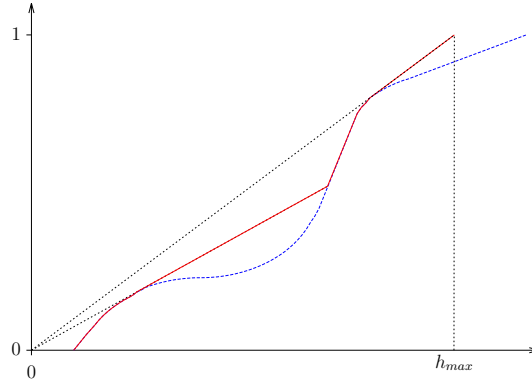


Figure 4.4: Example of ν (---) and ν_I (—)

4.7 Wavelet leaders and the associated formalism

Let us recall that the thermodynamic formalism fails for the detection of decreasing spectra. In the context of wavelet-based multifractal formalism, more accurate results can be obtained when, rather than relying directly on wavelet coefficients, one relies on alternative quantities, namely the wavelet leaders. This was possible thanks to the specificity of wavelet leaders: given a scale, they take into account a specific family of coefficients of smaller scales and located at the same place. Let us be more precise about the introduction of those quantities.

As presented in Section 4.5, the thermodynamic formalism relies on the estimation of the sums

$$2^{-j} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p. \quad (4.4)$$

The behavior of this sum for large j and positive p is related to the increasing part of the spectrum. The decreasing part is connected with the asymptotic behavior of this sum for negative p . Nevertheless, (4.4) is totally unstable for negative p due to the presence of small wavelet coefficients: indeed, when they are taken to a negative power, they can be extremely large. Therefore, it does not allow to estimate the decreasing part of spectra. One way to stabilize these sums and eliminate this source of instability was proposed by Jaffard [92]. The idea was to replace in (4.4) the single value $|c_\lambda|$ by a supremum of the $|c_{\lambda'}|$ where λ' is close to λ . This idea is consistent with the purpose of estimating multifractal spectra. Indeed, a small coefficient is not the signature of a large Hölder exponent if it has a large coefficient in its immediate neighborhood. On the opposite, a small value of the supremum means that all wavelet coefficients close to each other take a small value, which is the signature of a smooth zone.

Remark 4.7.1. In order to overcome the issue of the instability for negative p , Arneodo et al. [3] proposed the wavelet transform modulus maxima method, using the notion of line of maxima in the continuous wavelet transform: the value of the continuous wavelet transform at a point is replaced by a supremum on all lines of maxima ending at the considered point [105]. This technique proved helpful in many practical and theoretical problems, but its theoretical contribution was limited. In particular, there is no underlying functional space.

Definition 4.7.2. The *wavelet leaders* of a function $f \in L^2(\mathbb{T})$ whose wavelet coefficients are given by the sequence \vec{c} are defined by

$$d_\lambda = \sup_{\lambda' \subseteq 3\lambda} |c_{\lambda'}|, \quad \forall \lambda \in \Lambda,$$

where 3λ denotes the dyadic interval with the same center as λ but three times larger.

Remark 4.7.3. In view of the periodization, we use the following conventions when the cube λ is at one of the boundaries of $[0, 1)$: $3\lambda(j, 0) = \lambda(j, 2^j - 1) \cup \lambda(j, 0) \cup \lambda(j, 1)$ and $3\lambda(j, 2^j - 1) = \lambda(j, 2^j - 2) \cup \lambda(j, 2^j - 1) \cup \lambda(j, 0)$.

If f is bounded, then

$$|c_\lambda| \leq 2^j \int_0^1 |f(x)| |\psi_\lambda(x)| dx \leq C \sup_{x \in [0,1]} |f(x)|$$

for some $C > 0$, so that the wavelet leaders are finite.

Wavelet leaders also appeared to be very interesting in the study of the pointwise Hölder regularity. Indeed, their decay properties are directly related with the Hölder exponent. If $x_0 \in [0, 1)$, there is a unique dyadic interval λ of length 2^{-j} which contains x_0 . Let $\lambda_j(x_0)$ denotes this dyadic interval. Then, we set

$$d_j(x_0) := d_{\lambda_j(x_0)} = \sup_{\lambda' \subseteq 3\lambda_j(x_0)} |c_{\lambda'}|.$$

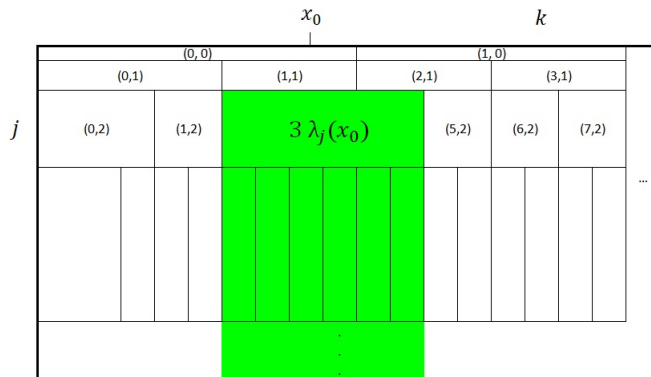


Figure 4.5: Representation of the dyadic intervals which are involved in the computation of $d_j(x)$.

Proposition 4.7.4. [91] *If f is a uniformly Hölder function, then*

$$h_f(x_0) = \liminf_{j \rightarrow +\infty} \frac{\log d_j(x_0)}{\log 2^{-j}}.$$

Therefore, Hölder exponents can be recovered from wavelet leaders by local log-log plot regressions, see [91]. Comparing this result with (4.1), one can see that the wavelet leaders are a more adequate tool to study the Hölder regularity than the wavelet coefficients. Moreover, the heuristic underlying the Frisch-Parisi conjecture and presented in Section 4.5 is based on the fact that if $h_f(x_0) = h$, then $|c_{\lambda_j(x_0)}| \sim 2^{-h_j}$. Actually, this

estimation only holds if f has *cusplike* singularities (i.e. which have a behavior similar to $|x - x_0|^h$ around x_0 , see [112]). In the case of wavelet leaders, from Proposition 4.7.4, we really have $|d_j(x_0)| \sim 2^{-hj}$ for large j under the unique assumption that f is uniformly Hölder.

This argument leads to the wavelet leaders method. By mimicking the thermodynamic formalism, one sets

$$W_f(j, p) = 2^{-j} \sum_{\lambda \in \Lambda_j}^* d_\lambda^p, \quad \forall p \in \mathbb{R}$$

where the symbol $\sum_{\lambda \in \Lambda_j}^*$ means that the sum is restricted to the intervals $\lambda \in \Lambda_j$ such that $d_\lambda \neq 0$. From this, one sets

$$\tilde{\eta}_f(p) = \liminf_{j \rightarrow +\infty} \frac{\log W_f(j, p)}{\log 2^{-j}}.$$

The multifractal formalism associated to the wavelet leaders, called the *wavelet leaders method*, is based on the estimation

$$d_f(h) = \inf_{p \in \mathbb{R}} (hp - \tilde{\eta}_f(p) + 1). \quad (4.5)$$

Again, Jaffard [91] proved that the function $\tilde{\eta}_f$ does not depend on the chosen wavelet basis and that if f is uniformly Hölder,

$$d_f(h) \leq \inf_{p \in \mathbb{R}} (hp - \tilde{\eta}_f(p) + 1).$$

One can show that $\eta_f(p) = \tilde{\eta}_f(p)$ if $p < p_c$ [92] and therefore, compared with the thermodynamic multifractal formalism, this upper bound is sharpened since it is taken on every p . Moreover, equality (4.5) holds for large classes of models such as fractional Brownian motions, cascades, Lévy processes... [2, 16, 91].

Let us make two remarks. First, the estimation (4.5) only gives concave spectra. Secondly, the wavelet leaders give rise to some generalization of the Besov spaces called the Oscillation spaces [91]. These spaces are a particular case of Oscillation spaces $\mathcal{O}_p^{s, s'}$ considered in [87, 92]. In the case $p > 0$, as for Besov spaces, $\tilde{\eta}_f(p)$ determines which oscillation spaces f belongs to.

Definition 4.7.5. For $s \in \mathbb{R}$ and $p > 0$, a sequence $\vec{c} \in \Omega$ belongs to \mathcal{O}_p^s if

$$\sup_{j \in \mathbb{N}_0} 2^{(s - \frac{1}{p})j} \left(\sum_{k=0}^{2^j - 1} d_{j,k}^p \right)^{\frac{1}{p}} < +\infty.$$

With this definition, it is direct to see that

$$\tilde{\eta}_f(p) = \sup\{s \in \mathbb{R} : \vec{c} \in \mathcal{O}_p^s\}, \quad \forall p > 0.$$

For $s \in \mathbb{R}$ and $p > 0$, we endow the space \mathcal{O}_p^s with the $(1 \wedge p)$ -norm

$$\|\vec{c}\|_{\mathcal{O}_p^s} := \sup_{j \in \mathbb{N}_0} 2^{(s - \frac{1}{p})j} \left(\sum_{k=0}^{2^j - 1} d_{j,k}^p \right)^{\frac{1}{p}},$$

so that $(\mathcal{O}_p^s, \|\cdot\|_{\mathcal{O}_p^s})$ is a complete topological vector space. When p is negative, an analogous definition exists but these Oscillation spaces are not vector spaces anymore. Therefore, prevalent results as those obtained for Besov spaces (see Theorem 4.5.2) do not make sense. However, one could wonder if there could be generic results in some other sense. Up to now, no appropriate topology has been proposed on these spaces to achieve this purpose.

Chapter 5

Wavelet leaders profile

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5.1 Introduction

As presented in the previous chapter, the Frisch-Parisi conjecture, classically used for estimating the multifractal spectrum of a function, can only lead to recover the increasing and concave hull of spectra. It appeared that more accurate information concerning the pointwise regularity can be obtained when relying on wavelet leaders, which are local suprema of wavelet coefficients. First, the wavelet leaders allow to stabilize the coefficients which can take a small value “accidentally”. Moreover, they give an easier characterization of the pointwise regularity than wavelet coefficients (see Proposition 4.7.4) so that Hölder exponents can be recovered from wavelet leaders by local log-log plot regressions. In this context, the wavelet leaders method has been introduced as generalization of the thermodynamic multifractal formalism using wavelet leaders. In particular, this method allows to recover increasing and decreasing parts of spectra. Nevertheless, this method is still limited to concave spectra.

In order to get a suitable context to obtain multifractal results in the non-concave case, \mathcal{S}^ν spaces have then been introduced. As presented in Chapter 4, several positive

results for the estimation of non-concave spectra have been obtained. However, those spaces can only detect the increasing part of spectra. Moreover, it is limited to spectra with increasing-visibility since it is based on the estimation of them by the function

$$h \in (0, h_{\max}] \mapsto h \sup_{h' \in (0, h]} \frac{\nu_f(h')}{h'}$$

where $h_{\max} = \inf_{h \geq \alpha_{\min}} \frac{h}{\nu_f(h)}$. Let us however mention a particular case where this transformation of ν_f into a function with increasing-visibility is not necessary in order to estimate the multifractal spectrum: let us assume that f is a function whose wavelet coefficients are given by $c_\lambda = \mu(\lambda)$ where μ is a finite Borel measure on $[0, 1]$. This method of prescribing wavelet coefficients using measures has been proposed by Barral and Seuret [20]. Moreover, for every $\beta > 0$, let f_β denote the function whose wavelet coefficients are given by $2^{-\beta j} c_\lambda$. In this case, a direct computation shows that $d_{f_\beta}(h) = d_f(h - \beta)$ for all $h \geq \beta$. Remark that the same relation also holds for the wavelet profile, that is to say $\nu_{f_\beta}(h) = \nu_f(h - \beta)$ for all $h \geq \beta$. If one has

$$\inf \left\{ \frac{\nu_f(x) - \nu_f(y)}{x - y} : x, y \in [h_{\min}, h'_{\max}], x < y \right\} > 0,$$

where $h_{\min} = \inf\{\alpha : \nu_f(\alpha) \geq 0\}$, $h'_{\max} = \inf\{\alpha : \nu_f(\alpha) = 1\}$, then there exists $\beta > 0$ such that the function ν_{f_β} is with increasing-visibility on $[h_{\min}, h'_{\max}]$. In this case, the wavelet profile ν_{f_β} gives an approximation for the multifractal spectrum d_{f_β} of f_β . Therefore, since $d_{f_\beta}(h) = d_f(h - \beta)$ and $\nu_{f_\beta}(h) = \nu_f(h - \beta)$, the increasing part of the multifractal spectrum d_f of f can be approximated by the wavelet profile ν_f .

Remark that in this case, there is a nice decreasing property in the repartition of the wavelet coefficients: if $\lambda' \subseteq \lambda$, then $|c_{\lambda'}| \leq |c_\lambda|$. Using the wavelet leaders instead of the wavelet coefficients, we get this decreasing property for any function. Therefore, one can hope that the definition of a profile with the wavelet leaders will directly give (without any transformation) an estimation of the multifractal spectrum.

The leaders profile method presented in this chapter aims at combining the advantages of the two previous methods. With the use of the wavelet leaders, one can consider the entire spectrum, while the profile function allows to recover non-concave spectra. Moreover, this combination also gives estimations for spectra which are not with increasing-visibility. Let us mention that this method is being studied in practice in [2, 64].

This chapter is structured as follows. In Section 5.2, we introduce a quantity based on the distribution of the wavelet leaders of a function and we show that it gives an upper bound for its multifractal spectrum. Nevertheless, we show that its definition can depend on the wavelet basis chosen to compute it. Moreover, its definition is numerically instable because it is based on a double limit. That is why we derive in Section 5.3 another large deviation type quantity based on the wavelet leaders which still yields an upper bound for the spectrum and which is robust. It allows to propose a new multifractal formalism: the leaders profile method. In Section 5.4, we illustrate this formalism on classical models. We end this chapter in Section 5.5 with a theoretical comparison of this method with the wavelet leaders method (presented in Section 4.7 of Chapter 4) and the wavelet profile method (based on \mathcal{S}^ν spaces and presented in Section 4.6 of Chapter 4). Most of the results presented in this chapter have been gathered in the papers [24] and [64]. Let us note that in this chapter, \vec{c} represents the sequence of wavelet coefficients of a bounded function f on \mathbb{T} and \vec{d} its sequence of wavelet leaders.

5.2 Upper bound for the multifractal spectrum

In this section, we define a large deviation spectrum based on the wavelet leaders of a signal in a given wavelet basis, and we show that this quantity yields an upper bound for the multifractal spectrum of the signal. Note that a similar approach has been developed in [19] using *oscillations* of the function (i.e. the difference between the supremum and the infimum of the function on an interval) instead of wavelet leaders. The advantage of wavelet leaders is that they allow to deal with Hölder exponents of order larger than 1.

Definition 5.2.1. The *wavelet leaders density* $\tilde{\rho}_{\vec{c}}$ of $\vec{c} \in C^0$ is defined for every $\alpha \geq 0$ by

$$\tilde{\rho}_{\vec{c}}(\alpha) := \lim_{\varepsilon \rightarrow 0^+} \limsup_{j \rightarrow +\infty} \frac{\log \#\{\lambda \in \Lambda_j : 2^{-(\alpha+\varepsilon)j} \leq d_\lambda < 2^{-(\alpha-\varepsilon)j}\}}{\log 2^j}$$

and for $\alpha = +\infty$ by

$$\tilde{\rho}_{\vec{c}}(+\infty) := \lim_{A \rightarrow +\infty} \liminf_{j \rightarrow +\infty} \frac{\log \#\{\lambda \in \Lambda_j : d_\lambda \leq 2^{-Aj}\}}{\log 2^j}.$$

This definition formalizes the idea that there are approximately $2^{\tilde{\rho}_{\vec{c}}(\alpha)j}$ wavelet leaders of size $2^{-\alpha j}$ at large scales j . From typical properties of large deviation spectra, we get the following property of the wavelet leaders density.

Proposition 5.2.2. *The wavelet leaders density of a sequence $\vec{c} \in C^0$ is upper semi-continuous on $[0, +\infty)$, i.e. for every $\alpha_0 \in [0, +\infty)$, one has*

$$\limsup_{\alpha \rightarrow \alpha_0} \tilde{\rho}_{\vec{c}}(\alpha) \leq \tilde{\rho}_{\vec{c}}(\alpha_0).$$

Proof. From the definition, for every $\gamma > 0$, there is $\varepsilon > 0$ such that

$$\liminf_{j \rightarrow +\infty} \frac{\log \#\{\lambda \in \Lambda_j : 2^{-(\alpha_0+2\varepsilon)j} \leq d_\lambda < 2^{-(\alpha_0-2\varepsilon)j}\}}{\log 2^j} \leq \tilde{\rho}_{\vec{c}}(\alpha_0) + \gamma.$$

On the other hand, if $|\alpha - \alpha_0| < \varepsilon$, we have

$$\#\{\lambda \in \Lambda_j : 2^{-(\alpha_0+2\varepsilon)j} \leq d_\lambda < 2^{-(\alpha_0-2\varepsilon)j}\} \geq \#\{\lambda \in \Lambda_j : 2^{-(\alpha+\varepsilon)j} \leq d_\lambda < 2^{-(\alpha-\varepsilon)j}\}$$

so that

$$\liminf_{j \rightarrow +\infty} \frac{\log \#\{\lambda \in \Lambda_j : 2^{-(\alpha+\varepsilon)j} \leq d_\lambda < 2^{-(\alpha-\varepsilon)j}\}}{\log 2^j} \leq \tilde{\rho}_{\vec{c}}(\alpha_0) + \gamma.$$

Consequently, $\tilde{\rho}_{\vec{c}}(\alpha) \leq \tilde{\rho}_{\vec{c}}(\alpha_0) + \gamma$ if $|\alpha - \alpha_0| < \varepsilon$ so that

$$\limsup_{\alpha \rightarrow \alpha_0} \tilde{\rho}_{\vec{c}}(\alpha) = \inf_{\varepsilon > 0} \sup_{\alpha_0 < \alpha < \alpha_0 + \varepsilon} \tilde{\rho}_{\vec{c}}(\alpha) \leq \tilde{\rho}_{\vec{c}}(\alpha_0) + \gamma.$$

We get the conclusion since $\gamma > 0$ is arbitrary. \square

Let f be a uniformly Hölder function. We denote by \vec{c} its sequence of wavelet coefficients in a fixed wavelet basis. Let us consider the points x_0 such that $h_f(x_0) = h$. Using Proposition 4.7.4, we know that $d_j(x_0) \sim 2^{-hj}$ and from the definition of the wavelet leaders density, there are about $2^{\rho_{\vec{c}}(h)j}$ such dyadic intervals. Moreover, if we cover each singularity x_0 such that $h_f(x_0) = h$ by dyadic intervals of size 2^{-j} , from the

definition of the Hausdorff dimension, there are about $2^{d_f(h)j}$ such intervals. This large deviation-type argument shows that one can hope to have the equality $\tilde{\rho}_{\tilde{c}}(h) = d_f(h)$.

In general, this equality is not verified, but we will prove that $\tilde{\rho}_{\tilde{c}}$ yields an upper bound for the multifractal spectrum of f . First, let us introduce some notations. For every $\alpha \in \mathbb{R}$, we set

$$F^j(\alpha) = \{k \in \{0, \dots, 2^j - 1\} : d_{j,k} \geq 2^{-\alpha j}\} \quad \text{and} \quad E^j(\alpha) = \bigcup_{k \in F^j(\alpha)} \lambda_{j,k}.$$

We also define

$$E(\alpha) = \limsup_{j \rightarrow +\infty} E^j(\alpha) = \bigcap_{j \in \mathbb{N}_0} \bigcup_{m \geq j} E^m(\alpha).$$

Remark that, since f is uniformly Hölder, there exist $\alpha_0 > 0$ and $C > 0$ such that

$$\forall j \in \mathbb{N}_0, k \in \{0, \dots, 2^j - 1\}, \quad |c_{j,k}| \leq C2^{-\alpha_0 j}.$$

Therefore, $E(\alpha) = \emptyset$ if $\alpha < \alpha_0$.

Lemma 5.2.3. [24] *Let f be a uniformly Hölder function and let $\alpha \in [0, +\infty)$.*

1. *If $x_0 \in E(\alpha)$, then $h_f(x_0) \leq \alpha$.*
2. *If $h_f(x_0) < \alpha$, then $x_0 \in E(\alpha)$.*

Proof. 1. Let us assume that $x_0 \in E(\alpha)$. Then for every j , there exist $m_j \geq j$ and $k_j \in F^{m_j}(\alpha)$ such that $x_0 \in \lambda_{m_j, k_j}$. This means that $d_{m_j}(x_0) = d_{m_j, k_j} \geq 2^{-\alpha m_j}$. It follows that

$$h_f(x_0) = \liminf_{j \rightarrow +\infty} \frac{\log d_j(x_0)}{\log 2^{-j}} \leq \lim_{j \rightarrow +\infty} \frac{\log d_{m_j, k_j}}{\log 2^{-m_j}} \leq \lim_{j \rightarrow +\infty} \frac{\log 2^{-\alpha m_j}}{\log 2^{-m_j}} = \alpha.$$

2. Let us assume that $h_f(x_0) < \alpha$. Then, there exists an increasing sequence $(j_n)_{n \in \mathbb{N}}$ of natural numbers such that

$$\frac{\log d_{j_n}(x_0)}{\log 2^{-j_n}} < \alpha.$$

Consequently, $d_{j_n}(x_0) > 2^{-\alpha j_n}$ and $x_0 \in E^{j_n}(\alpha)$. □

Lemma 5.2.4. [24] *Let f be a uniformly Hölder function. Then, for every $h \in [0, +\infty)$.*

$$\{x_0 : h_f(x_0) = h\} = \bigcap_{\varepsilon > 0} E(h + \varepsilon) \setminus E(h - \varepsilon).$$

Proof. The result is obtained directly from Lemma 5.2.3. □

Theorem 5.2.5. [24] *Let f be a uniformly Hölder function. Then its multifractal spectrum satisfies*

$$d_f(h) \leq \tilde{\rho}_{\tilde{c}}(h), \quad \forall h \in [0, +\infty].$$

Proof. We first assume that $h \in [0, +\infty)$. Because of Lemma 5.2.4, we have to show that

$$\dim_{\mathcal{H}} \left(\bigcap_{\varepsilon > 0} E(h + \varepsilon) \setminus E(h - \varepsilon) \right) \leq \tilde{\rho}_{\tilde{c}}(h).$$

Let us consider $\delta > 0$. From the definition of $\tilde{\rho}_{\bar{c}}$, there exist $\varepsilon_0 > 0$ and $j_0 \in \mathbb{N}_0$ such that

$$\#\left\{\lambda \in \Lambda_j : 2^{-(h+\varepsilon)j} \leq d_\lambda < 2^{-(h-\varepsilon)j}\right\} \leq 2^{(\tilde{\rho}_{\bar{c}}(h)+\delta)j}, \quad \forall j \geq j_0.$$

If we set

$$E_{\varepsilon_0}^j(h) = E^j(h + \varepsilon_0) \setminus E^j(h - \varepsilon_0), \quad \forall j \in \mathbb{N}_0,$$

then

$$E(h + \varepsilon_0) \setminus E(h - \varepsilon_0) \subseteq \bigcap_{J \in \mathbb{N}_0} \bigcup_{j \geq J} E_{\varepsilon_0}^j(h).$$

Let us show that

$$\mathcal{H}^s \left(\bigcap_{J \in \mathbb{N}_0} \bigcup_{j \geq J} E_{\varepsilon_0}^j(h) \right) < +\infty$$

where $s = \tilde{\rho}_{\bar{c}}(h) + 2\delta$. Remark that for every $j \in \mathbb{N}_0$, the set $E_{\varepsilon_0}^j(h)$ is covered by $\#\{\lambda \in \Lambda_j : 2^{-(h+\varepsilon_0)j} \leq d_\lambda < 2^{-(h-\varepsilon_0)j}\}$ intervals of length 2^{-j} . For every $\eta > 0$, there is $J(\eta) \geq j_0$ such that $2^{-j} \leq \eta$ if $j \geq J(\eta)$. Then, we have

$$\begin{aligned} \mathcal{H}_\eta^s \left(\bigcap_{J \in \mathbb{N}_0} \bigcup_{j \geq J} E_{\varepsilon_0}^j(h) \right) &\leq \mathcal{H}_\eta^s \left(\bigcup_{j \geq J(\eta)} E_{\varepsilon_0}^j(h) \right) \\ &\leq \sum_{j \geq J(\eta)} (\#\{\lambda \in \Lambda_j : 2^{-(h+\varepsilon)j} \leq d_\lambda < 2^{-(h-\varepsilon)j}\}) 2^{-sj} \\ &\leq \sum_{j \geq J(\eta)} 2^{(\tilde{\rho}_{\bar{c}}(h)+\delta)j} 2^{-sj} \leq \sum_{j \in \mathbb{N}_0} 2^{-\delta j} < +\infty. \end{aligned}$$

Consequently,

$$\mathcal{H}^s \left(\bigcap_{J \in \mathbb{N}_0} \bigcup_{j \geq J} E_{\varepsilon_0}^j(h) \right) = \lim_{\eta \rightarrow 0^+} \mathcal{H}_\eta^s \left(\bigcap_{J \in \mathbb{N}_0} \bigcup_{j \geq J} E_{\varepsilon_0}^j(h) \right) \leq \sum_{j \in \mathbb{N}_0} 2^{-\delta j} < +\infty$$

and it follows that

$$\dim_{\mathcal{H}} \left(\bigcap_{\varepsilon > 0} E(h + \varepsilon) \setminus E(h - \varepsilon) \right) \leq \dim_{\mathcal{H}} \left(\bigcap_{J \in \mathbb{N}_0} \bigcup_{j \geq J} E_{\varepsilon_0}^j(h) \right) \leq s = \tilde{\rho}_{\bar{c}}(h) + 2\delta.$$

Since $\delta > 0$ is arbitrary, we finally get

$$\dim_{\mathcal{H}} \left(\bigcap_{\varepsilon > 0} E(h + \varepsilon) \setminus E(h - \varepsilon) \right) \leq \tilde{\rho}_{\bar{c}}(h)$$

which leads to the conclusion.

We consider now the case $h = +\infty$. For every $A > 0$, let

$$B_A(j) = \bigcup_{\lambda : d_\lambda \leq 2^{-Aj}} \lambda.$$

Then it follows from Proposition 4.7.4 that

$$\{x_0 : h_f(x_0) = +\infty\} = \bigcap_{A > 0} \bigcup_{J \in \mathbb{N}_0} \bigcap_{j \geq J} B_A(j).$$

The result follows as previously from the definition of $\tilde{\rho}_{\vec{c}}(+\infty)$. Indeed, let us fix $A > 0$. Then, for every $\varepsilon > 0$, there are $A > 0$ and a subsequence $(j_n)_{n \in \mathbb{N}}$ such that

$$\#\{\lambda \in \Lambda_{j_n} : d_\lambda \leq 2^{-Aj_n}\} \leq 2^{(\tilde{\rho}_{\vec{c}}(+\infty) + \varepsilon)j_n}$$

for every $n \in \mathbb{N}$. Therefore, the set $B_A(j_n)$ is covered by less than $2^{(\tilde{\rho}_{\vec{c}}(+\infty) + \varepsilon)j_n}$ intervals of length 2^{-j_n} . Let us fix $\eta > 0$. For every $J \in \mathbb{N}_0$, there is $n \in \mathbb{N}$ such that $j_n \geq J$ and $2^{-j_n} \leq \eta$. Then, we have

$$\mathcal{H}_\eta^s \left(\bigcap_{j \geq J} B_A(j) \right) \leq \mathcal{H}_\eta^s(B_A(j_n)) \leq 2^{(\tilde{\rho}_{\vec{c}}(+\infty) + \varepsilon)j_n} 2^{-j_n s}$$

so that

$$\mathcal{H}_\eta^s \left(\bigcup_{J \in \mathbb{N}_0} \bigcap_{j \geq J} B_A(j) \right) \leq \sum_{n \in \mathbb{N}} 2^{(\tilde{\rho}_{\vec{c}}(+\infty) + \varepsilon)j_n} 2^{-j_n s} < +\infty$$

if $s > \tilde{\rho}_{\vec{c}}(+\infty) + \varepsilon$. It follows that $\dim_{\mathcal{H}}(\{x_0 : h_f(x_0) = +\infty\}) \leq \tilde{\rho}_{\vec{c}}(+\infty) + \varepsilon$. The real number $\varepsilon > 0$ is arbitrary, hence the conclusion. \square

One drawback when dealing with wavelet leaders is that the suprema corresponding to two neighbors dyadic intervals overlap. For instance, in a probabilistic framework, this will create correlations between wavelet leaders, even if they don't exist between wavelet coefficients. Therefore it is natural to wonder if the developments that we pursued could be developed in a simpler framework where wavelet leaders are replaced by *restricted wavelet leaders* defined by

$$e_\lambda := \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|, \quad \forall \lambda \in \Lambda.$$

As before, we can consider the function

$$\tilde{\rho}_{\vec{c}}^*(\alpha) := \lim_{\varepsilon \rightarrow 0^+} \limsup_{j \rightarrow +\infty} \frac{\log \#\{\lambda \in \Lambda_j : 2^{-(\alpha+\varepsilon)j} \leq e_\lambda < 2^{-(\alpha-\varepsilon)j}\}}{\log 2^j}.$$

Let us show that $\tilde{\rho}_{\vec{c}} \leq \tilde{\rho}_{\vec{c}}^*$ but that these functions do not necessarily coincide.

First, remark that $d_\lambda = \max\{e_\mu : \mu \in N(\lambda)\}$ where $N(\lambda)$ denotes the set of the 3 “neighbors” of λ in Λ_j (i.e. the dyadic intervals of length 2^{-j} , whose boundary intersects the boundary of λ , with the usual conventions). Therefore,

$$\#\{\lambda \in \Lambda_j : 2^{-(\alpha+\varepsilon)j} \leq d_\lambda < 2^{-(\alpha-\varepsilon)j}\} \leq 3\#\{\lambda \in \Lambda_j : 2^{-(\alpha+\varepsilon)j} \leq e_\lambda < 2^{-(\alpha-\varepsilon)j}\}$$

and it follows that for any sequence \vec{c} , we have $\tilde{\rho}_{\vec{c}} \leq \tilde{\rho}_{\vec{c}}^*$.

Let us now check that these two quantities can differ. Consider the Cantor set of ratio $\frac{1}{4}$: we start with the interval $[0, 1]$, and, at each step in the standard Cantor set construction, we keep the two outer dyadic intervals whose length is $\frac{1}{4}$ times the length of the parent interval. We denote by C_n the subset of $[0, 1]$ obtained at step n and we denote the Cantor set by

$$C \left(\frac{1}{4} \right) = \bigcap_{n \in \mathbb{N}} C_n.$$

Note that the dyadic intervals that appear in the construction (we will call them the “fundamental intervals”) correspond to scales j that are even.

Let us now define a wavelet sequence as follows. Let $0 < \gamma < \alpha$.

- Let j be even. If $\lambda_{j,k}$ is a fundamental interval, we set $c_{j,k} := 2^{-\gamma j}$. If $\lambda_{j,k}$ is a subinterval of a fundamental interval of the generation $j-2$ (we will call them the “secondary intervals”), then we set $c_{j,k} := 2^{-\alpha j}$. Otherwise, we set $c_{j,k} := 0$.
- Let now j be odd. If $\lambda_{j,k}$ is a subinterval of a fundamental interval of the generation $j-1$, we set $c_{j,k} := 2^{-\gamma j}$. Otherwise, we set $c_{j,k} := 0$.

One easily checks that all wavelet leaders are either equal to $2^{-\gamma j}$ or 0, while restricted leaders associated to a secondary interval are equal to $2^{-\alpha j}$ (indeed, this is the value of the corresponding wavelet coefficients, and all wavelet coefficients associated to proper subintervals vanish). Consequently, $\tilde{\rho}_{\vec{c}}(\alpha) \neq \tilde{\rho}_{\vec{c}}^*(\alpha)$.

Remark 5.2.6. From Theorem 5.2.5, if f is a uniformly Hölder function, we have $d_f(h) \leq \tilde{\rho}_{\vec{c}}(h)$ for all $h \in [0, +\infty]$. So, we also have $d_f(h) \leq \tilde{\rho}_{\vec{c}}^*(h)$ for all $h \in [0, +\infty]$.

The example that we have just exposed shows that the upper bound of the spectrum supplied by $\tilde{\rho}_{\vec{c}}^*(h)$ can be sharpened using wavelet leaders. This explains why one prefers the definition using wavelet leaders (see however Proposition 5.3.6 below, which shows that some quantities derived from these notions actually coincide).

The wavelet leaders density of a signal is defined through its wavelet coefficients. The independence from the sufficiently smooth wavelet basis which is chosen is a natural requirement. In practice, as presented in Chapter 4, one can use the notion of robustness.

Proposition 5.2.7. [24] *The definition of the wavelet leaders density of a function is not robust.*

Proof. We consider again the Cantor set of ratio $\frac{1}{4}$ and as before, we denote by C_n the subset obtained at step n in its construction. We define the subset Γ of $\Lambda \times \Lambda$ by

$$\Gamma := \left\{ (\lambda, \lambda') : \exists n \in \mathbb{N} \text{ such that } \lambda' \in C_n, [(k'+3)2^{-j'}, (k'+4)2^{-j'}] \in C_n \right. \\ \left. \text{and } j = j' + 1, k = 2k' + 3 \right\}.$$

Let us fix $\beta > \alpha > 0$ and let us define the infinite matrix A indexed by dyadic intervals by setting

$$A(\lambda, \lambda') := \begin{cases} 1 & \text{if } \lambda = \lambda', \\ 2^{-\beta j} 2^{\alpha j'} & \text{if } (\lambda, \lambda') \in \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix A is of the form $Id + R$. Remark that if $(\lambda, \lambda') \in \Gamma$, then $(\lambda', \lambda'') \notin \Gamma$ for any dyadic interval λ'' and it follows that $R^2 = 0$. This implies that A is invertible, with inverse $Id - R$.

Clearly, the matrices A and A^{-1} belong to \mathcal{A}^γ for every $\gamma > 0$. Let us fix $\delta > \beta$ and let us define the sequence \vec{c} by

$$c_\lambda := \begin{cases} 2^{-\alpha j} & \text{if there is } n \in \mathbb{N} \text{ such that } \lambda \in C_n, \\ 0 & \text{if there exists } \lambda' \text{ such that } (\lambda, \lambda') \in \Gamma, \\ 2^{-\delta j} & \text{otherwise.} \end{cases}$$

It is straightforward to see that $\tilde{\rho}_{\vec{c}}(\beta) = -\infty$. Let us now consider the image \vec{x} of \vec{c} by the matrix A , that is to say

$$x_\lambda = \sum_{\lambda' \in \Lambda} A(\lambda, \lambda') c_{\lambda'}, \quad \forall \lambda \in \Lambda.$$

Then, we have

$$x_\lambda = \begin{cases} 2^{-\alpha j} & \text{if there is } n \in \mathbb{N} \text{ such that } \lambda \in C_n, \\ 2^{-\beta j} & \text{if there exists } \lambda' \text{ such that } (\lambda, \lambda') \in \Gamma, \\ 2^{-\delta j} & \text{otherwise;} \end{cases}$$

hence $\tilde{\rho}_{\tilde{x}}(\beta) \geq \frac{1}{2}$. \square

This counter-example motivates the introduction, in the next section, of another notion based on the wavelet leaders of a signal which will be shown to be robust.

5.3 Wavelet leaders profile

A theoretical drawback when working with the wavelet leaders density is that it is not a robust quantity. Consequently, it may lead to quantities that are not intrinsic, and therefore not reliable for classification purposes. On the computational side, another drawback comes from the double limit in Definition 5.2.1. In practice, when dealing with real-life data, one can never really “pass to the limit” several times consecutively, and one must therefore make simultaneously ε become small and j large, and therefore introduce some dependency between j and ε . However, on the mathematical side, it is easy to check that, as soon as such a dependency between j and ε is introduced in Definition 5.2.1, the value of the corresponding limit can change radically (see for example [46, 47, 139]). In other words, this definition is numerically extremely unstable and, in practice, definitions that are based on a single limit are the only ones that can be used. Therefore, we will define another quantity based on the wavelet leaders density which will turn out to be robust and which still yields an upper bound for the multifractal spectrum.

Definition 5.3.1. The *increasing wavelet leaders profile* of a sequence $\vec{c} \in C^0$ is defined for every $\alpha \in [0, +\infty]$ by

$$\tilde{\nu}_{\vec{c}}^+(\alpha) := \lim_{\varepsilon \rightarrow 0^+} \limsup_{j \rightarrow +\infty} \frac{\log \#\{\lambda \in \Lambda_j : d_\lambda \geq 2^{-(\alpha+\varepsilon)j}\}}{\log 2^j}.$$

Similarly, the *decreasing wavelet leaders profile* of \vec{c} is defined for every $\alpha \geq 0$ by

$$\tilde{\nu}_{\vec{c}}^-(\alpha) := \lim_{\varepsilon \rightarrow 0^+} \limsup_{j \rightarrow +\infty} \frac{\log \#\{\lambda \in \Lambda_j : d_\lambda \leq 2^{-(\alpha-\varepsilon)j}\}}{\log 2^j}$$

and for $\alpha = +\infty$ by

$$\tilde{\nu}_{\vec{c}}^-(+\infty) := \lim_{A \rightarrow +\infty} \liminf_{j \rightarrow +\infty} \frac{\log \#\{\lambda \in \Lambda_j : d_\lambda \leq 2^{-Aj}\}}{\log 2^j}.$$

These definitions formalize the idea that there are about $2^{\tilde{\nu}_{\vec{c}}^+(\alpha)j}$ (resp. $2^{\tilde{\nu}_{\vec{c}}^-(\alpha)j}$) wavelet leaders larger (resp. smaller) than $2^{-\alpha j}$ at large scales j . Remark that thanks to the limit over ε , the inequalities that appear in the definitions can be chosen strict or not.

Remark 5.3.2. The limit over ε which appears in the definition of the increasing and decreasing wavelet leaders profiles is required in order to derive some mathematical properties that will be useful in the sequel; however, it is not taken into account in applications, and the definition therefore boils down to a single limit, as required, see [64].

The next result gives the properties of the increasing and decreasing wavelet leaders profiles of a sequence $\vec{c} \in C^0$.

Proposition 5.3.3.

1. The increasing wavelet leaders profile of a sequence $\vec{c} \in C^0$ is increasing and right-continuous on $[0, +\infty]$, and takes values in $\{-\infty\} \cup [0, 1]$. Moreover, it satisfies $\tilde{\nu}_{\vec{c}}^+(+\infty) = 1$.
2. The decreasing wavelet leaders profile of a sequence $\vec{c} \in C^0$ is decreasing and left-continuous on $[0, +\infty)$, and takes values in $\{-\infty\} \cup [0, 1]$. Moreover, it satisfies $\tilde{\nu}_{\vec{c}}^-(0) = 1$.
3. If $\vec{c} \in C^0$, then the function

$$\alpha \in (0, +\infty) \mapsto \frac{\tilde{\nu}_{\vec{c}}^-(\alpha) - 1}{\alpha}$$

is decreasing, i.e. the function $1 - \tilde{\nu}_{\vec{c}}^-$ is with increasing-visibility on $[0, +\infty)$.

Proof. The two first points are immediate. Let us prove the last one. We fix α, α' such that $0 < \alpha' < \alpha$. From the definition of the decreasing wavelet leaders profile of \vec{c} , we know that for every $\delta > 0$, there is $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, there is a sequence $(j_n)_{n \in \mathbb{N}}$ which satisfies

$$\#\{\lambda \in \Lambda_{j_n} : d_\lambda \leq 2^{-(\alpha-\varepsilon)j_n}\} \geq 2^{\tilde{\nu}_{\vec{c}}^-(\alpha)-\delta} j_n, \quad \forall n \in \mathbb{N}.$$

Then, if $j \geq j_n$, we also have

$$\#\{\lambda \in \Lambda_j : d_\lambda \leq 2^{-(\alpha-\varepsilon)j_n}\} \geq 2^{j-j_n} 2^{\tilde{\nu}_{\vec{c}}^-(\alpha)-\delta} j_n$$

since $d_\lambda \leq d_{\lambda_n}$ if $\lambda \subseteq \lambda_n$. For every $n \in \mathbb{N}$, let us set

$$J_n = \left\lfloor \frac{\alpha - \varepsilon}{\alpha' - \varepsilon} j_n \right\rfloor.$$

If n is large enough, $J_n \geq j_n$ and we obtain

$$\begin{aligned} \#\{\lambda \in \Lambda_{J_n} : d_\lambda \leq 2^{-(\alpha'-\varepsilon)J_n}\} &\geq \#\{\lambda \in \Lambda_{J_n} : d_\lambda \leq 2^{-(\alpha-\varepsilon)j_n}\} \\ &\geq 2^{J_n-j_n} 2^{\tilde{\nu}_{\vec{c}}^-(\alpha)-\delta} j_n. \end{aligned}$$

Consequently,

$$\begin{aligned} \limsup_{j \rightarrow \infty} \frac{\log \#\{\lambda \in \Lambda_j : d_\lambda \leq 2^{-(\alpha'-\varepsilon)j}\}}{\log 2^j} &\geq \lim_{n \rightarrow \infty} \frac{\log \#\{\lambda \in \Lambda_{J_n} : d_\lambda \leq 2^{-(\alpha'-\varepsilon)J_n}\}}{\log 2^{J_n}} \\ &\geq \lim_{n \rightarrow \infty} \left(1 + (\tilde{\nu}_{\vec{c}}^-(\alpha) - \delta - 1) \frac{j_n}{J_n} \right) \\ &\geq 1 + (\tilde{\nu}_{\vec{c}}^-(\alpha) - \delta - 1) \frac{\alpha' - \varepsilon}{\alpha - \varepsilon} \end{aligned}$$

and it follows that

$$\tilde{\nu}_{\vec{c}}^-(\alpha') \geq 1 + (\tilde{\nu}_{\vec{c}}^-(\alpha) - \delta - 1) \frac{\alpha'}{\alpha}.$$

Since $\delta > 0$ is arbitrary, we get that

$$\frac{\tilde{\nu}_{\vec{c}}^-(\alpha') - 1}{\alpha'} \geq \frac{\tilde{\nu}_{\vec{c}}^-(\alpha) - 1}{\alpha}$$

hence the conclusion. \square

Additionally, if there exist $\alpha_0 > 0$ and $C_0 > 0$ (resp. $\alpha_1 > 0$ and $C_1 > 0$) such that

$$|c_\lambda| \leq C_0 2^{-\alpha_0 j} \quad (\text{resp. } d_\lambda \geq C_1 2^{-\alpha_1 j})$$

for every $j \in \mathbb{N}_0$, $\lambda \in \Lambda_j$, then $\tilde{\nu}_{\vec{c}}^+$ is identically equal to $-\infty$ on $(-\infty, \alpha_0)$ (resp. $\tilde{\nu}_{\vec{c}}^-$ is identically equal to $-\infty$ on $(\alpha_1, +\infty)$). Moreover, the increasing and decreasing wavelet leaders profiles of a sequence of wavelet coefficients still yield an upper bound for the spectrum of the corresponding function, as stated in the next result.

Proposition 5.3.4. [24] *Let f be a uniformly Hölder function, and let \vec{c} be the sequence of its wavelet coefficients in a given wavelet basis. The multifractal spectrum of f satisfies*

$$d_f(h) \leq \min\{\tilde{\nu}_{\vec{c}}^+(h), \tilde{\nu}_{\vec{c}}^-(h)\}, \quad \forall h \in [0, +\infty].$$

Proof. It is clear that $\tilde{\nu}_{\vec{c}}^+(h) \geq \tilde{\rho}_{\vec{c}}(h)$ and $\tilde{\nu}_{\vec{c}}^-(h) \geq \tilde{\rho}_{\vec{c}}(h)$ for every h . The result follows then directly from Theorem 5.2.5. \square

The following lemma shows the link between the wavelet leaders density of a sequence and its increasing and decreasing wavelet leaders profiles.

Lemma 5.3.5. [24]

1. *If $\vec{c} \in C^0$, then*

$$\tilde{\nu}_{\vec{c}}^+(\alpha) = \sup_{\alpha' \leq \alpha} \tilde{\rho}_{\vec{c}}(\alpha'), \quad \forall \alpha \in [0, +\infty).$$

2. *Assume that $\vec{c} \in C^0$ is a sequence for which there are $\alpha_1 > 0$ and $C_1 > 0$ such that $d_{j,k} \geq C_1 2^{-\alpha_1 j}$ for every $j \in \mathbb{N}_0$, $k \in \{0, \dots, 2^j - 1\}$. Then*

$$\tilde{\nu}_{\vec{c}}^-(\alpha) = \sup_{\alpha' \geq \alpha} \tilde{\rho}_{\vec{c}}(\alpha'), \quad \forall \alpha \in [0, +\infty].$$

Proof. 1. Let $\alpha_0 = \inf\{\alpha \geq 0 : \tilde{\nu}_{\vec{c}}^+ \geq 0\}$. The result is clear if $\alpha < \alpha_0$. So, let us assume that $\alpha \geq \alpha_0$. Of course, we have $\tilde{\nu}_{\vec{c}}^+(\alpha) \geq \tilde{\rho}_{\vec{c}}(\alpha)$. Since $\tilde{\nu}_{\vec{c}}^+$ is increasing, we get that

$$\tilde{\nu}_{\vec{c}}^+(\alpha) \geq \sup_{\alpha' \leq \alpha} \tilde{\rho}_{\vec{c}}(\alpha').$$

For the other inequality, let us fix $\varepsilon > 0$. By definition of $\tilde{\rho}_{\vec{c}}$, for every $\alpha' \leq \alpha + \varepsilon$, there exist $r(\alpha') > 0$ and $J(\alpha') \in \mathbb{N}$ such that

$$\#\left\{\lambda \in \Lambda_j : d_\lambda \geq 2^{-(\alpha' + r(\alpha'))j}\right\} - \#\left\{\lambda \in \Lambda_j : d_\lambda \geq 2^{-(\alpha' - r(\alpha'))j}\right\} \leq 2^{(\tilde{\rho}_{\vec{c}}(\alpha') + \varepsilon)j}$$

for every $j \geq J(\alpha')$. From the covering of the compact $[\alpha_0, \alpha + \varepsilon]$ by the open sets $(\alpha' - r(\alpha'), \alpha' + r(\alpha'))$, we extract a finite subcovering

$$\{(\alpha'_i - r(\alpha'_i), \alpha'_i + r(\alpha'_i)) : i \in \{1, \dots, n\}\}.$$

Fix $J \geq \max_{1 \leq i \leq n} J(\alpha'_i)$. For every $j \geq J$, we have

$$\begin{aligned} & \#\{\lambda \in \Lambda_j : d_\lambda \geq 2^{-(\alpha+\varepsilon)j}\} \\ & \leq \sum_{i=1}^n \left(\#\{\lambda \in \Lambda_j : d_\lambda \geq 2^{-(\alpha'_i+r(\alpha'_i))j}\} - \#\{\lambda \in \Lambda_j : d_\lambda \geq 2^{-(\alpha'_i-r(\alpha'_i))j}\} \right) \\ & \leq \sum_{i=1}^n 2^{(\tilde{\rho}_\varepsilon(\alpha'_i)+\varepsilon)j} \leq n2^{(\sup_{\alpha' \leq \alpha} \tilde{\rho}_\varepsilon(\alpha')+\varepsilon)j}. \end{aligned}$$

It follows directly that $\tilde{\nu}_\varepsilon^+(\alpha) \leq \sup_{\alpha' \leq \alpha} \tilde{\rho}_\varepsilon(\alpha')$.

2. The proof of the second part is very similar. The result is obvious if $\alpha > \alpha_1$. The case $\alpha = +\infty$ follows from the definitions of $\tilde{\nu}_\varepsilon^+(+\infty)$ and $\tilde{\rho}_\varepsilon(+\infty)$. So it remains to study the case $\alpha \leq \alpha_1$. As done previously, since $\tilde{\nu}_\varepsilon^-$ is decreasing, we get that

$$\tilde{\nu}_\varepsilon^-(\alpha) \geq \sup_{\alpha' \geq \alpha} \tilde{\rho}_\varepsilon(\alpha').$$

For the other inequality, let us fix $\varepsilon > 0$. Again, for every $\alpha' \geq \alpha + \varepsilon$, there exist $r(\alpha') > 0$ and $J(\alpha') \in \mathbb{N}$ such that

$$\#\{\lambda \in \Lambda_j : d_\lambda \geq 2^{-(\alpha'+r(\alpha'))j}\} - \#\{\lambda \in \Lambda_j : d_\lambda \geq 2^{-(\alpha'-r(\alpha'))j}\} \leq 2^{(\tilde{\rho}_\varepsilon(\alpha')+\varepsilon)j}$$

for every $j \geq J(\alpha')$. Taking a finite subcovering $\{(\alpha'_i - r(\alpha'_i), \alpha'_i + r(\alpha'_i)) : i \in \{1, \dots, n\}\}$ of $[\alpha + \varepsilon, \alpha_1]$, we get

$$\begin{aligned} & \#\{\lambda \in \Lambda_j : d_\lambda \leq 2^{-(\alpha+\varepsilon)j}\} \\ & \leq \sum_{i=1}^n \left(\#\{\lambda \in \Lambda_j : d_\lambda \geq 2^{-(\alpha'_i+r(\alpha'_i))j}\} - \#\{\lambda \in \Lambda_j : d_\lambda \geq 2^{-(\alpha'_i-r(\alpha'_i))j}\} \right) \\ & \leq \sum_{i=1}^n 2^{(\tilde{\rho}_\varepsilon(\alpha'_i)+\varepsilon)j} \leq n2^{(\sup_{\alpha' \geq \alpha} \tilde{\rho}_\varepsilon(\alpha')+\varepsilon)j}, \end{aligned}$$

hence the conclusion. \square

The next proposition shows that, unlike the wavelet leaders density, we can define the increasing and decreasing wavelet leaders profiles of a sequence using the restricted wavelet leaders e_λ instead of the wavelet leaders d_λ .

Proposition 5.3.6. [24] *If $\vec{c} \in C^0$, then for every $\alpha \in [0, +\infty]$,*

$$\tilde{\nu}_\vec{c}^+(\alpha) = \lim_{\varepsilon \rightarrow 0^+} \limsup_{j \rightarrow +\infty} \frac{\log \#\{\lambda \in \Lambda_j : e_\lambda \geq 2^{-(\alpha+\varepsilon)j}\}}{\log 2^j}. \quad (5.1)$$

Moreover, for every $\alpha \in [0, +\infty)$,

$$\tilde{\nu}_\vec{c}^-(\alpha) = \lim_{\varepsilon \rightarrow 0^+} \limsup_{j \rightarrow +\infty} \frac{\log \#\{\lambda \in \Lambda_j : e_\lambda \leq 2^{-(\alpha-\varepsilon)j}\}}{\log 2^j}, \quad (5.2)$$

and for $\alpha = +\infty$,

$$\tilde{\nu}_\vec{c}^-(+\infty) = \lim_{A \rightarrow +\infty} \liminf_{j \rightarrow +\infty} \frac{\log \#\{\lambda \in \Lambda_j : e_\lambda \leq 2^{-Aj}\}}{\log 2^j}. \quad (5.3)$$

Proof. 1. Let us define $\tilde{\nu}_{\vec{c}}^{+,*}(\alpha)$ as the right hand side of (5.1). Then it is clear that $\tilde{\nu}_{\vec{c}}^{+,*}(\alpha) \leq \tilde{\nu}_{\vec{c}}^+(\alpha)$ for every $\alpha \in \mathbb{R}$ since $e_\lambda \leq d_\lambda$ for every dyadic interval λ . For the other inequality, let us fix $\alpha \in \mathbb{R}$ and $\delta > 0$. By definition, there exist $J \geq 0$ and $\varepsilon > 0$ such that

$$\#\{\lambda \in \Lambda_j : e_\lambda \geq 2^{-(\alpha+\varepsilon)j}\} \leq 2^{(\tilde{\nu}_{\vec{c}}^{+,*}(\alpha)+\delta)j}, \quad \forall j \geq J.$$

Let us fix $j \geq J$. As before, for all $\lambda \in \Lambda_j$, we denote $N(\lambda)$ the set of the 3 “neighbors” of λ in Λ_j . Then we have $d_\lambda = \max\{e_\mu : \mu \in N(\lambda)\}$ and it follows that, for j large enough,

$$\begin{aligned} \#\{\lambda \in \Lambda_j : d_\lambda \geq 2^{-(\alpha+\varepsilon)j}\} &\leq 3 \cdot \#\{\lambda \in \Lambda_j : e_\lambda \geq 2^{-(\alpha+\varepsilon)j}\} \\ &\leq 3 \cdot 2^{(\tilde{\nu}_{\vec{c}}^{+,*}(\alpha)+\delta)j}. \end{aligned}$$

Thus $\tilde{\nu}_{\vec{c}}^+(\alpha) \leq \tilde{\nu}_{\vec{c}}^{+,*}(\alpha) + \delta$ and since $\delta > 0$ is arbitrary, we get the conclusion if α is finite. The result is also true for $\alpha = +\infty$ because these two functions take the value one for $\alpha = +\infty$.

2. The proof of the second point is similar. Of course, we always have

$$\#\{\lambda \in \Lambda_j : e_\lambda \leq 2^{-(\alpha-\varepsilon)j}\} \geq \#\{\lambda \in \Lambda_j : d_\lambda \leq 2^{-(\alpha-\varepsilon)j}\}.$$

Moreover, for any dyadic interval $\lambda \in \Lambda_j$, there exists $\lambda' \in \Lambda_{j+2}$ such that $3\lambda' \subseteq \lambda$. Therefore $d_{\lambda'} \leq e_\lambda$ and

$$\begin{aligned} \#\{\lambda \in \Lambda_j : e_\lambda \leq 2^{-(\alpha-\varepsilon)j}\} &\leq \#\{\lambda \in \Lambda_{j+2} : d_\lambda \leq 2^{-(\alpha-\varepsilon)j}\} \\ &\leq \#\{\lambda \in \Lambda_{j+2} : d_\lambda \leq 2^{-(\alpha-2\varepsilon)(j+2)}\} \end{aligned}$$

if j is large enough. Hence the result for $\alpha \in [0, +\infty)$. The same argument holds for $\alpha = +\infty$. \square

This equivalence of the definitions of the wavelet leaders profiles is important because the formulation using restricted wavelet leaders is more suited for the study of its properties: indeed, the suprema at a given scale are taken on non-overlapping intervals. Consequently, from now on, we will often work with restricted wavelet leaders instead of wavelet leaders. Both functions $\tilde{\nu}_{\vec{c}}^+$ and $\tilde{\nu}_{\vec{c}}^{+,*}$ will be denoted by $\tilde{\nu}_{\vec{c}}^+$. We use similar notations for the decreasing profile.

Proposition 5.3.7. *Let $\vec{c} \in C^r$ for some $r > 0$. The definitions of the increasing and the decreasing wavelet leaders profiles of \vec{c} are robust.*

The robustness of the increasing and decreasing wavelet leaders profiles is proved in Appendix A. Consequently, given a uniformly Hölder function f , we can define its increasing (resp. decreasing) wavelet leaders profile $\tilde{\nu}_f^+$ (resp $\tilde{\nu}_f^-$) by setting $\tilde{\nu}_f^+ = \tilde{\nu}_{\vec{c}}^+$ (resp. $\tilde{\nu}_f^- = \tilde{\nu}_{\vec{c}}^-$), where \vec{c} is the sequence of wavelet coefficients of f in a given wavelet basis.

5.4 Examples

From Proposition 5.3.7, we know that the increasing and decreasing wavelet leaders profiles of a uniformly Hölder function do not depend on the chosen wavelet basis. Moreover, from Proposition 5.3.4, we know that they give an upper bound for the

multifractal spectrum of this function. They are therefore a “good candidate” to propose a new multifractal formalism based on wavelet leaders. In practice, in order to estimate the multifractal spectrum of a function using its wavelet leaders profile, one can proceed as follows.

Definition 5.4.1. Let f be a uniformly Hölder function and let us denote by α_s the smallest positive number such that $\tilde{\nu}_f^+(\alpha_s) = 1$. The *wavelet leaders profile* of f is defined by

$$\tilde{\nu}_f(\alpha) := \begin{cases} \tilde{\nu}_f^+(\alpha) & \text{if } \alpha \in [0, \alpha_s], \\ \tilde{\nu}_f^-(\alpha) & \text{if } \alpha \in [\alpha_s, +\infty]. \end{cases}$$

The *leaders profile method* is based on the estimation of the multifractal spectrum of f by the function $\tilde{\nu}_f$.

Remark that this definition makes sense: indeed, it is easy to see that we also have $\tilde{\nu}_f^-(\alpha_s) = 1$ (by using the wavelet leaders density). In this section, we show that the leaders profile method holds for some classical models used in applications.

5.4.1 Fractional Brownian motion

A classical process is the fractional Brownian motion [108]. It is a Gaussian stochastic process which is self-similar with stationary increments and which depends on a parameter $\beta \in (0, 1)$, called the *Hurst index*. Its main property is the existence of a long-range correlation which introduces a weak dependence between the points of a realization. Such dependences are detected in many experimental observations and that is why fractional Brownian motions model many monofractal phenomena [15, 56, 108, 116]. The regularity of sample paths of fractional Brownian motions is well known and is recalled in the following theorem.

Theorem 5.4.2. [97] *With probability one, a sample path B_β of a fractional Brownian motion of parameter $\beta \in (0, 1)$ has everywhere the Hölder exponent β , so that its multifractal spectrum satisfies*

$$d_{B_\beta}(\alpha) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ -\infty & \text{if } \alpha \neq \beta. \end{cases}$$

In particular, with probability one, a sample path of a fractional Brownian motion is uniformly Hölder.

Let ψ be a mother wavelet in the Schwartz class and let ψ_α be defined by

$$\widehat{\psi}_\alpha(\xi) = \frac{1}{|\xi|^\alpha} \widehat{\psi}(\xi).$$

Then ψ_α and $\psi_{-\alpha}$ form biorthogonal wavelet bases [113]. The use of these bases gives a decorrelation of the wavelet coefficients of a fractional Brownian motion. More precisely, if B_β is a fractional Brownian motion of index $\beta \in (0, 1)$, it can be written as

$$B_\beta(x) = \sum_{j \in \mathbb{N}} \sum_{k=0}^{2^j-1} 2^{-\beta j} \xi_{j,k} \psi_{\beta+1/2}(2^j x - k) + R(x), \quad x \in [0, 1], \quad (5.4)$$

where R is a C^∞ function and where the $\xi_{j,k}$ are independent standard centered Gaussian [1, 114].

The proof of the following lemma is inspired from Jaffard [90], Jaffard et al. [94].

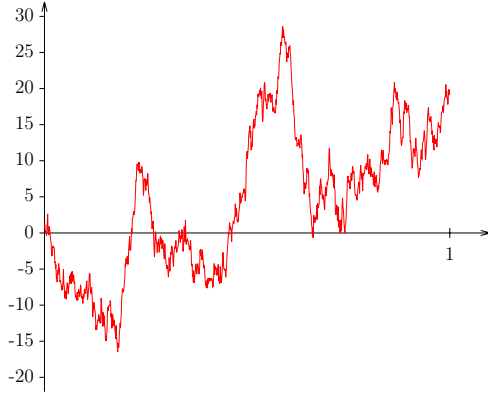


Figure 5.1: Sample path of a fractional Brownian motion of parameter $\beta = 0.5$

Lemma 5.4.3. *Let B_β be a sample path of a fractional Brownian motion of index $\beta \in (0, 1)$ and \vec{c} be the sequence given by the decomposition (5.4). With probability one, there is $J \in \mathbb{N}$ such that $d_{j,k} \geq j^{-4\beta} 2^{-\beta j}$ and $d_{j,k} \leq 2^{-\beta j} j$ for every $j \geq J$ and every $k \in \{0, \dots, 2^j - 1\}$.*

Proof. Let us fix a dyadic cube λ of length 2^{-j} . We use the wavelet decomposition given by (5.4). Let us set

$$j_0 = j + \lfloor \log_2(j^2) \rfloor + 1.$$

The number of dyadic cubes at the scale j_0 included in λ is greater than or equal to j^2 . Consequently,

$$\begin{aligned} \mathbb{P}[d_\lambda \leq j^{-4\beta} 2^{-\beta j}] &\leq \mathbb{P}[e_\lambda \leq j^{-4\beta} 2^{-\beta j}] \\ &\leq \mathbb{P}[|c_{\lambda_0}| \leq j^{-4\beta} 2^{-\beta j}, \forall \lambda_0 \subseteq \lambda, \lambda_0 \in \Lambda_{j_0}] \\ &= \prod_{\lambda_0 \subseteq \lambda, \lambda_0 \in \Lambda_{j_0}} \mathbb{P}[|c_{\lambda_0}| \leq j^{-4\beta} 2^{-\beta j}] \\ &= \prod_{\lambda_0 \subseteq \lambda, \lambda_0 \in \Lambda_{j_0}} \mathbb{P}[|\xi_{\lambda_0}| \leq 2^{\beta j_0} j^{-4\beta} 2^{-\beta j}] \\ &= \prod_{\lambda_0 \subseteq \lambda, \lambda_0 \in \Lambda_{j_0}} \sqrt{\frac{2}{\pi}} \int_0^{j^{-4\beta} 2^{(\beta j_0 - j)\beta}} e^{-\frac{t^2}{2}} dt \\ &\leq \prod_{\lambda_0 \subseteq \lambda, \lambda_0 \in \Lambda_{j_0}} \sqrt{\frac{2}{\pi}} j^{-4\beta} 2^{(\beta j_0 - j)\beta} \\ &\leq \left(\sqrt{\frac{2}{\pi}} j^{-4\beta} 2^{(\beta j_0 - j)\beta} \right)^{j^2} \\ &\leq \left(\sqrt{\frac{2}{\pi}} j^{-2\beta} 2^\beta \right)^{j^2} \\ &\leq e^{-j^2} \end{aligned}$$

if j is large enough. Let us denote by A_j the event “there is $k \in \{0, \dots, 2^j - 1\}$ such

that $d_{j,k} \leq j^{-4\beta} 2^{-\beta j}$. Then, we have

$$\mathbb{P}[A_j] \leq \sum_{k=0}^{2^j-1} \mathbb{P}[d_{j,k} \leq j^{-4\beta} 2^{-\beta j}] \leq 2^j e^{-j^2} \leq e^{-\frac{j^2}{2}}$$

if j is large enough. It follows that the series $\sum_{j \in \mathbb{N}} \mathbb{P}[A_j]$ converges. Using the Borel Cantelli lemma, we get

$$\mathbb{P} \left[\bigcup_{J \in \mathbb{N}} \bigcap_{j \geq J} \mathcal{C}_{A_j} \right] = 1$$

i.e. with probability one, there is $J \in \mathbb{N}$ such that $d_{j,k} \geq j^{-4\beta} 2^{-\beta j}$ for every $j \geq J$ and every $k \in \{0, \dots, 2^j - 1\}$.

Let us show with the same method that with probability one, there exists $J \in \mathbb{N}$ such that $|c_{j,k}| \leq 2^{-\beta j} j$ for every $j \geq J$ and every $k \in \{0, \dots, 2^j - 1\}$. We have

$$\mathbb{P}[|c_\lambda| \geq 2^{-\beta j} j] = \mathbb{P}[|\xi_\lambda| \geq j] = \sqrt{\frac{2}{\pi}} \int_j^{+\infty} e^{-\frac{t^2}{2}} dt \leq \int_j^{+\infty} e^{-\frac{t}{2}} dt = 2e^{-\frac{j}{2}}.$$

If A'_j denotes the event “there is $k \in \{0, \dots, 2^j - 1\}$ such that $|c_{j,k}| \geq 2^{-\beta j} j$ ”, we have

$$\sum_{j \in \mathbb{N}} \mathbb{P}[A'_j] < +\infty.$$

The Borel Cantelli lemma gives the result. In particular, with probability one, we have $d_{j,k} \leq 2^{-\beta j} j$ for every j large enough and every $k \in \{0, \dots, 2^j - 1\}$ since the function $x \mapsto x 2^{-\beta x}$ is decreasing if $x > \beta^{-1}$. \square

Proposition 5.4.4. *Let B_β be a sample path of a fractional Brownian motion of index $\beta \in (0, 1)$ and \vec{c} be the sequence given by the decomposition (5.4). With probability one,*

$$\tilde{\rho}_{\vec{c}}(\alpha) = \begin{cases} -\infty & \text{if } \alpha \neq \beta, \\ 1 & \text{if } \alpha = \beta. \end{cases}$$

Proof. First, let us fix $\alpha > \beta$. We know from Lemma 5.4.3 that with probability one, there exists $J \in \mathbb{N}$ such that

$$d_{j,k} \geq j^{-4\beta} 2^{-\beta j}, \quad \forall j \geq J, \forall k \in \{0, \dots, 2^j - 1\}.$$

Let us fix $\varepsilon > 0$ so that $\alpha - \varepsilon > \beta$. Then for j large enough, we have $j^{-4\beta} 2^{-\beta j} \geq 2^{-(\alpha - \varepsilon)j}$. It follows that

$$\#\{k \in \{0, \dots, 2^j - 1\} : 2^{-(\alpha + \varepsilon)j} \leq d_{j,k} < 2^{-(\alpha - \varepsilon)j}\} = 0$$

and we get that $\tilde{\rho}_{\vec{c}}(\alpha) = -\infty$ with probability one.

Now, assume that $\alpha < \beta$. Using Lemma 5.4.3, we know that with probability one, there exists $J \in \mathbb{N}$ such that

$$d_{j,k} \leq 2^{-\beta j} j, \quad \forall j \geq J, \forall k \in \{0, \dots, 2^j - 1\}.$$

We choose $\varepsilon > 0$ small enough so that $\alpha + \varepsilon < \beta$. Then $2^{-\beta j} j < 2^{-(\alpha + \varepsilon)j}$ if j is sufficiently large. Consequently,

$$\#\{k \in \{0, \dots, 2^j - 1\} : 2^{-(\alpha + \varepsilon)j} \leq d_{j,k} < 2^{-(\alpha - \varepsilon)j}\} = 0$$

hence $\tilde{\rho}_{\vec{c}}(\alpha) = -\infty$ with probability one.

Finally, let us show that $\tilde{\rho}_{\vec{c}}(\beta) = 1$. We know that for any uniformly Hölder function f , $d_{B_\beta}(h) \leq \tilde{\rho}_{\vec{c}}(h)$ for every $h \geq 0$. Moreover, with probability one, we know that $d_{B_\beta}(\beta) = 1$ and it follows that $\tilde{\rho}_{\vec{c}}(\beta) = 1$. \square

The following result shows that the leaders profile method yields the correct spectrum.

Proposition 5.4.5. *Let B_β be a sample path of a fractional Brownian motion of index $\beta \in (0, 1)$. With probability one, $d_{B_\beta} = \tilde{\nu}_f$ on $[0, +\infty]$.*

Proof. The robustness of the wavelet leaders profile implies that $\tilde{\nu}_f = \tilde{\nu}_{\vec{c}}$, where \vec{c} is the sequence given by the decomposition (5.4). The result follows then directly from Lemma 5.3.5, Theorem 5.4.2 and Proposition 5.4.4. \square

5.4.2 Lacunary wavelet series

Lacunary wavelet series have been introduced by Jaffard [89]. They depend on two parameters $\eta \in (0, 1)$ and $\alpha > 0$ and are defined through their wavelet coefficients as follows. Let $(g_{j,k})_{j \in \mathbb{N}_0, k \in \{0, \dots, 2^j - 1\}}$ be a sequence of independent random variables in a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ whose laws are Bernoulli laws with parameter $2^{-(1-\eta)j}$, i.e. such that

$$g_{j,k} = \begin{cases} 1 & \text{with a probability } 2^{-(1-\eta)j}, \\ 0 & \text{with a probability } 1 - 2^{-(1-\eta)j}. \end{cases}$$

The wavelet coefficients of the lacunary wavelet series $R_{\alpha,\eta}$ are given by

$$c_{j,k} := g_{j,k} 2^{-\alpha j}, \quad \forall j \in \mathbb{N}_0, k \in \{0, \dots, 2^j - 1\}.$$

The regularity of lacunary wavelet series have been studied by Jaffard.

Theorem 5.4.6. [89] *With probability one, the multifractal spectrum of a sample path of the lacunary wavelet series $R_{\alpha,\eta}$ is given by*

$$d_{R_{\alpha,\eta}}(h) = \begin{cases} \frac{h\eta}{\alpha} & \text{if } h \in [\alpha, \frac{\alpha}{\eta}], \\ -\infty & \text{otherwise.} \end{cases}$$

In particular, almost every lacunary wavelet series is multifractal. Let us prove that with probability one, the leaders profile method gives the correct spectrum.

Lemma 5.4.7. *Let us fix $\eta' > \eta$. With probability one, there is $J \in \mathbb{N}$ such that*

$$\#\{k \in \{0, \dots, 2^j - 1\} : c_{j,k} = 2^{-\alpha j}\} \leq 2^{\eta' j}, \quad \forall j \geq J.$$

Proof. For every j , we denote by B_j the event

$$\#\{k \in \{0, \dots, 2^j - 1\} : c_{j,k} = 2^{-\alpha j}\} > 2^{\eta' j}.$$

Remark that at a given scale j , we count the number of successes of a binomial distribution of parameters $(2^j, 2^{-(1-\eta)j})$, where the success means “ $c_{j,k} = 2^{-\alpha j}$ ”. It follows that

$$\begin{aligned} \mathbb{P}[B_j] &= \sum_{2^{\eta' j} < m \leq 2^j} \binom{2^j}{m} (2^{-(1-\eta)j})^m (1 - 2^{-(1-\eta)j})^{2^j - m} \\ &\leq \sum_{2^{\eta' j} < m \leq 2^j} \frac{(2^j 2^{-(1-\eta)j})^m}{m!} \leq 2^j \frac{(2^{\eta j})^{2^{\eta' j}}}{\Gamma(2^{\eta' j} + 1)}. \end{aligned}$$

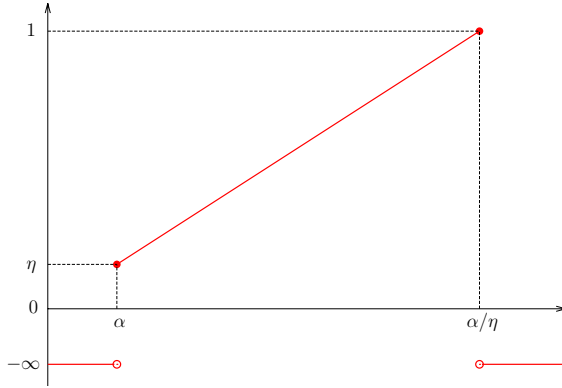


Figure 5.2: Almost sure multifractal spectrum of a sample path of a lacunary wavelet series

Using Stirling's formula, we obtain then that for j large enough,

$$2^j \frac{(2^{\eta j})^{2^{\eta' j}}}{\Gamma(2^{\eta' j} + 1)} \sim \frac{2^j}{\sqrt{2\pi}} \left(2^{(\eta - \eta')j} \frac{e}{\sqrt{2}} \right)^{2^{\eta' j}} \leq \frac{1}{\sqrt{2\pi}} \left(2^{(\eta - \eta')j} e \right)^{2^{\eta' j}}$$

since

$$2^j \leq (\sqrt{2})^{2^{\eta' j}}$$

if j is large enough. Since $\eta' > \eta$, this last term is the general term of a series which converges. We conclude the proof using the Borel Cantelli lemma. \square

Remark 5.4.8. In the next lemma, we will use the following fact: if $a \in (0, 1)$ and if $b > 0$, then

$$(1 - a)^b \leq e^{-ba}.$$

It suffices to use the Taylor development of the function $x \mapsto \log(1 - x)$ around 0 to prove it.

Lemma 5.4.9. *For every $\varepsilon > 0$, with probability one, there exists $J \in \mathbb{N}$ such that the wavelet leaders $d_{j,k}$ of $R_{\alpha,\eta}$ satisfy*

$$d_{j,k} \geq 2^{-j(\frac{\alpha}{\eta} + \varepsilon)}, \quad \forall j \geq J, k \in \{0, \dots, 2^j - 1\}.$$

Proof. Let us fix $\varepsilon > 0$. For every $j \in \mathbb{N}$, let us denote by A_j the event “there exists $k \in \{0, \dots, 2^j - 1\}$ such that $d_{j,k} < 2^{-(\frac{\alpha}{\eta} + \varepsilon)j}$ ”. By the Borel Cantelli lemma, it suffices to prove that

$$\sum_{j \in \mathbb{N}} \mathbb{P}[A_j] < +\infty.$$

Let us set $j_0 = \lfloor \frac{1}{\eta}(j + \log_2 j) \rfloor + 1$. We have

$$\begin{aligned} \mathbb{P}[A_j] &\leq \sum_{k=0}^{2^j-1} \mathbb{P}[d_{j,k} < 2^{-j(\frac{\alpha}{\eta}+\varepsilon)}] \\ &\leq \sum_{k=0}^{2^j-1} \mathbb{P}[e_{j,k} < 2^{-j(\frac{\alpha}{\eta}+\varepsilon)}] \\ &\leq \sum_{k=0}^{2^j-1} \prod_{\lambda_0 \subseteq \lambda(j,k), \lambda_0 \in \Lambda_{j_0}} \mathbb{P}[|c_{\lambda_0}| < 2^{-j(\frac{\alpha}{\eta}+\varepsilon)}]. \end{aligned}$$

From the definition of j_0 , we have $j_0 \leq j(\frac{1}{\eta} + \frac{\varepsilon}{\alpha})$ if j is large enough. Consequently, $|c_{\lambda_0}| < 2^{-j(\frac{\alpha}{\eta}+\varepsilon)}$ if and only if $|c_{\lambda_0}| = 0$. Therefore, we obtain

$$\begin{aligned} \mathbb{P}[A_j] &\leq \sum_{k=0}^{2^j-1} \left(1 - 2^{-(1-\eta)j_0}\right)^{2^{j_0-j}} \\ &\leq 2^j \left(1 - 2^{-(1-\eta)j_0}\right)^{2^{j_0-j}} \\ &\leq 2^j \exp\left(-2^{j_0-j} 2^{-(1-\eta)j_0}\right) \\ &\leq 2^j \exp(-j) = \left(\frac{2}{e}\right)^j \end{aligned}$$

where we have used Remark 5.4.8 and the relation $j_0 \geq \frac{1}{\eta}(j + 2 \log_2 j)$. This leads to the conclusion. \square

Proposition 5.4.10. *With probability one, $d_{R_{\alpha,\eta}} = \tilde{\nu}_{R_{\alpha,\eta}}$ on $[0, +\infty]$.*

Proof. First, let us fix $h < \alpha$. Of course, we have $d_{j,k} \leq 2^{-\alpha j}$ for every $j \in \mathbb{N}$ and every $k \in \{0, \dots, 2^j - 1\}$. Then, if $\varepsilon > 0$ is chosen small enough so that $h + \varepsilon < \alpha$, we have $d_{j,k} < 2^{-(h+\varepsilon)j}$ for every $j \in \mathbb{N}$, $k \in \{0, \dots, 2^j - 1\}$. It follows that $\tilde{\nu}_{R_{\alpha,\eta}}^+(h) = -\infty$.

Secondly, let us fix $h \in [\alpha, \frac{\alpha}{\eta}]$. We already know that $d_{R_{\alpha,\eta}}(h) \leq \tilde{\nu}_{R_{\alpha,\eta}}^+(h)$, so it suffices to show that $\tilde{\nu}_{R_{\alpha,\eta}}^+(h) \leq \frac{h\eta}{\alpha}$. Let us remark that

$$2^{-(h+\varepsilon)j} \leq 2^{-\alpha j'} \iff j' \leq \frac{h+\varepsilon}{\alpha} j.$$

Consequently, if $2^{-(h+\varepsilon)j} \leq d_{j,k}$, then there exists $\lambda' \subseteq 3\lambda(j,k)$ with $c_{\lambda'} = 2^{-\alpha j'}$ and $j' \leq \frac{h+\varepsilon}{\alpha} j$. Fix $\eta' > \eta$. Lemma 5.4.7 implies that

$$\begin{aligned} \#\{\lambda \in \Lambda_j : d_\lambda \geq 2^{-(h+\varepsilon)j}\} &\leq \sum_{j' \leq \frac{h+\varepsilon}{\alpha} j} \#\{\lambda' \in \Lambda_{j'} : c_{\lambda'} = 2^{-\alpha j'}\} \\ &\leq \sum_{j' \leq \frac{h+\varepsilon}{\alpha} j} 2^{\eta' j'} \\ &= \frac{2^{\eta' \lfloor \frac{h+\varepsilon}{\alpha} j \rfloor + 1} - 1}{2^{\eta'} - 1} \\ &\leq \frac{2^{\eta' \frac{h+\varepsilon}{\alpha} j + 1}}{2^{\eta'} - 1} \end{aligned}$$

with probability one. Therefore, we obtain

$$\begin{aligned} \limsup_{j \rightarrow +\infty} \frac{\log \#\{\lambda \in \Lambda_j : d_\lambda \geq 2^{-(h+\varepsilon)j}\}}{\log 2^j} &\leq \limsup_{j \rightarrow +\infty} \frac{\log \left(\frac{2^{\eta' \frac{h+\varepsilon}{\alpha} j + 1}}{2^{\eta' - 1}} \right)}{\log(2^j)} \\ &= \limsup_{j \rightarrow +\infty} \log_2 \left(\frac{2^{\eta' \frac{h+\varepsilon}{\alpha} + \frac{1}{j}}}{(2^{\eta' - 1})^{\frac{1}{j}}} \right) \\ &= \eta' \frac{h + \varepsilon}{\alpha} \end{aligned}$$

with probability one. Taking a sequence $(\eta_n)_{n \in \mathbb{N}}$ decreasing to η and a sequence $(\varepsilon_m)_{m \in \mathbb{N}}$ decreasing to 0, we get that $\tilde{\nu}_{R_{\alpha, \eta}}^+(h) \leq \frac{h\eta}{\alpha}$ with probability one.

Finally, from Lemma 5.4.9, we know that, with probability one, $\tilde{\nu}_{R_{\alpha, \eta}}^-(h) = -\infty$ for every $h \in (\frac{\alpha}{\eta}, +\infty]$.

Taking a dense sequence $(h_n)_{n \in \mathbb{N}}$ in \mathbb{R} and using the right-continuity (resp. left-continuity) of $\tilde{\nu}_{R_{\alpha, \eta}}^+$ (resp. $\tilde{\nu}_{R_{\alpha, \eta}}^-$), we get the conclusion. \square

5.4.3 Random wavelet series

The model we present in this section is a generalization of the lacunary wavelet series. It was introduced by Aubry and Jaffard [12]. The proofs of the results presented here are developed in Appendix B.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We suppose that, at a given scale j , the wavelet coefficients $c_{j,k}$ of the process are drawn independently with a given law. We denote by ρ_j the common probability measure of the 2^j random variables $x_{j,k} = -\log_2(|c_{j,k}|)/j$; the measure ρ_j thus satisfies

$$\mathbb{P}[|c_{j,k}| \geq 2^{-\alpha j}] = \rho_j((-\infty, \alpha]).$$

For $\alpha \geq 0$, we denote

$$\rho(\alpha) := \lim_{\varepsilon \rightarrow 0} \limsup_{j \rightarrow +\infty} \frac{\log(2^j \rho_j([\alpha - \varepsilon, \alpha + \varepsilon]))}{\log(2^j)}.$$

Definition 5.4.11. If there exists $\gamma > 0$ such that $\rho(\alpha) < 0$ for every $\alpha < \gamma$, we say that

$$f = \sum_{j \in \mathbb{N}_0} \sum_{k=0}^{2^j - 1} c_{j,k} \psi_{j,k}$$

is a *random wavelet series*. We will also assume that there is $\alpha > 0$ such that $\rho(\alpha) > 0$, so that

$$h_{\max} := \left(\sup_{\alpha > 0} \frac{\rho(\alpha)}{\alpha} \right)^{-1} < +\infty.$$

We define

$$W := \left\{ \alpha \geq 0 : \sum_{j \in \mathbb{N}_0} 2^j \rho_j([\alpha - \varepsilon, \alpha + \varepsilon]) = +\infty, \forall \varepsilon > 0 \right\}$$

and we set $h_{\min} := \inf(W)$.

Remark 5.4.12. If $\rho(\alpha) > 0$, then $\alpha \in W$. It follows directly that if $\nu(\alpha) > 0$, one has $\alpha \geq h_{\min}$.

Aubry and Jaffard have computed the almost sure multifractal spectrum of a random wavelet series.

Theorem 5.4.13. [12] *Let f be a random wavelet series. With probability one, the spectrum of singularities of f is given by*

$$d_f(h) = \begin{cases} h \sup_{\alpha \in (0, h]} \frac{\rho(\alpha)}{\alpha} & \text{if } h \in [h_{\min}, h_{\max}], \\ -\infty & \text{otherwise.} \end{cases}$$

Let us denote by $\tilde{\rho}_j$ the common probability measure of the 2^j random variables $-\log_2(|e_{j,k}|)/j$, where $e_{j,k}$ denotes the restricted wavelet leaders. Therefore, we have

$$\mathbb{P}[e_{j,k} \geq 2^{-\alpha j}] = \tilde{\rho}_j((-\infty, \alpha]).$$

Remark that one has

$$\mathbb{E}[\#\{k \in \{0, \dots, 2^j - 1\} : e_{j,k} \geq 2^{-\alpha j}\}] = 2^j \tilde{\rho}_j((-\infty, \alpha])$$

since $\#\{k \in \{0, \dots, 2^j - 1\} : e_{j,k} \geq 2^{-\alpha j}\}$ counts the number of successes of a binomial distribution of parameters $(2^j, \tilde{\rho}_j((-\infty, \alpha]))$, where the success means “ $e_{j,k} \geq 2^{-\alpha j}$ ”. For every $\alpha \geq 0$, we define

$$\tilde{\nu}^+(\alpha) := \lim_{\varepsilon \rightarrow 0^+} \limsup_{j \rightarrow +\infty} \frac{\log(2^j \tilde{\rho}_j((-\infty, \alpha + \varepsilon]))}{\log(2^j)}$$

and

$$\alpha_s := \inf \left\{ \alpha \geq 0 : \tilde{\nu}^+(\alpha) = 1 \right\}.$$

We also assume that for every $\varepsilon > 0$ and every $\delta > 0$, there is $J \in \mathbb{N}$ such that

$$\tilde{\rho}_j((-\infty, \alpha_s + \varepsilon]) \geq 2^{-\delta j}, \quad \forall j \geq J.$$

Proposition 5.4.14. *Let f be a random wavelet series.*

1. *With probability one,*

$$\tilde{\nu}_f^+(\alpha) = \begin{cases} \tilde{\nu}^+(\alpha) & \text{if } \alpha \geq h_{\min}, \\ -\infty & \text{otherwise.} \end{cases}$$

2. *With probability one, if $\alpha \in (\alpha_s, +\infty]$, $\tilde{\nu}_f^-(\alpha) = -\infty$.*

The next result shows that with probability one, a random wavelet series satisfies the formalism based on the wavelet leaders profile.

Theorem 5.4.15. *Let f be a random wavelet series. With probability one, we have $d_f = \tilde{\nu}_f$ on $[0, +\infty]$.*

5.4.4 Deterministic cascades

The next model we consider is a deterministic wavelet cascade; it is the simplest case of the famous cascade models which have been introduced as turbulence models, and are also used in financial modelling.

Let us consider the binomial measure μ of parameter $p \in (0, 1)$, which is the unique measure supported on $[0, 1]$ such that

$$\mu(\lambda_{j,k}) = p^{\phi(j,k)}(1-p)^{j-\phi(j,k)},$$

where $\phi(j, k)$ is the number of 1 among the j first coordinates in the dyadic decomposition of $k2^{-j}$. Following a general framework proposed by Barral and Seuret [20], let us construct the wavelet series F_μ by prescribing its wavelet coefficients in a given wavelet basis as follows: for every λ , we set $c_\lambda := \mu(\lambda)$. We will say that F_μ is a *deterministic Bernoulli cascade of parameter p* . It is known (see [20, 137] for example) that the wavelet series F_μ is well defined and its multifractal spectrum is given by

$$d_{F_\mu}(\alpha) = \begin{cases} -(\beta \log_2(\beta) + (1-\beta) \log_2(1-\beta)) & \text{if } \alpha \in (-\log_2(1-p), -\log_2(p)), \\ 0 & \text{if } \alpha \in \{-\log_2(1-p), -\log_2(p)\}, \\ -\infty & \text{otherwise,} \end{cases}$$

where

$$\beta = \frac{\alpha + \log_2(1-p)}{\log_2(1-p) - \log_2(p)}.$$

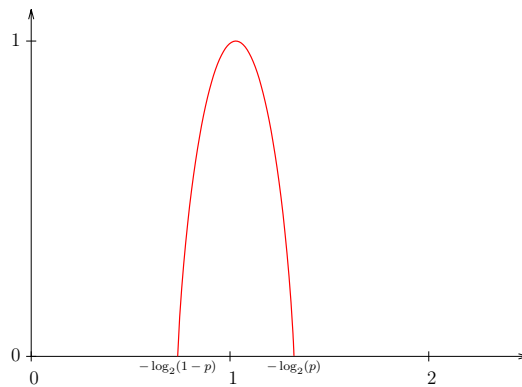


Figure 5.3: Multifractal spectrum of a deterministic Bernoulli cascade of parameter $p = 0.4$

Remark that the wavelet coefficients are simply defined recursively by

$$\begin{cases} c_{0,0} & := 1, \\ c_{j,2k} & := c_{j-1,k}(1-p), \\ c_{j,2k+1} & := c_{j-1,k}p, \end{cases} \quad (5.5)$$

for all $j \geq 1$ and $k \in \{0, \dots, 2^j - 1\}$. At each scale $j \in \mathbb{N}_0$, we have $\binom{j}{l}$ coefficients of size $p^l(1-p)^{j-l}$ for $l \in \{0, \dots, j\}$. Moreover, if $\lambda' \subseteq \lambda$, then $|c_{\lambda'}| \leq |c_\lambda|$ and therefore,

the wavelet coefficients are the restricted wavelet leaders. In order to avoid trivial cases, we will assume that $p \neq 1/2$. We will also assume that $p < 1/2$; the case $p > 1/2$ is similar.

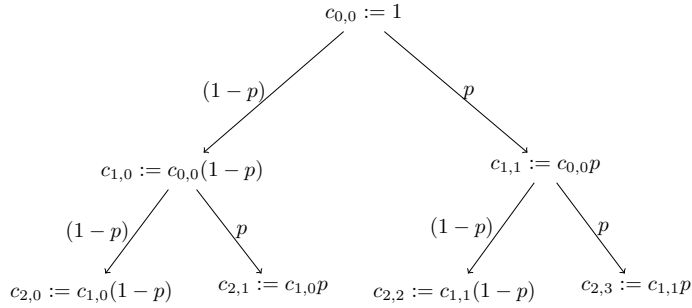


Figure 5.4: Construction of the wavelet coefficients of a deterministic Bernoulli cascade

The following proposition is a classical result in the theory of large deviation (see [104] for example).

Proposition 5.4.16. *If the wavelet series F_μ is a deterministic cascade of parameter p and if \vec{c} is its sequence of wavelet coefficients given by (5.5), then $\tilde{\rho}_{\vec{c}} = d_{F_\mu}$ on $[0, +\infty]$.*

Proof. From Theorem 5.2.5 and Remark 5.2.6, it suffices to show that $\tilde{\rho}_{\vec{c}}^*(\alpha) \leq d_{F_\mu}(\alpha)$ for every $\alpha \in [0, +\infty]$. First, if $\alpha < -\log_2(1-p)$, we fix $\varepsilon > 0$ such that $\alpha + \varepsilon < -\log_2(1-p)$. Then, we have $c_{j,k} \leq (1-p)^j < 2^{-(\alpha+\varepsilon)j}$ for every $j \in \mathbb{N}_0$, $k \in \{0, \dots, 2^j - 1\}$. It follows that

$$\#\{\lambda \in \Lambda_j : 2^{-(\alpha+\varepsilon)j} \leq e_\lambda < 2^{-(\alpha-\varepsilon)j}\} = \#\{\lambda \in \Lambda_j : 2^{-(\alpha+\varepsilon)j} \leq |c_\lambda| < 2^{-(\alpha-\varepsilon)j}\} = 0$$

and $\tilde{\rho}_{\vec{c}}^*(\alpha) = -\infty$. Similarly, if $\alpha > -\log_2(p)$, then for $\varepsilon > 0$ such that $\alpha - \varepsilon > -\log_2(p)$, we have $|c_{j,k}| \geq p^j > 2^{-(\alpha-\varepsilon)j}$ for every $j \in \mathbb{N}$, $k \in \{0, \dots, 2^j - 1\}$. Consequently, $\tilde{\rho}_{\vec{c}}^*(\alpha) = -\infty$.

Secondly, let us assume that $\alpha \in (-\log_2(1-p), -\log_2(p))$. Let us fix $\varepsilon > 0$. Remark that we have

$$2^{-(\alpha+\varepsilon)j} \leq p^l (1-p)^{j-l} < 2^{-(\alpha-\varepsilon)j} \Leftrightarrow \beta_\varepsilon^+ j \geq l > \beta_\varepsilon^- j,$$

where

$$\beta_\varepsilon^+ = \frac{\alpha + \varepsilon + \log_2(1-p)}{\log_2(1-p) - \log_2(p)} \quad \text{and} \quad \beta_\varepsilon^- = \frac{\alpha - \varepsilon + \log_2(1-p)}{\log_2(1-p) - \log_2(p)}.$$

With these notations, we have

$$\begin{aligned} & \frac{\log \#\{\lambda \in \Lambda_j : 2^{-(\alpha+\varepsilon)j} \leq e_\lambda < 2^{-(\alpha-\varepsilon)j}\}}{\log 2^j} \\ &= \frac{1}{j} \log_2 \left(\sum_{l=\lceil \beta_\varepsilon^- j \rceil}^{\lfloor \beta_\varepsilon^+ j \rfloor} \binom{j}{l} \right) \\ &\sim \frac{1}{j} \log_2 \left(\sum_{l=\lceil \beta_\varepsilon^- j \rceil}^{\lfloor \beta_\varepsilon^+ j \rfloor} \frac{\sqrt{2\pi j} \left(\frac{j}{e}\right)^j}{\sqrt{2\pi l} \left(\frac{l}{e}\right)^l \sqrt{2\pi(j-l)} \left(\frac{j-l}{e}\right)^{j-l}} \right) \end{aligned}$$

where we have used Stirling's formula. Moreover,

$$\begin{aligned}
& \frac{1}{j} \log_2 \left(\sum_{l=\lceil \beta_\varepsilon^- j \rceil}^{\lfloor \beta_\varepsilon^+ j \rfloor} \frac{\sqrt{2\pi j} \left(\frac{j}{e}\right)^j}{\sqrt{2\pi l} \left(\frac{l}{e}\right)^l \sqrt{2\pi(j-l)} \left(\frac{j-l}{e}\right)^{j-l}} \right) \\
&= \frac{1}{j} \log_2 \left(\sum_{l=\lceil \beta_\varepsilon^- j \rceil}^{\lfloor \beta_\varepsilon^+ j \rfloor} \frac{\sqrt{j}}{\sqrt{l} \left(\frac{l}{j}\right)^l \sqrt{2\pi(j-l)} \left(\frac{j-l}{j}\right)^{j-l}} \right) \\
&\leq \frac{1}{j} \log_2 \left(\sum_{l=\lceil \beta_\varepsilon^- j \rceil}^{\lfloor \beta_\varepsilon^+ j \rfloor} \frac{\sqrt{j}}{\sqrt{\beta_\varepsilon^- j} (\beta_\varepsilon^-)^{\beta_\varepsilon^- j} \sqrt{2\pi j} (1 - \beta_\varepsilon^+) (1 - \beta_\varepsilon^+)^{j(1 - \beta_\varepsilon^+)}} \right) \\
&\leq \frac{1}{j} \log_2 \left(\frac{\beta_\varepsilon^+ j - \beta_\varepsilon^- j + 1}{\sqrt{\beta_\varepsilon^-} 2\pi (1 - \beta_\varepsilon^+)} \right) - \frac{1}{2j} \log_2(j) \\
&\quad + \log_2 \left(\frac{1}{(\beta_\varepsilon^-)^{\beta_\varepsilon^-} (1 - \beta_\varepsilon^+)^{(1 - \beta_\varepsilon^+)}} \right)
\end{aligned}$$

that converges to $\log_2 \left(\frac{1}{(\beta_\varepsilon^-)^{\beta_\varepsilon^-} (1 - \beta_\varepsilon^+)^{(1 - \beta_\varepsilon^+)}} \right)$ as j tends to $+\infty$. It follows that

$$\tilde{\rho}_\varepsilon^*(\alpha) \leq \lim_{\varepsilon \rightarrow 0^+} \log_2 \left(\frac{1}{(\beta_\varepsilon^-)^{\beta_\varepsilon^-} (1 - \beta_\varepsilon^+)^{(1 - \beta_\varepsilon^+)}} \right) = -(\beta \log_2(\beta) + (1 - \beta) \log_2(1 - \beta))$$

where

$$\beta = \frac{\alpha + \log_2(1 - p)}{\log_2(1 - p) - \log_2(p)}.$$

This concludes this part of the proof. The case $\alpha \in \{-\log_2(1 - p), -\log_2(p)\}$ is very similar. \square

Remark that the function d_{F_μ} has a unique maximum realized at the point

$$\alpha_s = -\frac{1}{2} \log_2((1 - p)p).$$

The leaders profile method gives the correct spectrum, as stated in the next corollary.

Corollary 5.4.17. [24] *If the wavelet series F_μ is a deterministic cascade of parameter p , then we have $d_{F_\mu} = \tilde{\nu}_{F_\mu}$ on $[0, +\infty]$.*

Proof. The result follows directly from Proposition 5.4.16 and Lemma 5.3.5. \square

5.4.5 Thresholded deterministic cascades

Let f be a function whose wavelet coefficients in a wavelet basis $(\psi_\lambda)_{\lambda \in \Lambda}$ are given by \vec{c} and let $\gamma > 0$. Following [132], the wavelet series f^t defined by

$$f^t = \sum_{j \in \mathbb{N}_0} \sum_{\lambda \in \Lambda_j} c_\lambda^t \psi_\lambda \quad \text{where} \quad c_\lambda^t = c_\lambda \mathbf{1}_{|\cdot| \geq 2^{-\gamma j}}(c_\lambda),$$

is said to be *obtained from f after a threshold of order γ* . This method was introduced by Seuret in order to create functions with *oscillating singularities*. They also display non-concave multifractal spectra, as stated in the following proposition.

Proposition 5.4.18. [132] *Let F_μ be a deterministic Bernoulli cascade of parameter $p \in (0, 1/2)$. Let $\omega_t : [\gamma, -\log_2(p)] \rightarrow (0, +\infty)$ be the increasing function*

$$u \mapsto \gamma \frac{u + \log_2(1-p)}{\gamma + \log_2(1-p)}.$$

Let $h_{\max}^t = \omega_t(-\log_2(p))$. If $\gamma \in [-\log_2(1-p), -\log_2(p)]$, the multifractal spectrum of F_μ^t is equal to

$$d_{F_\mu^t}(h) = \begin{cases} d_f(h) & \text{if } h \in [-\log_2(1-p), \gamma], \\ d_f(\omega_t^{-1}(h)) & \text{if } h \in (\gamma, h_{\max}^t], \\ -\infty & \text{otherwise.} \end{cases}$$

Let us fix $\alpha_s = -\frac{1}{2} \log_2((1-p)p)$, the point at which d_{F_μ} is maximum. If $\gamma > \alpha_s$, the spectrum of F_μ^t is non-concave in its decreasing part (see [132] and Figure 5.5) and all the multifractal formalisms proposed up to now fail for F_μ^t . Let us show that the computation of the wavelet leaders profile of F_μ^t leads to the correct spectrum.

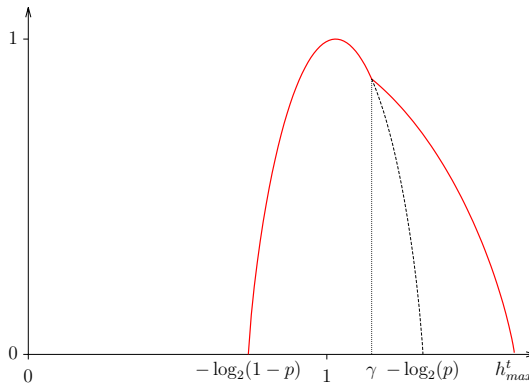


Figure 5.5: Multifractal spectrum of F_μ^t for $p = 0.4$ and $\gamma = 1.2$

Proposition 5.4.19. [24] *Let F_μ be a deterministic Bernoulli cascade of parameter $p \in (0, 1/2)$ and let F_μ^t be the wavelet series obtained from F_μ after a threshold of order $\gamma > \alpha_s$ where $\alpha_s = -\frac{1}{2} \log_2((1-p)p)$. Then we have $d_{F_\mu^t} = \tilde{\nu}_{F_\mu^t}$ on $[0, +\infty]$.*

Proof. From Proposition 5.3.4, it suffices to show that $\tilde{\nu}_{F_\mu^t} \leq d_{F_\mu^t}$. Let us denote by e_λ the restricted wavelet leaders of F_μ and by e_λ^t the restricted wavelet leaders of F_μ^t .

First assume that $\alpha < \alpha_s$. We clearly have $\tilde{\nu}_{F_\mu^t}(\alpha) \leq \tilde{\nu}_{F_\mu^+}(\alpha)$ and therefore, we get $\tilde{\nu}_{F_\mu^t}(\alpha) \leq d_{F_\mu^t}(\alpha)$ using Corollary 5.4.17.

Secondly, if $\alpha \in [\alpha_s, \gamma]$, we have

$$\#\{\lambda \in \Lambda_j : e_\lambda^t \leq 2^{-(\alpha-\varepsilon)j}\} \leq \#\{\lambda \in \Lambda_j : e_\lambda \leq 2^{-(\alpha-\varepsilon)j}\}.$$

Indeed, if $e_\lambda > 2^{-(\alpha-\varepsilon)j}$, then $c_\lambda = e_\lambda \geq 2^{-\gamma j}$. It follows that $c_\lambda^t = c_\lambda$ and that $e_\lambda^t = c_\lambda = e_\lambda$. Therefore, $e_\lambda^t > 2^{-(\alpha-\varepsilon)j}$ and $\tilde{\nu}_{F_\mu^t}(\alpha) \leq \tilde{\nu}_{F_\mu}(\alpha) = d_f(\alpha)$.

Finally, assume that $\alpha > \gamma$. Remark that if $\lambda \in \Lambda_j$ is such that $e_\lambda^t \leq 2^{-(\alpha-\varepsilon)j}$ with $\alpha - \varepsilon > \gamma$, then $e_\lambda^t = p^l(1-p)^{\lceil C_\gamma l \rceil - j}$, where

$$C_\gamma = \frac{\log_2(1-p) - \log_2(p)}{\gamma + \log_2(1-p)},$$

and $\beta_{\varepsilon,j} \leq l \leq j$, where

$$\beta_{\varepsilon,j} = \frac{\log_2(1-p) + (\alpha - \varepsilon)j}{(1 - C_\gamma) \log_2(1-p) - \log_2(p)} = \frac{(\log_2(1-p) + (\alpha - \varepsilon)j)(\gamma + \log_2(1-p))}{\gamma(\log_2(1-p) - \log_2(p))}.$$

Moreover, since $\alpha > \gamma > \alpha_s$, $\beta_{\varepsilon,j}$ is bigger than $j/2$ for j large enough. Therefore,

$$\#\{\lambda \in \Lambda_j : e_\lambda^t \leq 2^{-(\alpha-\varepsilon)j}\} \leq \sum_{l=\lfloor \beta_{\varepsilon,j} \rfloor}^j \binom{j}{l} \leq j \binom{j}{\lfloor \beta_{\varepsilon,j} \rfloor}$$

and it follows that

$$\begin{aligned} & \frac{\log \#\{\lambda \in \Lambda_j : e_\lambda^t \leq 2^{-(\alpha-\varepsilon)j}\}}{\log 2^j} \\ & \leq \frac{1}{j} \log_2 \left(j \binom{j}{\lfloor \beta_{\varepsilon,j} \rfloor} \right) \\ & \sim \frac{1}{j} \log_2 \left(\frac{j \sqrt{2\pi j} \left(\frac{j}{e}\right)^j}{\sqrt{2\pi \lfloor \beta_{\varepsilon,j} \rfloor} \left(\frac{j}{e}\right)^{\lfloor \beta_{\varepsilon,j} \rfloor} \sqrt{2\pi(j - \lfloor \beta_{\varepsilon,j} \rfloor)} \left(\frac{j - \lfloor \beta_{\varepsilon,j} \rfloor}{e}\right)^{j - \lfloor \beta_{\varepsilon,j} \rfloor}} \right) \end{aligned}$$

where we have used Stirling's formula. Moreover,

$$\begin{aligned} & \frac{1}{j} \log_2 \left(\frac{j \sqrt{2\pi j} \left(\frac{j}{e}\right)^j}{\sqrt{2\pi \lfloor \beta_{\varepsilon,j} \rfloor} \left(\frac{j}{e}\right)^{\lfloor \beta_{\varepsilon,j} \rfloor} \sqrt{2\pi(j - \lfloor \beta_{\varepsilon,j} \rfloor)} \left(\frac{j - \lfloor \beta_{\varepsilon,j} \rfloor}{e}\right)^{j - \lfloor \beta_{\varepsilon,j} \rfloor}} \right) \\ & = \frac{1}{j} \log_2 \left(\frac{j \sqrt{j}}{\left(\frac{\lfloor \beta_{\varepsilon,j} \rfloor}{j}\right)^{\lfloor \beta_{\varepsilon,j} \rfloor} \sqrt{2\pi \lfloor \beta_{\varepsilon,j} \rfloor} (j - \lfloor \beta_{\varepsilon,j} \rfloor) \left(\frac{j - \lfloor \beta_{\varepsilon,j} \rfloor}{j}\right)^{j - \lfloor \beta_{\varepsilon,j} \rfloor}} \right) \\ & = \frac{1}{j} \log_2 \left(\frac{\sqrt{j}}{\left(\frac{\lfloor \beta_{\varepsilon,j} \rfloor}{j}\right)^{\lfloor \beta_{\varepsilon,j} \rfloor} \sqrt{2\pi \frac{\lfloor \beta_{\varepsilon,j} \rfloor}{j} (1 - \frac{\lfloor \beta_{\varepsilon,j} \rfloor}{j})} \left(1 - \frac{\lfloor \beta_{\varepsilon,j} \rfloor}{j}\right)^{j - \lfloor \beta_{\varepsilon,j} \rfloor}} \right) \\ & \leq \frac{1}{j} \log_2 \left(\frac{\sqrt{j}}{\sqrt{2\pi \frac{\lfloor \beta_{\varepsilon,j} \rfloor}{j} (1 - \frac{\lfloor \beta_{\varepsilon,j} \rfloor}{j})}} \right) - \log_2 \left(\frac{\left(\frac{\lfloor \beta_{\varepsilon,j} \rfloor}{j}\right)^{\frac{\lfloor \beta_{\varepsilon,j} \rfloor}{j}}}{\left(1 - \frac{\lfloor \beta_{\varepsilon,j} \rfloor}{j}\right)^{1 - \frac{\lfloor \beta_{\varepsilon,j} \rfloor}{j}}} \right). \end{aligned}$$

If we compute the limit as $j \rightarrow +\infty$ and $\varepsilon \rightarrow 0^+$, we get that

$$\tilde{\nu}_{F_\mu^t}(\alpha) \leq -(\beta \log_2(\beta) + (1 - \beta) \log_2(1 - \beta))$$

with

$$\beta = \lim_{\varepsilon \rightarrow 0^+} \lim_{j \rightarrow +\infty} \frac{\beta_{\varepsilon,j}}{j} = \alpha \frac{\gamma + \log_2(1-p)}{\gamma(\log_2(1-p) - \log_2(p))}.$$

In particular, if $\alpha > h_{\max}^t$, then $\tilde{\nu}_{F_\mu^t}(\alpha) < 0$ so that $\tilde{\nu}_{F_\mu^t}(\alpha) = -\infty$. The conclusion follows from Proposition 5.4.18. \square

5.4.6 Sum of deterministic cascades

The last example we present is given by the sum of two deterministic Bernoulli cascades. In order to compute their multifractal spectrum, let us first give some definitions concerning the multifractal analysis of measures. If μ is a finite Borel measure on $[0, 1]$, the Hölder exponent of μ at $x_0 \in [0, 1]$ is defined by

$$h_\mu(x_0) = \liminf_{r \rightarrow 0^+} \frac{\log \mu(B(x, r))}{\log r}.$$

The multifractal spectrum of μ is then given by

$$d_\mu : [0, +\infty] \rightarrow \{-\infty\} \cup [0, 1] : h \mapsto \dim_{\mathcal{H}}(\{x \in \mathbb{R}^n : h_\mu(x) = h\}).$$

In [20, 133], it is proved that if μ is *uniformly regular* (i.e. there exist a constant $C > 0$ and an exponent $h_{\min} > 0$ such that $\mu(B(x, r)) \leq Cr^{h_{\min}}$ for any ball $B(x, r) \subseteq [0, 1]$) and if f is a function whose wavelet coefficients are given by $c_\lambda = \mu(\lambda)$ for every $\lambda \in \Lambda$, then $d_f = d_\mu$.

Theorem 5.4.20. [99] *Let us denote by F_{p_1} (resp. F_{p_2}) the deterministic Bernoulli cascade of parameter p_1 (resp. p_2), with $0 < p_1 < p_2 < 1/2$. The multifractal spectrum of the function $F = F_{p_1} + F_{p_2}$ is equal to*

$$d_F(h) = \begin{cases} d_{F_{p_1}}(h) & \text{if } h \leq h_0, \\ d_{F_{p_2}}(h) & \text{if } h \geq h_0, \end{cases}$$

for all $h \in [-\log_2(1 - p_1), -\log_2(p_2)]$, where

$$h_0 = \frac{\log_2(1 - p_1) \log_2(p_2) - \log_2(1 - p_2) \log_2(p_1)}{\log_2\left(\frac{1-p_2}{p_2}\right) - \log_2\left(\frac{1-p_1}{p_1}\right)}.$$

Moreover, $d_F(h) = -\infty$ if $h \notin [-\log_2(1 - p_1), -\log_2(p_2)]$.

Proof. Let us denote by μ_{p_1} and μ_{p_2} the binomial measures of parameter p_1 and p_2 respectively. If \vec{c} denotes the wavelet coefficients of F , then

$$c_\lambda = \mu_{p_1}(\lambda) + \mu_{p_2}(\lambda), \quad \forall \lambda \in \Lambda.$$

Therefore, the multifractal spectrum of F is the same as the multifractal spectrum of the measure

$$\mu = \mu_{p_1} + \mu_{p_2}.$$

It suffices then to compute d_μ . Let us first remark that for every $x \in [0, 1]$,

$$h_\mu(x) = \min\{h_{\mu_{p_1}}(x), h_{\mu_{p_2}}(x)\}.$$

Indeed, for every $\varepsilon > 0$, there is $0 < R < 1$ such that

$$\mu_{p_1}(B(x, r)) \leq r^{h_{\mu_{p_1}}(x) - \varepsilon} \quad \text{and} \quad \mu_{p_2}(B(x, r)) \leq r^{h_{\mu_{p_2}}(x) - \varepsilon}$$

for every $r < R$. One has

$$\begin{aligned} \frac{\log(\mu_{p_1}(B(x, r)) + \mu_{p_2}(B(x, r)))}{\log r} &\geq \frac{\log(r^{h_{\mu_{p_1}}(x) - \varepsilon} + r^{h_{\mu_{p_2}}(x) - \varepsilon})}{\log r} \\ &\geq \frac{\log(2r^{\min\{h_{\mu_{p_1}}(x), h_{\mu_{p_2}}(x)\} - \varepsilon})}{\log r}, \end{aligned}$$

for every $r < R$. It follows that $h_\mu(x) \geq \min\{h_{\mu_{p_1}}(x), h_{\mu_{p_2}}(x)\} - \varepsilon$ and since $\varepsilon > 0$ is arbitrary, we get that $h_\mu(x) \geq \min\{h_{\mu_{p_1}}(x), h_{\mu_{p_2}}(x)\}$.

In view of

$$\mu_{p_1}(B(x, r)) + \mu_{p_2}(B(x, r)) \geq \mu_{p_1}(B(x, r))$$

and

$$\mu_{p_1}(B(x, r)) + \mu_{p_2}(B(x, r)) \geq \mu_{p_2}(B(x, r)),$$

the other inequality is obvious.

For every $x \in [0, 1]$, let $S_n(x)$ denote the number of 0 appearing in the n first terms of the proper dyadic development of x . If μ_p is the binomial measure of parameter $p < 1/2$, it is direct to see that

$$\liminf_{n \rightarrow +\infty} \frac{S_n(x)}{n} = \frac{h_{\mu_p}(x) + \log_2(1-p)}{\log_2(1-p) - \log_2 p}.$$

Consequently, for every $x \in [0, 1]$, we have

$$\frac{h_{\mu_{p_1}}(x) + \log_2(1-p_1)}{\log_2(1-p_1) - \log_2 p_1} = \frac{h_{\mu_{p_2}}(x) + \log_2(1-p_2)}{\log_2(1-p_2) - \log_2 p_2}$$

and a simple computation shows that

$$h_\mu(x) = h_{\mu_{p_2}}(x) \iff h_{\mu_{p_1}}(x) \geq h_0,$$

where

$$h_0 = \frac{\log_2(1-p_1) \log_2(p_2) - \log_2(1-p_2) \log_2(p_1)}{\log_2\left(\frac{1-p_2}{p_2}\right) - \log_2\left(\frac{1-p_1}{p_1}\right)}.$$

The conclusion follows. □

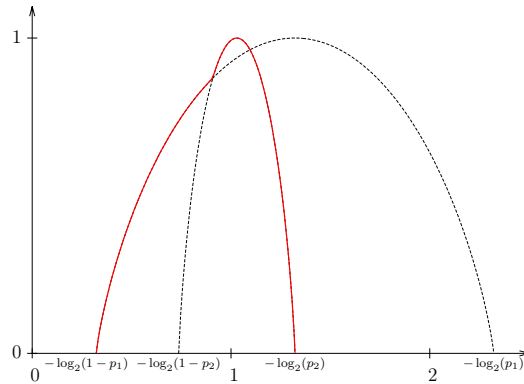


Figure 5.6: Multifractal spectrum the sum of two deterministic Bernoulli cascades of parameters $p_1 = 0.2$ and $p_2 = 0.4$

Remark 5.4.21. Let us notice that $d_{\mu_{p_1}}(h_0) = d_{\mu_{p_2}}(h_0)$ and that h_0 corresponds to the first intersection between the two graphs because we have

$$h_0 < -\frac{1}{2} \log_2((1-p_2)p_2).$$

Proposition 5.4.22. *Let us denote by F_{p_1} (resp. F_{p_2}) the deterministic Bernoulli cascade of parameter p_1 (resp. p_2), with $0 < p_1 < p_2 < 1/2$. If $F = F_{p_1} + F_{p_2}$, then $d_F = \tilde{\nu}_F$ on $[0, +\infty]$.*

Proof. Let us denote

$$h_s = -\frac{1}{2} \log_2((1-p_2)p_2).$$

Using Proposition 5.3.4, we already have the upper bound. Let us prove the other inequality. We denote by \tilde{c}^1 , \tilde{c}^2 and \tilde{c} the wavelet coefficients of F_{p_1} , F_{p_2} and F respectively. Remark that the restricted wavelet leaders of F_{p_1} , F_{p_2} and F are equal to their wavelet coefficients. In particular, $e_\lambda = e_\lambda^1 + e_\lambda^2$.

First, let us fix $\alpha \leq h_s$. For every $\delta > 0$, there is $J \in \mathbb{N}$ such that

$$\begin{aligned} \#\{\lambda \in \Lambda_j : e_\lambda \geq 2^{-(\alpha+\varepsilon)j}\} &\leq \#\{\lambda \in \Lambda_j : e_\lambda^1 \geq \frac{1}{2}2^{-(\alpha+\varepsilon)j}\} \\ &\quad + \#\{\lambda \in \Lambda_j : e_\lambda^2 \geq \frac{1}{2}2^{-(\alpha+\varepsilon)j}\} \\ &\leq 2^{(\tilde{\nu}_{F_{p_1}}^+(\alpha)+\delta)j} + 2^{(\tilde{\nu}_{F_{p_2}}^+(\alpha)+\delta)j} \\ &\leq 2 \cdot 2^{(\max\{\tilde{\nu}_{F_{p_1}}^+(\alpha), \tilde{\nu}_{F_{p_2}}^+(\alpha)\}+\delta)j} \end{aligned}$$

if $j \geq J$. It follows that $\tilde{\nu}_F^+(\alpha) \leq \max\{\tilde{\nu}_{F_{p_1}}^+(\alpha), \tilde{\nu}_{F_{p_2}}^+(\alpha)\}$. If $\alpha < h_0$, then using Corollary 5.4.17 and Theorem 5.4.20, a simple computation gives $\tilde{\nu}_{F_{p_1}}^+(\alpha) \geq \tilde{\nu}_{F_{p_2}}^+(\alpha)$ and $\tilde{\nu}_F^+(\alpha) \leq \tilde{\nu}_{F_{p_1}}^+(\alpha) = d_F(\alpha)$. Similarly, if $\alpha \in [h_0, h_s]$, then $\tilde{\nu}_F^+(\alpha) \leq \tilde{\nu}_{F_{p_2}}^+(\alpha) = d_F(\alpha)$.

Assume now that $\alpha > h_s$. It is clear that $\tilde{\nu}_F^-(\alpha) \leq \min\{\tilde{\nu}_{F_{p_1}}^-(\alpha), \tilde{\nu}_{F_{p_2}}^-(\alpha)\}$ since $e_\lambda \leq e_\lambda^1$ and $e_\lambda \leq e_\lambda^2$. Moreover, using again Corollary 5.4.17 and Theorem 5.4.20, a simple computation shows that $\tilde{\nu}_{F_{p_1}}^-(\alpha) \geq \tilde{\nu}_{F_{p_2}}^-(\alpha)$ and the conclusion follows. \square

5.5 Comparison of the formalisms

In this section, we compare from a theoretical point of view the leaders profile method with both the wavelet leaders method and the wavelet profile method, presented in Chapter 4. More precisely, we first show that while it is not concave, the function $\tilde{\nu}_f$ gives a sharper approximation of the spectrum of f than the wavelet leaders method. We then prove that if $\tilde{\nu}_f$ is not with increasing-visibility, the leaders profile method is also more efficient on the increasing part of the spectrum than the method based on the \mathcal{S}^ν spaces.

5.5.1 Comparison with the wavelet leaders method

Let us first recall the wavelet leaders method introduced in Chapter 4. First, one has to compute the function

$$\tilde{\eta}_f(p) = \liminf_{j \rightarrow +\infty} \frac{\log W_f(j, p)}{\log 2^{-j}} \quad \text{where} \quad W_f(j, p) = 2^{-j} \sum_{\lambda \in \Lambda_j}^* d_\lambda^p, \quad \forall p \in \mathbb{R}.$$

Then the method based on the estimation of the multifractal spectrum of f by the function

$$\inf_{p \in \mathbb{R}} (hp - \tilde{\eta}_f(p) + 1), \quad h \geq 0.$$

Let us start by the following lemma which gives a connection between the function $\tilde{\eta}_f$ and the wavelet leaders density.

Lemma 5.5.1. [64] *Let \vec{c} denote the wavelet coefficients of a locally bounded function f in a wavelet basis. If $\tilde{\rho}_{\vec{c}}$ takes the value $-\infty$ outside of a compact set of $(0, +\infty)$, then*

$$\tilde{\eta}_f(p) = \inf_{h \geq 0} (hp - \tilde{\rho}_{\vec{c}}(h) + 1), \quad \forall p \in \mathbb{R}.$$

Proof. Let $H_{\min}, H_{\max} \geq 0$ be such that $\tilde{\rho}_{\vec{c}}$ takes the value $-\infty$ outside of $[H_{\min}, H_{\max}]$. Then

$$\inf_{h \geq 0} (hp - \tilde{\rho}_{\vec{c}}(h) + 1) = \inf_{h \in [H_{\min}, H_{\max}]} (hp - \tilde{\rho}_{\vec{c}}(h) + 1).$$

Let us fix $h \in [H_{\min}, H_{\max}]$, $\delta > 0$ and $\varepsilon > 0$. Then, from the definition of the wavelet leaders density, there exists a subsequence $(j_m)_{m \in \mathbb{N}}$ such that

$$\#\{\lambda \in \Lambda_{j_m} : 2^{-(h+\varepsilon)j_m} \leq d_\lambda < 2^{-(h-\varepsilon)j_m}\} \geq 2^{(\tilde{\rho}_{\vec{c}}(h)-\delta)j_m}$$

for every $m \in \mathbb{N}$. It follows that

$$\begin{aligned} \tilde{\eta}_f(p) &\leq \lim_{m \rightarrow +\infty} \frac{\log \left(2^{-j_m} \sum_{\lambda \in \Lambda_{j_m}}^* d_\lambda^p \right)}{\log 2^{-j_m}} \\ &\leq \lim_{m \rightarrow +\infty} \frac{\log \left(2^{-j_m} 2^{(\tilde{\rho}_{\vec{c}}(h)-\delta)j_m} 2^{-(hp+\varepsilon|p|)j_m} \right)}{\log 2^{-j_m}} \\ &= 1 - \tilde{\rho}_{\vec{c}}(h) + \delta + hp + \varepsilon|p|. \end{aligned}$$

Since $\varepsilon > 0$ and $\delta > 0$ are arbitrary, we get that

$$\tilde{\eta}_f(p) \leq hp - \tilde{\rho}_{\vec{c}}(h) + 1.$$

This result is valid for every $h \in [H_{\min}, H_{\max}]$ and consequently,

$$\tilde{\eta}_f(p) \leq \inf_{h \in [H_{\min}, H_{\max}]} (hp - \tilde{\rho}_{\vec{c}}(h) + 1).$$

For the other inequality, let us fix $\delta > 0$ and $\varepsilon > 0$. For every $h \in [H_{\min}, H_{\max}]$, there exist $\varepsilon_h \leq \varepsilon$ and $J_h \in \mathbb{N}_0$ such that

$$\#\{\lambda \in \Lambda_j : 2^{-(h+\varepsilon_h)j} \leq d_\lambda < 2^{-(h-\varepsilon_h)j}\} \leq 2^{(\tilde{\rho}_{\vec{c}}(h)+\delta)j}$$

for every $j \geq J_h$. From the compactness of $[H_{\min}, H_{\max}]$, there exist $h_1 < \dots < h_N$ in $[H_{\min}, H_{\max}]$, $\varepsilon_1, \dots, \varepsilon_N \leq \varepsilon$ and $J \in \mathbb{N}_0$ such that the intervals $(h_i - \varepsilon_i, h_i + \varepsilon_i)$ cover $[H_{\min}, H_{\max}]$ and

$$\#\{\lambda \in \Lambda_j : 2^{-(h_i+\varepsilon_i)j} \leq d_\lambda < 2^{-(h_i-\varepsilon_i)j}\} \leq 2^{(\tilde{\rho}_{\vec{c}}(h_i)+\delta)j}$$

for every $i \in \{1, \dots, N\}$ and every $j \geq J$. Since $h_1 - \varepsilon_1 < H_{\min}$ and $h_N + \varepsilon_N > H_{\max}$, we can assume that

$$2^{-(h_N+\varepsilon_N)j} \leq d_\lambda \leq 2^{-(h_1-\varepsilon_1)j}$$

for every $j \geq J$ and $\lambda \in \Lambda_j$. It follows that

$$\begin{aligned} 2^{-j} \sum_{\lambda \in \Lambda_j}^* d_\lambda^p &\leq 2^{-j} \sum_{i=1}^N 2^{(\tilde{\rho}_{\vec{c}}(h_i) + \delta)j} 2^{-(h_i p - \varepsilon_i |p|)j} \\ &\leq 2^{(\delta + \varepsilon |p|)j} \sum_{i=1}^N 2^{-(h_i p - \tilde{\rho}_{\vec{c}}(h_i) + 1)j} \\ &\leq N 2^{(\delta + \varepsilon |p|)j} 2^{-\inf_{h \in \mathbb{R}} (hp - \tilde{\rho}_{\vec{c}}(h) + 1)j} \end{aligned}$$

and consequently,

$$\tilde{\eta}_f(p) \geq \inf_{h \geq 0} (hp - \tilde{\rho}_{\vec{c}}(h) + 1) - \delta - \varepsilon |p|.$$

The numbers $\delta > 0$ and $\varepsilon > 0$ being arbitrary, we get the conclusion. \square

Proposition 5.5.2. [64] *Let \vec{c} denote the wavelet coefficients of a function f in a wavelet basis. If $\tilde{\rho}_{\vec{c}}$ takes the value $-\infty$ outside of a compact set of $[0, +\infty)$, then*

$$\tilde{\nu}_f(h) \leq \inf_{p \in \mathbb{R}} (hp - \tilde{\eta}_f(p) + 1), \quad \forall h \geq 0$$

and the function $h \geq 0 \mapsto \inf_{p \in \mathbb{R}} (hp - \tilde{\eta}_f(p) + 1)$ is the concave hull of $\tilde{\nu}_f$.

Proof. By definition, $h \mapsto \inf_{p \in \mathbb{R}} (hp - \tilde{\eta}_f(p) + 1)$ is the Legendre transform of $\tilde{\eta}_f$. Moreover, we know from Lemma 5.5.1 that $\tilde{\eta}_f$ is the Legendre transform of $\tilde{\rho}_{\vec{c}}$. Using properties of this transform, we directly get that the function $h \mapsto \inf_{p \in \mathbb{R}} (hp - \tilde{\eta}_f(p) + 1)$ is the concave hull of $\tilde{\rho}_{\vec{c}}$. In particular, α_s is also the point at which the function $h \mapsto \inf_{p \in \mathbb{R}} (hp - \tilde{\eta}_f(p) + 1)$ is maximum. The function $\tilde{\nu}_f$ is the increasing hull of $\tilde{\rho}_{\vec{c}}$ on $[0, h_s]$ and its decreasing hull on $[h_s, +\infty)$ from Lemma 5.3.5, hence the conclusion. \square

Therefore, as one could hope, the leaders profile method gives a better theoretical approximation of spectra while there are not concave.

5.5.2 Comparison with the wavelet profile method

Let us recall that the formalism based on the \mathcal{S}^ν spaces, presented in Chapter 4, is based on the estimation of the multifractal spectrum of f by the function

$$\alpha \in [0, h_{\max}] \mapsto \alpha \sup_{\alpha' \in (0, \alpha]} \frac{\nu_f(\alpha')}{\alpha'},$$

where $h_{\max} = \inf_{\alpha \geq \alpha_{\min}} \frac{\alpha}{\nu_f(\alpha)}$. Let α_{\min} be defined as $\alpha_{\min} = \inf\{\alpha \geq 0 : \nu_f(\alpha) \geq 0\}$. We directly get that $\alpha_{\min} = \inf\{\alpha \geq 0 : \tilde{\nu}_f(\alpha) \geq 0\}$. Since the method based on the \mathcal{S}^ν spaces only allows to estimate increasing spectra, we will of course compare the leaders profile method with it on $[0, \alpha_s]$, i.e. the domain on which the function $\tilde{\nu}_f$ is increasing.

Proposition 5.5.3. [64] *If $\alpha_{\min} > 0$, then*

$$\tilde{\nu}_f^+(\alpha) \leq \alpha \sup_{\alpha' \in (0, \alpha]} \frac{\nu_f(\alpha')}{\alpha'}$$

for every $\alpha \in [0, \alpha_s]$. Moreover, the inequality becomes an equality on $[0, \alpha_s]$ if and only if $\tilde{\nu}_f$ is with increasing-visibility on $[\alpha_{\min}, \alpha_s]$.

Proof. Let us fix $\alpha \in [\alpha_{\min}, \alpha_s]$, $\delta > 0$, $0 < \alpha_0 < \alpha_{\min}$ and $\varepsilon > 0$ such that $\alpha_0 - \varepsilon > 0$. For every $\alpha' \leq \alpha + \varepsilon$, there exist $\varepsilon' \leq \varepsilon$ and $J \in \mathbb{N}_0$ such that

$$\#\{\lambda \in \Lambda_j : |c_\lambda| \geq 2^{-(\alpha+\varepsilon')j}\} \leq 2^{(\nu_f(\alpha')+\delta)j}, \quad \forall j \geq J.$$

Let us choose $\alpha_1, \dots, \alpha_N \leq h + \varepsilon$, $\varepsilon_1, \dots, \varepsilon_N \leq \varepsilon$ and $J \in \mathbb{N}_0$ such that the intervals $(\alpha_i, \alpha_i + \varepsilon_i)$ cover $[\alpha_0, \alpha + \varepsilon]$ and

$$\#\{\lambda \in \Lambda_j : |c_\lambda| \geq 2^{-(\alpha_i+\varepsilon_i)j}\} \leq 2^{(\nu_f(\alpha_i)+\delta)j}$$

for every $i \in \{1, \dots, N\}$ and every $j \geq J$. Remark that since $\alpha_0 < \alpha_{\min}$, we can assume that $|c_\lambda| \leq 2^{-\alpha_0 j}$ for every $j \geq J$. We obtain

$$\#\{\lambda \in \Lambda_j : e_\lambda \geq 2^{-(\alpha+\varepsilon)j}\} \leq \sum_{j \leq j' \leq \frac{\alpha+\varepsilon}{\alpha_0}j} \#\{\lambda' \in \Lambda_{j'} : |c_{\lambda'}| \geq 2^{-(\alpha+\varepsilon)j'}\}.$$

For every j' which appears in the sum, we have

$$\alpha_0 \leq \frac{(\alpha + \varepsilon)j}{j'} \leq \alpha + \varepsilon$$

and from the covering of $[\alpha_0, \alpha + \varepsilon]$, there exists $i \in \{1, \dots, N\}$ such that

$$\alpha_i j' \leq (\alpha + \varepsilon)j \leq (\alpha_i + \varepsilon_i)j'.$$

We get then

$$\begin{aligned} \#\{\lambda' \in \Lambda_{j'} : |c_{\lambda'}| \geq 2^{-(\alpha+\varepsilon)j'}\} &\leq \#\{\lambda' \in \Lambda_{j'} : |c_{\lambda'}| \geq 2^{-(\alpha_i+\varepsilon_i)j'}\} \\ &\leq 2^{(\nu_f(\alpha_i)+\delta)j'} \\ &\leq 2^{(\nu_f(\alpha_i)+\delta)\frac{\alpha+\varepsilon}{\alpha_i}j} \end{aligned}$$

and consequently,

$$\#\{\lambda \in \Lambda_j : e_\lambda \geq 2^{-(\alpha+\varepsilon)j}\} \leq \left(\left(\frac{\alpha + \varepsilon}{\alpha_0} - 1 \right) j + 1 \right) 2^{j(\alpha+\varepsilon) \sup_{\alpha' \in [\alpha_{\min}, \alpha+\varepsilon]} \frac{\nu_f(\alpha')+\delta}{\alpha'}}.$$

It follows that

$$\limsup_{j \rightarrow +\infty} \frac{\log \#\{\lambda \in \Lambda_j : e_\lambda \geq 2^{-(\alpha+\varepsilon)j}\}}{\log 2^j} \leq (\alpha + \varepsilon) \sup_{\alpha' \in [\alpha_{\min}, \alpha+\varepsilon]} \frac{\nu_f(\alpha') + \delta}{\alpha'}.$$

Taking the limit as $\varepsilon \rightarrow 0^+$ and using the right-continuity of ν_f , we get

$$\tilde{\nu}_f^+(\alpha) \leq \alpha \sup_{\alpha' \in [\alpha_{\min}, \alpha]} \frac{\nu_f(\alpha') + \delta}{\alpha'} \leq \alpha \sup_{\alpha' \in [\alpha_{\min}, \alpha]} \frac{\nu_f(\alpha')}{\alpha'} + \delta \frac{\alpha}{\alpha_{\min}}.$$

Since $\delta > 0$ is arbitrary, we obtain the first part of the proposition. Moreover, we have

$$\nu_f(\alpha) \leq \tilde{\nu}_f^+(\alpha) \leq \alpha \sup_{\alpha' \in (0, \alpha]} \frac{\nu_f(\alpha')}{\alpha'}$$

for every $\alpha \in [\alpha_{\min}, \alpha_s]$, which leads to the conclusion. \square

Remark 5.5.4. In particular, one always has $h_{\max} \leq \alpha_s$. Moreover, it becomes an equality if $\tilde{\nu}_f$ is with increasing-visibility on $[\alpha_{\min}, \alpha_s]$.

Consequently, from a theoretical point of view, the leaders profile method is better than the method based on the \mathcal{S}^ν spaces for two reasons. First, it allows to detect the decreasing part of spectra while the wavelet profile method is limited to increasing spectra. Secondly, it allows to detect spectra without increasing-visibility.

In practice, another drawback of the method based on the \mathcal{S}^ν spaces lies in the estimation of $\sup_{\alpha' \in (0, \alpha]} \frac{\nu_f(\alpha')}{\alpha'}$, once ν_f has been computed. Indeed, while dealing with numerical data, there are precision errors in the function ν_f which can introduce a bias in the computation of this upper bound.

Chapter 6

\mathcal{L}^ν spaces

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6.1 Introduction

In the previous chapter, we have introduced a new multifractal formalism based on the wavelet leaders which allows to detect non-concave and decreasing spectra. In this chapter, we present the underlying function space, denoted \mathcal{L}^ν . This new space is robust and encapsulates the information supplied by the increasing and decreasing wavelet leaders profiles. We investigate then which ones of the results proved in [13] in the wavelet coefficients setting can be extended in the wavelet leaders setting. In particular, we endow this space with a topology and obtain generic results about the form of the wavelet leaders profile of the functions in \mathcal{L}^ν . The main difference is that now the profile includes an increasing and a decreasing part, and is therefore much more realistic for most multifractal models, for which the decreasing part can prove crucial for identification, or model selection, see [101] for instance. Nevertheless, it implies that the underlying space is not a vector space.

Since the wavelet leaders profile is independent of the chosen wavelet basis, the same will hold for the space \mathcal{L}^ν . Therefore, as in the case of \mathcal{S}^ν spaces and Besov spaces, we

can consider \mathcal{L}^ν as a sequence space (and not as a function space). That is the point of view that we adopt here. If $\vec{c} \in \Omega$, we set

$$e_\lambda := \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|, \quad \lambda \in \Lambda_j, \quad j \in \mathbb{N}_0,$$

similarly to what was done in Chapter 4. In the present situation of sequence spaces, we call again $c_{j,k}$ or c_λ wavelet coefficients and $e_{j,k}$ or e_λ restricted wavelet leaders (even if these elements no longer depend on a function). Similar definitions are adopted for the wavelet leaders of a sequence. With this definition, it may happen that $e_\lambda = +\infty$. We will work usually with sequences in C^0 to avoid this situation. Moreover, in order to get a topology on \mathcal{L}^ν , we will also assume that the sequences have a given maximal regularity.

In this chapter, given a sequence $\vec{c} \in C^0$, we denote by \vec{e} its sequence of restricted wavelet leaders, unless explicitly stipulated. Similarly, given a sequence $(\vec{c}^{(n)})_{n \in \mathbb{N}}$ of sequences of C^0 , we denote by $\vec{e}^{(n)}$ the sequence of restricted wavelet leaders of $\vec{c}^{(n)}$.

This chapter is structured as follows. In Section 6.2, we construct sequences with a prescribed wavelet leaders profile. The spaces \mathcal{L}^ν are introduced in Section 6.3. In order to endow \mathcal{L}^ν with a topology, we consider in Section 6.4 and Section 6.5 two derived spaces, $\mathcal{L}^{\nu,+}$ and $\mathcal{L}^{\nu,-}$, respectively related to the increasing and the decreasing wavelet leaders profiles. The topology of \mathcal{L}^ν is presented in Section 6.6. Finally, we present generic results about the form of the wavelet leaders profile of sequences of \mathcal{L}^ν in Section 6.7.

6.2 Admissible profile

Given a sequence $\vec{c} \in C^0$, let us recall that we denote by α_s the smallest positive number such that $\tilde{\nu}_\vec{c}^+(\alpha_s) = 1$ and we define the wavelet leaders profile $\tilde{\nu}_\vec{c}$ of \vec{c} on $[0, +\infty]$ by

$$\tilde{\nu}_\vec{c}(\alpha) = \begin{cases} \tilde{\nu}_\vec{c}^+(\alpha) & \text{if } \alpha \in [0, \alpha_s], \\ \tilde{\nu}_\vec{c}^-(\alpha) & \text{if } \alpha \in [\alpha_s, +\infty]. \end{cases}$$

As presented in Chapter 5, this function takes values in $\{-\infty\} \cup [0, 1]$ and there exist $0 \leq \alpha_{\min} \leq \alpha_s \leq \alpha_{\max} \leq +\infty$ such that $\tilde{\nu}_\vec{c} = -\infty$ on $[0, \alpha_{\min}) \cup (\alpha_{\max}, +\infty]$, $\tilde{\nu}_\vec{c}(\alpha_s) = 1$, $\tilde{\nu}_\vec{c}$ is increasing and with values in $[0, 1]$ on $[\alpha_{\min}, \alpha_s]$ and $\tilde{\nu}_\vec{c}$ is decreasing and with values in $[0, 1]$ on $[\alpha_s, \alpha_{\max}]$. Moreover, $\nu_\vec{c}$ is right-continuous on $[0, \alpha_s]$ and left-continuous on $[\alpha_s, +\infty)$. Finally, we know that the function $1 - \tilde{\nu}_\vec{c}$ is with increasing-visibility on $[\alpha_s, \alpha_{\max}]$. In this chapter, we assume that $\alpha_{\max} < +\infty$. Let us show that these properties entirely characterize the wavelet leaders profiles, as stated in the next result.

Proposition 6.2.1. *Any function $\nu : [0, +\infty) \rightarrow \{-\infty\} \cup [0, 1]$ for which there exist $0 \leq \alpha_{\min} \leq \alpha_s \leq \alpha_{\max} < +\infty$ such that*

$$\left\{ \begin{array}{l} \nu = -\infty \text{ on } [0, \alpha_{\min}) \cup (\alpha_{\max}, +\infty), \\ \nu(\alpha_s) = 1, \\ \nu \text{ is increasing, right-continuous and with values in } [0, 1] \text{ on } [\alpha_{\min}, \alpha_s], \\ \nu \text{ is decreasing, left-continuous and with values in } [0, 1] \text{ on } [\alpha_s, \alpha_{\max}], \\ \alpha \mapsto \frac{\nu(\alpha)-1}{\alpha} \text{ is decreasing on } [\alpha_s, \alpha_{\max}], \end{array} \right.$$

is the wavelet leaders profile of a sequence of C^0 .

Proof. Let us consider a sequence $(\alpha_n)_{n \in \mathbb{N}}$ which forms a dense subset of $[\alpha_{\min}, \alpha_{\max}]$ and such that α_s appears in the sequence. Using the right-continuity (resp. the left-continuity) of ν and of the increasing wavelet leaders profile (resp. decreasing wavelet leaders profile) on $[\alpha_{\min}, \alpha_s]$ (resp. on $[\alpha_s, \alpha_{\max}]$), it suffices to construct a sequence \vec{c} such that $\tilde{\nu}_{\vec{c}}(\alpha_n) = \nu(\alpha_n)$ for every $n \in \mathbb{N}$ and such that $\tilde{\nu}_{\vec{c}}$ is identically equal to $-\infty$ on $[0, \alpha_{\min}) \cup (\alpha_{\max}, +\infty]$. First, we will construct sequences $\vec{c}^{(n)}$, $n \in \mathbb{N}$, such that $\tilde{\nu}_{\vec{c}^{(n)}}(\alpha_n) \geq \nu(\alpha_n)$.

1. If $\alpha_n < \alpha_s$

If $\nu(\alpha_n) = 0$, let $J_n = 0$ and otherwise, let J_n be the smallest integer such that

$$2^{\nu(\alpha_n)J_n} \geq \frac{2}{2^{1-\nu(\alpha_n)} - 1}. \quad (6.1)$$

Then, for every $j \geq J_n$, we have

$$\lfloor 2^{\nu(\alpha_n)j} \rfloor \leq 2 \lfloor 2^{\nu(\alpha_n)(j-1)} \rfloor.$$

Let us define $\vec{c}^{(n)}$ as follows: if $j < J_n$, we set $c_{j,k}^{(n)} := 0$ and if $j \geq J_n$, we set

$$c_{j,k}^{(n)} := \begin{cases} 2^{-\alpha_n(j+n)} & \text{for } \lfloor 2^{\nu(\alpha_n)j} \rfloor \text{ values of } k, \\ 2^{-\alpha_s(j+n)} & \text{otherwise,} \end{cases}$$

where the positions of the $k \in \{0, \dots, 2^j - 1\}$ such that $c_{j,k}^{(n)} = 2^{-\alpha_n(j+n)}$ are chosen first to fill entirely dyadic cubes of scale $j - 1$ whose coefficients equal $2^{-\alpha_n(j-1+n)}$. It follows that if $j \geq J_n$, the restricted wavelet leaders are the wavelet coefficients. In particular, one has $\tilde{\nu}_{\vec{c}^{(n)}}^+(\alpha_n) = \nu(\alpha_n)$.

2. If $\alpha_n > \alpha_s$ is such that $\nu(\alpha_n) < 1$

Let $j_0 = 0$ and for every $l \in \mathbb{N}_0$, let j_{l+1} be the smallest integer larger than j_l such that

$$\alpha_s(j_{l+1} + n) \geq \alpha_n(j_l + n).$$

In particular, if $j_l < j < j_{l+1}$, $\alpha_s(j + n) < \alpha_n(j_l + n)$. Let us define $\vec{c}^{(n)}$ as follows: if $j = j_l$, we set

$$c_{j_l,k}^{(n)} := \begin{cases} 2^{-\alpha_n(j_l+n)} & \text{for } \lfloor 2^{\nu(\alpha_n)j_l} \rfloor \text{ values of } k, \\ 2^{-\alpha_s(j_l+n)} & \text{otherwise,} \end{cases}$$

and if j is between j_l and j_{l+1} , we set

$$c_{j,k}^{(n)} := \begin{cases} 2^{-\alpha_n(j_l+n)} & \text{if } \lambda(j, k) \text{ is included in } \lambda_l \in \Lambda_{j_l} \text{ with } c_{\lambda_l} = 2^{-\alpha_n(j_l+n)}, \\ 2^{-\alpha_s(j+n)} & \text{otherwise.} \end{cases}$$

With this construction, the restricted wavelet leaders of $\vec{c}^{(n)}$ are its wavelet coefficients. Moreover,

$$\limsup_{j \rightarrow +\infty} \frac{\log \#\{\lambda \in \Lambda_j : e_{\lambda}^{(n)} \leq 2^{-(\alpha_s - \varepsilon)j}\}}{\log 2^j} \geq \lim_{l \rightarrow +\infty} \frac{\log \#\{\lambda \in \Lambda_{j_l} : e_{\lambda}^{(n)} \leq 2^{-(\alpha_s - \varepsilon)j_l}\}}{\log 2^{j_l}}$$

so that $\tilde{\nu}_{\vec{c}^{(n)}}^-(\alpha_n) \geq \nu(\alpha_n)$.

Let us also make two remarks that will be useful in the general construction. First, if $\alpha < \alpha_s$, then

$$\#\{\lambda \in \Lambda_{j-n} : e_\lambda^{(n)} \geq 2^{-\alpha j}\} = 0.$$

Indeed, if $j - n = j_l$, then $e_\lambda^{(n)} = 2^{-\alpha_s j} < 2^{-\alpha j}$ or $e_\lambda^{(n)} = 2^{-\alpha_n j} < 2^{-\alpha j}$. If $j - n$ is between j_l and j_{l+1} , then $e_\lambda^{(n)} = 2^{-\alpha_s j} < 2^{-\alpha j}$ or $e_\lambda^{(n)} = 2^{-\alpha_n(j_l+n)} < 2^{-\alpha_s j} < 2^{-\alpha j}$. Secondly, assume that $\alpha \in (\alpha_s, \alpha_n]$ and fix j such that $j - n$ is between j_l and j_{l+1} . If $2^{-\alpha_n j_l} > 2^{-\alpha j}$, then

$$\#\{\lambda \in \Lambda_{j-n} : e_\lambda^{(n)} \leq 2^{-\alpha j}\} = 0$$

and if $2^{-\alpha_n j_l} \leq 2^{-\alpha j}$, we have

$$\begin{aligned} \#\{\lambda \in \Lambda_{j-n} : e_\lambda^{(n)} \leq 2^{-\alpha j}\} &\leq 2^{j-n-j_l} 2^{\nu(\alpha_n)j_l} \\ &\leq 2^{j-n} 2^{(\nu(\alpha_n)-1)\frac{\alpha}{\alpha_n}j} \\ &\leq 2^{j-n} 2^{(\nu(\alpha)-1)j} \\ &\leq 2^{\nu(\alpha)j}, \end{aligned}$$

using the last property of the admissible profile ν .

3. If $\alpha_n \geq \alpha_s$ is such that $\nu(\alpha_n) = 1$

We consider the sequence $\vec{c}^{(n)}$ defined by

$$c_{j,k}^{(n)} = 2^{-\alpha_n(j+n)}, \quad \forall j \in \mathbb{N}, k \in \{0, \dots, 2^j - 1\}.$$

Then, $\tilde{\nu}_{\vec{c}^{(n)}}^-(\alpha_n) = \tilde{\nu}_{\vec{c}^{(n)}}^+(\alpha_n) = 1$.

4. General case

We consider the sequence \vec{c} defined using the sequences $\vec{c}^{(n)}$, $n \in \mathbb{N}$, as follows: we set $c_{0,0} := 0$ and for every $j \in \mathbb{N}$, $k \in \{0, \dots, 2^j - 1\}$, we set

$$c_{j,k} := \begin{cases} c_{j-1,k-2^{j-1}}^{(1)} & \text{if } k \in \{2^{j-1}, \dots, 2^j - 1\}, \\ c_{j-2,k-2^{j-2}}^{(2)} & \text{if } k \in \{2^{j-2}, \dots, 2^{j-1} - 1\}, \\ \vdots & \\ c_{1,k-2}^{(j-1)} & \text{if } k \in \{2, 3\}, \\ c_{0,0}^{(j)} & \text{if } k = 1, \\ 0 & \text{if } k = 0. \end{cases}$$

Clearly, we have $|c_{j,k}| \leq 2^{-\alpha_{\min}j}$ for every $j \in \mathbb{N}_0$, $k \in \{0, \dots, 2^j - 1\}$, and therefore $\vec{c} \in C^{\alpha_{\min}} \subseteq C^0$. In particular, if $\alpha < \alpha_{\min}$, then $\tilde{\nu}_{\vec{c}}^+(\alpha) = -\infty$. Similarly, if $\alpha > \alpha_{\max}$, $e_{j,k} > 2^{-\alpha j}$ for every j, k and we obtain $\tilde{\nu}_{\vec{c}}^-(\alpha) = -\infty$. Moreover, if $\alpha_n = \alpha_s$, we have $\tilde{\nu}_{\vec{c}}^-(\alpha_s) \geq \tilde{\nu}_{\vec{c}^{(n)}}^-(\alpha_s) = 1$. The same holds for the increasing wavelet leaders profile of \vec{c} , so that $\tilde{\nu}_{\vec{c}}^+(\alpha_s) = \tilde{\nu}_{\vec{c}^{(n)}}^+(\alpha_s) = 1$.

Let us now show that for every $n \in \mathbb{N}$ such that $\alpha_n < \alpha_s$, we have $\tilde{\nu}_{\vec{c}}^+(\alpha_n) = \nu(\alpha_n)$. By construction, it is clear that $\tilde{\nu}_{\vec{c}}^+(\alpha_n) \geq \tilde{\nu}_{\vec{c}^{(n)}}^+(\alpha_n) = \nu(\alpha_n)$ and we only have to prove

the other inequality. Let us fix $\varepsilon > 0$ such that $\alpha_n + \varepsilon < \alpha_s$. At a given scale $j \geq n$, we have to take into consideration the sequences $\tilde{c}^{(1)}, \dots, \tilde{c}^{(j)}$ and the restricted wavelet leaders corresponding to $k = 0$. More precisely,

$$\#\{\lambda \in \Lambda_j : e_\lambda \geq 2^{-(\alpha_n + \varepsilon)j}\} \leq \sum_{m=1}^j \#\{\lambda \in \Lambda_{j-m} : e_\lambda^{(m)} \geq 2^{-(\alpha_n + \varepsilon)j}\} + 1,$$

where we have added the case $k = 0$. By construction, we know that if $\alpha_m > \alpha_n + \varepsilon$, we have

$$\#\{\lambda \in \Lambda_{j-m} : e_\lambda^{(m)} \geq 2^{-(\alpha_n + \varepsilon)j}\} = 0.$$

Assume then that $\alpha_m \leq \alpha_n + \varepsilon < \alpha_s$. If $j - m \geq J_m$, we have $e_{j-m,k}^{(m)} = c_{j-m,k}^{(m)}$ and it follows that

$$\#\{\lambda \in \Lambda_{j-m} : e_\lambda^{(m)} \geq 2^{-(\alpha_n + \varepsilon)j}\} \leq 2^{\nu(\alpha_m)(j-m)} \leq 2^{\nu(\alpha_n + \varepsilon)j}.$$

If $j - m \leq J_m$, we have

$$\begin{aligned} \#\{\lambda \in \Lambda_{j-m} : e_\lambda^{(m)} \geq 2^{-(\alpha_n + \varepsilon)j}\} &\leq \lfloor 2^{\nu(\alpha_m)J_m} \rfloor \\ &\leq 2 \lfloor 2^{\nu(\alpha_m)(J_m - 1)} \rfloor \\ &\leq \frac{4}{2^{1 - \nu(\alpha_m)} - 1} \\ &\leq \frac{4}{2^{1 - \nu(\alpha_n + \varepsilon)} - 1} \end{aligned}$$

using the choice (6.1) of J_m . If j is large enough, we have

$$\frac{4}{2^{1 - \nu(\alpha_n + \varepsilon)} - 1} \leq 2^{\nu(\alpha_n + \varepsilon)j}$$

and it follows that

$$\#\{\lambda \in \Lambda_j : e_\lambda \geq 2^{-(\alpha_n + \varepsilon)j}\} \leq j 2^{\nu(\alpha_n + \varepsilon)j} + 1.$$

Therefore,

$$\tilde{\nu}_{\tilde{c}}^+(\alpha_n) \leq \lim_{\varepsilon \rightarrow 0^+} \nu(\alpha_n + \varepsilon) = \nu(\alpha_n)$$

using the right-continuity of ν .

Finally, let us show that for every $n \in \mathbb{N}$ such that $\alpha_n > \alpha_s$, $\tilde{\nu}_{\tilde{c}}^-(\alpha_n) = \nu(\alpha_n)$. As previously, $\tilde{\nu}_{\tilde{c}}^-(\alpha_n) \geq \tilde{\nu}_{\tilde{c}^{(n)}}^-(\alpha_n) \geq \nu(\alpha_n)$ and we only have to prove the other inequality. Let us fix $\varepsilon > 0$ such that $\alpha_n - \varepsilon > \alpha_s$. Remark that if $\alpha_m < \alpha_n - \varepsilon$, we have

$$\#\{\lambda \in \Lambda_{j-m} : e_\lambda^{(m)} \leq 2^{-(\alpha_n - \varepsilon)j}\} \leq \#\{\lambda \in \Lambda_{j-m} : c_\lambda^{(m)} \leq 2^{-(\alpha_n - \varepsilon)j}\} = 0.$$

If $\alpha_m \geq \alpha_n - \varepsilon$, we know that

$$\#\{\lambda \in \Lambda_{j-m} : e_\lambda^{(m)} \leq 2^{-(\alpha_n - \varepsilon)j}\} \leq 2^{\nu(\alpha_n - \varepsilon)j}.$$

In total,

$$\begin{aligned} \#\{\lambda \in \Lambda_j : e_\lambda \leq 2^{-(\alpha_n - \varepsilon)j}\} &\leq \sum_{m=1}^j \#\{\lambda \in \Lambda_{j-m} : e_\lambda^{(m)} \leq 2^{-(\alpha_n - \varepsilon)j}\} + 1 \\ &\leq j 2^{\nu(\alpha_n - \varepsilon)j} + 1 \end{aligned}$$

and it follows that

$$\tilde{\nu}_{\vec{c}}^-(\alpha_n) \leq \lim_{\varepsilon \rightarrow 0^+} \nu(\alpha_n - \varepsilon) = \nu(\alpha_n)$$

using the left-continuity of ν . □

Remark 6.2.2. If one considers the wavelet series f associated to this sequence \vec{c} , i.e. the function whose coefficients in a fixed wavelet basis are given by the sequence \vec{c} , then it does not satisfy the leaders profile method. Indeed, it is direct to check that, for example, $E^f(h) = \emptyset$ if $h < \alpha_s$ does not belong to $\{\alpha_n : n \in \mathbb{N}\}$. Let us mention that a function with prescribed wavelet leaders profile will also be constructed in Chapter 7. This construction will use the topology of \mathcal{L}^ν and it will give a function which satisfies the leaders profile method.

Following this result, we consider the next definition.

Definition 6.2.3. A function which satisfies the conditions of Proposition 6.2.1 is called an *admissible profile*.

Using the previous construction of a sequence with a prescribed wavelet leaders profile, it is possible to construct a subspace of C^0 with a maximal dimension whose elements have the same prescribed wavelet leaders profile ν . We will show in Section 6.7 that this subspace can be chosen to be dense in \mathcal{L}^ν .

Proposition 6.2.4. *Let ν be an admissible profile. The set of sequences $\vec{c} \in C^0$ such that $\tilde{\nu}_{\vec{c}} = \nu$ is \mathfrak{c} -lineable in C^0 .*

Proof. Let $\vec{c} \in C^0$ be a sequence such that $\tilde{\nu}_{\vec{c}} = \nu$, as constructed in Proposition 6.2.1. Let us denote by \vec{e} the sequence of restricted wavelet leaders of \vec{c} . For every $r > 0$, we define the sequence $\vec{x}^{(r)} \in C^0$ by setting

$$x_\lambda^{(r)} = \frac{1}{j^r} e_\lambda$$

for every $j \in \mathbb{N}_0$, $\lambda \in \Lambda_j$. Of course, the restricted wavelet leaders of $\vec{x}^{(r)}$ are its wavelet leaders.

Let us first prove that the sequences $\vec{x}^{(r)}$, $r > 0$, are linearly independent. Let $r_N > r_{N-1} > \dots > r_1 > 0$ ($N \in \mathbb{N}$) and let us assume that

$$\theta_1 \vec{x}^{(r_1)} + \dots + \theta_N \vec{x}^{(r_N)} = 0,$$

where $\theta_1, \dots, \theta_N \in \mathbb{C}$. Since $\tilde{\nu}_{\vec{c}} = \nu$, we know that $e_\lambda \neq 0$ for infinitely many $j \in \mathbb{N}_0$, $\lambda \in \Lambda_j$. Consequently,

$$\theta_1 \frac{1}{j^{r_1}} + \dots + \theta_N \frac{1}{j^{r_N}} = 0$$

for infinitely many $j \in \mathbb{N}_0$. If we multiply this relation by j^{r_1} , we get

$$\theta_1 + \theta_2 j^{r_1 - r_2} + \dots + \theta_N j^{r_1 - r_N} = 0$$

for infinitely many $j \in \mathbb{N}_0$. Taking $j \rightarrow +\infty$, we get that $\theta_1 = 0$ and recursively, $\theta_1 = \dots = \theta_N = 0$.

Secondly, let us prove that if

$$\vec{z} = \theta_1 \vec{x}^{(r_1)} + \dots + \theta_N \vec{x}^{(r_N)},$$

with $r_N > r_{N-1} > \dots > r_1 > 0$ ($N \in \mathbb{N}$) and $\theta_1, \dots, \theta_N \in \mathbb{C} \setminus \{0\}$, then $\tilde{\nu}_{\tilde{z}} = \nu$. Let us remark that

$$\begin{aligned} \sup_{\lambda' \subseteq \lambda} |z_{\lambda'}| &\leq \sup_{\lambda' \subseteq \lambda} \left(|\theta_1| \frac{1}{j^{r_1}} + \dots + |\theta_N| \frac{1}{j^{r_N}} \right) e_{\lambda'} \\ &= \left(|\theta_1| \frac{1}{j^{r_1}} + \dots + |\theta_N| \frac{1}{j^{r_N}} \right) e_\lambda \\ &\leq e_\lambda \end{aligned}$$

if j is large enough. Moreover, for every $\varepsilon > 0$, we have

$$\begin{aligned} \sup_{\lambda' \subseteq \lambda} |z_{\lambda'}| &\geq \left| \theta_1 \frac{1}{j^{r_1}} + \dots + \theta_N \frac{1}{j^{r_N}} \right| e_\lambda \\ &\geq \left(|\theta_1| \frac{1}{j^{r_1}} - |\theta_2| \frac{1}{j^{r_2}} - \dots - |\theta_N| \frac{1}{j^{r_N}} \right) e_\lambda \\ &\geq 2^{-\frac{\varepsilon}{2}j} e_\lambda \end{aligned}$$

if j is large enough. It follows that

$$\# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |z_{\lambda'}| \geq 2^{-(\alpha+\varepsilon)j} \right\} \leq \# \left\{ \lambda \in \Lambda_j : e_\lambda \geq 2^{-(\alpha+\varepsilon)j} \right\}$$

and

$$\# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |z_{\lambda'}| \geq 2^{-(\alpha+\varepsilon)j} \right\} \geq \# \left\{ \lambda \in \Lambda_j : e_\lambda \geq 2^{-(\alpha+\frac{\varepsilon}{2})j} \right\}$$

for every $\alpha \in [0, \alpha_s]$ if j is large enough. Consequently, we obtain $\tilde{\nu}_{\tilde{z}}(\alpha) = \tilde{\nu}_{\tilde{z}}(\alpha) = \nu(\alpha)$. Similarly, we directly get that $\tilde{\nu}_{\tilde{z}}(\alpha) = \tilde{\nu}_{\tilde{z}}(\alpha) = \nu(\alpha)$ if $\alpha \geq \alpha_s$. \square

6.3 Definition of \mathcal{L}^ν spaces

In this section, we introduce new sequence spaces which contain the information supplied by the wavelet leaders profiles.

Definition 6.3.1. Given an admissible profile ν , the space \mathcal{L}^ν is defined by the set of sequences $\tilde{c} \in C^0$ such that $\tilde{\nu}_{\tilde{c}} \leq \nu$ on $[0, +\infty)$, i.e. such that

$$\begin{cases} \tilde{\nu}_{\tilde{c}}^+(\alpha) \leq \nu(\alpha), & \forall \alpha \leq \alpha_s, \\ \tilde{\nu}_{\tilde{c}}^-(\alpha) \leq \nu(\alpha), & \forall \alpha \geq \alpha_s. \end{cases}$$

Using Proposition 5.3.7, we directly get that the space \mathcal{L}^ν is robust. Therefore, one can define an associated function space: a locally bounded function f belongs to the function space \mathcal{L}^ν if its sequence of wavelet coefficients \tilde{c} in a given wavelet basis belongs to the sequence space \mathcal{L}^ν . Remark that if the admissible profile ν is such that $\alpha_{\min} > 0$, then all the functions of \mathcal{L}^ν are uniformly Hölder. The following proposition gives a useful characterization of the space \mathcal{L}^ν .

Proposition 6.3.2. *A sequence $\vec{c} \in C^0$ belongs to \mathcal{L}^ν if and only if for every $\alpha \geq 0$, $\varepsilon > 0$ and $C > 0$, there exists $J \in \mathbb{N}_0$ such that for every $j \geq J$*

$$\begin{cases} \#\tilde{E}_j^+(C, \alpha)(\vec{c}) \leq 2^{(\nu(\alpha)+\varepsilon)j} & \text{if } \alpha < \alpha_s, \\ \#\tilde{E}_j^-(C, \alpha)(\vec{c}) \leq 2^{(\nu(\alpha)+\varepsilon)j} & \text{if } \alpha > \alpha_s, \end{cases}$$

where

$$\tilde{E}_j^+(C, \alpha)(\vec{c}) = \{\lambda \in \Lambda_j : e_\lambda \geq C2^{-\alpha j}\} \text{ and } \tilde{E}_j^-(C, \alpha)(\vec{c}) = \{\lambda \in \Lambda_j : e_\lambda \leq C2^{-\alpha j}\}.$$

Proof. This proof is a simple adaptation of the proof of Lemma 2.3 in [13]. Assume that $\vec{c} \in \mathcal{L}^\nu$. Let us fix $\alpha \in [0, \alpha_s)$, $\varepsilon > 0$ and $C > 0$. From the definition of $\tilde{\nu}_\vec{c}^+(\alpha)$, there exist $J \in \mathbb{N}$ and $\eta > 0$ such that $C \geq 2^{-\eta j}$ for every $j \geq J$ and

$$\#\tilde{E}_j^+(1, \alpha + \eta)(\vec{c}) \leq 2^{(\tilde{\nu}_\vec{c}^+(\alpha)+\varepsilon)j}, \quad \forall j \geq J.$$

From the choice of η , we have $\#\tilde{E}_j^+(C, \alpha)(\vec{c}) \leq \#\tilde{E}_j^+(1, \alpha + \eta)(\vec{c})$ for every $j \geq J$ and since $\tilde{\nu}_\vec{c}^+(\alpha) \leq \nu(\alpha)$, we get

$$\#\tilde{E}_j^+(C, \alpha)(\vec{c}) \leq 2^{(\nu(\alpha)+\varepsilon)j}, \quad \forall j \geq J.$$

The proof for the decreasing part is similar.

Assume now that \vec{c} satisfies the assumptions of the proposition. Let $\alpha \in [0, \alpha_s)$ and $\varepsilon > 0$ be such that $\alpha + \varepsilon < \alpha_s$. Then, there is $J \in \mathbb{N}$ such that

$$\#\tilde{E}_j^+(1, \alpha + \varepsilon)(\vec{c}) \leq 2^{(\nu(\alpha+\varepsilon)+\varepsilon)j}, \quad \forall j \geq J$$

so that

$$\limsup_{j \rightarrow +\infty} \frac{\log \#\tilde{E}_j^+(1, \alpha + \varepsilon)(\vec{c})}{\log 2^j} \leq \nu(\alpha + \varepsilon) + \varepsilon.$$

The right-continuity of ν gives

$$\lim_{\varepsilon \rightarrow 0} \limsup_{j \rightarrow +\infty} \frac{\log \#\tilde{E}_j^+(1, \alpha + \varepsilon)(\vec{c})}{\log 2^j} \leq \nu(\alpha).$$

If $\alpha \in (\alpha_s, +\infty)$, the proof is similar. If $\alpha = \alpha_s$, the result is direct since $\nu(\alpha_s) = 1$. \square

Remark 6.3.3. Of course, from Proposition 5.3.6, the same result holds if we replace the restricted wavelet leaders by the wavelet leaders.

Let us remark that, in general, the space \mathcal{L}^ν is not a vector space since 0 does not belong to \mathcal{L}^ν .

Definition 6.3.4. Given an admissible profile ν , we define the spaces $\mathcal{L}^{\nu,+}$ and $\mathcal{L}^{\nu,-}$ as follows:

- the space $\mathcal{L}^{\nu,+}$ is the set of sequences $\vec{c} \in C^0$ such that for every $\alpha \in [0, \alpha_s)$, $\varepsilon > 0$ and $C > 0$, there exists $J \in \mathbb{N}_0$ such that

$$\#\tilde{E}_j^+(C, \alpha)(\vec{c}) \leq 2^{(\nu(\alpha)+\varepsilon)j}, \quad \forall j \geq J,$$

- the space $\mathcal{L}^{\nu,-}$ is the set of sequences $\vec{c} \in \Omega$ such that for every $\alpha \in (\alpha_s, +\infty)$, $\varepsilon > 0$ and $C > 0$, there exists $J \in \mathbb{N}_0$ such that

$$\#\widetilde{E}_j^-(C, \alpha)(\vec{c}) \leq 2^{(\nu(\alpha)+\varepsilon)j}, \quad \forall j \geq J.$$

Remark that we allow the wavelet leaders of a sequence of $\mathcal{L}^{\nu,-}$ to be infinite. Of course, we have

$$\mathcal{L}^\nu = \mathcal{L}^{\nu,+} \cap \mathcal{L}^{\nu,-}.$$

Moreover, as we will see, the space $\mathcal{L}^{\nu,+}$ is a vector space. Obviously, the notation $\mathcal{L}^{\nu,+}$ is used to refer to the fact that we consider only the increasing part of the admissible profile ν . Similarly, in the definition of $\mathcal{L}^{\nu,-}$, we only consider the decreasing part of ν .

Remark 6.3.5. If one considers an admissible profile such that $\nu(\alpha) = 1$ for every $\alpha \geq \alpha_s$, it is easy to see that $\mathcal{L}^{\nu,-} = C^0$ so that $\mathcal{L}^\nu = \mathcal{L}^{\nu,+}$ and \mathcal{L}^ν is a vector space. Remark that it is the notion of admissible profile used in the case of \mathcal{S}^ν spaces.

6.4 $\mathcal{L}^{\nu,+}$ spaces

In this section, we show that $\mathcal{L}^{\nu,+}$ is a vector space and we endow it with a distance $\widetilde{\delta}^+$. We study then some basic properties of this topological vector space. Most of the results and proofs presented in this section are derived from [25] and are similar to the corresponding results in \mathcal{S}^ν spaces, see [13].

Let us recall that $\mathcal{L}^{\nu,+}$ is the set of sequences $\vec{c} \in C^0$ such that for every $\alpha \in [0, \alpha_s)$, $\varepsilon > 0$ and $C > 0$, there exists $J \in \mathbb{N}_0$ such that

$$\#\widetilde{E}_j^+(C, \alpha)(\vec{c}) \leq 2^{(\nu(\alpha)+\varepsilon)j} \quad \forall j \geq J,$$

where

$$\widetilde{E}_j^+(C, \alpha)(\vec{c}) = \{\lambda \in \Lambda_j : e_\lambda \geq C2^{-\alpha j}\}.$$

From Proposition 6.3.2, we know that a sequence $\vec{c} \in C^0$ belongs to $\mathcal{L}^{\nu,+}$ if and only if $\widetilde{\nu}_{\vec{c}}^+ \leq \nu$ on $[0, \alpha_s]$.

Proposition 6.4.1. [25] *The space $\mathcal{L}^{\nu,+}$ is a vector space.*

Proof. We trivially have $\vec{0} \in \mathcal{L}^{\nu,+}$. Moreover, if $\vec{c} \in \mathcal{L}^{\nu,+}$ and $\theta \in \mathbb{C} \setminus \{0\}$, then $\theta\vec{c} \in \mathcal{L}^{\nu,+}$ since $\widetilde{E}_j^+(C, \alpha)(\theta\vec{c}) = \widetilde{E}_j^+(\frac{C}{|\theta|}, \alpha)(\vec{c})$. Let $\vec{c}, \vec{c}' \in \mathcal{L}^{\nu,+}$. Let us fix $\alpha \in [0, \alpha_s)$, $\varepsilon > 0$ and $C > 0$. There exists $J \in \mathbb{N}_0$ such that

$$\#\widetilde{E}_j^+\left(\frac{C}{2}, \alpha\right)(\vec{c}) \leq 2^{(\nu(\alpha)+\frac{\varepsilon}{2})j}, \quad \#\widetilde{E}_j^+\left(\frac{C}{2}, \alpha\right)(\vec{c}') \leq 2^{(\nu(\alpha)+\frac{\varepsilon}{2})j} \quad \text{and} \quad 2 \cdot 2^{\frac{\varepsilon}{2}j} \leq 2^{\varepsilon j}$$

for every $j \geq J$. We have

$$e_\lambda + e'_\lambda \geq \sup_{\lambda' \subseteq \lambda} |e_{\lambda'} + e'_{\lambda'}| \geq C2^{-\alpha j} \quad \Rightarrow \quad e_\lambda \geq \frac{C}{2} 2^{-\alpha j} \quad \text{or} \quad e'_\lambda \geq \frac{C}{2} 2^{-\alpha j}$$

and so

$$\begin{aligned} \#\widetilde{E}_j^+(C, \alpha)(\vec{c} + \vec{c}') &\leq \#\widetilde{E}_j^+\left(\frac{C}{2}, \alpha\right)(\vec{c}) + \#\widetilde{E}_j^+\left(\frac{C}{2}, \alpha\right)(\vec{c}') \\ &\leq 2^{\nu(\alpha)j} 2 \cdot 2^{\frac{\varepsilon}{2}j} \\ &\leq 2^{(\nu(\alpha)+\varepsilon)j} \end{aligned}$$

for every $j \geq J$. Thus $\vec{c} + \vec{c}' \in \mathcal{L}^{\nu,+}$. □

6.4.1 Auxiliary spaces $\tilde{A}^+(\alpha, \beta)$

As for the case of the \mathcal{S}^ν spaces, a useful description can also be obtained by the introduction of auxiliary spaces. These new spaces will then be used to define a topology on $\mathcal{L}^{\nu,+}$.

Definition 6.4.2. Let $\alpha \geq 0$ and $\beta \in \{-\infty\} \cup [0, +\infty)$. A sequence $\vec{c} \in C^0$ belongs to the auxiliary space $\tilde{A}^+(\alpha, \beta)$ if there exist $C, C' \geq 0$ such that

$$\#\tilde{E}_j^+(C, \alpha)(\vec{c}) \leq C'2^{\beta j}, \quad \forall j \in \mathbb{N}_0.$$

Let us remark that a simple adaptation of the proof of Proposition 6.4.1 shows that the auxiliary spaces are vector spaces.

Remark 6.4.3.

1. If $\beta = -\infty$, then $\tilde{A}^+(\alpha, \beta)$ is the set of the sequences $\vec{c} \in \Omega$ satisfying

$$\sup_{j \in \mathbb{N}_0} \sup_{k \in \{0, \dots, 2^j - 1\}} 2^{j\alpha} e_{j,k} < +\infty.$$

Let us remark that it is the Hölder space C^α . Indeed, since $\alpha \geq 0$, this result follows from the fact that if there is $R > 0$ such that $\sup_{j,k} 2^{\alpha j} |c_{j,k}| \leq R$ then $|c_{j',k'}| \leq 2^{-\alpha j'} R \leq 2^{-\alpha j} R$ for every $j' \geq j, k' \in \{0, \dots, 2^{j'} - 1\}$. Hence, we get $e_{j,k} \leq R2^{-\alpha j}$ for every $j \in \mathbb{N}_0$.

2. If $\beta \geq 1$, then $\tilde{A}^+(\alpha, \beta) = \Omega$.

Let us now define a distance on these auxiliary spaces.

Definition 6.4.4. Let $\alpha \geq 0$ and $\beta \in \{-\infty\} \cup [0, +\infty)$. For $\vec{c}, \vec{c}' \in \tilde{A}^+(\alpha, \beta)$, we write

$$\tilde{\delta}_{\alpha,\beta}^+(\vec{c}, \vec{c}') := \inf \left\{ C + C' : C, C' \geq 0 \text{ and } \#\tilde{E}_j^+(C, \alpha)(\vec{c} - \vec{c}') \leq C'2^{\beta j}, \forall j \in \mathbb{N}_0 \right\}.$$

Lemma 6.4.5. [25] For every $\alpha \geq 0$ and $\beta \in \{-\infty\} \cup [0, +\infty)$, $\tilde{\delta}_{\alpha,\beta}^+$ is a distance on $\tilde{A}^+(\alpha, \beta)$ which is invariant by translation and which satisfies

$$\tilde{\delta}_{\alpha,\beta}^+(\theta\vec{c}, \vec{0}) \leq \sup\{1, |\theta|\} \tilde{\delta}_{\alpha,\beta}^+(\vec{c}, \vec{0})$$

for all $\vec{c} \in \tilde{A}^+(\alpha, \beta)$ and $\theta \in \mathbb{C}$.

Proof. If $\beta = -\infty$, it is immediate to check that $(\tilde{A}^+(\alpha, \beta), \tilde{\delta}_{\alpha,\beta}^+)$ is the normed space $(C^\alpha, \|\cdot\|_{C^\alpha})$ and all properties are then satisfied. So, let us assume that $\beta \geq 0$. By definition, it is clear that $\tilde{\delta}_{\alpha,\beta}^+$ is invariant by translation. Moreover, if $|\theta| \leq 1$, one has $\#\tilde{E}_j^+(C, \alpha)(\theta\vec{c}) \leq \#\tilde{E}_j^+(C, \alpha)(\vec{c})$ so that $\tilde{\delta}_{\alpha,\beta}^+(\theta\vec{c}, \vec{0}) \leq \tilde{\delta}_{\alpha,\beta}^+(\vec{c}, \vec{0})$. If $\theta > 1$, $\#\tilde{E}_j^+(C, \alpha)(\vec{c}) \leq \#\tilde{E}_j^+(|\theta|C, \alpha)(\theta\vec{c})$ and it follows that $\tilde{\delta}_{\alpha,\beta}^+(\theta\vec{c}, \vec{0}) \leq |\theta| \tilde{\delta}_{\alpha,\beta}^+(\vec{c}, \vec{0})$.

Let us now show that $\tilde{\delta}_{\alpha,\beta}^+$ is a distance on $\tilde{A}^+(\alpha, \beta)$.

- The positivity and the symmetry of $\tilde{\delta}_{\alpha,\beta}^+$ are immediate.

- Assume that $\tilde{\delta}_{\alpha,\beta}^+(\vec{c}, \vec{0}) = 0$. Let us consider $0 < \varepsilon < 1$. Then there is $C, C' > 0$ such that $C + C' \leq \varepsilon$ and

$$\#\tilde{E}_j^+(C, \alpha)(\vec{c}) \leq C'2^{\beta j}, \quad \forall j \in \mathbb{N}_0.$$

In particular, for $j = 0$,

$$\#\tilde{E}_0^+(C, \alpha)(\vec{c}) \leq C' \leq \varepsilon < 1$$

so that $e_{0,0} = \sup_{\lambda \in \Lambda} |c_\lambda| < C2^{-\alpha} \leq \varepsilon 2^{-\alpha}$. Since $0 < \varepsilon < 1$ is arbitrary, we get $c_\lambda = 0$ for every $\lambda \in \Lambda$.

- Let us now prove the triangle inequality. Because of the invariance by translation, it suffices to show that

$$\tilde{\delta}_{\alpha,\beta}^+(\vec{c} - \vec{c}', \vec{0}) \leq \tilde{\delta}_{\alpha,\beta}^+(\vec{c}, \vec{0}) + \tilde{\delta}_{\alpha,\beta}^+(\vec{c}', \vec{0})$$

Let us fix $\eta > 0$. Then there are $C_1, C'_1, C_2, C'_2 > 0$ such that

$$C_1 + C'_1 < \tilde{\delta}_{\alpha,\beta}^+(\vec{c}, \vec{0}) + \eta, \quad C_2 + C'_2 < \tilde{\delta}_{\alpha,\beta}^+(\vec{c}', \vec{0}) + \eta$$

and

$$\#\tilde{E}_j^+(C_1, \alpha)(\vec{c}) \leq C'_1 2^{\beta j}, \quad \#\tilde{E}_j^+(C_2, \alpha)(\vec{c}') \leq C'_2 2^{\beta j}$$

for all $j \in \mathbb{N}_0$. If $\lambda \in \Lambda_j$ does not belong to $\tilde{E}_j^+(C_1, \alpha)(\vec{c}) \cup \tilde{E}_j^+(C_2, \alpha)(\vec{c}')$, then

$$\sup_{\lambda' \subseteq \lambda} |c_{\lambda'} - c'_{\lambda'}| \leq \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| + \sup_{\lambda' \subseteq \lambda} |c'_{\lambda'}| < (C_1 + C_2)2^{-\alpha j}$$

and consequently, for every $j \in \mathbb{N}_0$,

$$\#\tilde{E}_j^+(C_1 + C_2, \alpha)(\vec{c} - \vec{c}') \leq \#\tilde{E}_j^+(C_1, \alpha)(\vec{c}) + \#\tilde{E}_j^+(C_2, \alpha)(\vec{c}') \leq (C'_1 + C'_2)2^{\beta j}.$$

The triangle inequality follows. □

Remark 6.4.6. It is direct to check that the distance defined by

$$\tilde{\delta}_{\alpha,\beta}^{+,*}(\vec{c}, \vec{c}') := \inf \left\{ C \geq 0 : \#\tilde{E}_j^+(C, \alpha)(\vec{c} - \vec{c}') \leq C2^{\beta j}, \forall j \in \mathbb{N}_0 \right\}$$

leads to the same topology.

In the following proposition, we get that the topology defined by $\tilde{\delta}_{\alpha,\beta}^+$ is stronger than the *uniform topology*, i.e. the topology defined by the norm of C^0 . The equivalence with the uniform topology happens when $\beta > 1$.

Proposition 6.4.7. [25] *Let $\alpha \geq 0$ and $\beta \in \{-\infty\} \cup [0, +\infty)$.*

1. *The addition is continuous on $(\tilde{A}^+(\alpha, \beta), \tilde{\delta}_{\alpha,\beta}^+)$.*
2. *The space $(\tilde{A}^+(\alpha, \beta), \tilde{\delta}_{\alpha,\beta}^+)$ has a stronger topology than the uniform topology. Moreover, every Cauchy sequence in $(\tilde{A}^+(\alpha, \beta), \tilde{\delta}_{\alpha,\beta}^+)$ is also a uniform Cauchy sequence.*

3. If $\beta > 1$, the topology defined by the distance $\tilde{\delta}_{\alpha,\beta}^+$ is equivalent to the uniform topology.

4. (a) If B is a bounded set of $(\tilde{A}^+(\alpha, \beta), \tilde{\delta}_{\alpha,\beta}^+)$, then there exists $r > 0$ such that

$$\begin{aligned} B &\subseteq \{ \vec{c} \in \Omega : \#\{\lambda \in \Lambda_j : e_\lambda \geq r 2^{-\alpha j}\} \leq r 2^{\beta j}, \forall j \in \mathbb{N}_0 \} \\ &\subseteq \{ \vec{c} \in \Omega : \#\{\lambda \in \Lambda_j : e_\lambda > r 2^{-\alpha j}\} \leq r 2^{\beta j}, \forall j \in \mathbb{N}_0 \}. \end{aligned}$$

(b) Let $r, r' \geq 0$, $\alpha' \geq \alpha$ and $\beta' \leq \beta$. The set

$$B = \left\{ \vec{c} \in \Omega : \#\{\lambda \in \Lambda_j : e_\lambda > r 2^{-\alpha' j}\} \leq r' 2^{\beta' j}, \forall j \in \mathbb{N}_0 \right\}$$

is a bounded set of $(\tilde{A}^+(\alpha, \beta), \tilde{\delta}_{\alpha,\beta}^+)$. Moreover, B is closed for the uniform convergence.

Proof. 1. The first point is obvious using the triangular inequality and the invariance by translation of the distance $\tilde{\delta}_{\alpha,\beta}^+$.

2. Let $(\vec{c}^{(m)})_{m \in \mathbb{N}}$ be a sequence of elements of $\tilde{A}^+(\alpha, \beta)$ which converges to \vec{c} in $(\tilde{A}^+(\alpha, \beta), \tilde{\delta}_{\alpha,\beta}^+)$. If $\beta = -\infty$, it suffices to observe that C^α is included continuously in the space C^0 . Let us consider now the case $\beta \geq 0$. Let $\varepsilon > 0$ and $\eta := \min\{\frac{1}{2}, \varepsilon\}$. There exists $M \in \mathbb{N}$ such that

$$\#\left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}^{(m)} - c_{\lambda'}| \geq \eta 2^{-\alpha j} \right\} \leq \eta 2^{\beta j}$$

for all $j \in \mathbb{N}_0$ and $m \geq M$. Consequently, taking $j = 0$, we obtain for all $m \geq M$,

$$\sup_{\lambda \in \Lambda} |c_\lambda^{(m)} - c_\lambda| < \eta \leq \varepsilon.$$

The proof is similar for the Cauchy sequences.

3. From the previous point, it only remains to show that, if $\beta > 1$, the uniform topology is stronger than the topology defined by the distance $\tilde{\delta}_{\alpha,\beta}^+$. Let $(\vec{c}^{(m)})_{m \in \mathbb{N}}$ be a sequence of $\tilde{A}^+(\alpha, \beta) = \Omega$ which converges uniformly to \vec{c} and let $\varepsilon > 0$. Since $\beta > 1$, there exists $J \in \mathbb{N}_0$ such that $2^j \leq \varepsilon 2^{\beta j}$ for every $j > J$ and then we have

$$\#\left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}^{(m)} - c_{\lambda'}| \geq \varepsilon 2^{-\alpha j} \right\} \leq 2^j \leq \varepsilon 2^{\beta j}$$

for every $j > J$ and $m \in \mathbb{N}$. Using the uniform convergence, there exists $M \in \mathbb{N}$ such that

$$\sup_{\lambda' \subseteq \lambda} |c_{\lambda'}^{(m)} - c_{\lambda'}| < \varepsilon 2^{-\alpha j}$$

for every $\lambda \in \Lambda_j$ with $j \in \{0, \dots, J\}$ and $m \geq M$. So, for every $m \geq M$, we have

$$\#\left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}^{(m)} - c_{\lambda'}| \geq \varepsilon 2^{-\alpha j} \right\} = 0 \leq \varepsilon 2^{\beta j}, \quad \forall j \in \{0, \dots, J\}.$$

It follows that $(\vec{c}^{(m)})_{m \in \mathbb{N}}$ converges to \vec{c} in $(\tilde{A}^+(\alpha, \beta), \tilde{\delta}_{\alpha,\beta}^+)$.

4. The properties related to the boundedness are immediate. Let us show that B is closed for the uniform convergence. Let $(\vec{c}^{(m)})_{m \in \mathbb{N}}$ be a sequence of B which converges uniformly to \vec{c} and let $\varepsilon > 0$. Then there exists $M \in \mathbb{N}$ such that

$$\sup_{\lambda \in \Lambda} |c_\lambda^{(m)} - c_\lambda| < \varepsilon$$

for all $m \geq M$. Let us fix $j \in \mathbb{N}_0$ and $\lambda \in \Lambda_j$. We have

$$e_\lambda > r2^{-\alpha'j} \quad \Rightarrow \quad e_\lambda^{(M)} > r2^{-\alpha'j}.$$

Otherwise, $e_\lambda^{(M)} \leq r2^{-\alpha'j}$ and then, by taking ε smaller if needed, we have

$$r2^{-\alpha'j} < e_\lambda - \varepsilon \leq \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}^{(M)} - c_{\lambda'}| + e_\lambda^{(M)} - \varepsilon \leq r2^{-\alpha'j},$$

which is impossible. So, we have

$$\#\{\lambda \in \Lambda : e_\lambda > r2^{-\alpha'j}\} \leq \#\{\lambda \in \Lambda : e_\lambda^{(M)} > r2^{-\alpha'j}\} \leq r'2^{\beta'j}, \quad \forall j \in \mathbb{N}_0$$

and $\vec{c} \in B$. □

Remark 6.4.8. If $\beta \in [0, 1]$ and $\alpha > 0$, the scalar multiplication

$$(\theta, \vec{c}) \in \mathbb{C} \times \tilde{A}^+(\alpha, \beta) \mapsto \theta \vec{c} \in \tilde{A}^+(\alpha, \beta)$$

is not continuous and consequently, the space $(\tilde{A}^+(\alpha, \beta), \tilde{\delta}_{\alpha, \beta}^+)$ is not a topological vector space. Indeed, let \vec{c} be the sequence of $\tilde{A}^+(\alpha, \beta)$ defined by

$$c_{j,k} := \begin{cases} j2^{-\alpha j} & \text{if } k \in \{0, \dots, \lfloor 2^{\beta j} \rfloor - 1\}, \\ 0 & \text{if } k \in \{\lfloor 2^{\beta j} \rfloor, \dots, 2^j - 1\}, \end{cases}$$

for $j \in \mathbb{N}_0$. For large j , we have $\lfloor 2^{\beta(j+1)} \rfloor / 2 \leq \lfloor 2^{\beta j} \rfloor$ and then $e_{j,k} = c_{j,k}$ for every $k \in \{0, \dots, 2^j - 1\}$. Following the proof of Proposition 3.5 in [13], the sequence $(\vec{c}/m)_{m \in \mathbb{N}}$ does not converge to $\vec{0}$ in $(\tilde{A}^+(\alpha, \beta), \tilde{\delta}_{\alpha, \beta}^+)$. Indeed, otherwise, one could find $M \in \mathbb{N}$ such that

$$\#\left\{\lambda \in \Lambda_j : \left| \frac{c_\lambda}{m} \right| \geq \frac{1}{2}2^{-\alpha j}\right\} \leq \frac{1}{2}2^{\beta j}, \quad \forall j \in \mathbb{N}_0, m \geq M.$$

Taking $j = m$, we get

$$\#\left\{\lambda \in \Lambda_j : \left| \frac{c_\lambda}{m} \right| \geq \frac{1}{2}2^{-\alpha m}\right\} \leq \frac{1}{2}2^{\beta m}$$

which leads to a contradiction if m is large enough. This counterexample also shows that the topology defined by $\tilde{\delta}_{\alpha, \beta}^+$ and the uniform topology are not equivalent for such α and β .

Proposition 6.4.9. [25] *The space $(\tilde{A}^+(\alpha, \beta), \tilde{\delta}_{\alpha, \beta}^+)$ is a complete metric space.*

Proof. Let $(\vec{c}^{(m)})_{m \in \mathbb{N}}$ be a Cauchy sequence in $(\tilde{A}^+(\alpha, \beta), \tilde{\delta}_{\alpha, \beta}^+)$. From the point 2 of Proposition 6.4.7, $(\vec{c}^{(m)})_{m \in \mathbb{N}}$ is also a uniform Cauchy sequence and thus it converges

uniformly. Let us denote by \vec{c} this limit. By assumption, if $\eta > 0$, there exists $M \in \mathbb{N}$ such that

$$\# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}^{(p)} - c_{\lambda'}^{(q)}| > \eta 2^{-\alpha j} \right\} \leq \eta 2^{\beta j}, \quad \forall j \in \mathbb{N}_0, \forall p, q \geq M.$$

It follows from the point 4 of Proposition 6.4.7 that we have

$$\# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}^{(p)} - c_{\lambda'}| > \eta 2^{-\alpha j} \right\} \leq \eta 2^{\beta j}, \quad \forall j \in \mathbb{N}_0, \forall p \geq M$$

and the conclusion follows. \square

Let us end with some relations between auxiliary spaces. The second part will be useful to obtain the continuity of the scalar multiplication in $\mathcal{L}^{\nu,+}$.

Lemma 6.4.10. [25]

1. If $\alpha \geq \alpha'$ and $\beta \leq \beta'$, then

$$\tilde{A}^+(\alpha, \beta) \subseteq \tilde{A}^+(\alpha', \beta') \quad \text{and} \quad \tilde{\delta}_{\alpha', \beta'}^+ \leq \tilde{\delta}_{\alpha, \beta}^+.$$

2. Let $\alpha' > \alpha$ and $\beta' < \beta$. If the sequence $(\theta_m)_{m \in \mathbb{N}}$ converges to θ in \mathbb{C} and if the sequence $(\vec{c}^{(m)})_{m \in \mathbb{N}}$ converges to \vec{c} in $(\tilde{A}^+(\alpha, \beta), \tilde{\delta}_{\alpha, \beta}^+)$ with $\vec{c} \in \tilde{A}^+(\alpha', \beta')$, then the sequence $(\theta_m \vec{c}^{(m)})_{m \in \mathbb{N}}$ converges to $\theta \vec{c}$ in $(\tilde{A}^+(\alpha, \beta), \tilde{\delta}_{\alpha, \beta}^+)$.

Proof. The first item is obvious. Let us prove the second one. Since the sequence $(\theta_m)_{m \in \mathbb{N}}$ converges to θ in \mathbb{C} , there exists $D > 0$ such that $|\theta_m - \theta| \leq D$ for all $m \in \mathbb{N}$. We have

$$\theta_m \vec{c}^{(m)} - \theta \vec{c} = (\theta_m - \theta)(\vec{c}^{(m)} - \vec{c}) + \theta(\vec{c}^{(m)} - \vec{c}) + (\theta_m - \theta)\vec{c}$$

and then

$$\tilde{\delta}_{\alpha, \beta}^+(\theta_m \vec{c}^{(m)}, \theta \vec{c}) \leq \sup\{1, D\} \tilde{\delta}_{\alpha, \beta}^+(\vec{c}^{(m)}, \vec{c}) + \sup\{1, |\theta|\} \tilde{\delta}_{\alpha, \beta}^+(\vec{c}^{(m)}, \vec{c}) + \tilde{\delta}_{\alpha, \beta}^+((\theta_m - \theta)\vec{c}, \vec{0})$$

thanks to Lemma 6.4.5. The two first terms converge to 0 by assumption. Let us consider the convergence of the third term. Since $\vec{c} \in \tilde{A}^+(\alpha', \beta')$, there exist $C, C' \geq 0$ such that

$$\# \left\{ \lambda \in \Lambda_j : e_\lambda \geq C 2^{-\alpha' j} \right\} \leq C' 2^{\beta' j}, \quad \forall j \in \mathbb{N}_0.$$

Let us consider $\eta > 0$. Then there exists $J \in \mathbb{N}_0$ such that $DC 2^{-j(\alpha' - \alpha)} \leq \eta$ and $C' 2^{-j(\beta - \beta')} \leq \eta$ for every $j \geq J$. Consequently, we have

$$\# \left\{ \lambda \in \Lambda_j : |\theta_m - \theta| e_\lambda \geq \eta 2^{-\alpha j} \right\} \leq \eta 2^{\beta j}, \quad \forall j \geq J, m \in \mathbb{N},$$

since $|\theta_m - \theta| \leq D$ for every $m \in \mathbb{N}$. Moreover, since the sequence $(\theta_m)_{m \in \mathbb{N}}$ converges to θ , there exists $M \in \mathbb{N}$ such that

$$|\theta_m - \theta| e_\lambda < \eta 2^{-\alpha j}$$

for every $m \geq M$, $j \in \{0, \dots, J-1\}$ and $\lambda \in \Lambda_j$. Hence $\tilde{\delta}_{\alpha, \beta}^+((\theta_m - \theta)\vec{c}, \vec{0}) \leq 2\eta$ for every $m \geq M$ and we get the conclusion. \square

Remark 6.4.11. Of course, if $\beta = \beta' = -\infty$, this lemma remains true.

6.4.2 Topology on $\mathcal{L}^{\nu,+}$ spaces

Let us now present the connection between $\mathcal{L}^{\nu,+}$ and the auxiliary spaces $\tilde{A}^+(\alpha, \beta)$. We will use it to endow $\mathcal{L}^{\nu,+}$ with a distance.

Proposition 6.4.12. [25] *For any dense sequence $\alpha = (\alpha_n)_{n \in \mathbb{N}}$ in $[0, \alpha_s]$ and any sequence $\varepsilon = (\varepsilon_m)_{m \in \mathbb{N}}$ of $(0, +\infty)$ which converges to 0, we have*

$$\mathcal{L}^{\nu,+} = \bigcap_{\varepsilon > 0} \bigcap_{\alpha \in [0, \alpha_s]} \tilde{A}^+(\alpha, \nu(\alpha) + \varepsilon) = \bigcap_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \tilde{A}^+(\alpha_n, \nu(\alpha_n) + \varepsilon_m).$$

Proof. Remark that a sequence \vec{c} belongs to $\mathcal{L}^{\nu,+}$ (resp. $\tilde{A}^+(\alpha, \beta)$) if and only if \vec{c} belongs to \mathcal{S}^ν (resp. $A(\alpha, \beta)$). The result follows then from Proposition 4.6.3. For the convenience of the reader, let us nevertheless develop it, independently of \mathcal{S}^ν . It is direct to see that

$$\mathcal{L}^{\nu,+} \subseteq \bigcap_{\varepsilon > 0} \bigcap_{\alpha \in [0, \alpha_s]} \tilde{A}^+(\alpha, \nu(\alpha) + \varepsilon) \subseteq \bigcap_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \tilde{A}^+(\alpha_n, \nu(\alpha_n) + \varepsilon_m).$$

So, let us consider

$$\vec{c} \in \bigcap_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \tilde{A}^+(\alpha_n, \nu(\alpha_n) + \varepsilon_m)$$

and let us fix $\alpha \in [0, \alpha_s]$, $\varepsilon > 0$ and $C > 0$. We will show that there is $J \in \mathbb{N}$ such that

$$\#\tilde{E}_j^+(C, \alpha)(\vec{c}) \leq 2^{(\nu(\alpha) + \varepsilon)j}, \quad \forall j \geq J.$$

If $\nu(\alpha) = -\infty$, then there is $\alpha_n \geq \alpha$ such that $\nu(\alpha_n) = -\infty$. It follows that

$$\vec{c} \in \tilde{A}^+(\alpha_n, \nu(\alpha_n) + \varepsilon_m) = C^{\alpha_n} \subseteq C^\alpha = \tilde{A}^+(\alpha, \nu(\alpha) + \varepsilon).$$

Assume now that $\nu(\alpha) \geq 0$. Using the right-continuity of ν , there are $n, m \in \mathbb{N}$ such that $\alpha_n > \alpha$, $3\varepsilon_m \leq \varepsilon$ and $\nu(\alpha) \leq \nu(\alpha_n) \leq \nu(\alpha) + \varepsilon_m$. Since $\vec{c} \in \tilde{A}^+(\alpha_n, \nu(\alpha_n) + \varepsilon_m)$, there are $C_0, C'_0 > 0$ such that

$$\#\tilde{E}_j^+(C_0, \alpha_n)(\vec{c}) \leq C'_0 2^{(\nu(\alpha_n) + \varepsilon_m)j}, \quad \forall j \in \mathbb{N}_0.$$

Let us fix $J \in \mathbb{N}$ such that $C_0 2^{-\alpha_n j} \leq C 2^{-\alpha j}$ and $C'_0 \leq 2^{\frac{5}{3}j}$ for all $j \geq J$. Consequently,

$$\#\tilde{E}_j^+(C, \alpha)(\vec{c}) \leq \#\tilde{E}_j^+(C_0, \alpha_n)(\vec{c}) \leq C'_0 2^{(\nu(\alpha_n) + \varepsilon_m)j} \leq 2^{(\nu(\alpha) + \varepsilon)j}$$

for every $j \geq J$, which concludes the proof. \square

As in the case of \mathcal{S}^ν spaces, this description allows to obtain a structure of complete metric space on $\mathcal{L}^{\nu,+}$. Let us recall the following classical result (see [95] for example).

Proposition 6.4.13. *Let E_m ($m \in \mathbb{N}$) be spaces endowed with the topologies defined by the distances d_m and set $E = \bigcap_{m \in \mathbb{N}} E_m$. On E , let us consider the topology τ defined as follows: for every $e \in E$, a basis of neighborhoods of e is given by the family of sets*

$$\bigcap_{(m)} \{f \in E : d_m(e, f) \leq r_m\}$$

where $r_m > 0$ for every $m \in \mathbb{N}$ and (m) means that it is an intersection on a finite number of values of m . Then, this topology satisfies the following properties.

1. For every $m \in \mathbb{N}$, the identity $i : (E, \tau) \rightarrow (E_m, d_m)$ is continuous and τ is the weakest topology on E which verifies this property.
2. The topology τ is equivalent to the topology defined on E by the distance d given by

$$d(e, f) := \sum_{m=1}^{+\infty} 2^{-m} \frac{d_m(e, f)}{1 + d_m(e, f)}, \quad e, f \in E.$$

3. A sequence is a Cauchy sequence in (E, τ) if and only if it is a Cauchy sequence in (E_m, d_m) for every $m \in \mathbb{N}$.
4. A sequence converges to e in (E, τ) if and only if it converges to e in (E_m, d_m) for every $m \in \mathbb{N}$.

Using this Proposition 6.4.13, we can define a distance on the spaces $\mathcal{L}^{\nu,+}$.

Definition 6.4.14. Let $\alpha := (\alpha_n)_{n \in \mathbb{N}}$ be a dense sequence in $[0, \alpha_s]$ and $\varepsilon := (\varepsilon_m)_{m \in \mathbb{N}}$ be a sequence of $(0, +\infty)$ which converges to 0. For $m, n \in \mathbb{N}$, we write

$$\tilde{\delta}_{m,n}^+ := \tilde{\delta}_{\alpha_n, \nu(\alpha_n) + \varepsilon_m}^+ \quad \text{and} \quad \tilde{A}^+(m, n) := \tilde{A}^+(\alpha_n, \nu(\alpha_n) + \varepsilon_m).$$

Then, for $m \in \mathbb{N}$, we denote

$$\tilde{\delta}_m^+ := \sum_{n=1}^{+\infty} 2^{-n} \frac{\tilde{\delta}_{m,n}^+}{1 + \tilde{\delta}_{m,n}^+} \quad \text{and} \quad \tilde{\delta}_{\alpha, \varepsilon}^+ := \sum_{m=1}^{+\infty} 2^{-m} \tilde{\delta}_m^+.$$

Proposition 6.4.15. [25] For every sequences α and ε chosen as above, $\tilde{\delta}_{\alpha, \varepsilon}^+$ is a distance on $\mathcal{L}^{\nu,+}$. All these distances define the same topology.

Proof. Thanks to Lemma 6.4.5 and Proposition 6.4.13, it is clear that $\tilde{\delta}_{\alpha, \varepsilon}^+$ is a distance on $\mathcal{L}^{\nu,+}$. In order to show that all these distances define the same topology, it is sufficient to prove that if $(\tilde{c}^{(m)})_{m \in \mathbb{N}}$ is a sequence of $\mathcal{L}^{\nu,+}$ which converges to \tilde{c} for the distance $\tilde{\delta}_{\alpha, \varepsilon}^+$, then for every $\alpha \in [0, \alpha_s]$ and every $\varepsilon > 0$, the sequence $(\tilde{c}^{(m)})_{m \in \mathbb{N}}$ converges to \tilde{c} for the distance $\tilde{\delta}_{\alpha, \nu(\alpha) + \varepsilon}^+$.

If $\nu(\alpha) = -\infty$, we can find $\alpha_n \geq \alpha$ such that $\nu(\alpha_n) = -\infty$. It follows then directly that $\tilde{\delta}_{\alpha, \nu(\alpha) + \varepsilon}^+ \leq \tilde{\delta}_{m,n}^+$.

If $\nu(\alpha) \geq 0$, let $m, n \in \mathbb{N}$ be such that $2\varepsilon_m \leq \varepsilon$, $\alpha_n \geq \alpha$ and $\nu(\alpha_n) < \nu(\alpha) + \varepsilon_m$. Then, we have $\nu(\alpha_n) + \varepsilon_m < \nu(\alpha) + \varepsilon$ and using Lemma 6.4.10, we get $\tilde{\delta}_{\alpha, \nu(\alpha) + \varepsilon}^+ \leq \tilde{\delta}_{m,n}^+$. \square

In view of this result, we write this distance $\tilde{\delta}^+$ independently of these sequences α and ε . In fact, this result can be seen as a direct consequence of the closed graph theorem since the distance defines a complete topological vector space, as we will see in Propositions 6.4.16 and 6.4.18.

Agreement. From now on, $\alpha := (\alpha_n)_{n \in \mathbb{N}}$ denotes a dense sequence in $[0, \alpha_s]$ and $\varepsilon := (\varepsilon_m)_{m \in \mathbb{N}}$ denotes a sequence of $(0, +\infty)$ which converges to 0.

As a direct consequence of Proposition 6.4.13, we get the following result.

Proposition 6.4.16. [25]

1. The topology defined by $\tilde{\delta}^+$ on $\mathcal{L}^{\nu,+}$ is the weakest topology such that, for every $m, n \in \mathbb{N}$, the identity $i : \mathcal{L}^{\nu,+} \rightarrow \tilde{A}^+(m, n)$ is continuous.

2. A sequence of $\mathcal{L}^{\nu,+}$ is a Cauchy sequence in $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$ if and only if, for every $m, n \in \mathbb{N}$, it is a Cauchy sequence in $(\tilde{A}^+(m, n), \tilde{\delta}_{m,n}^+)$.
3. A sequence of $\mathcal{L}^{\nu,+}$ converges in $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$ if and only if, for every $m, n \in \mathbb{N}$, it converges in $(\tilde{A}^+(m, n), \tilde{\delta}_{m,n}^+)$.

Proposition 6.4.17. [25] *The space $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$ is a complete topological vector space.*

Proof. From Proposition 6.4.7, we know that the addition is continuous on each of the spaces $\tilde{A}^+(m, n)$. Using Proposition 6.4.16, it is also continuous on $\mathcal{L}^{\nu,+}$. Let us prove that the scalar multiplication in $\mathcal{L}^{\nu,+}$ is continuous. Let $(\theta_l)_{l \in \mathbb{N}}$ be a sequence in \mathbb{C} which converges to $\theta \in \mathbb{C}$, and let $(\tilde{c}^{(l)})_{l \in \mathbb{N}}$ be a sequence of $\mathcal{L}^{\nu,+}$ which converges to \tilde{c} in $\mathcal{L}^{\nu,+}$. From the properties of $\tilde{\delta}$, it suffices to prove that $(\lambda^l \tilde{c}^{(l)})_{l \in \mathbb{N}}$ converges to $\lambda \tilde{c}$ in $\tilde{A}^+(\alpha, \nu(\alpha) + \varepsilon)$ for every $\alpha \in [0, \alpha_s]$ and every $\varepsilon > 0$. If $\nu(\alpha) = -\infty$, it is immediate since $\tilde{A}^+(\alpha, \nu(\alpha) + \varepsilon) = C^\alpha$ as normed spaces. Assume that $\nu(\alpha) \geq 0$. Using the right-continuity of ν , let us consider $m, n \in \mathbb{N}$ such that $\alpha_n > \alpha$ and $\nu(\alpha_n) + \varepsilon_m < \nu(\alpha) + \varepsilon$. The result follows then directly from Lemma 6.4.10 and Proposition 6.4.12.

Let us now prove that the space $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$ is complete. Let $(\tilde{c}^{(l)})_{l \in \mathbb{N}}$ be a Cauchy sequence in $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$. From Proposition 6.4.16, it is a Cauchy sequence in the spaces $(\tilde{A}^+(m, n), \tilde{\delta}_{m,n}^+)$ for every $m, n \in \mathbb{N}$. We know from Proposition 6.4.9 that these spaces are complete and thus, there exists $\tilde{c}_{m,n} \in \tilde{A}^+(m, n)$ such that $(\tilde{c}^{(l)})_{l \in \mathbb{N}}$ converges to $\tilde{c}_{m,n}$ in $(\tilde{A}^+(m, n), \tilde{\delta}_{m,n}^+)$. By Proposition 6.4.7, we get that $\tilde{c}_{m,n} = \tilde{c}$ is unique since it is the uniform limit of $(\tilde{c}^{(l)})_{l \in \mathbb{N}}$. The conclusion follows using again Proposition 6.4.16. \square

Proposition 6.4.18. [25] *If $\tilde{\delta}_1^+$ and $\tilde{\delta}_2^+$ define complete topologies on $\mathcal{L}^{\nu,+}$ which are stronger than the pointwise topology, then these topologies are equivalent.*

Proof. It is a direct consequence of the closed graph theorem and Proposition 6.4.16. \square

Remark 6.4.19. Let us note that the inclusion $\mathcal{L}^{\nu,+} \subseteq C^0$ is continuous by combining Proposition 6.4.16 (item 3) and Proposition 6.4.7 (item 2).

As already proved before, the definition of the space $\mathcal{L}^{\nu,+}$ does not depend on the chosen wavelet basis. Therefore, it can be seen as a function space. Let us now show that the topology defined on this space is also a “good topology”, in the sense that it is also independent of the chosen wavelet basis. This allows to consider the space $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$ as a topological function space.

Proposition 6.4.20. *Let A be a quasisiagonal matrix. If $\alpha_{\min} > 0$, the application*

$$A : (\mathcal{L}^{\nu,+}, \tilde{\delta}^+) \rightarrow (\mathcal{L}^{\nu,+}, \tilde{\delta}^+) : \tilde{c} \mapsto A\tilde{c}$$

is continuous.

Proof. The result of robustness of Proposition 5.3.7 ensures that A maps $\mathcal{L}^{\nu,+}$ into $\mathcal{L}^{\nu,+}$. As the operator A is a linear operator between complete metrizable topological vector spaces whose topologies are stronger than the pointwise topology, the continuity is obtained using the closed graph theorem. \square

6.4.3 Compact subsets of $\mathcal{L}^{\nu,+}$ spaces

Let us continue with the characterization of compact subsets of $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$. It will only hold if $\alpha_{\min} > 0$. Let us start with some observations useful to obtain this characterization.

Lemma 6.4.21. [25]

1. Let $\alpha > 0$ and let B be a bounded set of $(C^\alpha, \|\cdot\|_{C^\alpha})$. If $(\vec{c}^{(l)})_{l \in \mathbb{N}}$ is a sequence of B which converges pointwise to \vec{c} , then it converges uniformly to \vec{c} .
2. Let $\alpha_0 > 0$, $\beta_0 \geq 0$ and let B be a bounded set of $(\tilde{A}^+(\alpha_0, \beta_0), \tilde{\delta}_{\alpha_0, \beta_0}^+)$. If $(\vec{c}^{(l)})_{l \in \mathbb{N}}$ is a sequence of B which converges uniformly to \vec{c} , then it converges to \vec{c} in $(\tilde{A}^+(\alpha, \beta), \tilde{\delta}_{\alpha, \beta}^+)$ for all α and β such that $\alpha < \alpha_0$ and $\beta > \beta_0$.
3. Let $\alpha_0 > 0$ and let B be a bounded set of $(C^{\alpha_0}, \|\cdot\|_{C^{\alpha_0}})$. If $(\vec{c}^{(l)})_{l \in \mathbb{N}}$ is a sequence of B which converges uniformly to \vec{c} , then it converges to \vec{c} in $(C^\alpha, \|\cdot\|_{C^\alpha})$ for all $\alpha < \alpha_0$.

Proof. 1. By assumption, there exists $R > 0$ such that $|c_{j,k}^{(l)} - c_{j,k}| \leq R2^{-\alpha j}$ for every $j \in \mathbb{N}_0$, $k \in \{0, \dots, 2^j - 1\}$ and every $l \in \mathbb{N}$. Let $\eta > 0$. On one hand, since $\alpha > 0$, there exists $J \in \mathbb{N}_0$ such that $R2^{-\alpha j} < \eta$ for every $j > J$ and then

$$|c_{j,k}^{(l)} - c_{j,k}| < \eta, \quad \forall l \in \mathbb{N}, j > J, k \in \{0, \dots, 2^j - 1\}.$$

On the other hand, thanks to the pointwise convergence, there exists $L \in \mathbb{N}$ (which only depends on η) such that

$$\sup_{j \in \{0, \dots, J\}} \sup_{k \in \{0, \dots, 2^j - 1\}} |c_{j,k}^{(l)} - c_{j,k}| < \eta, \quad \forall l \geq L.$$

Thus

$$\sup_{j \in \mathbb{N}_0} \sup_{k \in \{0, \dots, 2^j - 1\}} |c_{j,k}^{(l)} - c_{j,k}| < \eta, \quad \forall l \geq L.$$

2. Since the sequence $(\vec{c}^{(l)} - \vec{c})_{l \in \mathbb{N}}$ is bounded in $(\tilde{A}^+(\alpha_0, \beta_0), \tilde{\delta}_{\alpha_0, \beta_0}^+)$, there exist $R, R' \geq 0$ such that

$$\#\left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}^{(l)} - c_{\lambda'}| > R2^{-\alpha_0 j} \right\} \leq R' 2^{\beta_0 j}, \quad \forall j \in \mathbb{N}_0, l \in \mathbb{N},$$

using Lemma 6.4.7 (item 4). Let $\eta > 0$. Since $\alpha < \alpha_0$ and $\beta > \beta_0$, there exists $J \in \mathbb{N}_0$ such that $R2^{-\alpha_0 j} < \eta 2^{-\alpha j}$ and $R' 2^{\beta_0 j} < \eta 2^{\beta j}$ for every $j > J$ and then

$$\#\left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}^{(l)} - c_{\lambda'}| \geq \eta 2^{-\alpha j} \right\} \leq \eta 2^{\beta j}, \quad \forall l \in \mathbb{N}, j > J.$$

Moreover, thanks to the uniform convergence, there exists $L \in \mathbb{N}$ (which only depends on η) such that

$$\sup_{\lambda' \subseteq \lambda} |c_{\lambda'}^{(l)} - c_{\lambda'}| < \eta 2^{-\alpha j}, \quad \forall j \in \{0, \dots, J\}, \lambda \in \Lambda_j, l \geq L$$

and then

$$\#\left\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}^{(l)} - c_{\lambda'}| \geq \eta 2^{-\alpha j}\right\} = 0 \leq \eta 2^{\beta j}, \quad \forall j \in \{0, \dots, J\}, l \geq L.$$

Thus, we have $\tilde{\delta}_{\alpha,\beta}^+(\vec{c}^{(l)}, \vec{c}) \leq 2\eta$ for every $l \geq L$.

3. The proof of this item is similar to the two previous ones. \square

Lemma 6.4.22. [25] *Assume that $\alpha_{\min} > 0$. For $m, n \in \mathbb{N}$, let $C(m, n)$ and $C'(m, n)$ be positive constants and let us consider the set $\tilde{K}_{m,n}^+$ defined by*

$$\left\{\vec{c} \in C^0 : \#\{\lambda \in \Lambda_j : e_\lambda > C(m, n) 2^{-\alpha n j}\} \leq C'(m, n) 2^{(\nu(\alpha_n) + \varepsilon_m)j}, \forall j \in \mathbb{N}_0\right\}.$$

We write

$$\tilde{K}^+ := \bigcap_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \tilde{K}_{m,n}^+.$$

From all sequences of \tilde{K}^+ , we can extract a subsequence which converges pointwise.

Proof. Let $(\vec{c}^{(l)})_{l \in \mathbb{N}}$ be a sequence of \tilde{K}^+ . There exists $n \in \mathbb{N}_0$ such that $\alpha_n < \alpha_{\min}$ and then we have

$$|c_{j,k}^{(l)}| \leq 2^{-\alpha_n} C(m, n), \quad \forall l \in \mathbb{N}, j \in \mathbb{N}_0, k \in \{0, \dots, 2^j - 1\}.$$

This means that the sequence $(\vec{c}^{(l)})_{l \in \mathbb{N}}$ is pointwise bounded in \mathbb{C} and we can thus extract a pointwise convergent subsequence. \square

In what follows, \tilde{K}^+ will denote any subset defined as in Lemma 6.4.22. Let us remark that from Proposition 6.4.12, it is clear that $\tilde{K}^+ \subseteq \mathcal{L}^{\nu,+}$.

Proposition 6.4.23. [25] *Let us assume that $\alpha_{\min} > 0$. A set is a compact subset of $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$ if and only if it is closed in $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$ and included in some \tilde{K}^+ .*

Proof. Since any compact set of a metric space is closed and bounded, the condition is obviously necessary. Let us show that the set \tilde{K}^+ is compact. Let $(\vec{c}^{(l)})_{l \in \mathbb{N}}$ be a sequence of \tilde{K}^+ . By Lemma 6.4.22, we can extract a subsequence which converges pointwise. Let us note again $(\vec{c}^{(l)})_{l \in \mathbb{N}}$ the subsequence and \vec{c} its pointwise limit. Let us show that $(\vec{c}^{(l)})_{l \in \mathbb{N}}$ converges to \vec{c} in $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$.

As $\alpha_{\min} > 0$, there exists $n_0 \in \mathbb{N}$ such that $0 < \alpha_{n_0} < \alpha_{\min}$. By construction, we know that $\vec{c}^{(l)} \in \tilde{K}_{m,n_0}^+$ for all $l \in \mathbb{N}$ and $m \in \mathbb{N}$. Moreover, \tilde{K}_{m,n_0}^+ is bounded in $(C^{\alpha_{n_0}}, \|\cdot\|_{C^{\alpha_{n_0}}})$. Using Lemma 6.4.21 (item 1), we get that $(\vec{c}^{(l)})_{l \in \mathbb{N}}$ converges uniformly to \vec{c} .

Let $\alpha \in [0, \alpha_s]$ and $\varepsilon > 0$. If $\nu(\alpha) \in \mathbb{R}$, the right-continuity of ν gives $n, m \in \mathbb{N}$ such that

$$\varepsilon_m \leq \varepsilon, \quad \alpha_n > \alpha \text{ and } \nu(\alpha_n) + \varepsilon_m < \nu(\alpha) + \varepsilon.$$

Lemma 6.4.21 (item 2) implies that the sequence $(\vec{c}^{(l)})_{l \in \mathbb{N}}$ converges to \vec{c} in the space $(\tilde{A}^+(\alpha, \nu(\alpha) + \varepsilon), \tilde{\delta}_{\alpha, \nu(\alpha) + \varepsilon}^+)$. If $\nu(\alpha) = -\infty$, there exists $n \in \mathbb{N}$ such that $\alpha_n > \alpha$ and $\nu(\alpha_n) = -\infty$. By Lemma 6.4.21 (item 3), $(\vec{c}^{(l)})_{l \in \mathbb{N}}$ converges to \vec{c} in the space $(\tilde{A}^+(\alpha, \nu(\alpha) + \varepsilon), \tilde{\delta}_{\alpha, \nu(\alpha) + \varepsilon}^+)$. Proposition 6.4.16 gives the conclusion. \square

Let us remark that we have also obtained within this last proof the following result.

Corollary 6.4.24. [25] Every sequence of \tilde{K}^+ which converges pointwise converges also in $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$ to an element of \tilde{K}^+ .

Remark 6.4.25. The characterization is not longer valid in the case $\alpha_{\min} = 0$. Indeed, let ν be the admissible profile defined by

$$\nu(\alpha) = \begin{cases} -\infty & \text{if } \alpha < 0, \\ 1 & \text{if } \alpha \geq 0. \end{cases}$$

It is direct to see that $\mathcal{L}^{\nu,+} = C^0$. If we assume that we have this characterization, then the unit ball of C^0 would be compact and therefore the space would be finite dimensional. This leads to a contradiction.

6.4.4 Comparison of $\mathcal{L}^{\nu,+}$ spaces with Oscillation spaces

As presented in Chapter 4, for positive values of p , the leader scaling function $\tilde{\eta}_{\vec{c}}(p)$ of a sequence \vec{c} gives information about which Oscillation spaces contain \vec{c} . More precisely, we have

$$\vec{c} \in \bigcap_{p>0} \bigcap_{\varepsilon>0} \mathcal{O}_p^{\frac{\tilde{\eta}_{\vec{c}}(p)}{p} - \varepsilon}.$$

One could therefore wonder if the knowledge of the $\mathcal{L}^{\nu,+}$ spaces which contain \vec{c} would give more information than the knowledge of the Oscillation spaces to which \vec{c} belongs. From Proposition 5.5.2, one could expect that it will be the case if ν is not concave. Let us prove it.

Let us note that a sequence \vec{c} belongs to $\mathcal{L}^{\nu,+}$ (resp. \mathcal{O}_p^s) if and only if the sequence defined by the wavelet leaders of \vec{c} belongs to \mathcal{S}^ν (resp. $b_{p,\infty}^s$). Proposition 4.6.5 implies then directly the following embedding result, where we recall that the concave conjugate η of ν is defined by

$$\eta(p) = \inf_{\alpha \geq \alpha_{\min}} (\alpha p - \nu(\alpha) + 1), \quad p > 0.$$

Proposition 6.4.26. [25] Let η be the concave conjugate of ν . For any dense sequence $(p_n)_{n \in \mathbb{N}}$ in $(0, +\infty)$ and for any sequence $(\varepsilon_m)_{m \in \mathbb{N}}$ of $(0, +\infty)$ which converges to 0, we have

$$\mathcal{L}^{\nu,+} \subseteq \bigcap_{p>0} \bigcap_{\varepsilon>0} \mathcal{O}_p^{\frac{\eta(p)}{p} - \varepsilon} = \bigcap_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \mathcal{O}_{p_n}^{\frac{\eta(p_n)}{p_n} - \varepsilon_m}.$$

Moreover, the inclusion becomes an equality if ν is concave on $[\alpha_{\min}, \alpha_s]$.

Moreover, we have the following result concerning the topology of $\mathcal{L}^{\nu,+}$ if ν is concave.

Theorem 6.4.27. [25] Let us assume that ν is concave on $[\alpha_{\min}, \alpha_s]$ and let η be the concave conjugate of ν . If $(p_n)_{n \in \mathbb{N}}$ is a dense sequence in $(0, +\infty)$ and if $(\varepsilon_m)_{m \in \mathbb{N}}$ is a sequence of $(0, +\infty)$ which converges to 0, then

$$\mathcal{L}^{\nu,+} = \bigcap_{p>0} \bigcap_{\varepsilon>0} \mathcal{O}_p^{\frac{\eta(p)}{p} - \varepsilon} = \bigcap_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \mathcal{O}_{p_n}^{\frac{\eta(p_n)}{p_n} - \varepsilon_m}$$

and the topology $\tilde{\tau}^+$ on $\mathcal{L}^{\nu,+}$ defined as the weakest one such that each identity map $i : (\mathcal{L}^{\nu,+}, \tilde{\tau}^+) \rightarrow \mathcal{O}_{p_n}^{\frac{\eta(p_n)}{p_n} - \varepsilon_m}$ is continuous, is equivalent to $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$.

Proof. The algebraic result is a consequence of Proposition 6.4.26. For every $n, m \in \mathbb{N}$, the topology of $\mathcal{O}_{p_n}^{\frac{\eta(p_n)}{p_n} - \varepsilon_m}$ is metrizable, complete and stronger than the topology of pointwise convergence. From Proposition 6.4.13, $\tilde{\tau}^+$ is metrizable, complete and stronger than the pointwise topology. The closed graph theorem leads to conclusion. \square

Let us now show that, as in the case of \mathcal{S}^ν , the concavity of ν is also a necessary condition to the equality between $\mathcal{L}^{\nu,+}$ and the intersection of Oscillation spaces. Let $\underline{\nu}$ be the concave hull of ν on $[\alpha_{\min}, \alpha_s]$, i.e. the smallest concave function F on this interval which satisfies $F \geq \nu$ on this interval. This function is defined, continuous, non-decreasing on $[\alpha_{\min}, \alpha_s]$ and with values in $[0, 1]$. Moreover, from Proposition 8.10 of [13], we know that

$$\eta(p) = \inf_{\alpha \in [\alpha_{\min}, \alpha_s]} (\alpha p - \underline{\nu}(\alpha) + 1), \quad p > 0,$$

where η is the concave conjugate of ν . For $\alpha < \alpha_{\min}$, we set $\underline{\nu}(\alpha) := -\infty$.

Proposition 6.4.28. [25] *If $\underline{\nu}$ is the concave hull of ν on $[\alpha_{\min}, \alpha_s]$, we have*

$$\mathcal{L}^{\underline{\nu},+} = \bigcap_{p>0} \bigcap_{\varepsilon>0} \mathcal{O}_p^{\frac{\eta(p)}{p} - \varepsilon}.$$

Proof. This result follows from Proposition 6.4.26. \square

Proposition 6.4.29. [25] *If ν is not concave on $[\alpha_{\min}, \alpha_s]$, then $\mathcal{L}^{\nu,+}$ is strictly included in $\mathcal{L}^{\underline{\nu},+}$.*

Proof. Using Proposition 6.2.1, let us consider $\vec{c} \in \mathcal{L}^{\underline{\nu},+}$ such that $\tilde{\nu}_{\vec{c}}^+ = \underline{\nu}$. By assumption, there is $\alpha \in [\alpha_{\min}, \alpha_s]$ such that $\nu(\alpha) < \underline{\nu}(\alpha) = \tilde{\nu}_{\vec{c}}^+(\alpha)$ and it follows that the sequence \vec{c} does not belong to $\mathcal{L}^{\nu,+}$. \square

6.4.5 Comparison of $\mathcal{L}^{\nu,+}$ spaces with \mathcal{S}^ν spaces

While studying $\mathcal{L}^{\nu,+}$ spaces, we work only with the increasing part of the wavelet leaders profile. Therefore, a natural question is to ask whether $\mathcal{L}^{\nu,+} = \mathcal{S}^\nu$. In view of Proposition 5.5.3 in Chapter 5, one could expect that these two spaces coincide if and only if ν is with increasing-visibility. That is what we will prove in this subsection.

From the definition of the wavelet leaders, it is direct to see that $\nu_{\vec{c}} \leq \tilde{\nu}_{\vec{c}}^+$ for any sequence $\vec{c} \in C^0$ since $|c_\lambda| \leq e_\lambda$ for every $\lambda \in \Lambda$. Given an admissible profile ν , we have then

$$\mathcal{L}^{\nu,+} \subseteq \mathcal{S}^\nu.$$

Here is a case where the inclusion is always strict.

Proposition 6.4.30. [25] *If ν is an admissible profile such that $\alpha_{\min} = 0$, then $\mathcal{L}^{\nu,+}$ is strictly included in \mathcal{S}^ν .*

Proof. Since $\mathcal{L}^{\nu,+}$ is always included in C^0 , it suffices to find an element of \mathcal{S}^ν which does not belong to C^0 . Such an element is given by the sequence $\vec{c} \in \Omega$ defined by $c_{j,0} := j$ and $c_{j,k} := 0$ for $k \neq 0$, at every scale $j \in \mathbb{N}_0$. \square

We already know that $\mathcal{L}^{\nu,+} \subseteq \mathcal{S}^\nu$ for all admissible profile ν . We can also easily compare the topologies of \mathcal{S}^ν and $\mathcal{L}^{\nu,+}$. The proof is direct.

Proposition 6.4.31. [25]

1. If $\alpha \geq 0$ and $\beta \in \{-\infty\} \cup [0, +\infty)$, then we have

$$\tilde{A}^+(\alpha, \beta) \subseteq A(\alpha, \beta) \quad \text{and} \quad \delta_{\alpha, \beta} \leq \tilde{\delta}_{\alpha, \beta}^+.$$

2. The space $(\mathcal{L}^{\nu, +}, \tilde{\delta}^+)$ has a stronger topology than the topology induced by the distance δ .

Let us now investigate under which conditions on the admissible profile ν one has $\mathcal{L}^{\nu, +} = \mathcal{S}^\nu$. As mentioned before, since the multifractal formalism based on the wavelet profile and on the wavelet leaders profile coincide if and only if the wavelet leaders profile is with increasing-visibility, one expects that the spaces coincide if and only if ν is with increasing-visibility. Since $\mathcal{L}^{\nu, +} \subsetneq \mathcal{S}^\nu$ if $\alpha_{\min} = 0$, we assume that $\alpha_{\min} > 0$ in the following.

Let ν be an admissible profile such that $\alpha_{\min} > 0$. We define the function ν_I by setting

$$\nu_I(\alpha) := \begin{cases} -\infty & \text{if } 0 \leq \alpha < \alpha_{\min}, \\ \alpha \sup_{\alpha' \in (0, \alpha]} \frac{\nu(\alpha')}{\alpha'} & \text{if } \alpha_{\min} \leq \alpha < h_{\max}, \\ 1 & \text{if } \alpha \geq h_{\max}, \end{cases}$$

where $h_{\max} := \inf_{h \geq \alpha_{\min}} \frac{h}{\nu(h)}$. Of course, we have $\nu \leq \nu_I$. Besides, $\nu = \nu_I$ if and only if ν is with increasing-visibility on $[\alpha_{\min}, \alpha_s]$ and in this case, $\alpha_s = h_{\max}$. Let us also remark that since ν is right-continuous, ν_I is also right-continuous. Therefore, ν_I is an admissible profile in the sense of \mathcal{S}^ν spaces. Moreover, from Proposition 4.6.6, we know that there is $\vec{c} \in \mathcal{S}^\nu$ such that $\nu_{\vec{c}} = \nu$ on $[0, +\infty)$ and

$$d_f(h) = \begin{cases} \nu_I(h) & \text{if } 0 \leq h \leq h_{\max}, \\ -\infty & \text{otherwise.} \end{cases}$$

Then, Proposition 5.5.3 directly implies the following.

Proposition 6.4.32. [25] *If ν is an admissible profile such that $\alpha_{\min} > 0$, there exists $\vec{c} \in \mathcal{S}^\nu$ such that $\nu_{\vec{c}} = \nu$ and $\tilde{\nu}_{\vec{c}}^+ = \nu_I$ on $[0, +\infty)$.*

Proposition 6.4.33. [25] *Let ν be an admissible profile such that $\alpha_{\min} > 0$. Then, we have $\mathcal{L}^{\nu, +} = \mathcal{S}^\nu$ if and only if ν is with increasing-visibility on $[\alpha_{\min}, \alpha_s]$, i.e. if and only if $\nu = \nu_I$ on $[0, +\infty)$.*

Proof. Let us first assume that $\mathcal{L}^{\nu, +} = \mathcal{S}^\nu$. From Proposition 6.4.32, we know that there is $\vec{c} \in \mathcal{S}^\nu$ such that $\nu_{\vec{c}} = \nu$ and $\tilde{\nu}_{\vec{c}}^+ = \nu_I$. Since $\vec{c} \in \mathcal{S}^\nu = \mathcal{L}^{\nu, +}$, we directly get that $\tilde{\nu}_{\vec{c}}^+ \leq \nu$ hence $\nu = \nu_I$. This means that ν is with increasing-visibility on $[\alpha_{\min}, \alpha_s]$.

Conversely, let us assume that ν is with increasing-visibility on $[\alpha_{\min}, \alpha_s]$. Then $\nu = \nu_I$ on $[0, +\infty)$. If $\vec{c} \in \mathcal{S}^\nu$, we have $\tilde{\nu}_{\vec{c}} \leq \nu_I = \nu$ from Proposition 5.5.3 and it follows that $\vec{c} \in \mathcal{L}^{\nu, +}$. \square

Proposition 6.4.34. [25] *When $\mathcal{L}^{\nu, +}$ is strictly included in \mathcal{S}^ν , the set $\mathcal{L}^{\nu, +}$ is not closed in the space \mathcal{S}^ν .*

Proof. First, let us remark that any sequence with only a finite number of non zero coefficients belongs to $\mathcal{L}^{\nu,+}$. Take now an element \vec{c} of \mathcal{S}^ν which is not in $\mathcal{L}^{\nu,+}$. The “truncated” sequence $(\vec{c}^{(N)})_{N \in \mathbb{N}}$ defined by

$$c_{j,k}^{(N)} := \begin{cases} c_{j,k} & \text{if } j \leq N \text{ and } k \in \{0, \dots, 2^j - 1\}, \\ 0 & \text{if } j > N \text{ and } k \in \{0, \dots, 2^j - 1\}, \end{cases}$$

for all $N \in \mathbb{N}$, converges to \vec{c} for the topology of \mathcal{S}^ν (see Lemma 6.3 in [13]) and each of its elements belongs to $\mathcal{L}^{\nu,+}$. Hence the conclusion. \square

Let us end this subsection by investigating for which admissible profiles ν' , we have the inclusion $\mathcal{S}^\nu \subseteq \mathcal{L}^{\nu',+}$.

Proposition 6.4.35. [25] *Let ν be an admissible profile such that $\alpha_{\min} > 0$. Then, we have $\mathcal{S}^\nu \subseteq \mathcal{L}^{\nu',+}$ if and only if $\nu' \geq \nu_I$ on $[0, +\infty)$ and in this case, the inclusion map is continuous.*

Proof. First, assume that $\mathcal{S}^\nu \subseteq \mathcal{L}^{\nu',+}$. Using Proposition 6.4.32, let $\vec{c} \in \mathcal{S}^\nu$ be such that $\nu_{\vec{c}} = \nu$ and $\tilde{\nu}_{\vec{c}}^+ = \nu_I$. Then $\vec{c} \in \mathcal{L}^{\nu',+}$ and it follows that $\nu_I = \tilde{\nu}_{\vec{c}}^+ \leq \nu'$. Reciprocally, it suffices to show that $\mathcal{S}^\nu \subseteq \mathcal{L}^{\nu_I,+}$. If $\vec{c} \in \mathcal{S}^\nu$, we know from Proposition 5.5.3 that $\tilde{\nu}_{\vec{c}}^+ \leq \nu_I$. This means that $\vec{c} \in \mathcal{L}^{\nu_I,+}$.

Both \mathcal{S}^ν and $\mathcal{L}^{\nu',+}$ are complete metrizable topological vector spaces whose topologies are stronger than the pointwise topology. The closed graph theorem gives the continuity. \square

Remark 6.4.36. If $\alpha_{\min} = 0$, the space \mathcal{S}^ν is not included in $\mathcal{L}^{\nu',+}$ for any admissible profile ν' since it is not included in C^0 .

6.5 $\mathcal{L}^{\nu,-}$ spaces

This section is devoted to the study of the $\mathcal{L}^{\nu,-}$ spaces. Unlike the $\mathcal{L}^{\nu,+}$ spaces, the $\mathcal{L}^{\nu,-}$ spaces are not vector spaces. Moreover, the definition of a distance on these spaces is more delicate. The approach we have chosen will be justified in this section.

Let us recall that given an admissible profile ν , the space $\mathcal{L}^{\nu,-}$ consists in the set of sequences $\vec{c} \in \Omega$ which satisfy the following: for every $\varepsilon > 0$, $C > 0$ and $\alpha \in (\alpha_s, +\infty)$, there exists $J \in \mathbb{N}$ such that

$$\#\widetilde{E}_j^-(C, \alpha)(\vec{c}) \leq 2^{(\nu(\alpha)+\varepsilon)j}, \quad \forall j \geq J,$$

where

$$\widetilde{E}_j^-(C, \alpha)(\vec{c}) = \{\lambda \in \Lambda_j : e_\lambda \leq C2^{-\alpha j}\}.$$

Let us also recall that the restricted wavelet leaders of a sequence of $\mathcal{L}^{\nu,-}$ can be infinite. Remark that, since we have assumed that $\alpha_{\max} < +\infty$, the restricted wavelet leaders e_λ are all different from 0 for every $\vec{c} \in \mathcal{L}^{\nu,-}$. In this case, we have

$$\#\{\lambda \in \Lambda_j : \frac{1}{e_\lambda} \geq \frac{1}{C}2^{\alpha j}\} \leq 2^{(\nu(\alpha)+\varepsilon)j}$$

for every $j \geq J$, where we use the convention that $\frac{1}{+\infty} := 0$. This means that the sequence $\frac{\vec{1}}{e}$ defined by

$$\left(\frac{\vec{1}}{e}\right)_\lambda := \frac{1}{e_\lambda}, \quad \forall \lambda \in \Lambda$$

belongs to \mathcal{S}^{ν^-} , where

$$\nu^-(\alpha) := \begin{cases} \nu(-\alpha) & \text{if } \alpha \leq -\alpha_s, \\ 1 & \text{if } \alpha \geq -\alpha_s. \end{cases}$$

Note that ν^- is increasing and right-continuous. Moreover, one has $\nu^-(\alpha) = -\infty$ for every $\alpha < -\alpha_{\max}$, so that ν^- is an admissible profile in the sense of \mathcal{S}^ν spaces, see [13] and Section 4.6 of Chapter 4. In other words, there is a natural application

$$T : \mathcal{L}^{\nu, -} \rightarrow \mathcal{S}^{\nu^-} : \vec{c} \mapsto \frac{\vec{1}}{e}.$$

This point of view will be adopted to define a topology on $\mathcal{L}^{\nu, -}$. As in the case of $\mathcal{L}^{\nu, +}$ spaces, let us start by defining auxiliary spaces.

6.5.1 Auxiliary spaces $\tilde{A}^-(\alpha, \beta)$

In this subsection, we introduce auxiliary spaces $\tilde{A}^-(\alpha, \beta)$ and endow them with a pseudo-distance.

Definition 6.5.1. Let $\alpha \geq 0$ and $\beta \in \{-\infty\} \cup [0, +\infty)$. A sequence $\vec{c} \in \Omega$ belongs to the auxiliary space $\tilde{A}^-(\alpha, \beta)$ if there exist $C, C' \geq 0$ such that

$$\# \tilde{E}_j^-(C, \alpha)(\vec{c}) \leq C' 2^{\beta j}, \quad \forall j \in \mathbb{N}_0$$

and if $e_\lambda \neq 0$ for every $\lambda \in \Lambda$.

Since $\tilde{A}^-(\alpha, \beta)$ is not a vector space, one cannot take as definition for the distance between \vec{c} and \vec{c}' the value

$$\inf \left\{ C + C' : C, C' \geq 0 \text{ and } \# \tilde{E}_j^-\left(\frac{1}{C}, \alpha\right)(\vec{c} - \vec{c}') \leq C' 2^{\beta j}, \forall j \in \mathbb{N}_0 \right\},$$

which would be in parallel with what is done for the auxiliary spaces $\tilde{A}^+(\alpha, \beta)$. We have thus to consider a different approach.

Remark that, if $A(\alpha, \beta)$ denotes the auxiliary spaces introduced in the study of the \mathcal{S}^ν spaces, see [13] and Section 4.6 of Chapter 4, then a sequence $\vec{c} \in \Omega$ belongs to $\tilde{A}^-(\alpha, \beta)$ if and only if $e_\lambda \neq 0$ for every $\lambda \in \Lambda$ and

$$\frac{\vec{1}}{e} \in A(-\alpha, \beta).$$

The properties of the auxiliary spaces $\tilde{A}^-(\alpha, \beta)$ will then follow directly from the properties of the auxiliary spaces $A(-\alpha, \beta)$. In particular, this leads to the following definition.

Definition 6.5.2. Let $\alpha \geq 0$ and $\beta \in \{-\infty\} \cup [0, +\infty)$. For $\vec{c}, \vec{c}' \in \tilde{A}^-(\alpha, \beta)$, we write

$$\tilde{\delta}_{\alpha, \beta}^-(\vec{c}, \vec{c}') := \delta_{-\alpha, \beta} \left(\frac{\vec{1}}{e}, \frac{\vec{1}}{e'} \right).$$

Proposition 6.5.3. For every $\alpha \geq 0$ and $\beta \in \{-\infty\} \cup [0, +\infty)$, $\tilde{\delta}_{\alpha, \beta}^-$ is a pseudo-distance on $\tilde{A}^-(\alpha, \beta)$.

Proof. Of course, $\tilde{\delta}_{\alpha,\beta}^-$ is positive and symmetric. Moreover, if $\vec{c}, \vec{c}', \vec{c}'' \in \tilde{A}^-(\alpha, \beta)$, then

$$\begin{aligned} \tilde{\delta}_{\alpha,\beta}^-(\vec{c}, \vec{c}') &= \delta_{-\alpha,\beta} \left(\frac{\vec{1}}{e}, \frac{\vec{1}}{e'} \right) \\ &\leq \delta_{-\alpha,\beta} \left(\frac{\vec{1}}{e}, \frac{\vec{1}}{e''} \right) + \delta_{-\alpha,\beta} \left(\frac{\vec{1}}{e''}, \frac{\vec{1}}{e'} \right) = \tilde{\delta}_{\alpha,\beta}^-(\vec{c}, \vec{c}'') + \tilde{\delta}_{\alpha,\beta}^-(\vec{c}'', \vec{c}') \end{aligned}$$

since $\delta_{-\alpha,\beta}$ is a distance on $A(-\alpha, \beta)$. \square

Remark 6.5.4. The definition of $\tilde{\delta}_{\alpha,\beta}^-$ does not give a distance. Indeed, if $\vec{c} \neq \vec{c}'$ are two elements of $\tilde{A}^-(\alpha, \beta)$ which have the same sequence of restricted wavelet leaders, then

$$\tilde{\delta}_{\alpha,\beta}^-(\vec{c}, \vec{c}') = 0.$$

Consequently, $(\tilde{A}^-(\alpha, \beta), \tilde{\delta}_{\alpha,\beta}^-)$ is not a Hausdorff space. The choice of this topology can appear unnatural since we compare the restricted wavelet leaders of sequences instead of their coefficients. Let us justify this approach: our goal is to get information about multifractal spectra of functions using their wavelet leaders profile. Both these notions depend only on the distribution of the wavelet leaders of the function. Two functions sharing the same restricted wavelet leaders have the same multifractal properties and the same wavelet leaders profile. In order to get results in this direction, this topology seems to be convenient. Nevertheless, we still don't know if this topology is independent of the chosen wavelet basis. Of course, one could consider the relation \mathcal{R} defined by

$$(\vec{c}, \vec{c}') \in \mathcal{R} \iff e_\lambda = e'_\lambda, \quad \forall \lambda \in \Lambda.$$

This relation is an equivalence relation. Then $\tilde{\delta}_{\alpha,\beta}^-$ defines a distance on

$$\tilde{A}^-(\alpha, \beta)/\mathcal{R}.$$

Nevertheless, we will not use this quotient space: indeed, when we will endow the entire space $\mathcal{L}^\nu = \mathcal{L}^{\nu,+} \cap \mathcal{L}^{\nu,-}$ with a topology, we will get a distance thanks to the topology of $\mathcal{L}^{\nu,+}$.

The next result follows directly from the definition of the distance on $\tilde{A}^-(\alpha, \beta)$.

Proposition 6.5.5. *Let $\alpha \geq 0$ and $\beta \in \{-\infty\} \cup [0, +\infty)$. A sequence $(\vec{c}^{(m)})_{m \in \mathbb{N}}$ of $\tilde{A}^-(\alpha, \beta)$ converges to \vec{c} (resp. is a Cauchy sequence) in $(\tilde{A}^-(\alpha, \beta), \tilde{\delta}_{\alpha,\beta}^-)$ if and only if the sequence $(\frac{\vec{1}}{e^{(m)}})_{m \in \mathbb{N}}$ converges to $\frac{\vec{1}}{e}$ (resp. is a Cauchy sequence) in $A(-\alpha, \beta)$.*

Let us mention the following property concerning the bounded sets of $\tilde{A}^-(\alpha, \beta)$. It follows from the characterization of bounded sets in $A(-\alpha, \beta)$, see Proposition 4.6.2.

Proposition 6.5.6. *Let $\alpha \geq 0$ and $\beta \in \{-\infty\} \cup [0, +\infty)$.*

1. *If B is a bounded set of $(\tilde{A}^-(\alpha, \beta), \tilde{\delta}_{\alpha,\beta}^-)$, then there exist $r, r' > 0$ such that*

$$B \subseteq \{ \vec{c} \in \Omega : \#\{ \lambda \in \Lambda_j : e_\lambda < r 2^{-\alpha j} \} \leq r' 2^{\beta j}, \forall j \in \mathbb{N}_0 \}.$$

2. Let $r, r' \geq 0$, $\alpha' \leq \alpha$ and $\beta' \leq \beta$. The set

$$B = \left\{ \vec{c} \in \Omega : \#\{\lambda \in \Lambda_j : e_\lambda < r 2^{-\alpha'j}\} \leq r' 2^{\beta'j}, \forall j \in \mathbb{N}_0 \right\}$$

is a bounded set of $(\tilde{A}^-(\alpha, \beta), \tilde{\delta}_{\alpha, \beta}^-)$.

We end with this result which compares auxiliary spaces $\tilde{A}^-(\alpha, \beta)$.

Lemma 6.5.7. *If $\alpha' \geq \alpha$ and $\beta' \geq \beta$, then*

$$\tilde{A}^-(\alpha, \beta) \subseteq \tilde{A}^-(\alpha', \beta') \quad \text{and} \quad \tilde{\delta}_{\alpha', \beta'}^- \leq \tilde{\delta}_{\alpha, \beta}^-.$$

6.5.2 Topology on $\mathcal{L}^{\nu, -}$ spaces

As in the case of the $\mathcal{L}^{\nu, +}$ spaces, we have a description of the $\mathcal{L}^{\nu, -}$ spaces in terms of auxiliary spaces $\tilde{A}^-(\alpha, \beta)$.

Proposition 6.5.8. *For any dense sequence $\alpha' = (\alpha'_p)_{p \in \mathbb{N}}$ in $[\alpha_s, +\infty)$ and any sequence $\varepsilon = (\varepsilon_m)_{m \in \mathbb{N}}$ of $(0, +\infty)$ which converges to 0, we have*

$$\mathcal{L}^{\nu, -} = \bigcap_{\varepsilon > 0} \bigcap_{\alpha \geq \alpha_s} \tilde{A}^-(\alpha, \nu(\alpha) + \varepsilon) = \bigcap_{m \in \mathbb{N}} \bigcap_{p \in \mathbb{N}} \tilde{A}^-(\alpha'_p, \nu(\alpha'_p) + \varepsilon_m).$$

Proof. We know that a sequence \vec{c} such that $e_\lambda \neq 0$ for every $\lambda \in \Lambda$ belongs to $\mathcal{L}^{\nu, -}$ (resp. $\tilde{A}^-(\alpha, \beta)$) if and only if $\frac{\vec{c}}{\varepsilon}$ belongs to $\mathcal{S}^{\nu, -}$ (resp. $A(-\alpha, \beta)$). The result follows then from Proposition 4.6.3. It can also be proved as in Proposition 6.4.12, using Lemma 6.5.7. \square

This result allows to define a pseudo-distance on the $\mathcal{L}^{\nu, -}$ spaces.

Definition 6.5.9. Let $\alpha' := (\alpha'_p)_{p \in \mathbb{N}}$ be a dense sequence in $[\alpha_s, +\infty)$ and $\varepsilon := (\varepsilon_m)_{m \in \mathbb{N}}$ be a sequence of $(0, +\infty)$ which converges to 0. For $m, p \in \mathbb{N}$, we write

$$\tilde{\delta}_{m, p}^- := \tilde{\delta}_{\alpha'_p, \nu(\alpha'_p) + \varepsilon_m}^- \quad \text{and} \quad \tilde{A}^-(m, p) := \tilde{A}^-(\alpha'_p, \nu(\alpha'_p) + \varepsilon_m).$$

Then, for $m \in \mathbb{N}$, we denote

$$\tilde{\delta}_m^- := \sum_{p=1}^{+\infty} 2^{-p} \frac{\tilde{\delta}_{m, p}^-}{1 + \tilde{\delta}_{m, p}^-} \quad \text{and} \quad \tilde{\delta}_{\alpha', \varepsilon}^- := \sum_{m=1}^{+\infty} 2^{-m} \tilde{\delta}_m^-.$$

Proposition 6.5.10. *For every sequences α' and ε chosen as above, $\tilde{\delta}_{\alpha', \varepsilon}^-$ is a pseudo-distance on $\mathcal{L}^{\nu, -}$. All these distances define the same topology.*

Proof. Thanks to Proposition 6.5.3, it is direct to see that $\tilde{\delta}_{\alpha', \varepsilon}^-$ is a pseudo-distance on $\mathcal{L}^{\nu, -}$. The second part of the proof is very similar to the proof of Proposition 6.4.15, using Lemma 6.5.7. \square

In view of this result, we write this pseudo-distance $\tilde{\delta}^-$ independently of these sequences α' and ε .

Agreement. From now on, $\alpha' := (\alpha'_p)_{p \in \mathbb{N}}$ denotes a dense sequence in $[\alpha_s, +\infty)$ and as previously, $\varepsilon := (\varepsilon_m)_{m \in \mathbb{N}}$ denotes a sequence of $(0, +\infty)$ which converges to 0.

Remark 6.5.11. Again, $\tilde{\delta}$ is not a distance on $\mathcal{L}^{\nu, -}$ and $(\mathcal{L}^{\nu, -}, \tilde{\delta})$ is not a Hausdorff space. It is a distance on the quotient space $\mathcal{L}^{\nu, -} / \mathcal{R}$.

A slight modification of Proposition 6.4.13 gives the following result.

Proposition 6.5.12.

1. The topology defined by $\tilde{\delta}^-$ on $\mathcal{L}^{\nu, -}$ is the weakest topology such that, for every $m, p \in \mathbb{N}$, the identity $i : \mathcal{L}^{\nu, -} \rightarrow \tilde{A}^-(m, p)$ is continuous.
2. A sequence of $\mathcal{L}^{\nu, -}$ is a Cauchy sequence in $(\mathcal{L}^{\nu, -}, \tilde{\delta}^-)$ if and only if, for every $m, p \in \mathbb{N}$, it is a Cauchy sequence in $(\tilde{A}^-(m, p), \tilde{\delta}_{m, p}^-)$.
3. A sequence of $\mathcal{L}^{\nu, -}$ converges in $(\mathcal{L}^{\nu, -}, \tilde{\delta}^-)$ if and only if, for every $m, p \in \mathbb{N}$, it converges in $(\tilde{A}^-(m, p), \tilde{\delta}_{m, p}^-)$.

This result and the description of the topology of the \mathcal{S}^ν spaces imply the next result.

Corollary 6.5.13. A sequence $(\vec{c}^{(m)})_{m \in \mathbb{N}}$ of $\mathcal{L}^{\nu, -}$ converges to \vec{c} (resp. is a Cauchy sequence) in $(\mathcal{L}^{\nu, -}, \tilde{\delta}^-)$ if and only if the sequence $\left(\frac{1}{e^{(m)}}\right)_{m \in \mathbb{N}}$ converges to $\frac{1}{e}$ (resp. is a Cauchy sequence) in $(\mathcal{S}^{\nu, -}, \delta)$, where

$$\nu^-(\alpha) := \begin{cases} \nu(-\alpha) & \text{if } \alpha \leq -\alpha_s, \\ 1 & \text{if } \alpha \geq -\alpha_s. \end{cases}$$

The last result we present in this section concerns the compact sets of $(\mathcal{L}^{\nu, -}, \tilde{\delta}^-)$ and will be helpful in order to prove the separability of the \mathcal{L}^ν spaces.

Lemma 6.5.14. For $m, p \in \mathbb{N}$, let $D(m, p)$ and $D'(m, p)$ be positive constants and let us define

$$\tilde{K}_{m, p}^- := \left\{ \vec{c} \in \Omega : \#\{\lambda \in \Lambda_j : e_\lambda < D(m, p) 2^{-\alpha'_p j}\} \leq D'(m, p) 2^{(\nu(\alpha'_p) + \varepsilon_m)j}, \forall j \in \mathbb{N}_0 \right\}.$$

Then, we set

$$\tilde{K}^- := \bigcap_{m \in \mathbb{N}} \bigcap_{p \in \mathbb{N}} \tilde{K}_{m, p}^-.$$

Every sequence of \tilde{K}^- which converges uniformly converges also in $(\mathcal{L}^{\nu, -}, \tilde{\delta}^-)$ to an element of \tilde{K}^- .

Proof. Let $(\vec{c}^{(l)})_{l \in \mathbb{N}}$ be a sequence of \tilde{K}^- which converges uniformly to \vec{c} . Let us consider $p \in \mathbb{N}$ such that $\alpha'_p > \alpha_{\max}$. Then, if $m \in \mathbb{N}$, we have

$$\#\{\lambda \in \Lambda_j : e_\lambda^{(l)} < D(m, p) 2^{-\alpha'_p j}\} = 0$$

for every $l \in \mathbb{N}$ and every $j \in \mathbb{N}_0$, so that $e_\lambda^{(l)} \geq D(m, p) 2^{-\alpha'_p j} > 0$ for every $l \in \mathbb{N}$ and every $\lambda \in \Lambda_j$ with $j \in \mathbb{N}_0$. From Proposition 6.5.8, we get that $\tilde{K}^- \subseteq \mathcal{L}^\nu$. Moreover, since $(\vec{c}^{(l)})_{l \in \mathbb{N}}$ converges uniformly to \vec{c} , we also have $e_\lambda \geq D(m, p) 2^{-\alpha'_p j} > 0$ for every $l \in \mathbb{N}$ and every $\lambda \in \Lambda_j$ with $j \in \mathbb{N}_0$. Moreover, for every $l \in \mathbb{N}$, $\vec{c}^{(l)} \in \tilde{K}^-$ and it follows that the sequence $\frac{1}{e^{(l)}}$ belongs to the set K defined by

$$\bigcap_{m \in \mathbb{N}} \bigcap_{p \in \mathbb{N}} \left\{ \vec{x} \in \Omega : \#\{\lambda \in \Lambda_j : |x_\lambda| > \frac{1}{D(m, p)} 2^{\alpha'_p j}\} \leq D'(m, p) 2^{(\nu(\alpha'_p) + \varepsilon_m)j}, \forall j \in \mathbb{N}_0 \right\}.$$

Since $\left(\frac{1}{\varepsilon^{(l)}}\right)_{l \in \mathbb{N}}$ converges pointwise to $\frac{1}{\varepsilon}$, Proposition 4.6.4 implies that $\left(\frac{1}{\varepsilon^{(l)}}\right)_{l \in \mathbb{N}}$ converges to $\frac{1}{\varepsilon}$ in \mathcal{S}^{ν^-} and that $\frac{1}{\varepsilon} \in K$. Then, $\vec{c} \in \tilde{K}^-$ and the conclusion follows from Corollary 6.5.13. \square

6.6 Topology on \mathcal{L}^ν spaces

In this section, we endow the space \mathcal{L}^ν with a distance $\tilde{\delta}$ and we show that the space $(\mathcal{L}^\nu, \tilde{\delta})$ is complete and separable. Let us recall that, given an admissible profile ν , one has

$$\mathcal{L}^\nu = \mathcal{L}^{\nu,+} \cap \mathcal{L}^{\nu,-}.$$

Since the spaces $\mathcal{L}^{\nu,+}$ and $\mathcal{L}^{\nu,-}$ have been endowed with a topology, we directly get a topology on \mathcal{L}^ν . Let us also recall that $\alpha := (\alpha_n)_{n \in \mathbb{N}}$ denotes a dense sequence in $[0, \alpha_s]$, $\alpha' := (\alpha'_p)_{p \in \mathbb{N}}$ denotes a dense sequence in $[\alpha_s, +\infty)$ and $\varepsilon := (\varepsilon_m)_{m \in \mathbb{N}}$ denotes a sequence of $(0, +\infty)$ which converges to 0.

Proposition 6.6.1. *For every sequences α , α' and ε chosen as above,*

$$\tilde{\delta}_{\alpha, \alpha', \varepsilon} := \tilde{\delta}_{\alpha, \varepsilon} + \tilde{\delta}_{\alpha', \varepsilon}$$

is a distance on \mathcal{L}^ν . All these distances define the same topology.

Proof. Using Propositions 6.4.15 and 6.5.3, it is clear that $\tilde{\delta}_{\alpha, \alpha', \varepsilon}$ is a distance on \mathcal{L}^ν . The independence on the choice of the sequences follows from the same propositions. \square

This result allows us to write this distance $\tilde{\delta}$ independently of these sequences α , α' and ε . We directly get the following result.

Proposition 6.6.2.

1. *The topology defined by $\tilde{\delta}$ on \mathcal{L}^ν is the weakest topology such that the inclusions $\mathcal{L}^\nu \rightarrow \mathcal{L}^{\nu,+}$ and $\mathcal{L}^\nu \rightarrow \mathcal{L}^{\nu,-}$ are continuous.*
2. *A sequence of \mathcal{L}^ν is a Cauchy sequence in $(\mathcal{L}^\nu, \tilde{\delta})$ if and only if it is a Cauchy sequence in $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$ and in $(\mathcal{L}^{\nu,-}, \tilde{\delta}^-)$.*
3. *A sequence of \mathcal{L}^ν converges in $(\mathcal{L}^\nu, \tilde{\delta})$ if and only if it converges in $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$ and in $(\mathcal{L}^{\nu,-}, \tilde{\delta}^-)$.*

Thanks to Propositions 6.4.16 and 6.5.12, we have the next result.

Proposition 6.6.3.

1. *The topology defined by $\tilde{\delta}$ on \mathcal{L}^ν is the weakest topology such that, for every $m, n, p \in \mathbb{N}$, the inclusions $\mathcal{L}^\nu \rightarrow \tilde{A}^+(m, n)$ and $\mathcal{L}^\nu \rightarrow \tilde{A}^-(m, p)$ are continuous.*
2. *A sequence of \mathcal{L}^ν is a Cauchy sequence in $(\mathcal{L}^\nu, \tilde{\delta})$ if and only if it is a Cauchy sequence in $(\tilde{A}^+(m, n), \tilde{\delta}_{m,n}^+)$ and in $(\tilde{A}^-(m, p), \tilde{\delta}_{m,p}^-)$ for every $m, n, p \in \mathbb{N}$.*
3. *A sequence of \mathcal{L}^ν converges in $(\mathcal{L}^\nu, \tilde{\delta})$ if and only if it converges in $(\tilde{A}^+(m, n), \tilde{\delta}_{m,n}^+)$ and in $(\tilde{A}^-(m, p), \tilde{\delta}_{m,p}^-)$ for every $m, n, p \in \mathbb{N}$.*

Proposition 6.6.4. *The space $(\mathcal{L}^\nu, \tilde{\delta})$ is a complete metric space and thus a Baire space.*

Proof. Let $(\tilde{c}^{(l)})_{l \in \mathbb{N}}$ be a Cauchy sequence in $(\mathcal{L}^\nu, \tilde{\delta})$. Then, it is a Cauchy sequence in $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$ and from Proposition 6.4.17, it converges to \tilde{c} in $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$. Moreover, it is also a Cauchy sequence in $(\mathcal{L}^{\nu,-}, \tilde{\delta}^-)$ hence, from Corollary 6.5.13, $(\overrightarrow{\frac{1}{e^{(m)}}})_{m \in \mathbb{N}}$ is a Cauchy sequence in $(\mathcal{S}^{\nu-}, \delta)$. Since the space $(\mathcal{S}^{\nu-}, \delta)$ is complete, there exists $\tilde{x} \in \mathcal{S}^{\nu-}$ such that $(\overrightarrow{\frac{1}{e^{(l)}}})_{l \in \mathbb{N}}$ converges to \tilde{x} in $(\mathcal{S}^{\nu-}, \delta)$. Since the topology of $\mathcal{S}^{\nu-}$ is stronger than the pointwise topology, we know that

$$x_\lambda = \lim_{l \rightarrow +\infty} \frac{1}{e_\lambda^{(l)}}, \quad \forall \lambda \in \Lambda.$$

Moreover, since the convergence in $\mathcal{L}^{\nu,+}$ is stronger than the uniform convergence, we have

$$\sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| = \lim_{l \rightarrow +\infty} e_\lambda^{(l)}, \quad \forall \lambda \in \Lambda.$$

It follows that that

$$x_\lambda = \frac{1}{\sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|}, \quad \forall \lambda \in \Lambda.$$

In particular, $\sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \neq 0$ for every $\lambda \in \Lambda$ and $\tilde{c} \in \mathcal{L}^{\nu,-}$. Proposition 6.6.2 gives the conclusion. \square

Lemma 6.6.5. *Assume that $\alpha_{\min} > 0$. For every $m, n, p \in \mathbb{N}$, let $C(m, n)$, $C'(m, n)$, $D(m, p)$ and $D'(m, p)$ be positive constants. Let us define*

$$\tilde{K} = \tilde{K}^+ \cap \tilde{K}^-$$

where \tilde{K}^+ (resp. \tilde{K}^-) is defined as in Lemma 6.4.22 (resp. Lemma 6.5.14). Every sequence of \tilde{K} which converges pointwise converges also in $(\mathcal{L}^\nu, \tilde{\delta})$ to an element of \tilde{K} .

Proof. Since $\alpha_{\min} > 0$, there exists $n \in \mathbb{N}$ such that $0 < \alpha_n < \alpha_{\min}$. By construction, $\tilde{K} \subseteq \tilde{K}_{m,n}^+$ which is bounded in $(C^{\alpha_n}, \|\cdot\|_{C^{\alpha_n}})$. Using Lemma 6.4.21 (item 1), we get that if a sequence of \tilde{K} converges pointwise, it converge also uniformly. The result follows then from Corollary 6.4.24 and Lemma 6.5.14. \square

The next characterization of the compact sets of $(\mathcal{L}^\nu, \tilde{\delta})$ is immediate.

Proposition 6.6.6. *Assume that $\alpha_{\min} > 0$. A subset of \mathcal{L}^ν is compact in $(\mathcal{L}^\nu, \tilde{\delta})$ if and only if it is closed and included in some $\tilde{K} = \tilde{K}^+ \cap \tilde{K}^-$.*

Proof. Since any compact set of a metric space is closed and bounded, the condition is necessary using Propositions 6.4.7 and 6.5.6. It suffices to show that \tilde{K} is compact in $(\mathcal{L}^\nu, \tilde{\delta})$. Let us assume $(\tilde{c}^{(l)})_{l \in \mathbb{N}}$ is a sequence of \tilde{K} . By Lemma 6.4.22, since $\tilde{K} \subseteq \tilde{K}^+$, we can extract a subsequence which converges pointwise. The conclusion follows from Lemma 6.6.5. \square

Let us now study the separability of the space $(\mathcal{L}^\nu, \tilde{\delta})$. We will see that it holds only if $\alpha_{\min} > 0$. Let us recall that, while working with \mathcal{L}^ν as a function space, this condition means that all functions of the space are uniformly Hölder.

Lemma 6.6.7. *Assume that $\alpha_{\min} > 0$ and let $\vec{z} \in \mathcal{L}^\nu$. For every $\vec{c} \in \mathcal{L}^\nu$ and every $N \in \mathbb{N}$, we set*

$$c_{j,k}^{(N)} := \begin{cases} c_{j,k} & \text{if } j < N, \\ e_{j,k} & \text{if } j = N, \\ z_{j,k} & \text{if } j > N. \end{cases}$$

Then the sequence $(\vec{c}^{(N)})_{N \in \mathbb{N}}$ converges to \vec{c} in $(\mathcal{L}^\nu, \tilde{\delta})$.

Proof. It is clear that $\vec{c}^{(N)} \in \mathcal{L}^\nu$ for every $N \in \mathbb{N}$ since $c_{j,k}^{(N)} = z_{j,k}$ if $j > N$. Since $\vec{c}, \vec{z} \in \mathcal{L}^{\nu,+}$, there are $C(m, n), C'(m, n) > 0$ such that \vec{c} , and \vec{z} belong to

$$\left\{ \vec{x} \in \mathcal{L}^\nu : \#\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |x_{\lambda'}| > C(m, n)2^{-\alpha_n j}\} \leq C'(m, n)2^{(\nu(\alpha_n) + \varepsilon_m)j}, \forall j \in \mathbb{N}_0 \right\}$$

for every $m, n \in \mathbb{N}$. Let us remark that if $j \leq N$, then for every $\lambda \in \Lambda_j$, we have either $e_\lambda^{(N)} = e_\lambda$ or $e_\lambda^{(N)} = \sup_{\lambda' \subseteq \lambda_0} |z_{\lambda'}| \leq \sup_{\lambda' \subseteq \lambda} |z_{\lambda'}|$ where $\lambda_0 \subseteq \lambda$. It follows that for $m, n \in \mathbb{N}$ and $j \leq N$,

$$\begin{aligned} & \#\{\lambda \in \Lambda_j : e_\lambda^{(N)} > C(m, n)2^{-\alpha_n j}\} \\ & \leq \#\{\lambda \in \Lambda_j : e_\lambda > C(m, n)2^{-\alpha_n j}\} + \#\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |z_{\lambda'}| > C(m, n)2^{-\alpha_n j}\} \\ & \leq 2C'(m, n)2^{(\nu(\alpha_n) + \varepsilon_m)j}. \end{aligned}$$

Since $c_{j,k}^{(N)} = z_{j,k}$ if $j > N$, we get that $\vec{c}^{(N)} \in \tilde{K}^+$ where \tilde{K}^+ is the intersection over $m, n \in \mathbb{N}$ of the sets

$$\left\{ \vec{x} \in \mathcal{L}^\nu : \#\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |x_{\lambda'}| > C(m, n)2^{-\alpha_n j}\} \leq 2C'(m, n)2^{(\nu(\alpha_n) + \varepsilon_m)j}, \forall j \in \mathbb{N}_0 \right\}.$$

For the decreasing part, since $\vec{c}, \vec{z} \in \mathcal{L}^{\nu,-}$, there are $D(m, p), D'(m, p) > 0$ such that \vec{c} and \vec{z} belong to the set \tilde{K}^- given by the intersection over $m, p \in \mathbb{N}$ of the sets

$$\left\{ \vec{x} \in \mathcal{L}^\nu : \#\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |x_{\lambda'}| < D(m, p)2^{-\alpha'_p j}\} \leq D'(m, p)2^{(\nu(\alpha'_p) + \varepsilon_m)j}, \forall j \in \mathbb{N}_0 \right\}.$$

Let us show that $\vec{c}^{(N)} \in \tilde{K}^-$ for every N . Let us fix $m, p \in \mathbb{N}$ and $N \in \mathbb{N}$. If $j \leq N$, then $e_\lambda^{(N)} \geq e_\lambda$ for every $\lambda \in \Lambda_j$ so that

$$\#\{\lambda \in \Lambda_j : e_\lambda^{(N)} < D(m, p)2^{-\alpha'_p j}\} \leq D'(m, p)2^{(\nu(\alpha'_p) + \varepsilon_m)j}.$$

Moreover, if $j > N$, then $c_{j,k}^{(N)} = z_{j,k}$ and it follows that $\vec{c}^{(N)} \in \tilde{K}^-$.

Since $(\vec{c}^{(N)})_{N \in \mathbb{N}}$ converges pointwise to \vec{c} , Lemma 6.6.5 gives that $(\vec{c}^{(N)})_{N \in \mathbb{N}}$ converges to \vec{c} in $(\mathcal{L}^\nu, \tilde{\delta})$. \square

Lemma 6.6.8. *Assume that $\alpha_{\min} > 0$. Let B be a subset of \mathcal{L}^ν and $\vec{z} \in \mathcal{L}^\nu$. If B is pointwise bounded and if there exists $N \in \mathbb{N}$ such that $c_\lambda = z_\lambda$ for every $\lambda \in \Lambda_j$ with $j > N$ and every $\vec{c} \in B$, then B is included in a compact subset of $(\mathcal{L}^\nu, \tilde{\delta})$.*

Proof. Since $\vec{z} \in \mathcal{L}^\nu$, there are $C(m, n), C'(m, n) > 0$ such that

$$\#\left\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |z_{\lambda'}| > C(m, n)2^{-\alpha_n j}\right\} \leq C'(m, n)2^{(\nu(\alpha_n) + \varepsilon_m)j}, \quad \forall j \in \mathbb{N}_0.$$

There are also $D(m, p), D'(m, p) > 0$ such that

$$\#\left\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |z_{\lambda'}| < D(m, p)2^{-\alpha'_p j}\right\} \leq D'(m, p)2^{(\nu(\alpha'_p) + \varepsilon_m)j}, \quad \forall j \in \mathbb{N}_0.$$

If $j \leq N$ and $\vec{c} \in B$, one has

$$\begin{aligned} 2^{\alpha_n j} \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| &= 2^{\alpha_n j} \max \left\{ \sup_{\lambda' \subseteq \lambda, j' > N} |z_{\lambda'}|, \sup_{\lambda' \subseteq \lambda, j' \leq N} |c_{\lambda'}| \right\} \\ &\leq 2^{\alpha_n N} \max \left\{ \sup_{\lambda' \subseteq \lambda, j' > N} |z_{\lambda'}|, \sup_{\lambda' \subseteq \lambda, j' \leq N} |c_{\lambda'}| \right\} \end{aligned}$$

which is bounded by a constant independent of $\vec{c} \in B$ since B is pointwise bounded. Therefore, there is $C(n) > 0$ such that

$$\#\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| > C(n)2^{-\alpha_n j}\} = 0, \quad \forall j \leq N, \forall \vec{c} \in B.$$

Similarly, for every $\vec{c} \in B$, if $j \leq N$, one has

$$\sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \geq \sup_{\lambda' \subseteq \lambda, j' > N} |z_{\lambda'}| > 0, \quad \forall \lambda \in \Lambda_j$$

since $\vec{z} \in \mathcal{L}^\nu$. Consequently, for every $p \in \mathbb{N}$, there is a constant $D(p) > 0$ such that

$$\#\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| < D(p)2^{-\alpha'_p j}\} = 0, \quad \forall j \leq N, \forall \vec{c} \in B.$$

It suffices then to take the constants $\max\{C(n), C(m, n)\}, C'(m, n), \min\{D(p), D(m, p)\}$ and $D'(m, p)$ and to use Proposition 6.6.6. \square

We can now prove that, if $\alpha_{\min} > 0$, the space $(\mathcal{L}^\nu, \tilde{\delta})$ is separable.

Proposition 6.6.9. *Assume that $\alpha_{\min} > 0$. Let us fix $\vec{z} \in \mathcal{L}^\nu$ and let us consider the set U of sequences $\vec{c} \in \Omega$ for which there exists $N \in \mathbb{N}$ such that $|c_\lambda| = |z_\lambda|$ if $\lambda \in \Lambda_j$ with $j > N$, and $c_\lambda \in \mathbb{Q} + i\mathbb{Q}$ if $\lambda \in \Lambda_j$ with $j \leq N$. Then $U \subseteq \mathcal{L}^\nu$ and U is dense in $(\mathcal{L}^\nu, \tilde{\delta})$. In particular, the space $(\mathcal{L}^\nu, \tilde{\delta})$ is separable.*

Proof. It is clear that $U \subseteq \mathcal{L}^\nu$ since $\vec{z} \in \mathcal{L}^\nu$. Let us fix $\vec{c} \in \mathcal{L}^\nu$ and let us consider the sequence $(\vec{c}^{(N)})_{N \in \mathbb{N}}$ given by Lemma 6.6.7. For every $N \in \mathbb{N}$, it suffices to find a sequence of U which converges to $\vec{c}^{(N)}$ in $(\mathcal{L}^\nu, \tilde{\delta})$. Using the density of $\mathbb{Q} + i\mathbb{Q}$ in \mathbb{C} , there is a sequence $(\vec{q}^{(N, l)})_{l \in \mathbb{N}}$ of U which converges pointwise to $\vec{c}^{(N)}$. From Lemma 6.6.8, this sequence can be included in a compact set \tilde{K} of \mathcal{L}^ν and since it is pointwise convergent, it converges also to $\vec{c}^{(N)}$ in $(\mathcal{L}^\nu, \tilde{\delta})$ using Lemma 6.6.5. \square

Let us now consider the case where the admissible profile ν is such that $\alpha_{\min} = 0$. The previous result is no longer valid. Indeed, with the admissible profile considered in Remark 6.4.25, the space \mathcal{L}^ν is C^0 which is not separable. More generally, we have the following property.

Proposition 6.6.10. *If $\alpha_{\min} = 0$, the metric space $(\mathcal{L}^\nu, \tilde{\delta})$ is not separable.*

Proof. This result uses classical considerations concerning sup-norms. Indeed, let us consider the uncountable set A of sequences \vec{c} of C^0 such that for each scale $j \in \mathbb{N}_0$, $c_{j,0} \in \{0, 1\}$ and the other coefficients are equal to $2^{-\alpha_s j}$. Using the assumption on α_{\min} , we easily prove that A is a subset of \mathcal{L}^ν . Moreover, $\|\vec{c} - \vec{c}'\|_{C^0} = 1$ for all distinct elements \vec{c} and \vec{c}' of A .

Let D be a dense subset of $(\mathcal{L}^\nu, \tilde{\delta})$. For every $\vec{c} \in A$, there exists a sequence $(\vec{c}^{(m)})_{m \in \mathbb{N}}$ of elements of D which converges in $(\mathcal{L}^\nu, \tilde{\delta})$ to $\vec{c} \in \mathcal{L}^\nu$. From Remark 6.4.19, the convergence also holds in C^0 . Consequently, there exists $M \in \mathbb{N}$ such that

$$\|\vec{c} - \vec{c}^{(m)}\|_{C^0} < \frac{1}{2}, \quad \forall m \geq M.$$

In particular, there exists $\vec{a} \in D$ such that

$$\|\vec{c} - \vec{a}\|_{C^0} < \frac{1}{2}.$$

Since the C^0 norm between two distinct elements of A is equal to 1, D must contain at least as many elements as A and cannot be countable. \square

6.7 Generic results in \mathcal{L}^ν spaces

In this section, we study the form of the wavelet leaders profile of most of the sequences of \mathcal{L}^ν . Note that since \mathcal{L}^ν is not a vector space, prevalent results cannot be obtained.

Proposition 6.7.1. *Let ν be an admissible profile such that $\alpha_{\min} > 0$. The set of sequences $\vec{c} \in \mathcal{L}^\nu$ such that $\tilde{\nu}_{\vec{c}} = \nu$ is residual in $(\mathcal{L}^\nu, \tilde{\delta})$.*

Proof. From Proposition 6.2.1, we know that we can consider $\vec{z} \in \mathcal{L}^\nu$ such that $\tilde{\nu}_{\vec{z}} = \nu$. Then, using the definition of $\tilde{\nu}_{\vec{z}}$, for every $m, n \in \mathbb{N}$, there exists an infinite set $J_{m,n}^+$ such that

$$\# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |z_{\lambda'}| \geq 2^{-(\alpha_n + 2\varepsilon_m)j} \right\} \geq 2^{(\nu(\alpha_n) - \varepsilon_m)j}, \quad \forall j \in J_{m,n}^+.$$

Similarly, for every $m, p \in \mathbb{N}$, there exists an infinite set $J_{m,p}^-$ such that

$$\# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |z_{\lambda'}| \leq 2^{-(\alpha'_p - \frac{\varepsilon_m}{2})j} \right\} \geq 2^{(\nu(\alpha'_p) - \varepsilon_m)j}, \quad \forall j \in J_{m,p}^-.$$

We know that \mathcal{L}^ν is separable, and more precisely, the set $U = \{\vec{y}^{(l)} : l \in \mathbb{N}\}$ defined with \vec{z} as in Proposition 6.6.9 is dense in \mathcal{L}^ν . Moreover, by construction, for every $l \in \mathbb{N}$, there exists $j_l \in \mathbb{N}_0$ such that $y_\lambda^{(l)} = z_\lambda$ for every $\lambda \in \Lambda_j$ with $j \geq j_l$. For every $m, n, l \in \mathbb{N}$, we fix $j_{m,n,l} \in J_{m,n}^+$ such that

$$j_{m,n,l} \geq j_l \quad \text{and} \quad \varepsilon_m j_{m,n,l} > 1$$

and for every $m, p, l \in \mathbb{N}$, we fix $j'_{m,p,l} \in J_{m,p}^-$ such that

$$j'_{m,p,l} \geq j_l \quad \text{and} \quad \frac{\varepsilon_m}{2} j'_{m,p,l} > 1.$$

For every $m, n, L \in \mathbb{N}$, let us consider the set $U_{m,n,L}^+$ defined by

$$U_{m,n,L}^+ := \bigcup_{l \geq L} \tilde{B}_{m,n,l}^+$$

where $\tilde{B}_{m,n,l}^+$ is the open ball in the auxiliary space $\tilde{A}^+(\alpha_n, \nu(\alpha_n) + \varepsilon_m)$ formed by the sequences $\tilde{c} \in \mathcal{L}^\nu$ such that

$$\begin{aligned} \inf \left\{ C > 0 : \#\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'} - y_{\lambda'}^{(l)}| \geq C 2^{-\alpha_n j}\} \right. \\ \left. \leq C 2^{(\nu(\alpha_n + \varepsilon_m)j}, \forall j \in \mathbb{N}_0 \right\} < 2^{-3\varepsilon_m j_{m,n,l}}. \end{aligned}$$

Similarly, for every $m, p, L \in \mathbb{N}$, we consider the set $U_{m,p,L}^-$ defined by

$$U_{m,p,L}^- := \bigcup_{l \geq L} \tilde{B}_{m,p,l}^-$$

where $\tilde{B}_{m,p,l}^-$ is the open ball in the auxiliary space $\tilde{A}^-(\alpha'_p, \nu(\alpha'_p) + \varepsilon_m)$ formed by the sequences $\tilde{c} \in \mathcal{L}^\nu$ such that

$$\begin{aligned} \inf \left\{ C > 0 : \#\{\lambda \in \Lambda_j : \left| \frac{1}{\sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|} - \frac{1}{\sup_{\lambda' \subseteq \lambda} |y_{\lambda'}^{(l)}|} \right| \geq C 2^{\alpha'_p j} \right\} \\ \leq C 2^{(\nu(\alpha'_p + \varepsilon_m)j}, \forall j \in \mathbb{N}_0 \right\} < 2^{-3\varepsilon_m j'_{m,p,l}}. \end{aligned}$$

Remark that, for every $m, n, p, L \in \mathbb{N}$, the set $U_{m,n,L}^+ \cap U_{m,p,L}^-$ is dense in \mathcal{L}^ν since it contains the sequences $\tilde{y}^{(l)}$, $l \geq L$. Finally, the set

$$W := \bigcap_{m,n,p,L \in \mathbb{N}} U_{m,n,L}^+ \cap U_{m,p,L}^-$$

is a countable intersection of dense open sets of \mathcal{L}^ν . Let us show that if $\tilde{c} \in W$, then $\tilde{\nu}_{\tilde{c}} = \nu$. Since $W \subseteq \mathcal{L}^\nu$, we already know that $\tilde{\nu}_{\tilde{c}} \leq \nu$.

First, let us consider the increasing part. For every $m, n, L \in \mathbb{N}$, there is $l \geq L$ such that $\tilde{c} \in \tilde{B}_{m,n,l}^+$, so that

$$\#\left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'} - y_{\lambda'}^{(l)}| \geq 2^{-3\varepsilon_m j_{m,n,l}} 2^{-\alpha_n j} \right\} \leq 2^{-3\varepsilon_m j_{m,n,l}} 2^{(\nu(\alpha_n) + \varepsilon_m)j}$$

for every $j \in \mathbb{N}_0$. Then, for $j = j_{m,n,l}$, we obtain

$$\begin{aligned} & \#\left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \geq 2^{-(\alpha_n + 3\varepsilon_m)j} \right\} \\ &= \#\left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'} - y_{\lambda'}^{(l)} + y_{\lambda'}^{(l)}| \geq 2^{-(\alpha_n + 3\varepsilon_m)j} \right\} \\ &\geq \#\left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |y_{\lambda'}^{(l)}| - \sup_{\lambda' \subseteq \lambda} |c_{\lambda'} - y_{\lambda'}^{(l)}| \geq 2^{-(\alpha_n + 3\varepsilon_m)j} \right\} \\ &\geq \#\left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |y_{\lambda'}^{(l)}| \geq 2 \cdot 2^{-(\alpha_n + 3\varepsilon_m)j} \text{ and } \sup_{\lambda' \subseteq \lambda} |c_{\lambda'} - y_{\lambda'}^{(l)}| < 2^{-(\alpha_n + 3\varepsilon_m)j} \right\} \end{aligned}$$

hence

$$\begin{aligned}
 & \# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \geq 2^{-(\alpha_n + 3\varepsilon_m)j} \right\} \\
 \geq & \# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |y_{\lambda'}^{(l)}| \geq 2 \cdot 2^{-(\alpha_n + 3\varepsilon_m)j} \right\} \\
 & \quad - \# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'} - y_{\lambda'}^{(l)}| \geq 2^{-(\alpha_n + 3\varepsilon_m)j} \right\} \\
 \geq & \# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |z_{\lambda'}| \geq 2^{-(\alpha_n + 2\varepsilon_m)j} \right\} \\
 & \quad - \# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'} - y_{\lambda'}^{(l)}| \geq 2^{-(\alpha_n + 3\varepsilon_m)j} \right\} \\
 \geq & 2^{(\nu(\alpha_n) - \varepsilon_m)j} - 2^{(\nu(\alpha_n) - 2\varepsilon_m)j} \\
 \geq & 2^{(\nu(\alpha_n) - 2\varepsilon_m)j}
 \end{aligned}$$

using the choice of $j_{m,n,l}$. It follows that, for every $m, n \in \mathbb{N}$,

$$\limsup_{j \rightarrow +\infty} \frac{\log \# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \geq 2^{-(\alpha_n + \varepsilon_m)j} \right\}}{\log 2^j} \geq \nu(\alpha_n) - 2\varepsilon_m.$$

Taking the limit as $m \rightarrow +\infty$, we get $\tilde{\nu}_c^+(\alpha_n) \geq \nu(\alpha_n)$ for every $n \in \mathbb{N}$. The conclusion follows from the right-continuity of the functions $\tilde{\nu}_c^+$ and ν .

Let us now consider the decreasing part. For every $m, p, L \in \mathbb{N}$, there is $l \geq L$ such that $\tilde{c} \in \tilde{B}_{m,p,l}^-$, so that

$$\begin{aligned}
 \# \left\{ \lambda \in \Lambda_j : \left| \frac{1}{\sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|} - \frac{1}{\sup_{\lambda' \subseteq \lambda} |y_{\lambda'}^{(l)}|} \right| \geq 2^{-3\varepsilon_m j'_{m,p,l}} 2^{\alpha'_p j} \right\} \\
 \leq 2^{-3\varepsilon_m j'_{m,p,l}} 2^{(\nu(\alpha'_p) + \varepsilon_m)j}
 \end{aligned}$$

for every $j \in \mathbb{N}_0$. For $j = j'_{m,p,l}$, we obtain

$$\begin{aligned}
 & \# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \leq 2^{-(\alpha'_p - \varepsilon_m)j} \right\} \\
 = & \# \left\{ \lambda \in \Lambda_j : \frac{1}{\sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|} \geq 2^{(\alpha'_p - \varepsilon_m)j} \right\} \\
 = & \# \left\{ \lambda \in \Lambda_j : \left| \frac{1}{\sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|} + \frac{1}{\sup_{\lambda' \subseteq \lambda} |y_{\lambda'}^{(l)}|} - \frac{1}{\sup_{\lambda' \subseteq \lambda} |y_{\lambda'}^{(l)}|} \right| \geq 2^{(\alpha'_p - \varepsilon_m)j} \right\} \\
 \geq & \# \left\{ \lambda \in \Lambda_j : \frac{1}{\sup_{\lambda' \subseteq \lambda} |y_{\lambda'}^{(l)}|} - \left| \frac{1}{\sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|} - \frac{1}{\sup_{\lambda' \subseteq \lambda} |y_{\lambda'}^{(l)}|} \right| \geq 2^{(\alpha'_p - \varepsilon_m)j} \right\}.
 \end{aligned}$$

As done in the increasing part, we get

$$\begin{aligned}
 & \# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \leq 2^{-(\alpha'_p - \varepsilon_m)j} \right\} \\
 \geq & \# \left\{ \lambda \in \Lambda_j : \frac{1}{\sup_{\lambda' \subseteq \lambda} |y_{\lambda'}^{(l)}|} \geq 2 \cdot 2^{(\alpha'_p - \varepsilon_m)j} \right. \\
 & \quad \left. \text{and } \left| \frac{1}{\sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|} - \frac{1}{\sup_{\lambda' \subseteq \lambda} |y_{\lambda'}^{(l)}|} \right| < 2^{(\alpha'_p - \varepsilon_m)j} \right\} \\
 \geq & \# \left\{ \lambda \in \Lambda_j : \frac{1}{\sup_{\lambda' \subseteq \lambda} |y_{\lambda'}^{(l)}|} \geq 2 \cdot 2^{(\alpha'_p - \varepsilon_m)j} \right\} \\
 & \quad - \# \left\{ \lambda \in \Lambda_j : \left| \frac{1}{\sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|} - \frac{1}{\sup_{\lambda' \subseteq \lambda} |y_{\lambda'}^{(l)}|} \right| \geq 2^{(\alpha'_p - \varepsilon_m)j} \right\} \\
 \geq & \# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |z_{\lambda'}| \leq 2^{-(\alpha'_p - \frac{\varepsilon_m}{2})j} \right\} \\
 & \quad - \# \left\{ \lambda \in \Lambda_j : \left| \frac{1}{\sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|} - \frac{1}{\sup_{\lambda' \subseteq \lambda} |y_{\lambda'}^{(l)}|} \right| \geq 2^{(\alpha'_p - 3\varepsilon_m)j} \right\} \\
 \geq & 2^{(\nu(\alpha'_p) - \varepsilon_m)j} - 2^{(\nu(\alpha'_p) - 2\varepsilon_m)j} \\
 \geq & 2^{(\nu(\alpha'_p) - 2\varepsilon_m)j}.
 \end{aligned}$$

Then, for every $m, n \in \mathbb{N}$, we have

$$\limsup_{j \rightarrow +\infty} \frac{\log \# \left\{ \lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \leq 2^{-(\alpha'_p - \varepsilon_m)j} \right\}}{\log 2^j} \geq \nu(\alpha'_p) - 2\varepsilon_m.$$

Taking the limit as $m \rightarrow +\infty$, we get $\tilde{\nu}_{\vec{c}}^-(\alpha'_p) \geq \nu(\alpha'_p)$ for every $n \in \mathbb{N}$ and the conclusion follows from the left-continuity of the functions $\tilde{\nu}_{\vec{c}}^-$ and ν . \square

Let us now generalize Proposition 6.2.4 with the construction of a dense vector subspace of \mathcal{L}^ν whose elements \vec{c} satisfy $\tilde{\nu}_{\vec{c}} = \nu$.

Proposition 6.7.2. *Let ν be an admissible profile such that $\alpha_{\min} > 0$. The set of sequences $\vec{c} \in \mathcal{L}^\nu$ such that $\tilde{\nu}_{\vec{c}} = \nu$ is \mathbf{c} -dense lineable in $(\mathcal{L}^\nu, \tilde{\delta})$.*

Proof. Let us denote by $\vec{x}^{(r)}$, $r > 0$ the sequences constructed in Proposition 6.2.4 and such that $\tilde{\nu}_{\vec{x}^{(r)}} = \nu$. For every $r > 0$, consider the set U_r obtained from $\vec{x}^{(r)}$ as in Proposition 6.6.9. We know that these sets are dense in $(\mathcal{L}^\nu, \tilde{\delta})$. Then we consider the subspace \mathcal{D} of C^0 defined by

$$\mathcal{D} = \text{span} \{ \vec{c} \in C^0 : \exists r > 0 \text{ such that } \vec{c} \in U_r \}.$$

Of course, $\dim \mathcal{D} = \mathbf{c}$ since \mathcal{D} contains the sequences $\vec{x}^{(r)}$, $r > 0$. If $\vec{z} \in \mathcal{D} \setminus \{0\}$, then there are $J \in \mathbb{N}$, $r_1, \dots, r_N > 0$ ($N \in \mathbb{N}$) and $\theta_1, \dots, \theta_N \in \mathbb{C}$ not all equal to 0 such that

$$z_{j,k} = \theta_1 x_{j,k}^{(r_1)} + \dots + \theta_N x_{j,k}^{(r_N)}$$

for every $j \geq J$, $k \in \{0, \dots, 2^j - 1\}$. Consequently, \vec{z} has the same wavelet leaders profile as

$$\theta_1 \vec{x}^{(r_1)} + \dots + \theta_N \vec{x}^{(r_N)}$$

and the conclusion follows from the proof of Proposition 6.2.4. □

Chapter 7

Validity of the leaders profile method

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7.1 Introduction

In the previous chapters, we have introduced a new multifractal formalism, the leaders profile method, and the underlying function spaces, the \mathcal{L}^ν spaces. As for the other multifractal formalisms, this method never holds in complete generality, but we have proved that it yields an upper bound for the multifractal spectrum of the functions in the space \mathcal{L}^ν . This is the best that can be expected: usually, there are no non-trivial minorations for the multifractal spectrum of all functions in the space. Nevertheless, one can hope that for most of the functions in the space, that is to say for a generic subset of the space, the inequality becomes an equality. As a first step toward the proof of the generic validity of this new method, we construct in this chapter functions with prescribed multifractal spectra which satisfy the leaders profile method. More precisely, given an admissible profile ν , we construct a function $f \in \mathcal{L}^\nu$ which satisfies $d_f = \tilde{\nu}_f = \nu$ on $[0, +\infty]$.

This chapter is structured as follows. In Section 7.2, we present the construction of functions f with increasing affine spectrum of the form

$$d_f(h) = \begin{cases} \frac{\gamma h}{\beta} & \text{if } h \in [\alpha, \beta], \\ 1 & \text{if } h = +\infty, \\ -\infty & \text{otherwise,} \end{cases}$$

with $\gamma \in (0, 1)$ and $0 < \alpha < \beta < +\infty$. This allows to create in Section 7.3 functions with prescribed multifractal spectrum which satisfy the leaders profile method.

7.2 Lacunary wavelet series on a Cantor set

In this section, we present a model based on the lacunary wavelet series presented in Chapter 5. It allows to construct functions whose spectra are affine functions.

We denote by $C(r)$ the Cantor set with ratio of dissection $r < \frac{1}{2}$ given by the following iterative Cantor-like construction. Let $C_0 = [0, 1]$. We remove from C_0 the open middle interval of length $1 - 2r$, leaving two closed intervals of length r . We call C_1 the union of these intervals. At step N in the construction, if we have inductively constructed C_N as a union of 2^N closed intervals of length r^N , we remove the open middle interval of length $(1 - 2r)r^N$ from each of the intervals of the step N and we define C_{N+1} as the union of the remaining 2^{N+1} closed intervals of length r^{N+1} . Finally, we define the Cantor set $C(r)$ by

$$C(r) = \bigcap_{N \in \mathbb{N}} C_N.$$

The Hausdorff dimension of $C(r)$, denoted in what follows by γ , is given by

$$\gamma = \dim_{\mathcal{H}} C(r) = -\frac{\log 2}{\log r},$$

see for example [65, 109].

The model we consider is constructed as follows. Let $\alpha > 0$ and $0 < \eta < 1$ be two parameters. Let $(g_{j,k})_{j \in \mathbb{N}_0, k \in \{0, \dots, 2^j - 1\}}$ be a sequence of independent random variables in a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ whose laws are Bernoulli laws with parameter $2^{-(1-\eta)j}$, i.e. such that

$$g_{j,k} = \begin{cases} 1 & \text{with probability } 2^{-(1-\eta)j}, \\ 0 & \text{with probability } 1 - 2^{-(1-\eta)j}. \end{cases}$$

We consider then the random wavelet series $R_{\alpha, \eta, r}$ whose coefficients are given by

$$c_{j,k} = \begin{cases} g_{j,k} 2^{-\alpha j} & \text{if } k \in K_j, \\ 0 & \text{otherwise,} \end{cases} \quad (7.1)$$

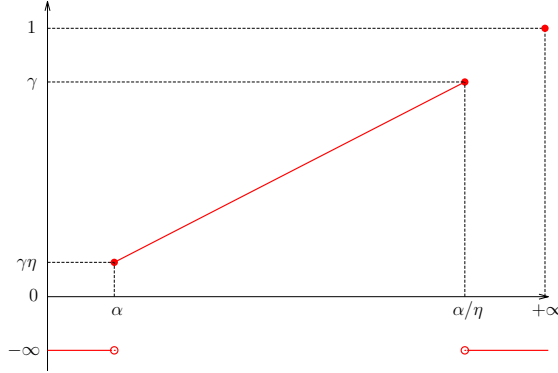
where K_j is the set of $k \in \{0, \dots, 2^j - 1\}$ for which there is $j' \leq j$ and $\lambda' \in \Lambda_{j'}$ with $\eta(j-1) \leq j' + \log_2 j'$, $\lambda(j, k) \subseteq \lambda'$ and $C(r) \cap \lambda' \neq \emptyset$. Intuitively, $k \in K_j$ if at a scale $j' \leq j$ close to j , the dyadic interval of $\Lambda_{j'}$ which contains $\lambda(j, k)$ meets $C(r)$. Let us remark that it suffices to ask that it holds for the smallest integer $j' \leq j$ such that $\eta(j-1) \leq j' + \log_2 j'$.

7.2.1 Multifractal spectrum of $R_{\alpha, \eta, r}$

The aim of this section is to compute the multifractal spectrum of $R_{\alpha, \eta, r}$. It is given by the following theorem.

Theorem 7.2.1. *With probability one,*

$$d_{R_{\alpha, \eta, r}}(h) = \begin{cases} \gamma h \frac{\eta}{\alpha} & \text{if } h \in [\alpha, \frac{\alpha}{\eta}], \\ 1 & \text{if } h = +\infty, \\ -\infty & \text{otherwise.} \end{cases}$$


 Figure 7.1: Almost sure multifractal spectrum of $R_{\alpha, \eta, r}$

Let us first show, using Proposition 4.7.4, that the range for the possible values of the Hölder exponent is $[\alpha, \frac{\alpha}{\eta}] \cup \{+\infty\}$. Clearly, since $|c_{j,k}| \leq 2^{-\alpha j}$ for every j, k , we have that $h_{R_{\alpha, \eta, r}}(x) \geq \alpha$ for every $x \in [0, 1]$. Of course, if $x \notin C(r)$, then since $C(r)$ is closed, $3\lambda_j(x) \cap C(r) = \emptyset$ for every j large enough. Therefore, $d_j(x) = 0$ and $h_{R_{\alpha, \eta, r}}(x) = +\infty$. Let us now prove that if $x \in C(r)$, then $h_{R_{\alpha, \eta, r}}(x) \leq \frac{\alpha}{\eta}$.

Lemma 7.2.2. *With probability one, for every $\varepsilon > 0$, there exists $J \in \mathbb{N}$ such that*

$$e_\lambda \geq 2^{-j(\frac{\alpha}{\eta} + \varepsilon)}$$

for every $j \geq J$ and every $\lambda \in \Lambda_j$ such that $\lambda \cap C(r) \neq \emptyset$.

Proof. Let us fix $\varepsilon > 0$. For every $j \in \mathbb{N}$, let us denote by A_j the event “there exists $\lambda \in \Lambda_j$ such that $\lambda \cap C(r) \neq \emptyset$ and $e_\lambda < 2^{-j(\frac{\alpha}{\eta} + \varepsilon)}$ ”. Let us set

$$j_0 = \lfloor \frac{1}{\eta}(j + \log_2 j) \rfloor + 1.$$

For j large enough, we have $j \leq j_0 \leq j(\frac{1}{\eta} + \frac{\varepsilon}{\alpha})$. Moreover, if $\lambda \in \Lambda_j$ is such that $\lambda \cap C(r) \neq \emptyset$ and if $\lambda(j_0, k_0) \subseteq \lambda$, we have $k_0 \in K_{j_0}$ and $c_{j_0, k_0} = g_{j_0, k_0} 2^{-\alpha j_0}$. Since there are $2^{j_0 - j}$ such intervals, we obtain

$$\begin{aligned} \mathbb{P}[A_j] &\leq \sum_{\lambda: \lambda \cap C(r) \neq \emptyset} \mathbb{P}[e_\lambda < 2^{-j(\frac{\alpha}{\eta} + \varepsilon)}] \\ &\leq \sum_{\lambda: \lambda \cap C(r) \neq \emptyset} \prod_{\lambda_0 \subseteq \lambda, \lambda_0 \in \Lambda_{j_0}} \mathbb{P}[|c_{\lambda_0}| < 2^{-j(\frac{\alpha}{\eta} + \varepsilon)}] \\ &\leq \sum_{\lambda: \lambda \cap C(r) \neq \emptyset} \left(1 - 2^{-(1-\eta)j_0}\right)^{2^{j_0 - j}} \\ &\leq 2^j \exp\left(-2^{j_0 - j} 2^{-(1-\eta)j_0}\right) \\ &\leq \left(\frac{2}{e}\right)^j \end{aligned}$$

using Remark 5.4.8 and the relation $\eta j_0 \geq j + \log_2 j$. Therefore,

$$\sum_{j \in \mathbb{N}} \mathbb{P}[A_j] < +\infty,$$

and using the Borel Cantelli Lemma, we get that with probability one, there exists $J \in \mathbb{N}$ such that

$$e_\lambda \geq 2^{-j(\frac{\alpha}{\eta} + \varepsilon)}$$

for every $j \geq J$ and every $\lambda \in \Lambda_j$ such that $\lambda \cap C(r) \neq \emptyset$. Taking a decreasing sequence $(\varepsilon_m)_{m \in \mathbb{N}}$ that converges to 0, we obtain the conclusion. \square

Proposition 4.7.4 directly implies that if $x \in C(r)$, the Hölder exponent of $R_{\alpha, \eta, r}$ at x is smaller than $\frac{\alpha}{\eta}$.

Let us now describe the iso-Hölder sets of $R_{\alpha, \eta, r}$. Let us denote by $(j_n, k_n)_{n \in \mathbb{N}}$ the sequence of indexes for which $g_{j_n, k_n} = 1$, re-ordered so that $j_n \leq j_{n+1}$ for every $n \in \mathbb{N}$. For every $\delta \in (0, 1]$, we consider a sequence $(\delta_n)_{n \in \mathbb{N}}$ of $(0, \delta]$ which converges to δ and we set

$$E_\delta((\delta_n)_{n \in \mathbb{N}}) := \limsup_{n \rightarrow +\infty} (k_n 2^{-j_n} - 2^{-\delta_n j_n}, k_n 2^{-j_n} + 2^{-\delta_n j_n}).$$

In order to simplify the notations, we will write $E_\delta := E_\delta((\delta_n)_{n \in \mathbb{N}})$. Finally, we consider

$$G_\delta := \bigcap_{0 < \delta' < \delta} E_{\delta'} \setminus \bigcup_{\delta < \delta' \leq 1} E_{\delta'} \quad \text{if } \delta < 1, \quad G_1 := \bigcap_{0 < \delta' < 1} E_{\delta'}.$$

Lemma 7.2.3. *Let us fix $\delta \in (0, 1)$. If $x \in E_\delta$, then $h_{R_{\alpha, \eta, r}}(x) \leq \frac{\alpha}{\delta}$.*

Proof. If $x \in E_\delta$, there exist infinitely many $n \in \mathbb{N}$ such that $|x - k_n 2^{-j_n}| < 2^{-\delta_n j_n}$. For every $n \in \mathbb{N}$, we consider $J_n = \lfloor \delta_n j_n \rfloor - 1 < j_n$. Let us show that $d_{J_n}(x) \geq 2^{-\frac{\alpha}{\delta_n}(J_n+2)}$. Let K_n denote the element of $\{0, \dots, 2^{J_n} - 1\}$ such that $\lambda_{J_n}(x) = \lambda(J_n, K_n)$. Then

$$k_n 2^{-j_n} \geq x - 2^{-\delta_n j_n} \geq K_n 2^{-J_n} - 2^{-\delta_n j_n} \geq (K_n - 1) 2^{-J_n}$$

and

$$\begin{aligned} (k_n + 1) 2^{-j_n} \leq 2^{-\delta_n j_n} + x + 2^{-j_n} &\leq 2^{-\delta_n j_n} + (K_n + 1) 2^{-J_n} + 2^{-j_n} \\ &\leq 2 \cdot 2^{-\delta_n j_n} + (K_n + 1) 2^{-J_n} \\ &\leq (K_n + 2) 2^{-J_n} \end{aligned}$$

since $\delta_n j_n \geq J_n + 1$. It follows that $\lambda(j_n, k_n) \subseteq 3\lambda(J_n, K_n)$ and

$$d_{J_n}(x) \geq |c_{j_n, k_n}| = 2^{-\alpha j_n} \geq 2^{-\frac{\alpha}{\delta_n}(J_n+2)}.$$

Consequently,

$$h_{R_{\alpha, \eta, r}}(x) = \liminf_{j \rightarrow +\infty} \frac{\log d_j(x)}{\log 2^{-j}} \leq \lim_{n \rightarrow +\infty} \frac{\log d_{J_n}(x)}{\log 2^{-J_n}} \leq \frac{\alpha}{\delta}.$$

\square

Lemma 7.2.4. *Let us fix $\delta \in (0, 1]$. If $x \notin E_\delta$, then $h_{R_{\alpha, \eta, r}}(x) \geq \frac{\alpha}{\delta}$.*

Proof. Since $x \notin E_\delta$, there is $N \in \mathbb{N}$ such that for every $n \geq N$, $|x - k_n 2^{-j_n}| \geq 2^{-\delta_n j_n}$. Let us show that for $j \geq j_N$, we have $d_j(x) \leq 2^{-\frac{\alpha}{\delta}(j-2)}$. Assume it is not the case. Then there is $n \geq N$ with $j_n \geq j$ and such that $c_{j_n, k_n} = 2^{-\alpha j_n} > 2^{-\frac{\alpha}{\delta}(j-2)}$ and $\lambda(j_n, k_n) \subseteq 3\lambda_j(x)$. In particular, we have $j - 2 > \delta j_n$. Since x and $k_n 2^{-j_n}$ belong to $3\lambda_j(x)$, we have

$$|x - k_n 2^{-j_n}| < 3 \cdot 2^{-j} < 2^{-\delta j_n} \leq 2^{-\delta_n j_n},$$

hence a contradiction. Consequently, we obtain

$$h_{R_{\alpha,\eta,r}}(x) = \liminf_{j \rightarrow +\infty} \frac{\log d_j(x)}{\log 2^{-j}} \geq \frac{\alpha}{\delta}.$$

□

Proposition 7.2.5. *For every $\delta \in (0, 1]$, we have $G_\delta = \{x \in [0, 1] : h_{R_{\alpha,\eta,r}}(x) = \frac{\alpha}{\delta}\}$.*

Proof. It follows directly from Lemma 7.2.3 and Lemma 7.2.4. □

Let us remark that in particular, G_δ is independent of the sequences $(\delta'_n)_{n \in \mathbb{N}}$ chosen to define the sets $E_{\delta'}$, $\delta' \in (0, 1]$.

In order to get the multifractal spectrum of $R_{\alpha,\eta,r}$, it suffices now to compute the Hausdorff dimension of the sets G_δ . We consider the random sets

$$F_j := \{k \in \{0, \dots, 2^j - 1\} : c_{j,k} = 2^{-\alpha j}\}, \quad j \in \mathbb{N}_0.$$

Equivalently, F_j is the set of $k \in \{0, \dots, 2^j - 1\}$ for which there is $n \in \mathbb{N}$ with $k = k_n$ and $j = j_n$. An upper bound for the Hausdorff dimension of G_δ will be obtained thanks to an estimation of the cardinality of F_j . Let us start with the following remark.

Remark 7.2.6. For every $j' \in \mathbb{N}$, there is $N \in \mathbb{N}$ is such that $r^N < 2^{-j'} \leq r^{N-1}$. Since $r^N < 2^{-j'}$, the number of dyadic intervals of length $2^{-j'}$ which intersect C_N is at most $2 \cdot 2^N$. Moreover, since $2^{-j'} \leq r^{N-1}$, we get that the number of dyadic intervals of length $2^{-j'}$ which intersect the Cantor set is smaller than $2^{\gamma j'+2}$. Let $j \in \mathbb{N}$ and consider the smallest integer $j' \leq j$ such that $\eta(j-1) \leq j' + \log_2 j'$. Since there are $2^{j-j'}$ dyadic intervals of size 2^{-j} included in a dyadic interval of size $2^{-j'}$ for $j \geq j'$, we get that

$$\#K_j \leq 4 \cdot 2^{j-(1-\gamma)j'}.$$

Lemma 7.2.7. *With probability one, for every $\varepsilon > 0$, there is $J \in \mathbb{N}$ such that*

$$\#F_j \leq 2^{(\gamma\eta+\varepsilon)j}, \quad \forall j \geq J.$$

Proof. Let us fix $\varepsilon > 0$. For every $j \in \mathbb{N}_0$, we denote by B_j the event “ $\#F^j > 2^{(\gamma\eta+\varepsilon)j}$ ”. Remark that at a given scale j , we count the number of successes of a binomial distribution of parameters $(n_j, 2^{-(1-\eta)j})$, where the success means “ $c_{j,k} = 2^{-\alpha j}$ ” and where $n_j = \#K_j$. From Remark 7.2.6, we know that

$$n_j \leq 4 \cdot 2^{j-(1-\gamma)j'}$$

where j' is the smallest integer such that $j' \leq j$ and $\eta(j-1) \leq j' + \log_2 j'$. In particular, we have

$$n_j 2^{-(1-\eta)j} \leq 4 \cdot 2^{\eta j} 2^{-(1-\gamma)j'} \leq 4j^2 2^{(1-\gamma)\eta} 2^{\gamma \eta j} < 2^{(\gamma\eta+\frac{\varepsilon}{2})j}$$

for j large enough. We get

$$\begin{aligned} \mathbb{P}[B_j] &= \sum_{2^{(\gamma\eta+\varepsilon)j} < m \leq n_j} \binom{n_j}{m} (2^{-(1-\eta)j})^m (1 - 2^{-(1-\eta)j})^{n_j-m} \\ &\leq \sum_{2^{(\gamma\eta+\varepsilon)j} < m \leq n_j} \frac{(n_j 2^{-(1-\eta)j})^m}{m!} \\ &\leq \frac{(n_j 2^{-(1-\eta)j})^{2^{(\gamma\eta+\varepsilon)j}}}{\Gamma(2^{(\gamma\eta+\varepsilon)j} + 1)}. \end{aligned}$$

Using Stirling's formula, we obtain then that for j large enough,

$$\begin{aligned} n_j \frac{(n_j 2^{-(1-\eta)j})^{2^{(\eta\gamma+\varepsilon)j}}}{\Gamma(2^{(\eta\gamma+\varepsilon)j} + 1)} &\sim \frac{n_j}{\sqrt{2\pi}} \left(n_j 2^{-(1-\eta)j} 2^{-(\eta\gamma+\varepsilon)j} \frac{e}{\sqrt{2}} \right)^{2^{(\eta\gamma+\varepsilon)j}} \\ &\leq \frac{2^j}{\sqrt{2\pi}} \left(2^{-\frac{\varepsilon}{2}j} \frac{e}{\sqrt{2}} \right)^{2^{(\eta\gamma+\varepsilon)j}} \\ &\leq \frac{1}{\sqrt{2\pi}} (2^{-\frac{\varepsilon}{2}j} e)^{2^{(\eta\gamma+\varepsilon)j}} \end{aligned}$$

since $2^j < \sqrt{2}^{2^{(\eta\gamma+\varepsilon)j}}$ if j is large enough. Therefore, $\mathbb{P}[B_j]$ is the general term of a converging series. We conclude the proof by using the Borel Cantelli lemma and by taking a sequence $(\varepsilon_m)_{m \in \mathbb{N}}$ which decreases to 0. \square

Proposition 7.2.8. *With probability one, for every $\delta \in (0, 1]$, $\dim_{\mathcal{H}}(G_\delta) \leq \gamma \frac{\eta}{\delta}$.*

Proof. From Lemma 7.2.7, we know that with probability one, for every $\varepsilon > 0$, there is $J \in \mathbb{N}$ such that

$$\#F_j \leq 2^{(\gamma\eta+\varepsilon)j}, \quad \forall j \geq J.$$

Remark that for every $N \in \mathbb{N}$ and every $\delta' < \delta$,

$$\bigcup_{n \geq N} (k_n 2^{-j_n} - 2^{-\delta'_n j_n}, k_n 2^{-j_n} + 2^{-\delta'_n j_n})$$

is a covering of G_δ with intervals of diameter smaller than $2 \cdot 2^{-\inf_{n \geq N} \delta'_n j_n}$. Let us fix $N_0 \in \mathbb{N}$. Then, for every $\kappa > 0$, there is $N \geq N_0$ such that $2 \cdot 2^{-\inf_{n \geq N} \delta'_n j_n} < \kappa$. It follows that with probability one, we have

$$\begin{aligned} \mathcal{H}_\kappa^s(G_\delta) &\leq \sum_{n \geq N} \left(\text{diam} (k_n 2^{-j_n} - 2^{-\delta'_n j_n}, k_n 2^{-j_n} + 2^{-\delta'_n j_n}) \right)^s \\ &= 2^s \sum_{j \geq j_N} \#F_j \cdot 2^{(\gamma\eta+\varepsilon)j} 2^{-s \inf_{n \geq N_0} \delta'_n j} \\ &\leq 2^s \sum_{j \geq j_N} 2^{(\gamma\eta+\varepsilon)j} 2^{-s \inf_{n \geq N_0} \delta'_n j}. \end{aligned}$$

If $s > \frac{1}{\inf_{n \geq N_0} \delta'_n} (\eta\gamma + \varepsilon)$, we obtain that

$$\mathcal{H}^s(G_\delta) \leq \sum_{j \in \mathbb{N}} 2^{(\gamma\eta+\varepsilon)j} 2^{-s \inf_{n \geq N_0} \delta'_n j} < +\infty$$

and therefore, $\dim_{\mathcal{H}}(G_\delta) \leq s$. Since $N_0 \in \mathbb{N}$ and $\varepsilon > 0$ are arbitrary, we get the conclusion. \square

Remark 7.2.9. With the same arguments, we also have that with probability one, for every $\delta \in (0, 1]$, $\dim_{\mathcal{H}}(E_\delta) \leq \gamma \frac{\eta}{\delta}$. Moreover, this result does not depend on the chosen sequence $(\delta_n)_{n \in \mathbb{N}}$ which converges to δ .

Obtaining a lower bound for the Hausdorff dimension of G_δ is more delicate. We will use the following result of Beresnevich and Velani [31]. It is simplified for the particular application we have in mind.

Theorem 7.2.10 (General mass transference principle). [31] *Let X be a compact set in \mathbb{R}^n and assume that there exist $s \leq n$ and $a, b, \varepsilon_0 > 0$ such that*

$$a\varepsilon^s \leq \mathcal{H}^s(B \cap X) \leq b\varepsilon^s$$

for any ball B of center $x \in X$ and of radius $\varepsilon \leq \varepsilon_0$. Let $s' > 0$. Given a ball $B = B(x, \varepsilon)$ with center in X , we set

$$B^{s'} = B\left(x, \varepsilon^{\frac{s'}{s}}\right).$$

Assume that $(B_n)_{n \in \mathbb{N}}$ is a sequence of balls with center in X and radius ε_n such that $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$. If

$$\mathcal{H}^s\left(X \cap \limsup_{n \rightarrow +\infty} B_n^{s'}\right) = \mathcal{H}^s(X),$$

then

$$\mathcal{H}^{s'}\left(X \cap \limsup_{n \rightarrow +\infty} B_n\right) = \mathcal{H}^{s'}(X).$$

Before applying this theorem in our case, let us prove the following lemma. It is a deeper result than Lemma 7.2.2.

Lemma 7.2.11. *With probability one, there is a sequence $(\eta_n)_{n \in \mathbb{N}}$ of real numbers smaller than η which converges to η such that*

$$C(r) \subseteq \limsup_{n \rightarrow +\infty} (k_n 2^{-j_n} - 2^{-\eta_n j_n}, k_n 2^{-j_n} + 2^{-\eta_n j_n}).$$

Proof. Let us first define a decreasing sequence $(\varepsilon_m)_{m \in \mathbb{N}}$ which converges to 0. For every $j_0 \in \mathbb{N}$, if there is $j \leq j_0$ such that

$$j_0 = \lfloor \frac{1}{\eta}(j + \log_2 j) \rfloor + 1,$$

we consider

$$\varepsilon_{j_0} > \frac{\log_2 j + 2\eta}{j + \log_2 j + 2\eta}.$$

Since the second member converges to 0, we can assume that ε_{j_0} tends to 0 as j_0 tends to infinity. We complete then this subsequence to obtain a sequence $(\varepsilon_m)_{m \in \mathbb{N}}$ of positive numbers which converges to 0.

Using the Borel Cantelli lemma, let us show that there exists $J \in \mathbb{N}$ such that for every $j \geq J$ and every $\lambda \in \Lambda_j$ for which $\lambda \cap C(r) \neq \emptyset$, there is $n \in \mathbb{N}$ such that $\lambda(j_n, k_n) \subseteq \lambda$ and $j_n \eta(1 - \varepsilon_{j_n}) < j$. For every $j \in \mathbb{N}$, let us denote by A_j the event “there exists $\lambda \in \Lambda_j$ such that $\lambda \cap C(r) \neq \emptyset$ and for every $n \in \mathbb{N}$ such that $\lambda(j_n, k_n) \subseteq \lambda$, $j_n \eta(1 - \varepsilon_{j_n}) \geq j$ ”. Equivalently, A_j is the event “there exists $\lambda \in \Lambda_j$ such that $\lambda \cap C(r) \neq \emptyset$ and for every $j' \geq j$ such that $\eta j'(1 - \varepsilon_{j'}) < j$, if $\lambda(j', k') \subseteq \lambda$, then $c_{j', k'} = 0$ ”. Let us consider

$$j_0 = \lfloor \frac{1}{\eta}(j + \log_2 j) \rfloor + 1.$$

Remark that $j_0 \geq j$ and, from our choice of ε_{j_0} , we have $\eta j_0(1 - \varepsilon_{j_0}) \leq j$. Therefore, we can proceed as in Lemma 7.2.2 to get that

$$\begin{aligned} \mathbb{P}[A_j] &\leq \sum_{\lambda: \lambda \cap C(r) \neq \emptyset} \prod_{\lambda_0 \subseteq \lambda, \lambda_0 \in \Lambda_{j_0}} \mathbb{P}[|c_{\lambda_0}| = 0] \\ &\leq \left(\frac{2}{e}\right)^j \end{aligned}$$

which is the general term of a converging series. The Borel Cantelli lemma implies that there is $J \in \mathbb{N}$ such that for every $j \geq J$ and every $\lambda \in \Lambda_j$ for which $\lambda \cap C(r) \neq \emptyset$, there is $n \in \mathbb{N}$ such that $\lambda(j_n, k_n) \subseteq \lambda$ and $j_n \eta(1 - \varepsilon_{j_n}) < j$.

Let us now consider $x \in C(r)$. Then, $\lambda_j(x) \cap C(r) \neq \emptyset$ and if $j \geq J$, there is (j_n, k_n) such that $\lambda(j_n, k_n) \subseteq \lambda_j(x)$ and $j_n \eta(1 - \varepsilon_{j_n}) < j$. This event happens with probability one, independently of the choice of $x \in C(r)$. Since x and $k_n 2^{-j_n}$ belong to $\lambda_j(x)$, it follows that

$$|x - k_n 2^{-j_n}| < 2^{-j} < 2^{-j_n \eta(1 - \varepsilon_{j_n})}.$$

We get the conclusion taking the sequence $(\eta_n)_{n \in \mathbb{N}}$ defined by $\eta_n = \eta(1 - \varepsilon_{j_n})$. \square

We can now obtain a lower bound for the Hausdorff dimension of G_δ .

Proposition 7.2.12. *With probability one, for every $\delta \in [\eta, 1]$, $\dim_{\mathcal{H}}(G_\delta) \geq \gamma \frac{\eta}{\delta}$.*

Proof. This result is a simple application of Theorem 7.2.10. With probability one, from Lemma 7.2.11, there is a sequence $(\eta_n)_{n \in \mathbb{N}}$ which converges to η such that $\eta_n \leq \eta$ for every $n \in \mathbb{N}$ and

$$C(r) \subseteq \limsup_{n \rightarrow +\infty} (k_n 2^{-j_n} - 2^{-\eta_n j_n}, k_n 2^{-j_n} + 2^{-\eta_n j_n}).$$

For every $n \in \mathbb{N}$, there is $x_n \in C(r)$ such that

$$(k_n 2^{-j_n} - 2^{-\eta_n j_n}, k_n 2^{-j_n} + 2^{-\eta_n j_n}) \cap C(r) \subseteq (x_n - 2 \cdot 2^{-\eta_n j_n}, x_n + 2 \cdot 2^{-\eta_n j_n}) \cap C(r).$$

Then, if we set $\eta'_n = \eta_n - \frac{1}{j_n}$ for every $n \in \mathbb{N}$, the previous inclusions give

$$C(r) = \limsup_{n \rightarrow +\infty} (x_n - 2^{-\eta'_n j_n}, x_n + 2^{-\eta'_n j_n}) \cap C(r)$$

with probability one. Consequently, we have

$$\mathcal{H}^\gamma \left(C(r) \cap \limsup_{n \rightarrow +\infty} (x_n - 2^{-\eta'_n j_n}, x_n + 2^{-\eta'_n j_n}) \right) = \mathcal{H}^\gamma(C(r)).$$

Moreover, it is known that $\mathcal{H}^\gamma(C(r)) = 1$ and that there are $a, b > 0$ such that

$$a\varepsilon^\gamma \leq \mathcal{H}^\gamma(B(x, \varepsilon) \cap C(r)) \leq b\varepsilon^\gamma$$

for every $x \in C(r)$ and $0 < \varepsilon \leq 1$, see [65, 109] for example. Let us fix $\delta \in [\eta, 1]$ and let us consider the sequence $(\delta_n)_{n \in \mathbb{N}}$ defined by

$$\delta_n = \delta \frac{\eta'_n}{\eta}, \quad \forall n \in \mathbb{N}.$$

By taking $s' = \gamma \frac{\eta}{\delta} \leq \gamma$, Theorem 7.2.10 gives that

$$\mathcal{H}^{\gamma \frac{\eta}{\delta}} \left(C(r) \cap \limsup_{n \rightarrow +\infty} (x_n - 2^{-\delta_n j_n}, x_n + 2^{-\delta_n j_n}) \right) = \mathcal{H}^{\gamma \frac{\eta}{\delta}}(C(r)) > 0.$$

If the set E_δ is defined with the increasing sequence $(\delta_n)_{n \in \mathbb{N}}$ which converges to δ , it follows that

$$\dim_{\mathcal{H}} E_\delta \geq \gamma \frac{\eta}{\delta}.$$

If $\delta < 1$, we have

$$G_\delta = \bigcap_{0 < \delta' < \delta} E_{\delta'} \setminus \bigcup_{\delta < \delta' \leq 1} E_{\delta'}.$$

Remark that $E_{\delta''} \subseteq E_{\delta'}$ if $\delta'' \geq \delta'$. Therefore, the union and the intersection that appear in the definition of G_δ can be taken countable by considering subsequences converging to δ . If $s = \gamma \frac{\eta}{\delta}$, let us show that $\mathcal{H}^s(G_\delta) > 0$. With probability one, we have

$$\mathcal{H}^s(G_\delta) = \mathcal{H}^s\left(\bigcap_{\delta' < \delta} E_{\delta'}\right) - \mathcal{H}^s\left(\bigcup_{\delta' > \delta} E_{\delta'}\right) = \mathcal{H}^s\left(\bigcap_{\delta' < \delta} E_{\delta'}\right)$$

since the s -dimensional Hausdorff measure of $E_{\delta'}$ vanishes if $\delta' > \delta$ using Remark 7.2.9. It follows that

$$\mathcal{H}^s(G_\delta) = \mathcal{H}^s\left(\bigcap_{\delta' < \delta} E_{\delta'}\right) \geq \mathcal{H}^s(E_\delta) > 0,$$

and it follows that $\dim_{\mathcal{H}}(G_\delta) \geq \gamma \frac{\eta}{\delta}$.

If $\delta = 1$,

$$\mathcal{H}^{\gamma\eta}(G_\delta) = \mathcal{H}^{\gamma\eta}\left(\bigcap_{\delta' < 1} E_{\delta'}\right) \geq \mathcal{H}^s(E_1) > 0,$$

hence the conclusion. \square

Combining Propositions 7.2.5, 7.2.8 and 7.2.12 with the possible values for the Hölder exponent, we get the announced Theorem 7.2.1.

7.2.2 Wavelet leaders density of $R_{\alpha,\eta,r}$

In this Section, we compute the almost sure wavelet leaders density of $R_{\alpha,\eta,r}$.

Proposition 7.2.13. *Let \vec{c} denote the sequence of wavelet coefficients of $R_{\alpha,\eta,r}$ given by (7.1). With probability one, $\tilde{\rho}_{\vec{c}} = d_{R_{\alpha,\eta,r}}$ on $[0, +\infty]$.*

Proof. From Theorem 5.2.5 and Remark 5.2.6, it suffices to show that $\tilde{\rho}_{\vec{c}}^*(h) \leq d_{R_{\alpha,\eta,r}}(h)$ for every $h \in [0, +\infty]$. Of course, we have $\tilde{\rho}_{\vec{c}}^*(+\infty) \leq 1$. So, we can assume that $h < +\infty$.

1. Assume that $h \in [0, \alpha)$. We know that $|c_\lambda| \leq 2^{-\alpha j}$ for every $j \in \mathbb{N}$, $\lambda \in \Lambda_j$. Hence, $\tilde{\rho}_{\vec{c}}^*(h) = -\infty = d_{R_{\alpha,\eta,r}}(h)$.

2. Assume that $h \in [\alpha, \frac{\alpha}{\eta}]$. With probability one, we know from Lemma 7.2.7 that for every $\varepsilon > 0$, there is $J \in \mathbb{N}$ such that

$$\#F_j = \#\{\lambda \in \Lambda_j : c_\lambda = 2^{-\alpha j}\} \leq 2^{(\gamma\eta + \varepsilon)j}$$

for every $j \geq J$. Moreover, the wavelet coefficients c_λ of $R_{\alpha,\eta,r}$ at a scale j only take the values $2^{-\alpha j}$ or 0. Therefore, we get that with probability one, for every $\varepsilon > 0$, $\nu_{R_{\alpha,\eta,r}}(h) \leq \eta\gamma + \varepsilon$, where $\nu_{R_{\alpha,\eta,r}}$ denotes the wavelet profile of $R_{\alpha,\eta,r}$. Consequently,

$$h \sup_{h' \in (0, h]} \frac{\nu_{R_{\alpha,\eta,r}}(h')}{h'} \leq h \sup_{h' \in [\alpha, h]} \frac{\eta\gamma + \varepsilon}{h'} = h \frac{\eta\gamma + \varepsilon}{\alpha}$$

Since $\varepsilon > 0$ is arbitrary, we get that with probability one,

$$h \sup_{h' \in (0, h]} \frac{\nu_{R_{\alpha,\eta,r}}(h')}{h'} \leq h \frac{\eta\gamma}{\alpha} = d_{R_{\alpha,\eta,r}}(h).$$

Using Proposition 5.5.3, we obtain then

$$d_{R_{\alpha,\eta,r}}(h) \leq \tilde{\rho}_{\tilde{c}}^*(h) \leq \tilde{\nu}_{R_{\alpha,\eta,r}}^+(h) \leq h \sup_{h' \in (0,h]} \frac{\nu_{R_{\alpha,\eta,r}}(h')}{h'} \leq d_{R_{\alpha,\eta,r}}(h).$$

This events happens with probability one, independently of the choice of $h \in [\alpha, \frac{\alpha}{\eta}]$, so that we have the announced equality on $[\alpha, \frac{\alpha}{\eta}]$.

3. Finally, assume that $h \in (\frac{\alpha}{\eta}, +\infty)$ and let us show that

$$\# \{ \lambda \in \Lambda_j : 0 < e_\lambda < 2^{-hj} \} = 0$$

for every j large enough. From Lemma 7.2.2, with probability one, we have

$$\# \{ \lambda \in \Lambda_j : 0 < e_\lambda < 2^{-hj} \} = \# \{ \lambda \in \Lambda_j : \lambda \cap C(r) = \emptyset \text{ and } 0 < e_\lambda < 2^{-hj} \}$$

for every $h \in (\frac{\alpha}{\eta}, +\infty)$, since $e_\lambda \geq 2^{-hj}$ if $\lambda \cap C(r) \neq \emptyset$ and if j is large enough. From the construction of $R_{\alpha,\eta,r}$, if $\lambda \cap C(r) = \emptyset$, we have either $e_\lambda = 0$ or $e_\lambda = 2^{-\alpha j_0}$ where $j_0 \geq j$ is such that $\eta(j_0 - 1) < j + \log_2 j$. In this last case, we have $e_\lambda > 2^{-hj}$ if j is large enough and the conclusion follows. \square

Let us remark that we also get that $\tilde{\nu}_{R_{\alpha,\eta,r}}^+ = d_{R_{\alpha,\eta,r}}$ on $[0, \frac{\alpha}{\eta}] \cup \{+\infty\}$.

7.3 Prescribed multifractal spectrum

Given an admissible profile ν such that $\alpha_{\min} > 0$, the aim of this section is to construct a function f such that

$$\tilde{\nu}_f = \nu = d_f \quad \text{on} \quad [0, +\infty].$$

We assume that $\nu(\alpha) > 0$ if $\alpha \in (\alpha_{\min}, \alpha_{\max})$. We will consider separately the increasing and decreasing part of the admissible profile. More precisely, in Proposition 7.3.6, we construct a function f^+ such that

$$\tilde{\nu}_{f^+}(h) = d_{f^+}(h) = \begin{cases} \nu(h) & \text{if } h \in [0, \alpha_s], \\ -\infty & \text{if } h \in (\alpha_s, +\infty]. \end{cases}$$

Similarly, in Proposition 7.3.9, we construct a function f^- such that

$$\tilde{\nu}_{f^-}(h) = d_{f^-}(h) = \begin{cases} -\infty & \text{if } h \in [0, \alpha_s), \\ \nu(h) & \text{if } h \in [\alpha_s, +\infty]. \end{cases}$$

If we consider the function $f = f^+ \circ l^+ + f^- \circ l^-$, where l^+ (reps. l^-) is the unique affine increasing map from $[0, 1]$ to $[\frac{1}{2}, 1]$ (resp $[0, \frac{1}{2}]$), we will then obtain the following theorem.

Theorem 7.3.1. *Let ν be an admissible profile with $\alpha_{\min} > 0$. If $\nu(\alpha) > 0$ for every $\alpha \in (\alpha_{\min}, \alpha_{\max})$, then there is a function f such that*

$$\tilde{\nu}_f = \nu = d_f \quad \text{on} \quad [0, +\infty].$$

Corollary 7.3.2. *Let ν be an admissible profile such that $\alpha_{\min} > 0$ and $\nu(\alpha) > 0$ for every $\alpha \in (\alpha_{\min}, \alpha_{\max})$. The set of functions $f \in \mathcal{L}^\nu$ such that $\tilde{\nu}_f = \nu = d_f$ on $[0, +\infty]$ is \mathfrak{c} -dense lineable in $(\mathcal{L}^\nu, \tilde{\delta})$.*

Proof. Using Theorem 7.3.1, we consider a function f such that $\tilde{\nu}_f = \nu = d_f$. Let \vec{c} denote the sequence of wavelet coefficients of f in a given wavelet basis. We construct then the same vector space \mathcal{D} as in Proposition 6.7.2, using the the sequence \vec{c} . We already know that \mathcal{D} is dense in $(\mathcal{L}^\nu, \delta)$, has dimension \mathfrak{c} and that for every $\vec{z} \in \mathcal{D} \setminus \{0\}$, $\tilde{\nu}_{\vec{z}} = \nu$. Let us now prove that, if the wavelet coefficients of a function g are given by $\vec{z} \in \mathcal{D} \setminus \{0\}$, then $d_g = \nu$. By construction of \mathcal{D} , there are $J \in \mathbb{N}$, $r_1, \dots, r_N > 0$ ($N \in \mathbb{N}$) and $\theta_1, \dots, \theta_N \in \mathbb{C}$ not all equal to 0 such that

$$z_\lambda = \left(\theta_1 \frac{1}{j^{r_1}} + \dots + \theta_N \frac{1}{j^{r_N}} \right) \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|$$

for every $j \geq J$, $\lambda \in \Lambda_j$. Consequently, as done in Proposition 6.2.4, for every $\varepsilon > 0$, there is $J_0 \geq J$ such that

$$\sup_{\lambda' \subseteq 3\lambda} |z_\lambda| \leq \sup_{\lambda' \subseteq 3\lambda} |c_{\lambda'}| \quad \text{and} \quad \sup_{\lambda' \subseteq 3\lambda} |z_\lambda| \geq 2^{-\frac{\varepsilon}{2}j} \sup_{\lambda' \subseteq 3\lambda} |c_{\lambda'}|$$

if $j \geq J_0$. Using Proposition 4.7.4, we get that the Hölder exponents of g are the same as those of f , and therefore, $d_g = d_f = \nu$. This concludes the proof. \square

7.3.1 Increasing part

Given an admissible profile ν , our goal is to construct a function f^+ such that

$$\tilde{\nu}_{f^+}^+ = \nu = d_{f^+} \quad \text{on} \quad [0, \alpha_s] \quad \text{and} \quad \tilde{\nu}_{f^+}^- = d_{f^+} = -\infty \quad \text{on} \quad (\alpha_s, +\infty].$$

Lemma 7.3.3. *Let $\alpha_s \in (0, +\infty)$. We denote by $\mathcal{A}^+(\alpha_s)$ the set of functions θ of the form*

$$\theta(h) = \begin{cases} \frac{\gamma h}{\beta} & \text{if } h \in [\alpha, \beta], \\ 1 & \text{if } h = \alpha_s, \\ -\infty & \text{otherwise,} \end{cases}$$

with $\gamma \in (0, 1)$ and $0 < \alpha < \beta < \alpha_s$. For every $\theta \in \mathcal{A}^+(\alpha_s)$, there is a function f such that $d_f = \theta$ on $[0, +\infty]$ and

$$\tilde{\nu}_f^+(h) = \begin{cases} -\infty & \text{if } h < \alpha, \\ \frac{\gamma h}{\beta} & \text{if } h \in [\alpha, \beta], \\ \gamma & \text{if } h \in [\beta, \alpha_s), \\ 1 & \text{if } h \geq \alpha_s. \end{cases}$$

Proof. Let us consider $\theta \in \mathcal{A}^+(\alpha_s)$. We fix $\eta \in (0, 1)$ such that $\frac{\alpha}{\eta} = \beta$ and $r \in (0, \frac{1}{2})$ such that $\gamma = -\frac{\log 2}{\log r}$. We will slightly modify the function $R_{\alpha, \eta, r}$ constructed in Section 7.2 into a function $R_{\alpha, \eta, r}^+$ as follows. Let \vec{c} be the sequence of wavelet coefficients of $R_{\alpha, \eta, r}$ given by (7.1). The sequence \vec{c}^+ of wavelet coefficients of $R_{\alpha, \eta, r}^+$ is defined by

$$c_{j,k}^+ := \begin{cases} c_{j,k} & \text{if } c_{j,k} \neq 0, \\ 2^{-\alpha_s j} & \text{if } c_{j,k} = 0. \end{cases}$$

With probability one, we know from Lemma 7.2.2 that for every $\varepsilon > 0$, $d_\lambda \geq 2^{-j(\frac{\alpha}{\eta} + \varepsilon)}$ at large scales j if $\lambda \cap C(r) \neq \emptyset$. If $\varepsilon > 0$ is chosen small enough, we have

$$2^{-j(\frac{\alpha}{\eta} + \varepsilon)} \geq 2^{-\alpha_s j}$$

and the wavelet leader d_λ is not modified by our construction, i.e. $d_\lambda^+ = d_\lambda$. It follows that the Hölder exponents of $R_{\alpha, \eta, r}^+$ at points of the Cantor set are not modified. If $x \notin C(r)$, then $3\lambda_j(x) \cap C(r) = \emptyset$ for large j and it follows that the Hölder exponent of $R_{\alpha, \eta, r}^+$ at x is α_s . Following the results of Section 7.2, we get that with probability one,

$$d_{R_{\alpha, \eta, r}^+}(h) = \begin{cases} \gamma h \frac{\eta}{\alpha} & \text{if } h \in [\alpha, \frac{\alpha}{\eta}], \\ 1 & \text{if } h = \alpha_s, \\ -\infty & \text{otherwise.} \end{cases}$$

Let us now compute the increasing wavelet leaders profile of $R_{\alpha, \eta, r}^+$. Since $c_{j,k}^+$ takes the value $2^{-\alpha_s j}$ or the value $2^{-\alpha j}$, we have $\tilde{\nu}_{R_{\alpha, \eta, r}^+}^+(h) = -\infty$ if $h < \alpha$. As done in Proposition 7.2.13, if $h \in [\alpha, \frac{\alpha}{\eta}]$, we have $\tilde{\nu}_{R_{\alpha, \eta, r}^+}^+(h) = \gamma h \frac{\eta}{\alpha}$ and if $h \in [\frac{\alpha}{\eta}, \alpha_s)$,

$$\# \{ \lambda \in \Lambda_j : 2^{-\alpha_s j} < d_\lambda \leq 2^{-h j} \} = 0$$

so that $\tilde{\nu}_{R_{\alpha, \eta, r}^+}^+(h) = \gamma$. It follows that the increasing wavelet leaders profile of $R_{\alpha, \eta, r}^+$ is given by

$$\tilde{\nu}_{R_{\alpha, \eta, r}^+}^+(h) = \begin{cases} -\infty & \text{if } h < \alpha, \\ \gamma h \frac{\eta}{\alpha} & \text{if } h \in [\alpha, \frac{\alpha}{\eta}], \\ \gamma & \text{if } h \in [\frac{\alpha}{\eta}, \alpha_s), \\ 1 & \text{if } h \geq \alpha_s. \end{cases}$$

□

Remark 7.3.4. We also include in $\mathcal{A}^+(\alpha_s)$ the degenerate cases where

$$\theta(h) = \begin{cases} 0 & \text{if } h = \alpha, \\ 1 & \text{if } h = \alpha_s, \\ -\infty & \text{otherwise,} \end{cases}$$

with $\alpha < \alpha_s$. Remark that in this case, if f is the function whose wavelet coefficients are given by $c_{j,k} := 2^{-\alpha j}$ if $k = 0$ and $c_{j,k} := 2^{-\alpha_s j}$ otherwise, then we have

$$\tilde{\nu}_f^+(h) = \begin{cases} -\infty & \text{if } h < \alpha, \\ 0 & \text{if } h \in [\alpha, \alpha_s), \\ 1 & \text{if } h \geq \alpha_s, \end{cases} \quad \text{and} \quad d_f(h) = \begin{cases} 0 & \text{if } h = \alpha, \\ 1 & \text{if } h = \alpha_s, \\ -\infty & \text{otherwise.} \end{cases}$$

Lemma 7.3.5. *Let $\alpha_s \in (0, +\infty)$. For every $\gamma \in (0, 1)$ and $\alpha_0 \in (0, \alpha_s)$, there is a sequence $(\theta_l)_{l \in \mathbb{N}}$ of $\mathcal{A}^+(\alpha_s)$ such that*

$$\sup_{l \in \mathbb{N}} \theta_l(h) = \begin{cases} -\infty & \text{if } h < \alpha_0, \\ \gamma & \text{if } h \in [\alpha_0, \alpha_s), \\ 1 & \text{if } h \geq \alpha_s. \end{cases}$$

Proof. Let us consider a sequence $(\beta_n)_{n \in \mathbb{N}}$ whose elements form a dense subset of (α_0, α_s) . For every $m, n \in \mathbb{N}$, we set $\alpha_{m,n} = \beta_n - \frac{1}{m}$ if $\beta_n - \frac{1}{m} \geq \alpha_0$, and $\alpha_{m,n} = \alpha_0$ if $\beta_n - \frac{1}{m} < \alpha_0$. We consider then the function $\theta_{m,n} \in \mathcal{A}^+(\alpha_s)$ defined by

$$\theta_{m,n}(h) := \begin{cases} \frac{\gamma h}{\beta_n} & \text{if } h \in [\alpha_{m,n}, \beta_n], \\ 1 & \text{if } h = \alpha_s, \\ -\infty & \text{otherwise.} \end{cases}$$

It suffices to show that if $h \in [\alpha_0, \alpha_s)$, then $\sup_{m,n \in \mathbb{N}} \theta_{m,n}(h) = \gamma$. Of course, we have $\sup_{m,n \in \mathbb{N}} \theta_{m,n}(h) \leq \gamma$. Moreover, for every $m \in \mathbb{N}$, there is $n \in \mathbb{N}$ such that $\beta_n \geq h \geq \beta_n - \frac{1}{m}$. Therefore,

$$\theta_{m,n}(h) = \frac{\gamma h}{\beta_n} \geq \gamma \frac{\beta_n - \frac{1}{m}}{\beta_n} \geq \gamma \left(1 - \frac{1}{m\alpha_0}\right)$$

hence the conclusion. \square

Proposition 7.3.6. *Let ν be an admissible profile with $\alpha_{\min} > 0$ and such that $\nu(\alpha) > 0$ if $\alpha \in (\alpha_{\min}, \alpha_{\max})$. Then, there is a function f such that*

$$\tilde{\nu}_f(\alpha) = d_f(\alpha) = \begin{cases} \nu(\alpha) & \text{if } \alpha \in [0, \alpha_s], \\ -\infty & \text{if } \alpha \in (\alpha_s, +\infty). \end{cases}$$

Proof. Since ν is right-continuous and increasing on $[\alpha_{\min}, \alpha_s]$, it is possible to find a sequence $(\gamma_m)_{m \in \mathbb{N}}$ of $(0, 1)$ and a sequence $(\alpha_m)_{m \in \mathbb{N}}$ of $(0, \alpha_s)$ such that

$$\nu = \sup_{m \in \mathbb{N}} \nu_m \quad \text{on } [0, \alpha_s]$$

where

$$\nu_m(h) = \begin{cases} -\infty & \text{if } h < \alpha_m, \\ \gamma_m & \text{if } h \in [\alpha_m, \alpha_s), \\ 1 & \text{if } h \geq \alpha_s. \end{cases}$$

Using Lemma 7.3.5, for every $m \in \mathbb{N}$, there is a sequence $(\theta_{m,l})_{l \in \mathbb{N}}$ of $\mathcal{A}^+(\alpha_s)$ such that

$$\nu_m = \sup_{l \in \mathbb{N}} \theta_{m,l}.$$

Reordering the sequence $(\theta_{m,l})_{m \in \mathbb{N}, l \in \mathbb{N}}$, we get a sequence $(\theta_n)_{n \in \mathbb{N}}$ of $\mathcal{A}^+(\alpha_s)$ such that

$$\nu = \sup_{n \in \mathbb{N}} \theta_n \quad \text{on } [0, \alpha_s].$$

For every $n \in \mathbb{N}$, using Lemma 7.3.3, we know that we can consider f_n such that $d_{f_n} = \theta_n$ on $[0, +\infty]$ and

$$\sup_{n \in \mathbb{N}} \tilde{\nu}_{f_n}^+ = \nu \quad \text{on } [0, \alpha_s].$$

In particular, $f_n \in \mathcal{L}^{\nu,+}$. We denote by $\bar{c}^{(n)}$ the wavelet coefficients of f_n . For every $n \in \mathbb{N}$, there is a unique affine increasing map from $[0, 1]$ to $[2^{-n}, 2^{-n+1}]$. We consider then the function $f_n \circ l_n$; its wavelet coefficient at position (j, k) is given by $c_{j-n, k-2^{j-n}}^{(n)}$ if $j \geq n$ and $k \in \{2^{j-n}, \dots, 2^{j-n+1} - 1\}$, and 0 otherwise. Of course, we still have

$f_n \circ l_n \in \mathcal{L}^{\nu,+}$. Using the continuity of the scalar multiplication in $\mathcal{L}^{\nu,+}$, there is $\varepsilon_n > 0$ such that

$$\tilde{\delta}^+(\varepsilon_n f_n \circ l_n, 0) < \frac{1}{n^2}.$$

For every $n \in \mathbb{N}$, we define then the function F^n by setting

$$F^n = \varepsilon_1 f_1 \circ l_1 + \cdots + \varepsilon_n f_n \circ l_n.$$

The wavelet coefficients of F^n , denoted by $\vec{C}^{(n)}$, are given at scales $j \geq n$ by

$$C_{j,k}^{(n)} = \begin{cases} \varepsilon_1 c_{j-1,k-2^{j-1}}^{(1)} & \text{if } k \in \{2^{j-1}, \dots, 2^j - 1\}, \\ \varepsilon_2 c_{j-2,k-2^{j-2}}^{(2)} & \text{if } k \in \{2^{j-2}, \dots, 2^{j-1} - 1\}, \\ \vdots & \\ \varepsilon_n c_{j-n,k-2^{j-n}}^{(n)} & \text{if } k \in \{2^{j-n}, \dots, 2^{j-n+1} - 1\}, \\ 0 & \text{if } k \in \{0, \dots, 2^{j-n} - 1\}. \end{cases}$$

Let us prove that $(F^n)_{n \in \mathbb{N}}$ converges in $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$. Using the invariance by translation of the distance in $\mathcal{L}^{\nu,+}$, we have

$$\tilde{\delta}^+(F^P, F^Q) \leq \sum_{N=P+1}^Q \tilde{\delta}^+(F^{N-1}, F^N) = \sum_{N=P+1}^Q \tilde{\delta}^+(\varepsilon_N f_N \circ l_N, 0) \leq \sum_{N=P+1}^Q \frac{1}{N^2}$$

and it follows that the sequence $(F^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$. From Proposition 6.4.17, we know that the space $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$ is complete and it follows that $(F^n)_{n \in \mathbb{N}}$ converges in $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$. Let us denote by F this limit. In fact, if \vec{c} denotes the sequence of wavelet coefficients of F , we have

$$c_{j,k} = \begin{cases} \varepsilon_1 c_{j-1,k-2^{j-1}}^{(1)} & \text{if } k \in \{2^{j-1}, \dots, 2^j - 1\}, \\ \varepsilon_2 c_{j-2,k-2^{j-2}}^{(2)} & \text{if } k \in \{2^{j-2}, \dots, 2^{j-1} - 1\}, \\ \vdots & \\ \varepsilon_{j-1} c_{1,k-2}^{(j-1)} & \text{if } k \in \{2, 3\}, \\ \varepsilon_j c_{0,0}^{(j)} & \text{if } k = 1, \\ 0 & \text{if } k = 0, \end{cases}$$

for every $j \in \mathbb{N}_0$, similarly to what was done in Proposition 6.2.1, since the convergence in $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$ is stronger than the pointwise convergence. As we have $F \in \mathcal{L}^{\nu,+}$, we get $\tilde{\nu}_F^+ \leq \nu$ on $[0, \alpha_s]$. Moreover, since affine mappings do not modify the multifractal spectrum (see Proposition 4.2.5) and since the intervals $(2^{-n}, 2^{-n+1})$ are disjoint, it is clear that

$$d_F = \sup_{n \in \mathbb{N}} d_{f_n} = \nu \quad \text{on } [0, \alpha_s].$$

Remark that problems may possibly occur at the points 2^{-n} , $n \in \mathbb{N}$, and at 0. Nevertheless, even if the Hölder exponent is modified at those points, it will still be in $[\alpha_{\min}, \alpha_s]$

and since those points form a countable set, using Proposition 4.2.4, it does not change the Hausdorff dimension of the iso-Hölder sets. Since the wavelet leaders profile gives an upper bound for the multifractal spectrum, we get

$$\tilde{\nu}_F^+ = d_F = \nu \quad \text{on} \quad [0, \alpha_s].$$

It remains to modify slightly the function F in order to have

$$\tilde{\nu}_F^- = d_F = -\infty \quad \text{on} \quad (\alpha_s, +\infty].$$

We consider the function F^* whose wavelet coefficients at scale $j \in \mathbb{N}$ are given by

$$c_\lambda^* = \max(d_\lambda, 2^{-\alpha_s j}), \quad \forall \lambda \in \Lambda_j,$$

where d_λ denote the wavelet leaders of F . Let us show that the wavelet leaders d_λ^* of F^* are its wavelet coefficients.

- Assume that $c_\lambda^* = d_\lambda > 2^{-\alpha_s j}$ and let $\lambda' \subseteq 3\lambda$. If $d_{\lambda'} \geq 2^{-\alpha_s j'}$, then we have $c_{\lambda'}^* = d_{\lambda'} \leq d_\lambda = c_\lambda^*$ and if $d_{\lambda'} < 2^{-\alpha_s j'}$, then $c_{\lambda'}^* = 2^{-\alpha_s j'} \leq 2^{-\alpha_s j} < c_\lambda^*$.
- Assume that $c_\lambda^* = 2^{-\alpha_s j} \geq d_\lambda$ and let $\lambda' \subseteq 3\lambda$. If $d_{\lambda'} \geq 2^{-\alpha_s j'}$, then we have $c_{\lambda'}^* = d_{\lambda'} < d_\lambda \leq c_\lambda^*$ and if $d_{\lambda'} < 2^{-\alpha_s j'}$, then $c_{\lambda'}^* = 2^{-\alpha_s j'} \leq 2^{-\alpha_s j} = c_\lambda^*$.

Clearly, the increasing wavelet leaders profile of F and its multifractal spectrum are not modified on $[0, \alpha_s]$. Moreover, we have $d_\lambda^* \geq 2^{-\alpha_s j}$ for every $\lambda \in \Lambda_j$ so that

$$\tilde{\nu}_{F^*}^- = d_{F^*} = -\infty \quad \text{on} \quad (\alpha_s, +\infty],$$

hence the conclusion. □

7.3.2 Decreasing part

Similarly to what is done in the previous subsection, given an admissible profile ν , our aim is to construct a function f^- such that

$$\tilde{\nu}_{f^-}^+ = d_{f^-} = -\infty \quad \text{on} \quad [0, \alpha_s) \quad \text{and} \quad \tilde{\nu}_{f^-}^- = \nu = d_{f^-} \quad \text{on} \quad [\alpha_s, +\infty].$$

Lemma 7.3.7. *Let $\alpha_s \in (0, +\infty)$. We denote by $\mathcal{A}^-(\alpha_s)$ the set of functions θ of the form*

$$\theta(h) = \begin{cases} \frac{\gamma h}{\beta} & \text{if } h \in [\alpha, \beta], \\ 1 & \text{if } h = \alpha_s, \\ -\infty & \text{otherwise,} \end{cases}$$

with $\gamma \in (0, 1)$ and $\alpha_s < \alpha < \beta < +\infty$. For every $\theta \in \mathcal{A}^-(\alpha_s)$, there is a function f such that $d_f = \theta$ on $[0, +\infty]$ and

$$\tilde{\nu}_f^-(h) = \begin{cases} 1 & \text{if } h \leq \alpha_s, \\ 1 + \frac{h}{\beta}(\gamma - 1) & \text{if } h \in (\alpha_s, \beta], \\ -\infty & \text{if } h > \beta. \end{cases}$$

Proof. Let us fix $\eta \in (0, 1)$ such that $\frac{\alpha}{\eta} = \beta$ and $r \in (0, \frac{1}{2})$ such that $\gamma = -\frac{\log 2}{\log r}$. Let us again modify the function $R_{\alpha, \eta, r}$ constructed in Section 7.2 into a function $R_{\alpha, \eta, r}^-$. The function $R_{\alpha, \eta, r}^-$ is defined through its sequence \tilde{c}^- of wavelet coefficients, with

$$\tilde{c}_{j,k}^- := \begin{cases} c_{j,k} & \text{if } k \in K_j, \\ 0 & \text{if } k \notin K_j \text{ and if there is } \lambda' : \lambda' \cap C(r) \neq \emptyset, \lambda \subseteq 3\lambda' \text{ and } \alpha_s j \leq \frac{\alpha}{\eta} j', \\ 2^{-\alpha_s j} & \text{otherwise,} \end{cases}$$

where $c_{j,k}$ denote the wavelet coefficients of $R_{\alpha, \eta, r}$ given by (7.1). With probability one, if $x \in C(r)$, we know that the Hölder exponent of $R_{\alpha, \eta, r}$ at x is in $[\alpha, \frac{\alpha}{\eta}]$. From our construction, if $\lambda_0 \subseteq 3\lambda_j(x)$ is such that $c_{\lambda_0} = 2^{-\alpha_s j_0}$, we have $\alpha_s j_0 > \frac{\alpha}{\eta} j$. Consequently, we obtain that the Hölder exponent of $R_{\alpha, \eta, r}^-$ at x is the same as the Hölder exponent of $R_{\alpha, \eta, r}$ at x . If x is not in the Cantor set, then $3\lambda_j(x) \cap C(r) = \emptyset$ for large j and it follows that the Hölder exponent of $R_{\alpha, \eta, r}^-$ at x is α_s . Following the results of Section 7.2, we obtain that with probability one,

$$d_{R_{\alpha, \eta, r}^-}(h) = \begin{cases} 1 & \text{if } h = \alpha_s, \\ \gamma h \frac{\eta}{\alpha} & \text{if } h \in [\alpha, \frac{\alpha}{\eta}], \\ -\infty & \text{otherwise.} \end{cases}$$

Let us now compute the decreasing wavelet leaders profile of $R_{\alpha, \eta, r}^-$. We have $\tilde{\nu}_{R_{\alpha, \eta, r}^-}^-(\alpha_s) \geq d_{R_{\alpha, \eta, r}^-}(\alpha_s) = 1$ so that $\tilde{\nu}_{R_{\alpha, \eta, r}^-}^-(h) = 1$ if $h \leq \alpha_s$. It suffices then to study the case $h > \alpha_s$.

First, let us assume that $h > \frac{\alpha}{\eta}$. Let us consider three different possibilities.

- If $\lambda \cap C(r) \neq \emptyset$, since $d_{\lambda}^- \geq d_{\lambda}$, Lemma 7.2.2 gives that with probability one, for every $h > \frac{\alpha}{\eta}$, $d_{\lambda}^- > 2^{-hj}$ if j is large enough.
- If $\lambda \cap C(r) = \emptyset$ and if for every λ' such that $\lambda' \cap C(r) \neq \emptyset$ and $\lambda \subseteq 3\lambda'$, we have $\alpha_s j > \frac{\alpha}{\eta} j'$, then $d_{\lambda}^- = 2^{-\alpha_s j} > 2^{-hj}$.
- In the last case, $\lambda \cap C(r) = \emptyset$ and there is λ' such that $\lambda' \cap C(r) \neq \emptyset$, $\lambda \subseteq 3\lambda'$ and $\alpha_s j \leq \frac{\alpha}{\eta} j'$. Let $j' < j$ be the maximal integer such that such a $\lambda' \in \Lambda_{j'}$ exists. In this case, $d_{\lambda}^- \geq 2^{-\alpha_s j_0}$ where j_0 is the smallest integer such that $\alpha_s j_0 > \frac{\alpha}{\eta} j'$. In particular, $\alpha_s(j_0 - 1) \leq \frac{\alpha}{\eta} j'$ and we get that

$$d_{\lambda}^- \geq 2^{-\frac{\alpha}{\eta} j'} 2^{-\alpha_s} \geq 2^{-\frac{\alpha}{\eta} j} 2^{-\alpha_s} > 2^{-hj}$$

if j is large enough.

It follows that if $h > \frac{\alpha}{\eta}$, then $\tilde{\nu}_{R_{\alpha, \eta, r}^-}^-(h) = -\infty$.

Secondly, let us assume that $h \in (\alpha_s, \frac{\alpha}{\eta}]$. We know that

$$d_{R_{\alpha, \eta, r}^-} \left(\frac{\alpha}{\eta} \right) = \gamma \leq \tilde{\nu}_{R_{\alpha, \eta, r}^-}^-\left(\frac{\alpha}{\eta} \right).$$

Using Proposition 5.3.3, we obtain

$$\frac{\tilde{\nu}_{R_{\alpha, \eta, r}^-}^-(h) - 1}{h} \geq \frac{\tilde{\nu}_{R_{\alpha, \eta, r}^-}^-\left(\frac{\alpha}{\eta}\right) - 1}{\frac{\alpha}{\eta}} \geq \frac{\gamma - 1}{\frac{\alpha}{\eta}}$$

so that

$$\tilde{\nu}_{R_{\alpha, \eta, r}^-}^-(h) \geq 1 + h \frac{\eta}{\alpha} (\gamma - 1).$$

It suffices then to prove the other inequality. Let us fix $\varepsilon > 0$ such that $h - \varepsilon > \alpha_s$ and let us estimate the number of wavelet leaders d_{λ}^- smaller than $2^{-(h-\varepsilon)j}$.

- If $\lambda \cap C(r) \neq \emptyset$, then $d_{\lambda}^- \geq d_{\lambda}$. We know that for every $\delta > 0$, there are less than $2^{(\gamma h \frac{\eta}{\alpha} + \delta)j}$ dyadic intervals λ which intersect $C(r)$ with $d_{\lambda} \leq 2^{-(h-\varepsilon)j}$ if j is large enough. Then, the same holds for the wavelet leaders d_{λ}^- .
- If $\lambda \cap C(r) = \emptyset$ and if for every λ' such that $\lambda' \cap C(r) \neq \emptyset$ and $\lambda \subseteq 3\lambda'$, we have $\alpha_s j > \frac{\alpha}{\eta} j'$, then by construction, $d_{\lambda}^- = c_{\lambda}^- = 2^{-\alpha_s j} > 2^{-(h-\varepsilon)j}$.
- In the last case, $\lambda \cap C(r) = \emptyset$ and there is λ' such that $\lambda' \cap C(r) \neq \emptyset$, $\lambda \subseteq 3\lambda'$ and $\alpha_s j \leq \frac{\alpha}{\eta} j'$. Let $j' < j$ be the maximal integer such that such a $\lambda' \in \Lambda_{j'}$ exists. Then, $d_{\lambda}^- \geq 2^{-\alpha_s j_0}$ where j_0 is the smallest integer such that $\alpha_s j_0 > \frac{\alpha}{\eta} j'$. In particular, $\alpha_s(j_0 - 1) \leq \frac{\alpha}{\eta} j'$ and $d_{\lambda}^- \geq 2^{-\frac{\alpha}{\eta} j' - \alpha_s}$. As done in Remark 7.2.6,

$$\#\{\lambda' \in \Lambda_{j'} : \lambda' \cap C(r) \neq \emptyset\} \leq 2^{\gamma j' + 2}.$$

It follows that the number of dyadic intervals λ of that case with $d_{\lambda} \leq 2^{-(h-\varepsilon)j}$ is smaller than

$$\begin{aligned} & \sum_{j' \leq j : \frac{\alpha}{\eta} j' + \alpha_s > (h-\varepsilon)j} 3 \cdot 2^{j-j'} \#\{\lambda' \in \Lambda_{j'} : \lambda' \cap C(r) \neq \emptyset\} \\ & \leq 12 \cdot 2^j \sum_{j' \leq j : \frac{\alpha}{\eta} j' + \alpha_s > (h-\varepsilon)j} 2^{(\gamma-1)j'} \\ & \leq 12j 2^j 2^{(\gamma-1)((h-\varepsilon)j - \alpha_s) \frac{\eta}{\alpha}} \\ & = 12j 2^{(1+(\gamma-1)(h-\varepsilon)\frac{\eta}{\alpha})j} 2^{-(\gamma-1)\alpha_s \frac{\eta}{\alpha}}. \end{aligned}$$

The combination of these three possibilities gives the conclusion. \square

Remark 7.3.8. We also consider in $\mathcal{A}^-(\alpha_s)$ the degenerate cases where

$$\theta(h) = \begin{cases} 0 & \text{if } h = \alpha, \\ 1 & \text{if } h = \alpha_s, \\ -\infty & \text{otherwise,} \end{cases}$$

with $\alpha > \alpha_s$. As done in the second case of Proposition 6.2.1, we set $j_0 = 0$ and for every $l \in \mathbb{N}_0$, we consider j_{l+1} the smallest integer larger than j_l such that

$$\alpha_s j_{l+1} \geq \alpha j_l.$$

Then, we define \vec{c} as follows: if $j = j_l$, we set

$$c_{j_l, k} := \begin{cases} 2^{-\alpha j_l} & \text{if } k = 0, \\ 2^{-\alpha_s j_l} & \text{otherwise,} \end{cases}$$

and if j is between j_l and j_{l+1} , we set

$$c_{j, k} := \begin{cases} 2^{-\alpha j_l} & \text{if } \lambda(j, k) \text{ is included in } \lambda(j_l, 0), \\ 2^{-\alpha_s j} & \text{otherwise.} \end{cases}$$

The function f whose wavelet coefficients are given by \vec{c} satisfies

$$\tilde{\nu}_f^-(h) = \begin{cases} 1 & \text{if } h \leq \alpha_s, \\ 1 - \frac{h}{\beta} & \text{if } h \in (\alpha_s, \alpha], \\ -\infty & \text{if } h > \alpha, \end{cases} \quad \text{and} \quad d_f(h) = \begin{cases} 0 & \text{if } h = \alpha, \\ 1 & \text{if } h = \alpha_s, \\ -\infty & \text{otherwise.} \end{cases}$$

Before stating the next result, let us remark that if $\alpha_{\max} < +\infty$, then $\nu(\alpha) > 0$ for every $\alpha \in [\alpha_s, \alpha_{\max}]$. Indeed, from the properties of ν , we know that

$$\frac{\nu(\alpha) - 1}{\alpha} \geq \frac{\nu(\alpha_{\max}) - 1}{\alpha_{\max}} \geq \frac{-1}{\alpha_{\max}}$$

so that

$$\nu(\alpha) \geq 1 - \frac{\alpha}{\alpha_{\max}} > 0$$

for every $\alpha \in [\alpha_s, \alpha_{\max}]$.

Proposition 7.3.9. *Let ν be an admissible profile with $\alpha_{\max} < +\infty$. Then, there is a function f such that*

$$\tilde{\nu}_f(h) = d_f(h) = \begin{cases} -\infty & \text{if } h \in [0, \alpha_s), \\ \nu(h) & \text{if } h \in [\alpha_s, +\infty]. \end{cases}$$

Proof. We proceed as in the proofs of Proposition 7.3.6 and Lemma 7.3.5. Since ν is left-continuous and decreasing on $[\alpha_s, \alpha_{\max}]$, it is possible to find a sequence of functions $(f_n)_{n \in \mathbb{N}}$ such that $d_{f_n} \in \mathcal{A}^-(\alpha_s)$ with

$$\nu = \sup_{n \in \mathbb{N}} d_{f_n} \quad \text{on} \quad [\alpha_s, \alpha_{\max}]$$

and such that there is $\beta_n \in (\alpha_s, \alpha_{\max}]$ with

$$\tilde{\nu}_{f_n}^-(h) = \begin{cases} 1 & \text{if } h \leq \alpha_s, \\ 1 + \frac{h}{\beta_n}(\gamma_n - 1) & \text{if } h \in (\alpha_s, \beta_n], \\ -\infty & \text{if } h > \beta_n, \end{cases}$$

with $\gamma_n \leq \nu(\beta_n)$. Since the decreasing wavelet leaders profile give an upper bound for the multifractal spectrum, we know that

$$\nu = \sup_{n \in \mathbb{N}} d_{f_n} \leq \sup_{n \in \mathbb{N}} \tilde{\nu}_{f_n}^- \quad \text{on} \quad [\alpha_s, \alpha_{\max}],$$

and let us show that the converse inequality is also true. Let us fix $h \in [\alpha_s, \alpha_{\max}]$ and $n \in \mathbb{N}$. If $h > \beta_n$, then $\tilde{\nu}_{f_n}^-(h) = -\infty \leq \nu(h)$. If $h \in (\alpha_s, \beta_n]$, then

$$\tilde{\nu}_{f_n}^-(h) = 1 + \frac{h}{\beta_n}(\gamma_n - 1) \leq 1 + \frac{h}{\beta_n}(\nu(\beta_n) - 1) \leq \nu(h)$$

from the properties of an admissible profile. The equality at $h = \alpha_s$ is obvious. In particular, if $\vec{c}^{(n)}$ denotes the wavelet coefficients of f_n , then $\vec{c}^{(n)} \in \mathcal{L}^{\nu, -}$.

For every $n \in \mathbb{N}$, we denote by $\vec{e}^{(n)}$ the restricted wavelet leaders of $\vec{c}^{(n)}$ and we consider the sequences $\vec{a}^{(n)}$ and $\vec{b}^{(n)}$ given by

$$a_{j,k}^{(n)} := \begin{cases} 0 & \text{if } k \in \{2^{j-n+1}, \dots, 2^j - 1\}, \\ \frac{1}{e_{j-n,k-2^{j-n}}^{(n)}} & \text{if } k \in \{2^{j-n}, \dots, 2^{j-n+1} - 1\}, \\ 2^{\alpha_s j} & \text{if } k \in \{0, \dots, 2^{j-n} - 1\}, \end{cases}$$

and

$$b_{j,k}^{(n)} := \begin{cases} 0 & \text{if } k \in \{2^{j-n+1}, \dots, 2^j - 1\}, \\ 2^{\alpha_s j} & \text{if } k \in \{0, \dots, 2^{j-n+1} - 1\}, \end{cases}$$

for every $j \geq n$, and $a_{j,k}^{(n)} = b_{j,k}^{(n)} := 0$ if $j < n$. Since $\vec{c}^{(n)} \in \mathcal{L}^{\nu,-}$, we have $\vec{a}^{(n)} \in \mathcal{S}^{\nu,-}$. Moreover, it is clear that $\vec{b}^{(n)} \in \mathcal{S}^{\nu,-}$. Using the continuity of the scalar multiplication in $\mathcal{S}^{\nu,-}$, there is $\varepsilon_n > 0$ such that

$$\delta(\varepsilon_n \vec{a}^{(n)}, \vec{0}) \leq \frac{1}{2n^2} \quad \text{and} \quad \delta(\varepsilon_n \vec{b}^{(n+1)}, \vec{0}) \leq \frac{1}{2(n+1)^2}.$$

Moreover, we can assume that $\varepsilon_n < \varepsilon_{n-1} 2^{-\alpha_s}$ for every $n \geq 2$. We define then the sequence $\vec{C}^{(n)}$ as follows: at scales $j < n$,

$$C_{j,k}^{(n)} := \begin{cases} \frac{1}{\varepsilon_1} c_{j-1,k-2^{j-1}}^{(1)} & \text{if } k \in \{2^{j-1}, \dots, 2^j - 1\}, \\ \frac{1}{\varepsilon_2} c_{j-2,k-2^{j-2}}^{(2)} & \text{if } k \in \{2^{j-2}, \dots, 2^{j-1} - 1\}, \\ \vdots & \\ \frac{1}{\varepsilon_{j-1}} c_{1,k-2}^{(j-1)} & \text{if } k \in \{2, 3\}, \\ \frac{1}{\varepsilon_j} c_{0,0}^{(j)} & \text{if } k = 1, \\ \frac{1}{\varepsilon_j} 2^{-\alpha_s j} & \text{if } k = 0, \end{cases}$$

and at scales $j \geq n$,

$$C_{j,k}^{(n)} := \begin{cases} \frac{1}{\varepsilon_1} c_{j-1,k-2^{j-1}}^{(1)} & \text{if } k \in \{2^{j-1}, \dots, 2^j - 1\}, \\ \frac{1}{\varepsilon_2} c_{j-2,k-2^{j-2}}^{(2)} & \text{if } k \in \{2^{j-2}, \dots, 2^{j-1} - 1\}, \\ \vdots & \\ \frac{1}{\varepsilon_n} c_{j-n,k-2^{j-n}}^{(n)} & \text{if } k \in \{2^{j-n}, \dots, 2^{j-n+1} - 1\}, \\ \frac{1}{\varepsilon_n} 2^{-\alpha_s j} & \text{if } k \in \{0, \dots, 2^{j-n} - 1\}. \end{cases}$$

Let us remark that if $\vec{E}^{(n)}$ denotes the sequence of restricted wavelet leaders of $\vec{C}^{(n)}$, we have

$$\frac{\overrightarrow{1}}{E^{(n)}} - \frac{\overrightarrow{1}}{E^{(n-1)}} = \varepsilon_n \vec{a}^{(n)} - \varepsilon_{n-1} \vec{b}^{(n)}.$$

Using the invariance by translation of the distance in $\mathcal{S}^{\nu,-}$, we get

$$\delta\left(\frac{\overrightarrow{1}}{E^{(n)}}, \frac{\overrightarrow{1}}{E^{(n-1)}}\right) \leq \delta(\varepsilon_n \vec{a}^{(n)}, \vec{0}) + \delta(\varepsilon_{n-1} \vec{b}^{(n)}, \vec{0}) \leq \frac{1}{n^2}$$

so that the sequence $(\overrightarrow{\frac{1}{E^{(n)}}})_{n \in \mathbb{N}}$ is a Cauchy sequence in $(\mathcal{S}^{\nu^-}, \delta)$. Since this space is complete, there is $\vec{x} \in \mathcal{S}^{\nu^-}$ such that the sequence converges to \vec{x} in $(\mathcal{S}^{\nu^-}, \delta)$. We know that the convergence in $(\mathcal{S}^{\nu^-}, \delta)$ is stronger than the pointwise convergence. It follows that

$$x_\lambda = \frac{1}{\sup_{\lambda' \subseteq \lambda} |c_\lambda|}, \quad \forall \lambda \in \Lambda,$$

where

$$c_{j,k} = \begin{cases} \frac{1}{\varepsilon_1} c_{j-1, k-2^{j-1}}^{(1)} & \text{if } k \in \{2^{j-1}, \dots, 2^j - 1\}, \\ \frac{1}{\varepsilon_2} c_{j-2, k-2^{j-2}}^{(2)} & \text{if } k \in \{2^{j-2}, \dots, 2^{j-1} - 1\}, \\ \vdots & \\ \frac{1}{\varepsilon_{j-1}} c_{1, k-2}^{(j-1)} & \text{if } k \in \{2, 3\}, \\ \frac{1}{\varepsilon_j} c_{0,0}^{(j)} & \text{if } k = 1, \\ \frac{1}{\varepsilon_j} 2^{-\alpha_s j} & \text{if } k = 0, \end{cases}$$

for every $j \in \mathbb{N}_0$. In particular, $\vec{c} \in \mathcal{L}^{\nu, -}$. Let us consider the function F^* whose wavelet coefficients at scale $j \in \mathbb{N}$ are given by

$$c_\lambda^* = \min(d_\lambda, 2^{-\alpha_s j}), \quad \forall \lambda \in \Lambda_j.$$

As done in Proposition 7.3.6, a simple computation shows that the wavelet leaders of F^* are its wavelet coefficients. Therefore, the wavelet leaders of F^* smaller than $2^{-\alpha_s j}$ are the same as those of \vec{c} and the others equal $2^{-\alpha_s j}$. As in Proposition 7.3.6, we directly get

$$\tilde{\nu}_{F^*}^+ = d_{F^*} = -\infty \quad \text{on} \quad [0, \alpha_s)$$

and

$$\tilde{\nu}_{F^*}^- = d_{F^*} = \nu \quad \text{on} \quad [\alpha_s, +\infty].$$

This gives to the conclusion. □

Appendix A

Robustness of the wavelet leaders profile

In this appendix, we show that the increasing and decreasing wavelet leaders profiles of a uniformly Hölder function are robust. In other words, let $\vec{c} \in C^r$ for some $r > 0$ and let A be a quasisdiagonal matrix. If \vec{x} is the image of \vec{c} by the matrix A , i.e. if

$$x_\lambda = \sum_{\lambda' \in \Lambda} A(\lambda, \lambda') c_{\lambda'}$$

for every dyadic interval λ , the aim of this appendix is to prove that $\tilde{\nu}_\vec{c}^+ = \tilde{\nu}_\vec{x}^+$ and $\tilde{\nu}_\vec{c}^- = \tilde{\nu}_\vec{x}^-$ on $[0, +\infty]$.

A.1 Increasing wavelet leaders profile

In this section, we show that $\tilde{\nu}_\vec{c}^+ = \tilde{\nu}_\vec{x}^+$ on $[0, +\infty]$. Let us first recall the following lemma.

Lemma A.1.1. *There exists a constant \tilde{C} such that if $\gamma > |\alpha|$ and $A \in \mathcal{A}^\gamma$,*

$$|c_{j,k}| \leq C2^{-\alpha j}, \quad \forall j, k \Rightarrow |x_{j,k}| \leq \tilde{C} \|A\|_\gamma C2^{-\alpha j}, \quad \forall j, k.$$

This lemma expresses the fact that operators whose matrix in a wavelet basis belongs to \mathcal{A}^γ are continuous on $C^\alpha(\mathbb{T})$ if $|\alpha| < \gamma$. It is a straightforward consequence of the proof of Schur's lemma (Lemma 4 in Chapter 8, [111]).

Definition A.1.2. Let $\varepsilon > 0$ and let λ be a dyadic interval. The ε -neighborhood of λ , denoted by $N^\varepsilon(\lambda)$, is the set of dyadic intervals λ' such that

$$\begin{cases} |j - j'| & \leq \varepsilon j, \\ \left| \frac{k}{2^j} - \frac{k'}{2^{j'}} \right| & \leq 2^{2\varepsilon j} 2^{-j}. \end{cases}$$

Remark A.1.3. Note that if λ' does not belong to $N^\varepsilon(\lambda)$ and if $\gamma \geq \varepsilon^{-2}$, a computation leads to

$$\omega_{2\gamma}(\lambda, \lambda') \leq \omega_\gamma(\lambda, \lambda') 2^{-j/\varepsilon}.$$

Proposition A.1.4. [24] *Let $\vec{c} \in C^r$ for some $r > 0$. The definition of the increasing wavelet leaders profile of \vec{c} is robust.*

Proof. Let A be a quasideagonal matrix and consider

$$x_\lambda = \sum_{\lambda' \in \Lambda} A(\lambda, \lambda') c_{\lambda'}$$

for every dyadic interval λ . Let us first show that $\tilde{\nu}_x^+(\alpha) \leq \tilde{\nu}_c^+(\alpha)$ for every $\alpha \geq 0$. Let us denote $\alpha_{\min} = \inf\{\alpha : \tilde{\nu}_c^+(\alpha) \geq 0\}$. Since $\vec{c} \in C^r(\mathbb{T})$, we know that $\alpha_{\min} > 0$.

1. Assume that $\alpha < \alpha_{\min}$.

If $\varepsilon > 0$ is such that $\alpha + \varepsilon < \alpha_{\min}$, there is $C_1 > 0$ with

$$|c_{j,k}| \leq C_1 2^{-(\alpha+\varepsilon)j}, \quad \forall j \in \mathbb{N}_0, k \in \{0, \dots, 2^j - 1\}.$$

Therefore, Lemma A.1.1 implies that $|x_{j,k}| \leq \tilde{C} \|A\|_\gamma C_1 2^{-(\alpha+\varepsilon)j}$ for $\gamma \geq \alpha + \varepsilon$ and for every $j \in \mathbb{N}_0$ and $k \in \{0, \dots, 2^j - 1\}$. We then directly obtain that $\tilde{\nu}_x^+(\alpha) = -\infty$ for every $\alpha < \alpha_{\min}$.

2. Assume that $\alpha \geq \alpha_{\min} > 0$.

Let us fix $\delta > 0$. We will prove that there exist $J \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\#\left\{\lambda \in \Lambda_l : \sup_{\lambda' \subseteq \lambda} |x_{\lambda'}| \geq 2^{-(\alpha+\varepsilon)l}\right\} \leq 2^{(\tilde{\nu}_c^+(\alpha)+8\delta)l}$$

for all $l \geq J$. Since $\delta > 0$ is arbitrary, we will get that $\tilde{\nu}_x^+(\alpha) \leq \tilde{\nu}_c^+(\alpha)$.

Using the right continuity of $\tilde{\nu}_c^+$, we choose $\varepsilon > 0$ such that $\varepsilon < \delta$, $\alpha - \alpha_{\min} < \varepsilon^{-1}$ and $\tilde{\nu}_c^+\left(\frac{\alpha}{1-\varepsilon}\right) \leq \tilde{\nu}_c^+(\alpha) + \delta$. The definition of $\tilde{\nu}_c^+$ gives $\varepsilon_0 > 0$ and $J \in \mathbb{N}_0$ such that

$$\#\left\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \geq 2^{-(\frac{\alpha}{1-\varepsilon}+2\varepsilon_0)j}\right\} \leq 2^{(\tilde{\nu}_c^+(\frac{\alpha}{1-\varepsilon})+\delta)j}$$

for every $j \geq J$. Of course, we can also assume that ε_0 is small enough so that it satisfies $\alpha + \varepsilon_0(1-\varepsilon) - \alpha_{\min} < \varepsilon^{-1}$. For every $l \in \mathbb{N}_0$, we define

$$\Lambda_1 := \Lambda_1(l, \varepsilon) = \bigcup_{(1-\varepsilon)l \leq j \leq (1+\varepsilon)l} \left\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \geq \frac{1}{4\tilde{C}\|A\|_\alpha} 2^{-(\frac{\alpha}{1-\varepsilon}+\varepsilon_0)j}\right\}$$

and

$$\Lambda_2 := \Lambda_2(l, \varepsilon) = \left\{\lambda \in \Lambda_l : \exists \lambda_1 \in \Lambda_1 \text{ with } \left|\frac{k_1}{2^{j_1}} - \frac{k}{2^l}\right| \leq 2^{-l+1} 2^{2\varepsilon l}\right\}.$$

a) *Let us show that if $\lambda_0 \notin \Lambda_2(l, \varepsilon)$ is of size 2^{-l} , then $\sup_{\lambda \subseteq \lambda_0} |x_\lambda| < 2^{-(\alpha+\varepsilon_0(1-\varepsilon))l}$.*

It suffices to show that if $\lambda \subseteq \lambda_0$, then

$$|x_\lambda| = \left| \sum_{\lambda' \in \Lambda} A(\lambda, \lambda') c_{\lambda'} \right| \leq \frac{1}{2} 2^{-(\alpha+\varepsilon_0(1-\varepsilon))l}.$$

So, let us fix $\lambda = \lambda(j, k) \subseteq \lambda_0 = \lambda(l, k_0)$. Remark that in particular, we have $j \geq l$. We set

$$y_\lambda := \sum_{\lambda' \not\subseteq \lambda_1 \forall \lambda_1 \in \Lambda_1 \text{ and } j' \geq (1-\varepsilon)l} A(\lambda, \lambda') c_{\lambda'}$$

and

$$z_\lambda := \sum_{\lambda' \subseteq \lambda_1 \text{ with } \lambda_1 \in \Lambda_1 \text{ or } j' < (1-\varepsilon)l} A(\lambda, \lambda') c_{\lambda'}$$

so that we have $|x_\lambda| \leq |y_\lambda| + |z_\lambda|$.

• If λ' is such that $\lambda' \not\subseteq \lambda_1$ for any $\lambda_1 \in \Lambda_1$ and if $j' \geq (1-\varepsilon)l$, then by definition of $\Lambda_1(l, \varepsilon)$, we have

$$|c_{\lambda'}| < \frac{1}{4\tilde{C}\|A\|_\alpha} 2^{-(\frac{\alpha}{1-\varepsilon} + \varepsilon_0)l(1-\varepsilon)} = \frac{1}{4\tilde{C}\|A\|_\alpha} 2^{-(\alpha + \varepsilon_0(1-\varepsilon))l}.$$

Using Lemma A.1.1 with $\alpha > 0$, we get that

$$|y_\lambda| \leq \frac{1}{4\tilde{C}\|A\|_\alpha} 2^{-(\alpha + \varepsilon_0(1-\varepsilon))l} \tilde{C}\|A\|_\alpha 2^{-0j} = \frac{1}{4} 2^{-(\alpha + \varepsilon_0(1-\varepsilon))l}.$$

• If λ' is such that there is $\lambda_1 \in \Lambda_1(l, \varepsilon)$ with $\lambda' \subseteq \lambda_1$ or if $j' < (1-\varepsilon)l$, let us show that $\lambda' \notin N^\varepsilon(\lambda)$. First, if $j' < (1-\varepsilon)l$, then $j' < (1-\varepsilon)j$ since $j \geq l$ and it follows that $\lambda' \notin N^\varepsilon(\lambda)$. So we can assume that $\lambda' \subseteq \lambda_1$ with $\lambda_1 \in \Lambda_1$. Since $\lambda_1 \in \Lambda_1$ and $\lambda_0 \notin \Lambda_2$, we know that

$$\left| \frac{k_0}{2^l} - \frac{k_1}{2^{j_1}} \right| > 2^{-l+1} 2^{2\varepsilon l}.$$

Let us first assume that

$$\frac{k_0}{2^l} - \frac{k_1}{2^{j_1}} > 2^{-l+1} 2^{2\varepsilon l}.$$

From the inclusions $\lambda \subseteq \lambda_0$ and $\lambda' \subseteq \lambda_1$, we have

$$\frac{k_0}{2^l} - \frac{k_1}{2^{j_1}} \leq \frac{k}{2^j} - \frac{k'}{2^{j'}} - \frac{1}{2^{j'}} + \frac{1}{2^{j_1}} \leq \frac{k}{2^j} - \frac{k'}{2^{j'}} + \frac{1}{2^{j_1}}$$

and consequently, from the previous relation,

$$\frac{k}{2^j} - \frac{k'}{2^{j'}} > 2^{-l+1} 2^{2\varepsilon l} - \frac{1}{2^{j_1}}.$$

Moreover, since $\lambda_1 \in \Lambda_1(l, \varepsilon)$, we have $j_1 \geq (1-\varepsilon)l$ and it follows that

$$\frac{k}{2^j} - \frac{k'}{2^{j'}} > 2^{-l+1} 2^{2\varepsilon l} - 2^{-((1-\varepsilon)l)} = 2^{(2\varepsilon-1)l} + 2^{(2\varepsilon-1)l} - 2^{-((1-\varepsilon)l)} \geq 2^{(2\varepsilon-1)l} \geq 2^{(2\varepsilon-1)j}$$

where the last inequality comes from the fact that $j \geq l$ and $2\varepsilon - 1 < 0$.

The second case is quite similar. Assume that

$$\frac{k_1}{2^{j_1}} - \frac{k_0}{2^l} > 2^{-l+1} 2^{2\varepsilon l}.$$

Using inclusions between dyadic intervals, we have

$$\frac{k_1}{2^{j_1}} - \frac{k_0}{2^l} \leq \frac{k'}{2^{j'}} - \frac{k}{2^j} - \frac{1}{2^j} + \frac{1}{2^l} \leq \frac{k'}{2^{j'}} - \frac{k}{2^j} + \frac{1}{2^l}$$

and it follows that

$$\frac{k'}{2^{j'}} - \frac{k}{2^j} > 2^{-l+1}2^{2\epsilon l} - \frac{1}{2^l} \geq 2^{(2\epsilon-1)l} \geq 2^{(2\epsilon-1)j}.$$

So, we have proved that $\lambda' \notin N^\epsilon(\lambda)$.

Consequently, using Remark A.1.3, we get

$$\begin{aligned} |z_\lambda| &\leq \sum_{\lambda': \lambda' \notin N^\epsilon(\lambda)} |A(\lambda, \lambda')| |c_{\lambda'}| \\ &\leq \sum_{\lambda': \lambda' \notin N^\epsilon(\lambda)} \|A\|_{2\epsilon-2} \omega_{2\epsilon-2}(\lambda, \lambda') |c_{\lambda'}| \\ &\leq \|A\|_{2\epsilon-2} \sum_{\lambda': \lambda' \notin N^\epsilon(\lambda)} \omega_{\epsilon-2}(\lambda, \lambda') 2^{-j\epsilon-1} |c_{\lambda'}| \\ &\leq \|A\|_{2\epsilon-2} 2^{-j\epsilon-1} C_1 \sum_{\lambda': \lambda' \notin N^\epsilon(\lambda)} \omega_{\epsilon-2}(\lambda, \lambda') 2^{-\alpha_0 j'} \end{aligned}$$

where $\alpha_0 < \alpha_{\min}$ is such that $\alpha + \epsilon_0(1 - \epsilon) - \alpha_0 < \epsilon^{-1}$ and the constant $C_1 > 0$ is such that $|c_{j,k}| \leq C_1 2^{-\alpha_0 j}$ for every $j \in \mathbb{N}$, $k \in \{0, \dots, 2^j - 1\}$. Lemma A.1.1 gives

$$|z_\lambda| \leq \|A\|_{2\epsilon-2} C_1 \tilde{C} 2^{-(\alpha_0 + \epsilon^{-1})j} \leq \|A\|_{2\epsilon-2} C_1 \tilde{C} 2^{-(\alpha_0 + \epsilon^{-1})l} < \frac{1}{4} 2^{-(\alpha + \epsilon_0(1 - \epsilon))l}$$

if l is large enough.

Finally, we have got

$$|x_\lambda| \leq |y_\lambda| + |z_\lambda| \leq \frac{1}{2} 2^{-(\alpha + \epsilon_0(1 - \epsilon))l}$$

if l is large enough. It follows that if $\lambda_0 \notin \Lambda_2(l, \epsilon)$ is of size 2^{-l} , then we have $\sup_{\lambda \subseteq \lambda_0} |x_\lambda| < 2^{-(\alpha + \epsilon_0(1 - \epsilon))l}$. So,

$$\# \left\{ \lambda_0 \in \Lambda_l : \sup_{\lambda \subseteq \lambda_0} |x_\lambda| \geq 2^{-(\alpha + \epsilon_0(1 - \epsilon))l} \right\} \leq \# \Lambda_2(l, \epsilon).$$

b) *Estimation of the cardinality of $\Lambda_2(l, \epsilon)$*

Remark first that if $\lambda_1 = \lambda(j_1, k_1) \in \Lambda_1(l, \epsilon)$ is fixed, we have

$$\begin{aligned} &\# \left\{ k \in \{0, \dots, 2^l - 1\} : \left| \frac{k}{2^l} - \frac{k_1}{2^{j_1}} \right| \leq 2^{-l+1}2^{2\epsilon l} \right\} \\ &= \# \left\{ k \in \{0, \dots, 2^l - 1\} : k_1 2^{l-j_1} - 2^{1+2\epsilon l} \leq k \leq k_1 2^{l-j_1} + 2^{1+2\epsilon l} \right\} \\ &\leq k_1 2^{l-j_1} + 2^{1+2\epsilon l} - k_1 2^{l-j_1} + 2^{1+2\epsilon l} + 1 = 2^{2\epsilon l+2} + 1 \\ &\leq 2^{3\epsilon l} \end{aligned}$$

if l is large enough. Therefore, we get

$$\# \Lambda_2(l, \epsilon) \leq \sum_{(1-\epsilon)l \leq j_1 \leq (1+\epsilon)l} \# \left\{ \lambda \in \Lambda_{j_1} : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \geq \frac{1}{4\tilde{C}\|A\|_\alpha} 2^{-\left(\frac{\alpha}{1-\epsilon} + \epsilon_0\right)j_1} \right\}.$$

Moreover,

$$\begin{aligned}
 & \# \left\{ \lambda \in \Lambda_{j_1} : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \geq \frac{1}{4\tilde{C}\|A\|_\alpha} 2^{-(\frac{\alpha}{1-\varepsilon} + \varepsilon_0)j_1} \right\} \\
 & \leq \# \left\{ \lambda \in \Lambda_{j_1} : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \geq 2^{-(\frac{\alpha}{1-\varepsilon} + 2\varepsilon_0)j_1} \right\} \\
 & \leq 2^{(\tilde{\nu}_\varepsilon^+(\frac{\alpha}{1-\varepsilon}) + \delta)j_1} \leq 2^{(\tilde{\nu}_\varepsilon^+(\alpha) + 2\delta)j_1}
 \end{aligned}$$

for j_1 (hence l) large enough. It follows that

$$\begin{aligned}
 & \# \left\{ \lambda_0 \in \Lambda_l : \sup_{\lambda \subseteq \lambda_0} |x_\lambda| \geq 2^{-(\alpha + \varepsilon_0(1-\varepsilon))l} \right\} \\
 & \leq \sum_{(1-\varepsilon)l \leq j_1 \leq (1+\varepsilon)l} 2^{(\tilde{\nu}_\varepsilon^+(\alpha) + 2\delta)j_1} 2^{3\varepsilon l} \\
 & \leq \sum_{(1-\varepsilon)l \leq j_1 \leq (1+\varepsilon)l} 2^{(\tilde{\nu}_\varepsilon^+(\alpha) + 2\delta)(1+\varepsilon)l} 2^{3\varepsilon l} \\
 & \leq (2\varepsilon l + 1) 2^{(\tilde{\nu}_\varepsilon^+(\alpha) + 2\delta + \varepsilon)(\tilde{\nu}_\varepsilon^+(\alpha) + 2\delta + 3)l} \\
 & \leq 2^{\varepsilon l} 2^{(\tilde{\nu}_\varepsilon^+(\alpha) + 2\delta + 5\varepsilon)l} \\
 & \leq 2^{(\tilde{\nu}_\varepsilon^+(\alpha) + 8\delta)l}
 \end{aligned}$$

if l is large enough.

So, we have proved that for every $\alpha \geq 0$, $\tilde{\nu}_x^+(\alpha) \leq \tilde{\nu}_\varepsilon^+(\alpha)$. We have also obtained that $\inf\{\alpha : \tilde{\nu}_x^+(\alpha) \geq 0\} \geq \alpha_{\min} > 0$. Since A^{-1} is also almost diagonal, the same proof shows that $\tilde{\nu}_\varepsilon^+(\alpha) \leq \tilde{\nu}_x^+(\alpha)$ for every $\alpha \geq 0$. The conclusion follows. \square

A.2 Decreasing wavelet leaders profile

In this section, we show that $\tilde{\nu}_\varepsilon^- = \tilde{\nu}_x^-$ on $[0, +\infty]$, where

$$x_\lambda = \sum_{\lambda' \in \Lambda} A(\lambda, \lambda') c_{\lambda'}, \quad \forall \lambda \in \Lambda,$$

for a quasidiagonal matrix A . Let us first introduce a new notation. Let us fix a dyadic interval $\lambda_0(l, k_0)$ and $\varepsilon > 0$. For $j \in \mathbb{N}_0$ and $k \in \{0, \dots, 2^j - 1\}$,

$$\lambda(j, k) \in \text{Cond}_\varepsilon(\lambda_0) \iff (1 - 2\varepsilon)j > l \quad \text{and} \quad 2^{(2\varepsilon-1)j} \leq \frac{k}{2^j} - \frac{k_0}{2^l} \leq 2^{-l} - 3 \cdot 2^{(2\varepsilon-1)j}.$$

Lemma A.2.1. [24] *Let us fix a dyadic interval $\lambda_0(k_0, l)$ and let us consider $\varepsilon > 0$. If $\lambda(j, k) \in \text{Cond}_\varepsilon(\lambda_0)$, then*

$$\lambda' \in N^\varepsilon(\lambda(j, k)) \implies \lambda' \subseteq \lambda_0.$$

Proof. First, we have

$$j' \geq (1 - \varepsilon)j \geq (1 - 2\varepsilon)j > l.$$

Moreover,

$$\frac{k'}{2^{j'}} \geq \frac{k}{2^j} - 2^{(2\varepsilon-1)j} \geq \frac{k_0}{2^l}.$$

Finally, we have

$$\begin{aligned} \frac{k'+1}{2^{j'}} &\leq \frac{1}{2^{j'}} + \frac{k}{2^j} + 2^{(2\varepsilon-1)j} \leq 2^{-j'} + \frac{k_0}{2^l} + 2^{-l} - 3 \cdot 2^{(2\varepsilon-1)j} + 2^{(2\varepsilon-1)j} \\ &\leq \frac{k_0+1}{2^l} - 2^{(2\varepsilon-1)j} \leq \frac{k_0+1}{2^l}. \end{aligned}$$

□

Proposition A.2.2. [24] *Let $\vec{c} \in C^r$ for some $r > 0$. The definition of the decreasing wavelet leaders profile of \vec{c} is robust.*

Proof. Let A be a quasisdiagonal matrix and consider

$$x_\lambda = \sum_{\lambda' \in \Lambda} A(\lambda, \lambda') c_{\lambda'}$$

for every dyadic interval λ . Let us first remark that, as done in the case of the proof of Proposition A.1.4, since $\vec{c} \in C^r$, there exists $C_1 > 0$ such that $|x_{j,k}| \leq C_1 2^{-rj}$ for every $j \in \mathbb{N}$, $k \in \{0, \dots, 2^j - 1\}$. In particular, we have $\tilde{v}_{\vec{c}}^-(\alpha) = \tilde{v}_{\vec{c}}^-(\alpha) = -\infty$ if $\alpha \leq r$.

Let us show that $\tilde{v}_{\vec{c}}^-(\alpha) \leq \tilde{v}_{\vec{c}}^-(\alpha)$ for every $\alpha > r$. Since A is quasisdiagonal, we will similarly obtain the other inequality. Let us fix $\varepsilon > 0$ small enough so that $\alpha - r < \varepsilon^{-1}$ and $\varepsilon < \frac{1}{2}$. Since $\alpha < r + \varepsilon^{-1}$, there is $J \in \mathbb{N}$ such that

$$\|A^{-1}\|_{2\varepsilon^{-2}} C_1 2^{-(r+\varepsilon^{-1})l} \leq \|A^{-1}\|_\alpha 2^{-\alpha l}$$

for every $l \geq J$. For $\varepsilon_0 > 0$ small enough, we have $\alpha - r + \varepsilon_0(1+\varepsilon) < \varepsilon^{-1}$ and for every $l \geq J$, we define

$$E_l = \left\{ \lambda_0 \in \Lambda_l : \sup_{\lambda \in \text{Cond}_\varepsilon(\lambda_0)} |c_\lambda| \leq 2\tilde{C} \|A^{-1}\|_\alpha 2^{-(\alpha-\varepsilon_0(1+\varepsilon))l} \right\}.$$

Let us show that if $\lambda_0 \in \Lambda_l$ is such that $\sup_{\lambda' \subseteq \lambda_0} |x_{\lambda'}| \leq 2^{-(\alpha-\varepsilon_0(1+\varepsilon))l}$, then $\lambda_0 \in E_l$. Let us fix $\lambda \in \text{Cond}_\varepsilon(\lambda_0)$. We have

$$|c_\lambda| \leq \left| \sum_{\lambda' \in N^\varepsilon(\lambda)} A^{-1}(\lambda, \lambda') x_{\lambda'} \right| + \left| \sum_{\lambda' \notin N^\varepsilon(\lambda)} A^{-1}(\lambda, \lambda') x_{\lambda'} \right|.$$

As done in Proposition A.1.4, using the Remark A.1.3, we have

$$\left| \sum_{\lambda' \notin N^\varepsilon(\lambda)} A^{-1}(\lambda, \lambda') x_{\lambda'} \right| \leq \|A^{-1}\|_{2\varepsilon^{-2}} C_1 \tilde{C} 2^{-(r+\varepsilon^{-1})l} \leq \tilde{C} \|A^{-1}\|_\alpha 2^{-(\alpha-\varepsilon_0(1+\varepsilon))l}.$$

Moreover, Lemma A.2.1 implies that if $\lambda' \in N^\varepsilon(\lambda)$, then $\lambda' \subseteq \lambda_0$. Consequently, we have $|x_{\lambda'}| \leq 2^{-(\alpha-\varepsilon_0(1+\varepsilon))l}$ and Lemma A.1.1 gives

$$\left| \sum_{\lambda' \in N^\varepsilon(\lambda)} A^{-1}(\lambda, \lambda') x_{\lambda'} \right| \leq \tilde{C} \|A^{-1}\|_\alpha 2^{-(\alpha-\varepsilon_0(1+\varepsilon))l}.$$

So, we get

$$|c_\lambda| \leq 2\tilde{C}\|A^{-1}\|_\alpha 2^{-(\alpha-\varepsilon_0(1+\varepsilon))l}$$

and

$$\sup_{\lambda \in \text{Cond}_\varepsilon(\lambda_0)} |c_\lambda| \leq 2\tilde{C}\|A^{-1}\|_\alpha 2^{-(\alpha-\varepsilon_0(1+\varepsilon))l}.$$

Consequently, we have

$$\begin{aligned} & \#\{\lambda_0 \in \Lambda_l : \sup_{\lambda' \subseteq \lambda_0} |x_{\lambda'}| \leq 2^{-(\alpha-\varepsilon_0(1+\varepsilon))l}\} \\ & \leq \#\left\{\lambda_0 \in \Lambda_l : \sup_{\lambda \in \text{Cond}_\varepsilon(\lambda_0)} |c_\lambda| \leq 2\tilde{C}\|A^{-1}\|_\alpha 2^{-(\alpha-\varepsilon_0(1+\varepsilon))l}\right\}. \end{aligned}$$

Let us choose $j_l \in \mathbb{N}$ such that $(1-2\varepsilon)j_l \geq l+3$ and $j_l \leq (1+\varepsilon)l$. For every $\lambda_0 = \lambda(l, k_0)$ of size 2^{-l} , we fix k such that

$$2^{(2\varepsilon-1)j_l} \leq \frac{k}{2^{j_l}} - \frac{k_0}{2^l} \leq 2^{-l} - 4 \cdot 2^{(2\varepsilon-1)j_l}.$$

Let us remark that in particular, we have $\lambda(j_l, k) \subseteq \lambda_0$ and therefore the $\lambda(j_l, k)$ are different for different λ_0 of size 2^{-l} . A simple computation shows that if $\lambda' \subseteq \lambda(j_l, k)$, then $\lambda' \in \text{Cond}_\varepsilon(\lambda_0)$. It follows that

$$\begin{aligned} & \#\left\{\lambda_0 \in \Lambda_l : \sup_{\lambda \in \text{Cond}_\varepsilon(\lambda_0)} |c_\lambda| \leq 2\tilde{C}\|A^{-1}\|_\alpha 2^{-(\alpha-\varepsilon_0(1+\varepsilon))l}\right\} \\ & \leq \#\{\lambda \in \Lambda_{j_l} : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \leq 2\tilde{C}\|A^{-1}\|_\alpha 2^{-(\alpha-\varepsilon_0(1+\varepsilon))l}\} \\ & \leq \#\{\lambda \in \Lambda_{j_l} : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \leq 2\tilde{C}\|A^{-1}\|_\alpha 2^{-(\frac{\alpha}{1+\varepsilon}-\varepsilon_0)j_l}\} \\ & \leq \#\{\lambda \in \Lambda_{j_l} : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \leq 2^{-(\frac{\alpha}{1+\varepsilon}-2\varepsilon_0)j_l}\} \end{aligned}$$

if l (hence j_l) is large enough. So, we have obtained

$$\#\{\lambda_0 \in \Lambda_l : \sup_{\lambda' \subseteq \lambda_0} |x_{\lambda'}| \leq 2^{-(\alpha-\varepsilon_0(1+\varepsilon))l}\} \leq \#\{\lambda \in \Lambda_{j_l} : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \leq 2^{-(\frac{\alpha}{1+\varepsilon}-2\varepsilon_0)j_l}\}.$$

This inequality, denoted (*), holds for every $\varepsilon \in (0, \frac{1}{2})$ such that $\alpha - r < \varepsilon^{-1}$ (using the corresponding ε_0 and j_l). Let us now consider three different cases. As usually, we denote

$$\alpha_{\max} = \sup\{\alpha \geq 0 : \tilde{\nu}_\varepsilon^-(\alpha) \geq 0\},$$

possibly equal to $+\infty$.

1. Assume that $\alpha \leq \alpha_{\max} < +\infty$.

Let us first fix $\delta > 0$. Using the left continuity of $\tilde{\nu}_\varepsilon^-$, we can assume that $\varepsilon > 0$ is small enough so that $\varepsilon < \delta$ and

$$\tilde{\nu}_\varepsilon^- \left(\frac{\alpha}{1+\varepsilon} \right) - \tilde{\nu}_\varepsilon^-(\alpha) \leq \delta.$$

From the definition of $\widetilde{\nu}_{\varepsilon}^{-}$, we can assume that

$$\#\left\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \leq 2^{-\left(\frac{\alpha}{1+\varepsilon} - 2\varepsilon_0\right)j}\right\} \leq 2^{\left(\widetilde{\nu}_{\varepsilon}^{-}\left(\frac{\alpha}{1+\varepsilon}\right) + \delta\right)j} \leq 2^{\left(\widetilde{\nu}_{\varepsilon}^{-}(\alpha) + 2\delta\right)j}$$

for $j \geq J$. Using (*), we get

$$\begin{aligned} \#\{\lambda_0 \in \Lambda_l : \sup_{\lambda' \subseteq \lambda_0} |x_{\lambda'}| \leq 2^{-(\alpha - \varepsilon_0(1+\varepsilon))l}\} &\leq 2^{\left(\widetilde{\nu}_{\varepsilon}^{-}(\alpha) + 2\delta\right)jl} \\ &\leq 2^{\left(\widetilde{\nu}_{\varepsilon}^{-}(\alpha) + 2\delta\right)(1+\varepsilon)l} = 2^{\left(\widetilde{\nu}_{\varepsilon}^{-}(\alpha) + 2\delta + \varepsilon \widetilde{\nu}_{\varepsilon}^{-}(\alpha) + 2\varepsilon\delta\right)l} \\ &\leq 2^{\left(\widetilde{\nu}_{\varepsilon}^{-}(\alpha) + 5\delta\right)l} \end{aligned}$$

for every l large enough and it follows that $\widetilde{\nu}_{\varepsilon}^{-}(\alpha) \leq \widetilde{\nu}_{\varepsilon}^{-}(\alpha)$.

2. Assume that $\alpha_{\max} < +\infty$ and $\alpha > \alpha_{\max}$

This case is immediate since

$$\#\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \leq 2^{-\left(\frac{\alpha}{1+\varepsilon} - 2\varepsilon_0\right)j}\} = 0$$

for every j large enough.

3. Assume that $\alpha_{\max} = +\infty$.

Let us fix $\delta > 0$. Again, we assume that $\varepsilon < \delta$. From the definition of $\widetilde{\nu}_{\varepsilon}^{-}(+\infty)$, for every α large enough, we have

$$\#\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \leq 2^{-\left(\frac{\alpha}{1+\varepsilon} - 2\varepsilon_0\right)j}\} \leq 2^{\left(\widetilde{\nu}_{\varepsilon}^{-}(+\infty) + \delta\right)j}$$

for infinitely many j . Given such a j , we consider $l \in \mathbb{N}$ such that $(1 - 2\varepsilon)j \geq l + 3$ and $j \leq (1 + \varepsilon)l$. Using (*), we get

$$\begin{aligned} \#\{\lambda_0 \in \Lambda_l : \sup_{\lambda' \subseteq \lambda_0} |x_{\lambda'}| \leq 2^{-(\alpha - \varepsilon_0(1+\varepsilon))l}\} &\leq \#\{\lambda \in \Lambda_j : \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}| \leq 2^{-\left(\frac{\alpha}{1+\varepsilon} - 2\varepsilon_0\right)j}\} \\ &\leq 2^{\left(\widetilde{\nu}_{\varepsilon}^{-}(+\infty) + \delta\right)j} \\ &\leq 2^{\left(\widetilde{\nu}_{\varepsilon}^{-}(+\infty) + 2\delta\right)(1+\varepsilon)l} \\ &\leq 2^{\left(\widetilde{\nu}_{\varepsilon}^{-}(+\infty) + 5\delta\right)l}. \end{aligned}$$

Since it holds for infinitely many l , this concludes the proof. \square

Appendix B

Random wavelet series

In this appendix, we prove the results presented in Chapter 5 related to random wavelet series. Let us recall the form of the multifractal spectrum of a random wavelet series.

Theorem B.1. [12] *Let f be a random wavelet series. With probability one, the spectrum of singularities of f is given by*

$$d_f(h) = \begin{cases} h \sup_{\alpha \in (0, h]} \frac{\rho(\alpha)}{\alpha} & \text{if } h \in [h_{\min}, h_{\max}], \\ -\infty & \text{otherwise,} \end{cases}$$

where

$$h_{\min} = \inf \left\{ \alpha \geq 0 : \sum_{j \in \mathbb{N}_0} 2^j \rho_j([\alpha - \varepsilon, \alpha + \varepsilon]) = +\infty, \forall \varepsilon > 0 \right\}$$

and $h_{\max} = \left(\sup_{\alpha > 0} \frac{\rho(\alpha)}{\alpha} \right)^{-1}$.

Let us recall that $\tilde{\rho}_j$ is the common probability measure of the 2^j random variables $-\log_2(|e_{j,k}|)/j$, where $e_{j,k}$ denote the restricted wavelet leaders.

Firs, we consider the increasing part of the wavelet leaders profile. For every $\alpha \geq 0$, let us define

$$\tilde{\nu}^+(\alpha) = \lim_{\varepsilon \rightarrow 0^+} \limsup_{j \rightarrow +\infty} \frac{\log(2^j \tilde{\rho}_j((-\infty, \alpha + \varepsilon]))}{\log(2^j)}$$

and

$$\alpha_s = \inf \left\{ \alpha \geq 0 : \tilde{\nu}^+(\alpha) = 1 \right\}.$$

We also assume that for every $\varepsilon > 0$ and every $\delta > 0$, there is $J \in \mathbb{N}$ such that

$$\tilde{\rho}_j((-\infty, \alpha_s + \varepsilon]) \geq 2^{-\delta j}, \quad \forall j \geq J. \tag{B.1}$$

For every α and every $j \in \mathbb{N}_0$, we consider the random set

$$F^j(\alpha) := \{k \in \{0, \dots, 2^j - 1\} : e_{j,k} \geq 2^{-\alpha j}\}.$$

Lemma B.2. *Let f be a random wavelet series. Let $\alpha \geq 0$ be such that $\tilde{\nu}^+(\alpha) > 0$. For every $\varepsilon > 0$ and every $\delta > 0$ such that $\tilde{\nu}^+(\alpha) - \delta > 0$, with probability one, there are infinitely many j satisfying*

$$\#F^j(\alpha + \varepsilon) \geq 2^{(\tilde{\nu}^+(\alpha) - \delta)j}.$$

Proof. From the definition of $\tilde{\nu}^+(\alpha)$, there is a sequence $(j_n)_{n \in \mathbb{N}}$ such that

$$2^{j_n} \tilde{\rho}_{j_n}((-\infty, \alpha + \varepsilon]) \geq 2^{\tilde{\nu}^+(\alpha) - \delta/2)j_n}, \quad \forall n \in \mathbb{N}.$$

Using the Borel Cantelli lemma, it suffices to show that $\mathbb{P}[A_n]$ is the general term of a series that converges, where A_n denotes the event “ $\#F^{j_n}(\alpha + \varepsilon) < 2^{\tilde{\nu}^+(\alpha) - \delta)j_n}$ ”. Remark that at a given scale j_n , since the $e_{j_n, k}$ are identically and independently distributed, we count the number of successes of a binomial distribution of parameters

$$(2^{j_n}, \tilde{\rho}_{j_n}((-\infty, \alpha + \varepsilon])),$$

where the success means “ $e_{j_n, k} \geq 2^{-\alpha j_n}$ ”. Therefore, if $j = j_n$, the probability of A_n is given by

$$\begin{aligned} & \sum_{0 \leq m < 2^{\tilde{\nu}^+(\alpha) - \delta)j} \binom{2^{j_n}}{m} (\tilde{\rho}_{j_n}((-\infty, \alpha + \varepsilon]))^m (1 - \tilde{\rho}_{j_n}((-\infty, \alpha + \varepsilon]))^{2^j - m} \\ & \leq \sum_{0 \leq m < 2^{\tilde{\nu}^+(\alpha) - \delta)j} \frac{(2^j \tilde{\rho}_j((-\infty, \alpha + \varepsilon]))^m}{m!} (1 - \tilde{\rho}_j((-\infty, \alpha + \varepsilon]))^{2^j - m} \\ & \leq \sum_{0 \leq m < 2^{\tilde{\nu}^+(\alpha) - \delta)j} \frac{(2^j \tilde{\rho}_j((-\infty, \alpha + \varepsilon]))^{2^{\tilde{\nu}^+(\alpha) - \delta)j}}{m!} (1 - \tilde{\rho}_j((-\infty, \alpha + \varepsilon]))^{2^j - 2^{\tilde{\nu}^+(\alpha) - \delta)j} \\ & \leq e (2^j \tilde{\rho}_j((-\infty, \alpha + \varepsilon]))^{2^{\tilde{\nu}^+(\alpha) - \delta)j} (1 - \tilde{\rho}_j((-\infty, \alpha + \varepsilon]))^{2^j - 2^{\tilde{\nu}^+(\alpha) - \delta)j} \\ & \leq e 2^{j 2^{\tilde{\nu}^+(\alpha) - \delta)j} (1 - \tilde{\rho}_j((-\infty, \alpha + \varepsilon]))^{\frac{3}{4} 2^j} \\ & \leq e 2^{j 2^{\tilde{\nu}^+(\alpha) - \delta)j} \exp\left(-\frac{3}{4} 2^j \tilde{\rho}_j((-\infty, \alpha + \varepsilon))\right) \\ & \leq e 2^{j 2^{\tilde{\nu}^+(\alpha) - \delta)j} \exp\left(-\frac{3}{4} 2^{\tilde{\nu}^+(\alpha) - \delta/2)j}\right) \\ & \leq \exp\left(-\frac{1}{2} 2^{\tilde{\nu}^+(\alpha) - \delta/2)j}\right) \end{aligned}$$

for n large enough, where we have used Remark 5.4.8. This concludes the proof. \square

Lemma B.3. *Let f be a random wavelet series, $\alpha \geq 0$ such that $\nu^+(\alpha) \geq 0$ and $\delta > 0$. We fix $\varepsilon > 0$ such that $2^j \tilde{\rho}_j((-\infty, \alpha + \varepsilon]) \leq 2^{\tilde{\nu}^+(\alpha) + \delta/2)j}$ for j large enough. With probability one, there is $J \in \mathbb{N}$ such that*

$$\#F^j(\alpha + \varepsilon) \leq 2^{\tilde{\nu}^+(\alpha) + \delta)j}, \quad \forall j \geq J.$$

Proof. For every j , we denote by B_j the event “ $\#F^j(\alpha + \varepsilon) > 2^{\tilde{\nu}^+(\alpha) + \delta)j}$ ”. As done in

the previous lemma, we have

$$\begin{aligned}
 \mathbb{P}[B_j] &= \sum_{2^{\tilde{\nu}^+(\alpha)+\delta}j < m \leq 2^j} \binom{2^j}{m} (\tilde{\rho}_j((-\infty, \alpha + \varepsilon]))^m (1 - \tilde{\rho}_j((-\infty, \alpha + \varepsilon]))^{2^j - m} \\
 &\leq \sum_{2^{\tilde{\nu}^+(\alpha)+\delta}j < m \leq 2^j} \frac{(2^j \tilde{\rho}_j((-\infty, \alpha + \varepsilon]))^m}{m!} \\
 &\leq \sum_{2^{\tilde{\nu}^+(\alpha)+\delta}j < m \leq 2^j} \frac{(2^j \tilde{\rho}_j((-\infty, \alpha + \varepsilon]))^{2^{\tilde{\nu}^+(\alpha)+\delta}j}}{\Gamma(2^{\tilde{\nu}^+(\alpha)+\delta}j + 1)} \\
 &\leq 2^j \frac{(2^j \tilde{\rho}_j((-\infty, \alpha + \varepsilon]))^{2^{\tilde{\nu}^+(\alpha)+\delta}j}}{\Gamma(2^{\tilde{\nu}^+(\alpha)+\delta}j + 1)}.
 \end{aligned}$$

Using Stirling's formula, we obtain then that for j large enough,

$$\begin{aligned}
 2^j \frac{(2^j \tilde{\rho}_j((-\infty, \alpha + \varepsilon]))^{2^{\tilde{\nu}^+(\alpha)+\delta}j}}{\Gamma(2^{\tilde{\nu}^+(\alpha)+\delta}j + 1)} &\sim 2^j \frac{(2^j \tilde{\rho}_j((-\infty, \alpha + \varepsilon])) e^{2^{-1/2}} 2^{2^{\tilde{\nu}^+(\alpha)+\delta}j}}{\sqrt{2\pi} (2^{\tilde{\nu}^+(\alpha)+\delta}j)^{2^{\tilde{\nu}^+(\alpha)+\delta}j}} \\
 &\leq 2^j \frac{(2^{\tilde{\nu}^+(\alpha)+\delta/2}j e^{2^{-1/2}})^{2^{\tilde{\nu}^+(\alpha)+\delta}j}}{\sqrt{2\pi} (2^{\tilde{\nu}^+(\alpha)+\delta}j)^{2^{\tilde{\nu}^+(\alpha)+\delta}j}} \\
 &= \frac{2^j}{\sqrt{2\pi}} \left(e^{2^{-1/2}} 2^{-j \frac{\delta}{2}} \right)^{2^{\tilde{\nu}^+(\alpha)+\delta}j} \\
 &\leq \frac{1}{\sqrt{2\pi}} \left(e^{2^{-j \frac{\delta}{2}}} \right)^{2^{\tilde{\nu}^+(\alpha)+\delta}j}
 \end{aligned}$$

since

$$2^j \leq \left(\sqrt{2} \right)^{2^{\tilde{\nu}^+(\alpha)+\delta}j}$$

if j is large enough. We conclude the proof using the Borel Cantelli lemma. \square

Proposition B.4. *Let f be a random wavelet series. With probability one,*

$$\tilde{\nu}_f^+(\alpha) = \begin{cases} -\infty & \text{if } \alpha \in [0, h_{\min}), \\ \tilde{\nu}^+(\alpha) & \text{if } \alpha \geq h_{\min}. \end{cases}$$

Proof. Let us fix $\alpha \geq h_{\min}$ such that $\tilde{\nu}^+(\alpha) > 0$. From Lemma B.3, with probability one, we have $\tilde{\nu}_f^+(\alpha) \leq \tilde{\nu}^+(\alpha) + \delta$. Taking a decreasing sequence $(\delta_n)_{n \in \mathbb{N}}$ converging to 0, we get that with probability one, $\tilde{\nu}_f^+(\alpha) \leq \tilde{\nu}(\alpha)$. Using Lemma B.2, we also obtain that $\tilde{\nu}_f^+(\alpha) \geq \tilde{\nu}^+(\alpha)$ with probability one, taking two sequences $(\varepsilon_n)_{n \in \mathbb{N}}$ and $(\delta_n)_{n \in \mathbb{N}}$ which converge to 0. Since $\tilde{\nu}^+$ and $\tilde{\nu}_f^+$ are right-continuous, we get the conclusion taking a dense sequence $(\alpha_n)_{n \in \mathbb{N}}$.

Let us now assume that $\alpha \geq h_{\min}$ is such that $\tilde{\nu}^+(\alpha) = 0$. In this case, Lemma B.3 gives that $\tilde{\nu}_f^+(\alpha) \leq 0$. If $\tilde{\nu}_f^+(\alpha) = -\infty$, then for $\varepsilon > 0$ small enough and $j \in \mathbb{N}$ large enough, we have

$$e_{j,k} \leq 2^{-(\alpha+\varepsilon)j}, \quad \forall k \in \{0, \dots, 2^j - 1\}.$$

In particular, using Proposition 4.7.4, the Hölder exponents of f are all strictly greater than α , which contradicts Theorem B.1 which asserts in particular that every value in $[h_{\min}, h_{\max}]$ is a Hölder exponent of f . Consequently, $\tilde{\nu}_f^+(\alpha) = \nu^+(\alpha) = 0$ with a probability one.

Finally, we know from Proposition 3.4 of [12] that with probability one, if $\alpha < h_{\min}$, then $|e_{j,k}| \leq 2^{-\alpha j}$ for every j large enough and every $k \in \{0, \dots, 2^j - 1\}$. It follows that $e_{j,k} \leq 2^{-\alpha j}$ for every j large enough and every $k \in \{0, \dots, 2^j - 1\}$ and $\tilde{\nu}_f^+(\alpha) = -\infty$. \square

In the following proof, we will use classes $\mathcal{G}^s(\mathbb{T})$ of sets of large intersection, defined in [67]. Let us recall that $\mathcal{G}^s(\mathbb{T})$ is the maximal class of G_σ -sets of Hausdorff dimension at least s , that is closed under countable intersections and similarities. The original setting was in \mathbb{R} . As done in [12], we make obvious modifications for working in \mathbb{T} .

Proposition B.5. *Let f be a random wavelet series. For every $\alpha \in [h_{\min}, \alpha_s]$, with probability one,*

$$d_f(\alpha) = \tilde{\nu}^+(\alpha).$$

Proof. We already know from Proposition 5.3.4 and Proposition B.4 that with probability one, for every $\alpha \in [h_{\min}, \alpha_s]$, $d_f(\alpha) \leq \tilde{\nu}^+(\alpha)$. In particular, if $\tilde{\nu}^+(\alpha) = 0$, then $d_f(\alpha) = 0$ and we have the announced equality. So, we can assume that $\tilde{\nu}^+(\alpha) > 0$. Let us fix $\varepsilon > 0$ and $\delta > 0$ such that $\tilde{\nu}^+(\alpha) - 2\delta > 0$. Using Lemma B.2, we know that with probability one, there exists a subsequence $(j_n)_{n \in \mathbb{N}}$ such that there are at least $2^{(\tilde{\nu}^+(\alpha) - \delta)j_n}$ restricted wavelet leaders such that

$$e_{j_n, k} \geq 2^{-(\alpha+\varepsilon)j_n}.$$

The locations of the restricted wavelet leaders satisfying this relation are picked at random among the 2^{j_n} possible locations. Applying Lemma 1 of [89], if $\gamma < \tilde{\nu}^+(\alpha) - \delta$, with probability one,

$$\mathbb{T} = \limsup_{n \rightarrow +\infty} \bigcup_{k \in F^{j_n}(\alpha+\varepsilon)} (k2^{-j_n} - 2^{-\gamma j_n}, k2^{-j_n} + 2^{-\gamma j_n}).$$

Consequently, applying Proposition 5.4 of [12], we get that for every $t \geq 1$,

$$E^t(\alpha + \varepsilon) := \limsup_{n \rightarrow +\infty} \bigcup_{k \in F^{j_n}(\alpha+\varepsilon)} (k2^{-j_n} - 2^{-\gamma t j_n}, k2^{-j_n} + 2^{-\gamma t j_n}) \in \mathcal{G}^{\frac{1}{t}}(\mathbb{T}).$$

Let us fix $t = \frac{1}{\tilde{\nu}^+(\alpha) - 2\delta} \geq 1$ and γ such that $1/t \leq \gamma < \tilde{\nu}^+(\alpha) - \delta$. We also set for every $\varepsilon > 0$,

$$G(\alpha - \varepsilon) := \limsup_{j \rightarrow +\infty} \bigcup_{k \in F^j(\alpha-\varepsilon)} (k2^{-j} - 2^{-(\tilde{\nu}^+(\alpha)+2\varepsilon)j}, k2^{-j} + 2^{-(\tilde{\nu}^+(\alpha)+2\varepsilon)j}) \in \mathcal{G}^{\frac{1}{t}}(\mathbb{T}).$$

Let us show that

$$\dim_{\mathcal{H}} \left(\bigcap_{\varepsilon > 0} E^t(\alpha + \varepsilon) \setminus G(\alpha - \varepsilon) \right) \geq \tilde{\nu}^+(\alpha).$$

Remark that for every $\varepsilon > 0$ we are in the conditions of Proposition 5.8 of [12] and therefore, with probability one,

$$E^t(\alpha + \varepsilon) \setminus G(\alpha - \varepsilon) \in \mathcal{G}^{\frac{1}{t}}(\mathbb{T}).$$

Since the intersection can be taken countable, we obtain that with probability one,

$$\bigcap_{\varepsilon > 0} (E^t(\alpha + \varepsilon) \setminus G(\alpha - \varepsilon)) \in \mathcal{G}^{\frac{1}{t}}(\mathbb{T}).$$

Consequently, with probability one,

$$\dim_{\mathcal{H}} \left(\bigcap_{\varepsilon > 0} E^t(\alpha + \varepsilon) \setminus G(\alpha - \varepsilon) \right) \geq \tilde{\nu}^+(\alpha) - 2\delta.$$

Since $\delta > 0$ is arbitrary, taking a sequence that decreases to 0, we get that

$$\dim_{\mathcal{H}} \left(\bigcap_{\varepsilon > 0} E^t(\alpha + \varepsilon) \setminus G(\alpha - \varepsilon) \right) \geq \tilde{\nu}^+(\alpha).$$

To conclude, it suffices to show that if x belongs to

$$\bigcap_{\varepsilon > 0} E^t(\alpha + \varepsilon) \setminus G(\alpha - \varepsilon),$$

then $h_f(x) = \alpha$. Let us fix x in this intersection. For every $\varepsilon > 0$, there are infinitely many n for which there is $k \in F^{j_n}(\alpha + \varepsilon)$ such that $x \in (k2^{-j_n} - 2^{-\gamma t j_n}, k2^{-j_n} + 2^{-\gamma t j_n})$. Since $t \geq 1/\gamma$, we get that $x \in ((k-1)2^{-j_n}, (k+2)2^{-j_n})$. Therefore, $\lambda(j_n, k) \subseteq 3\lambda_{j_n}(x)$. It follows that $d_{j_n}(x) \geq e_{j_n, k} \geq 2^{-(\alpha + \varepsilon)j_n}$, hence $h_f(x) \leq \alpha + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get that $h_f(x) \leq \alpha$.

Let us now fix $\varepsilon > 0$ small enough so that $\tilde{\nu}^+(\alpha) + 2\varepsilon < 1$. Fix k such that $x \in [k2^{-j}, (k+1)2^{-j}]$. Remark that for j large enough, since $\tilde{\nu}^+(\alpha) + 2\varepsilon < 1$, we have

$$[k2^{-j}, (k+1)2^{-j}] \subseteq \left(k2^{-j} - 2^{-\tilde{\nu}^+(\alpha) + 2\varepsilon j}, k2^{-j} + 2^{-\tilde{\nu}^+(\alpha) + 2\varepsilon j} \right)$$

and it follows that $e_{j, k} \leq 2^{-(\alpha - \varepsilon)j}$ since $x \notin G(\alpha - \varepsilon)$. Moreover, for j large enough,

$$[k2^{-j}, (k+1)2^{-j}] \subseteq \left((k-1)2^{-j} - 2^{-\tilde{\nu}^+(\alpha) + 2\varepsilon j}, (k-1)2^{-j} + 2^{-\tilde{\nu}^+(\alpha) + 2\varepsilon j} \right)$$

and

$$[k2^{-j}, (k+1)2^{-j}] \subseteq \left((k+1)2^{-j} - 2^{-\tilde{\nu}^+(\alpha) + 2\varepsilon j}, (k+1)2^{-j} + 2^{-\tilde{\nu}^+(\alpha) + 2\varepsilon j} \right).$$

It follows that $e_{j, k-1} \leq 2^{-(\alpha - \varepsilon)j}$ and $e_{j, k+1} \leq 2^{-(\alpha - \varepsilon)j}$. Therefore, we obtain that $d_j(x) = d_{j, k} \leq 2^{-(\alpha - \varepsilon)j}$ for j large enough. Consequently, $h_f(x) \geq \alpha - \varepsilon$ and we get the conclusion since $\varepsilon > 0$ is arbitrary small. \square

Remark B.6. In particular, with probability one, $\alpha_s = h_{\max}$. Indeed, by definition, $\alpha_s = \inf\{\alpha \geq 0 : \tilde{\nu}^+(\alpha) = 1\}$ and from Theorem B.1, we know that $d_f(h) = 1$ if and only if $h = h_{\max}$.

Corollary B.7. *Let f be a random wavelet series. With probability one,*

$$d_f(\alpha) = \tilde{\nu}_f^+(\alpha), \quad \forall \alpha \in [0, \alpha_s].$$

Proof. From Theorem B.1, Proposition B.4 and Proposition B.5, we know that for every $\alpha \in [0, \alpha_s]$, with probability one, $d_f(\alpha) = \tilde{\nu}_f^+(\alpha)$. We obtain the conclusion by taking a dense sequence $(\alpha_n)_{n \in \mathbb{N}}$ in $[0, \alpha_s]$ and by using the right continuity of the functions $\tilde{\nu}_f$ and d_f . \square

Remark B.8. Using the results of [12], this result can also be considered as a consequence of the comparison of the formalisms based on \mathcal{S}^ν spaces and on the leaders profile method, see Proposition 5.5.3.

Let us now study the decreasing wavelet leaders profile. With probability one, if $\alpha > \alpha_s = h_{\max}$, we know that $d_f(\alpha) = -\infty$. So, one can expect that the same holds for the profile $\tilde{\nu}_f^-$.

Proposition B.9. *Let f be a random wavelet series. With probability one,*

$$\tilde{\nu}_f^-(\alpha) = -\infty, \quad \forall \alpha \in (\alpha_s, +\infty].$$

Proof. Let $\alpha > \alpha_s$. It suffices to show that, with probability one, $e_{j,k} \geq 2^{-\alpha j}$ for every j large enough and $k \in \{0, \dots, 2^j - 1\}$. Fix $\varepsilon > 0$ and $\delta > 0$ such that $\alpha(1 - \delta) > \alpha_s + \varepsilon$. From the assumption B.1, we know that there is $J \in \mathbb{N}$ such that

$$\tilde{\rho}_j((-\infty, \alpha_s + \varepsilon]) \geq 2^{-\delta j}, \quad \forall j \geq J.$$

For every $j \in \mathbb{N}$, let us denote by A_j the event “there exists $k \in \{0, \dots, 2^j - 1\}$ such that $e_{j,k} < 2^{-\alpha j}$ ”. Let us set $j_0 = \lfloor \frac{1}{1-\delta}(j + \log_2 j) \rfloor + 1$. Remark that if j is large enough, then $(\alpha_s + \varepsilon)j_0 < \alpha j$. Then, using Remark 5.4.8, we have

$$\begin{aligned} \mathbb{P}[A_j] &\leq \sum_{k=0}^{2^j-1} \mathbb{P}[e_{j,k} < 2^{-\alpha j}] \\ &\leq \sum_{k=0}^{2^j-1} \prod_{\lambda_0 \subseteq \lambda, \lambda_0 \in \Lambda_{j_0}} \mathbb{P}[e_{\lambda_0} < 2^{-\alpha j}] \\ &\leq \sum_{k=0}^{2^j-1} \prod_{\lambda_0 \subseteq \lambda, \lambda_0 \in \Lambda_{j_0}} (1 - \tilde{\rho}_{j_0}((-\infty, \alpha_s + \varepsilon])) \\ &\leq \sum_{k=0}^{2^j-1} (1 - \tilde{\rho}_{j_0}((-\infty, \alpha_s + \varepsilon]))^{2^{j_0-j}} \\ &\leq \sum_{k=0}^{2^j-1} (1 - 2^{-\delta j_0})^{2^{j_0-j}} \\ &= 2^j (1 - 2^{-\delta j_0})^{2^{j_0-j}} \\ &\leq 2^j \exp(-2^{j_0-j} 2^{-\delta j_0}) \\ &\leq 2^j \exp(-2^{(1-\delta)j_0-j}) \\ &\leq 2^j \exp(-j) \\ &= \left(\frac{2}{e}\right)^j \end{aligned}$$

if j is large enough, using Remark 5.4.8. It follows that the series $\sum_{j \in \mathbb{N}} \mathbb{P}[A_j]$ converges. Using the Borel Cantelli lemma, we get that with a probability one, $\tilde{\nu}_f^-(\alpha) = -\infty$ if $\alpha > \alpha_s$. \square

A combination of the previous results gives the validity of the leaders profile method for almost every random wavelet series.

Theorem B.10. *Let f be a random wavelet series. With probability one, we have $d_f = \tilde{\nu}_f$ on $[0, +\infty]$.*

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List of symbols

Sets

\mathbb{N}	The set of strictly positive natural numbers
\mathbb{N}_0	The set of natural numbers
\mathbb{Z}	The set of entire numbers
\mathbb{Q}	The set of rational numbers
\mathbb{R}	The set of real numbers
\mathbb{C}	The set of complex numbers
\mathbb{T}	The unit torus \mathbb{R} / \mathbb{Z}
\mathbb{N}^n	The set of n -dimensional multi-indices
\mathbb{R}^n	The n -dimensional real Euclidean space
\mathbb{C}^n	The n -dimensional complex Euclidean space
$[a, b]$	Closed interval
(a, b)	Open interval
$[a, b), (a, b]$	Half open interval
Ω	Open set of \mathbb{R}^n in Part I The set of all complex sequences $(c_{j,k})_{j \in \mathbb{N}, k \in \{0, \dots, 2^j - 1\}}$ in Part II
K	Compact set of \mathbb{R}^n
\mathcal{H}	Hamel basis of \mathbb{R}
A^c	The complement of the set A
$E^f(h)$	Iso-Hölder set of f
λ	Dyadic interval
$\lambda_j(x_0)$	The unique dyadic interval of length 2^{-j} that contains x_0
Λ	The set of all dyadic intervals of $[0, 1)$
Λ_j	The set of all dyadic intervals of length 2^{-j} of $[0, 1)$

$E_j(C, \alpha)(\vec{c})$ $\{\lambda \in \Lambda_j : |c_\lambda| \geq C2^{-\alpha j}\}$

$\tilde{E}_j^+(C, \alpha)(\vec{c})$ $\{\lambda \in \Lambda_j : e_\lambda \geq C2^{-\alpha j}\}$

$\tilde{E}_j^-(C, \alpha)(\vec{c})$ $\{\lambda \in \Lambda_j : e_\lambda \leq C2^{-\alpha j}\}$

$C(r)$ Cantor set of ratio r

$\liminf_{n \rightarrow +\infty} A_n$ $\bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} A_n$

$\limsup_{n \rightarrow +\infty} A_n$ $\bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} A_n$

Spaces and sets of functions or sequences

$C^\infty(\Omega)$ The space of complex-valued smooth functions on Ω

$\mathcal{C}^\infty(K)$ The space of complex-valued smooth functions on the interior of K whose derivatives of any order can be continuously extended to K

$G^s(\Omega)$ The Gevrey class of order s

NG The set of nowhere Gevrey differentiable functions

$\mathcal{E}_{\{\cdot\}}(\Omega)$ Space of ultradifferentiable functions of Roumieu type

$\mathcal{E}_{(\cdot)}(\Omega)$ Space of ultradifferentiable functions of Beurling type

$\mathcal{E}_{[\cdot]}(\Omega)$ Space of ultradifferentiable functions

$\mathcal{E}'_{[\cdot]}(\Omega)$ The strong dual space of $\mathcal{E}_{[\cdot]}(\Omega)$

$\mathcal{A}_{(\omega)}(\Omega)$ $\underset{K \subseteq \Omega, n \in \mathbb{N}}{\text{ind}} \underset{K \subseteq \Omega, n \in \mathbb{N}}{\text{ind}} A(K, n)$

$\mathcal{A}_{\{\omega\}}(\Omega)$ $\underset{K \subseteq \Omega, n \in \mathbb{N}}{\text{ind}} \underset{K \subseteq \Omega, n \in \mathbb{N}}{\text{proj}} A(K, \frac{1}{n})$

$A(K, \lambda)$ The set of functions $f \in \mathcal{H}(\mathbb{C}^n)$ such that $|f|_{K, \lambda}^\omega < +\infty$

$\mathcal{H}(\mathbb{C}^n)$ The space of entire functions on \mathbb{C}^n

$\mathcal{S}(\mathbb{R})$ The Schwartz class on \mathbb{R}

$C^\alpha(x_0)$ The pointwise Hölder space

C^r The space of sequences \vec{c} such that $\|\vec{c}\|_{C^r} < +\infty$

$b_{p, \infty}^s$ Besov space

$A(\alpha, \beta)$ Auxiliary space (in the case of the \mathcal{S}^ν spaces)

\mathcal{O}_p^s Oscillation space

$\tilde{A}^+(\alpha, \beta)$ Auxiliary space (in the case of $\mathcal{L}^{\nu, +}$ spaces)

\tilde{K}^+	Compact subset of $(\mathcal{L}^{\nu,+}, \tilde{\delta}^+)$
\tilde{K}^-	Compact subset of $(\mathcal{L}^{\nu,-}, \tilde{\delta}^-)$
\tilde{K}	Compact subset of $(\mathcal{L}^{\nu}, \tilde{\delta})$

Symbols, operators and applications

$ \alpha $	The length of the multi-index $\alpha \in \mathbb{N}^n$, i.e. $\alpha_1 + \dots + \alpha_n$
$ x $	The absolute value of $x \in \mathbb{R}$
$ z $	The modulus of $z \in \mathbb{C}$
$\lfloor x \rfloor$	The largest integer less than or equal to $x \in \mathbb{R}$
$\Re z$	The real part of $z \in \mathbb{C}$
$\Im z$	The imaginary part of $z \in \mathbb{C}$
e_α	The function defined on \mathbb{R} by $e_\alpha(x) := \exp(\alpha x)$
χ_E	The indicator function of E
$\binom{n}{k}$	The binomial coefficient $\frac{n!}{k!(n-k)!}$
$D^\alpha f$	The derivative of f of order $\alpha \in \mathbb{N}_0^n$
$T(f, x_0)$	The Taylor series of f at x_0
$CS(f)$	The set of Cauchy singularities of f
$PS(f)$	The set of Pringsheim singularities of f
$*$	Convolution product of functions
\circ	Composition of functions
\mathfrak{c}	The continuum
$\lambda(M)$	Maximal cardinality of a subspace contained in a lineable set M
ind	Inductive limit
proj	Projective limit
span	Linear hull
M	Weight sequence
$M \preceq N$	$\sup_{k \in \mathbb{N}} \left(\frac{M_k}{N_k} \right)^{\frac{1}{k}} < +\infty$
$M \approx N$	$M \preceq N$ and $N \preceq M$

$M \triangleleft N$	$\lim_{k \rightarrow +\infty} \left(\frac{M_k}{N_k} \right)^{\frac{1}{k}} = 0$
ω	Weight function
$\omega \preceq \sigma$	$\sigma(t) = O(\omega(t))$ as $t \rightarrow +\infty$
$\omega \sim \sigma$	$\omega \preceq \sigma$ and $\sigma \preceq \omega$
$\omega \triangleleft \sigma$	$\sigma(t) = o(\omega(t))$ as $t \rightarrow +\infty$
\mathcal{M}	Weight matrix
$\mathcal{M} \triangleleft \mathcal{N}$	$M \triangleleft N$ for every $M \in \mathcal{M}$ and every $N \in \mathcal{N}$
h_K	Support functional of K
$f \sim g$	$\liminf_{t \rightarrow +\infty} \frac{f(t)}{g(t)} = 1$ (it is a limit in the case of Stirling's formula)
$h_f(x_0)$	Hölder exponent of f at x_0
d_f	Multifractal spectrum of f
ψ	Mother wavelet
$(\psi_\lambda)_{\lambda \in \Lambda}$	Orthogonal wavelet basis of $L^2(\mathbb{T})$
$c_{j,k}, c_\lambda$	Wavelet coefficients
$d_{j,k}, d_\lambda$	Wavelet leaders
$e_{j,k}, e_\lambda$	Restricted wavelet leaders
$\eta_{\vec{e}}, \eta_f$	Scaling function
$\tilde{\eta}_f$	Scaling function for the wavelet leaders method
$\nu_{\vec{e}}, \nu_f$	Wavelet profile
$\tilde{\rho}_{\vec{e}}$	Wavelet leaders density
$\tilde{\rho}_{\vec{e}}^*$	Wavelet leaders density defined using restricted wavelet leaders
$\tilde{\nu}_{\vec{e}}^+, \tilde{\nu}_f^+$	Increasing wavelet leaders profile
$\tilde{\nu}_{\vec{e}}^-, \tilde{\nu}_f^-$	Decreasing wavelet leaders profile
$\tilde{\nu}_{\vec{e}}, \tilde{\nu}_f$	Wavelet leaders profile
ν	Admissible profile
α	Dense sequence in $[0, \alpha_s]$
ε	Sequence of $(0, +\infty)$ which converges to 0
α'	Dense sequence in $[\alpha_s, +\infty)$

Norms, semi-norms and distances

$p_k(f)$	$\sup_{j \leq k} \sup_{x \in [0,1]} D^j f(x) $
$\ f\ _{K,h}^M$	$\sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} \frac{ D^\alpha f(x) }{h^{ \alpha } M^{ \alpha }}$
$\ f\ _{K,m}^\omega$	$\sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} D^\alpha f(x) \exp\left(-\frac{1}{m} \varphi_\omega^*(m \alpha)\right)$
$p_{K,m}^\omega(f)$	$\sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} D^\alpha f(x) \exp\left(-m \varphi_\omega^*\left(\frac{ \alpha }{m}\right)\right)$
$ f _{K,\lambda}^\omega$	$\sup_{z \in \mathbb{C}^n} f(z) \exp(-h_K(\Im z) - \lambda \omega(z))$
$\ \vec{c}\ _{C^r}$	$\sup_{j \in \mathbb{N}_0} \sup_{k \in \{0, \dots, 2^j - 1\}} 2^{rj} c_{j,k} $
$\delta_{\alpha,\beta}$	Distance on $A(\alpha, \beta)$
δ	Distance on \mathcal{S}^ν
$\tilde{\delta}_{\alpha,\beta}^+$	Distance on $\tilde{A}^+(\alpha, \beta)$
$\tilde{\delta}^+$	Distance on $\mathcal{L}^{\nu,+}$
$\tilde{\delta}_{\alpha,\beta}^-$	Pseudo-distance on $\tilde{A}^-(\alpha, \beta)$
$\tilde{\delta}^-$	Pseudo-distance on $\mathcal{L}^{\nu,-}$
$\tilde{\delta}$	Distance on \mathcal{L}^ν

Measures and dimensions

\mathcal{H}^s	The s -dimensional Hausdorff measure
\mathcal{L}	The Lebesgue measure
\mathcal{L}^n	The n -dimensional Lebesgue measure
\mathcal{L}_P	The Lebesgue measure supported by P
$\dim_{\mathcal{H}}$	The Hausdorff dimension
diam	The diameter
\mathbb{P}	The probability of an event

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