Sectional category of a class of maps

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Abstract. We propose a definition of 'sectional category of a class of maps'. This combines the notions of 'sectional category' of James, and 'category of a class of spaces' of Clapp and Puppe.

The category $\operatorname{cat} X$ of a space X in the sense of Lusternik and Schnirelmann is the smallest number n such that there exists an open covering $\{U_0, \ldots, U_n\}$ of X for which each inclusion $U_i \hookrightarrow X$ is nullhomotopic. In [**1**], M. Clapp and D. Puppe introduced the $\mathcal{A}$-category of X, where $\mathcal{A}$ is a class of spaces, replacing 'is nullhomotopic' in the previous definition by 'factors through some space of $\mathcal{A}$'. On the other hand, the sectional category $\operatorname{secat} p$ of a fibration $p\colon E \to X$, originally defined by Schwarz [**12**], is obtained by replacing 'each inclusion $U_i \hookrightarrow X$ is nullhomotopic' in the previous definition by 'p has a local section on each of the open sets U_i'. Here we gather these ideas by defining the sectional category of a class of maps with same target X.

We propose the Ganea and the Whitehead versions of this definition, as well as the open covering approach.

Sectional category earned its renown recently thanks to Farber's notion of topological complexity of a space A ([**6**]), which measures the difficulty of solving the motion planing problem: the topological complexity of A is the sectional category of the diagonal $\Delta\colon A \to A \times A$. Hence, particular attention is devoted to the sectional category of classes of maps with target $A \times A$ containing (or not) the diagonal.

Throughout this paper $\mathbb{T}$ will be a category of topological spaces and maps. It can be just topological spaces and continuous maps, but also pointed topological spaces and maps, G-equivariant topological spaces and maps, or else filtered topological spaces and maps. To assure that everything goes well, $\mathbb{T}$ should be a J-category in the sense of [**3**]. In [**11**], it is shown that different notions of sectional category are obtained for different J-structures, but coincide under reasonable conditions.

1. The Ganea point of view

Definition 1.1. For any finite sequence $\mathcal{S} = (\iota_0\colon A_0 \to X, \ldots, \iota_n\colon A_n \to X)$ of maps of $\mathbb{T}$, the *Ganea construction* of $\mathcal{S}$ is the following sequence of homotopy

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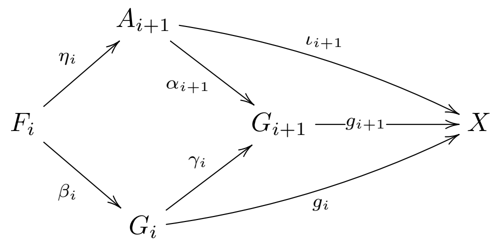
commutative diagrams ($0 \leqslant i < n$):



where the outside square is a homotopy pullback, the inside square is a homotopy pushout and the map $g_{i+1} = (g_i, \iota_{i+1})\colon G_{i+1} \to X$ is the whisker map induced by this homotopy pushout. The iteration starts with $g_0 = \iota_0\colon A_0 \to X$.

We can summarize all this by saying that g_n is the iterated join over X of all maps in $\mathcal{S}$.

We denote G_n by $G(\mathcal{S})$ and g_n by $g(\mathcal{S})$. We also write $g_n(\iota_X)$ instead of $g(\mathcal{S})$ when $\mathcal{S} = (\iota_X, \ldots, \iota_X)$.

DEFINITION 1.2. Let $\mathcal{A}$ be a class of maps of $\mathbb{T}$ with same target X. The *sectional category* of $\mathcal{A}$ is the least integer n such that there exists a sequence $\mathcal{S}$ of $n+1$ maps in $\mathcal{A}$, $g(\mathcal{S})\colon G(\mathcal{S}) \to X$ having a homotopy section, i.e. a map $\sigma\colon X \to G(\mathcal{S})$ such that $g(\mathcal{S}) \circ \sigma \simeq \mathrm{id}_X$.

We denote the sectional category by $\mathrm{secat}\,(\mathcal{A})$. We write $\mathrm{secat}\,(\iota_X) = \mathrm{secat}\,(\mathcal{A})$ when $\mathcal{A}$ is reduced to the single map $\iota_X\colon A \to X$. In this case, there is only one sequence of length $n+1$ of maps in $\mathcal{A}$ which is $(\iota_X, \ldots, \iota_X)$. If $\mathbb{T}$ is pointed with $*$ as zero object, we write $\mathrm{cat}\,(X) = \mathrm{secat}\,(\mathcal{A})$ when $\mathcal{A}$ is reduced to the single map $* \to X$. The integer $\mathrm{cat}\,(X)$ is the 'normalized' version of the Lusternik-Schnirelmann category.

We shall also write: $\mathrm{infcat}\,(\mathcal{A}) = \inf\{\mathrm{secat}\,(\iota) \mid \iota \in \mathcal{A}\}$.

REMARK 1.3. Clearly, for any class $\mathcal{A}$, $\mathrm{secat}\,(\mathcal{A}) \leqslant \mathrm{infcat}\,(\mathcal{A})$.

EXAMPLE 1.4. Let X be a fixed space in $\mathbb{T}$, and let $\mathcal{A}$ be a class of spaces in $\mathbb{T}$. Then $\mathcal{A}$-cat(X) in the sense of [**1**] is $\mathrm{secat}\,(\bar{\mathcal{A}})$ where $\bar{\mathcal{A}}$ is the class consisting of all maps from any space in $\mathcal{A}$ to X.

EXAMPLE 1.5. Let $\mathbb{T}$ be the category of stratified spaces and maps. Consider X a foliated manifold in $\mathbb{T}$ and let $\mathcal{A}$ be the class of all inclusions $A \hookrightarrow X$ where A is a transverse subspace of X, i.e. $A \cap F$ is at most countable for any leaf F of X. Then $\mathrm{secat}\,(\mathcal{A})$ is actually the *transverse LS-category* of X introduced by H. Colman [**2**] while $\mathrm{infcat}\,(\mathcal{A})$ is actually the *open LS-category* of X introduced by J.-P. Doeraene, E. Macias-Virgós and D. Tanré [**5**].

In fact, it appears that here $\mathrm{secat}\,(\mathcal{A}) = \mathrm{infcat}\,(\mathcal{A})$. Indeed in the light of Theorem 3.3, $\mathrm{secat}\,(\mathcal{A}) \leqslant n$ when there is a covering of X with open subspaces U_i ($0 \leqslant i \leqslant n$) which are each deformable in X to a transverse subspace A_i of X, in a stratified way. Then each U_i is deformable in a stratified way in X to $A = \bigcup\limits_{i=0}^{n} A_i$ which is also transverse. Hence $\mathrm{infcat}\,(\mathcal{A}) \leqslant n$.

Proposition 1.6. *Let $f\colon X \to Y$ be a map in $\mathbb{T}$ and assume that we have a sequence of homotopy commutative squares in $\mathbb{T}$ $(0 \leqslant i \leqslant n)$:*

$$(\dagger) \qquad \begin{array}{ccc} A & \xrightarrow{\iota_i} & X \\ \downarrow & & \downarrow f \\ B & \xrightarrow[\tau_i]{} & Y. \end{array}$$

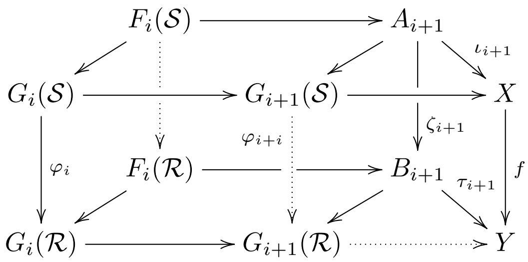
Then, for the corresponding sequences $\mathcal{S} = (\iota_0, \ldots, \iota_n)$ and $\mathcal{R} = (\tau_0, \ldots, \tau_n)$ of maps in $\mathbb{T}$, there is a homotopy commutative diagram

$$(\ddagger) \qquad \begin{array}{ccc} G(\mathcal{S}) & \xrightarrow{g(\mathcal{S})} & X \\ \downarrow & & \downarrow f \\ G(\mathcal{R}) & \xrightarrow[g(\mathcal{R})]{} & Y. \end{array}$$

In particular, if for any map $\iota\colon A \to X$ in a class $\mathcal{A}$ there exists a map $\tau\colon B \to Y$ in a class $\mathcal{B}$ with a homotopy square $(\dagger)$, and if f has a homotopy section, then $\operatorname{secat}(\mathcal{B}) \leqslant \operatorname{secat}(\mathcal{A})$.

On the other hand, if for any map $\tau\colon B \to Y$ in a class $\mathcal{B}$ the square $(\dagger)$ is a homotopy pull back and the map ι is in a class $\mathcal{A}$, then the diagram $(\ddagger)$ is also a homotopy pullback and in this case $\operatorname{secat}(\mathcal{A}) \leqslant \operatorname{secat}(\mathcal{B})$.

Proof. We can see that there is a map $\varphi\colon G(\mathcal{S}) \to G(\mathcal{R})$ such that $g(\mathcal{R}) \circ \varphi \simeq f \circ g(\mathcal{S})$, using the Join Theorem ([**4**, Theorem 51]) recursively in the following diagram:



beginning with $\varphi_0 = \zeta_0$ and ending with $\varphi = \varphi_n$.

Assume f has a homotopy section s. If $g(\mathcal{S})$ has a homotopy section σ, then $g(\mathcal{R})$ has a homotopy section $\varphi \circ \sigma \circ s$.

Assume the starting squares $(\dagger)$ are homotopy pullbacks. Then so is the front rightmost one in the above diagram for any $i < n$, thus $(\ddagger)$ is a homotopy pullback. If $g(\mathcal{R})$ has a homotopy section σ, then $g(\mathcal{S})$ has a homotopy section which is the induced map $(\sigma \circ f, \mathrm{id}_X)$. □

Definition 1.7. There is a preorder on maps of $\mathbb{T}$ with same target X defined by: $\iota\colon A \to X \succcurlyeq \tau\colon B \to X$ if ι factors through τ up to homotopy, i.e. there is a map $\zeta\colon A \to B$ such that $\tau \circ \zeta \simeq \iota$.

This preorder extends to classes of maps of $\mathbb{T}$ with same target X: we write $\mathcal{A} \succcurlyeq \mathcal{B}$ if each map of $\mathcal{A}$ factors through at least one map of $\mathcal{B}$ up to homotopy. We write $\mathcal{A} \approx \mathcal{B}$ if $\mathcal{A} \succcurlyeq \mathcal{B}$ and $\mathcal{B} \succcurlyeq \mathcal{A}$.

Remark 1.8. If $\hat{\mathcal{A}}$ is a subclass of $\mathcal{A}$, then $\hat{\mathcal{A}} \succcurlyeq \mathcal{A}$.

With $f = \mathrm{id}_X$ in Proposition 1.6 we get:

PROPOSITION 1.9. *For any classes $\mathcal{A}$ and $\mathcal{B}$ of maps in $\mathbb{T}$ with same target X:*

$$\mathcal{B} \preccurlyeq \mathcal{A} \implies \operatorname{secat}(\mathcal{B}) \leqslant \operatorname{secat}(\mathcal{A}).$$

COROLLARY 1.10. *Let $\mathbb{T}$ be pointed. For any class $\mathcal{A}$ of maps in $\mathbb{T}$ with same target X:*

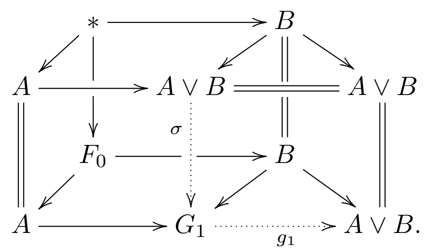
$$\operatorname{secat}(\mathcal{A}) \leqslant \operatorname{cat}(X).$$

COROLLARY 1.11. *For any class $\mathcal{A}$ of maps in $\mathbb{T}$ with same target X, and any subclass $\hat{\mathcal{A}}$ of $\mathcal{A}$, we have $\mathcal{A} \preccurlyeq \hat{\mathcal{A}}$ and* $\operatorname{secat}(\mathcal{A}) \leqslant \operatorname{secat}(\hat{\mathcal{A}})$. *If, moreover, each map of $\mathcal{A}$ factors up to homotopy through at least one map of $\hat{\mathcal{A}}$, then also $\hat{\mathcal{A}} \succcurlyeq \mathcal{A}$ and* $\operatorname{secat}(\hat{\mathcal{A}}) = \operatorname{secat}(\mathcal{A})$.

REMARK 1.12. From this fact, we may often replace a class $\mathcal{A}$ by a smaller or a greater one to compute $\operatorname{secat}(\mathcal{A})$. In particular, we can keep only one representative for each homotopy class of maps of $\mathcal{A}$. Conversely, we can always assume that all maps equivalent (for the relation $\approx$) to some map of $\mathcal{A}$ are also in $\mathcal{A}$.

COROLLARY 1.13. *For any class $\mathcal{A}$ of maps in $\mathbb{T}$ with same target X, if $\mathcal{A}$ contains a map $\tau\colon B \to X$ such that each map of $\mathcal{A}$ factors through τ, then* $\operatorname{secat}(\mathcal{A}) = \operatorname{infcat}(\mathcal{A}) = \operatorname{secat}(\tau)$.

EXAMPLE 1.14. Let $\mathbb{T}$ be pointed and let $\mathcal{A}$ be the set of the two maps $\mathrm{in}_1\colon A \hookrightarrow A \vee B$ and $\mathrm{in}_2\colon B \hookrightarrow A \vee B$. It is known that $\operatorname{secat}(\mathrm{in}_1) = \operatorname{cat}(B)$ and $\operatorname{secat}(\mathrm{in}_2) = \operatorname{cat}(A)$; hence $\operatorname{infcat}(\mathcal{A}) = \min\{\operatorname{cat} A, \operatorname{cat} B\}$. But $\operatorname{secat}(\mathcal{A}) = 1$ (or 0 if $A \simeq *$ or $B \simeq *$). Indeed apply the 'Whisker Maps inside a Cube' Lemma ([**4**, Lemma 49]) to the following diagram to get the section of $g(\mathrm{in}_1, \mathrm{in}_2)$:



This shows that $\operatorname{secat}(\mathcal{A})$ can be strictly less than $\operatorname{infcat}(\mathcal{A})$.

EXAMPLE 1.15. Let A and B be the homotopy cofibres of two applications $S^2 \to S^2$ of degrees relatively prime numbers p and q respectively, let $X = A \times B$; let $\mathcal{A}$ be the set of the two maps $\mathrm{in}_1\colon A \hookrightarrow A \times B$ and $\mathrm{in}_2\colon B \hookrightarrow A \times B$. It is known that $\operatorname{secat}(\mathrm{in}_1) = \operatorname{cat}(B)$ and $\operatorname{secat}(\mathrm{in}_2) = \operatorname{cat}(A)$. But A and B are suspensions, hence $\operatorname{cat} A = \operatorname{cat} B = 1$. Thus $\operatorname{secat}(\mathcal{A}) = 1$.

Now consider the map $\tau = g(\mathrm{in}_1, \mathrm{in}_2)$ which is a lower bound of $\mathcal{A}$ (for the preorder $\preccurlyeq$) by construction. This is the inclusion $A \vee B \hookrightarrow A \times B$. Then $H_2(A) = \mathbb{Z}_p$ and $H_2(B) = \mathbb{Z}_q$, hence by the Künneth formula, $H_*(\tau)$ is an isomorphism, and, by Whitehead's theorem, τ is a homotopy equivalence. Thus $\operatorname{secat}(\tau) = 0$.

This shows that $\operatorname{secat}\tau$, where τ is the join of two minimal maps of $\mathcal{A}$, can be strictly less than $\operatorname{secat}(\mathcal{A})$.

Proposition 1.16. *Let us denote by $\lfloor q \rfloor$ the integer part of any rational number q. For any class $\mathcal{A}$ of maps with same target X, consider the class $\mathcal{A}_k$ (respectively: $\mathcal{A}_{\leqslant k}$) of all maps $g(\mathcal{S})$ where $\mathcal{S}$ is any sequence of $k+1$ (respectively: at most $k+1$) maps of $\mathcal{A}$ (not necessarily distinct). Then:*

$$\operatorname{secat}(\mathcal{A}_{\leqslant k}) = \operatorname{secat}(\mathcal{A}_k) = \lfloor \frac{\operatorname{secat}(\mathcal{A})}{k+1} \rfloor.$$

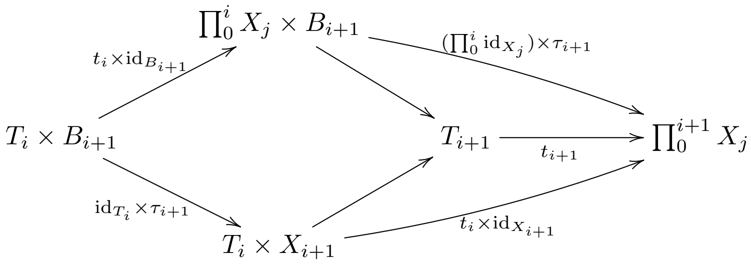
Proof. Any sequence of $n+1$ maps of $\mathcal{A}_k$ is a sequence $\mathcal{R} = (g(\mathcal{S}_0), \ldots, g(\mathcal{S}_n))$. By associativity of the join, $g(\mathcal{R}) \simeq g(\mathcal{S}_0 + \cdots + \mathcal{S}_n)$ where $\mathcal{S}_0 + \cdots + \mathcal{S}_n$ is the concatenation of the sequences $\mathcal{S}_i$, which is a sequence of $(n+1)(k+1)$ maps of $\mathcal{A}$. But $\operatorname{secat}(\mathcal{A}_k)$ is the least integer n such that there exists a sequence $\mathcal{R}$ of $n+1$ maps of $\mathcal{A}_k$ such that $g(\mathcal{R})$ has a homotopy section. Thus, if $\operatorname{secat}(\mathcal{A}) = m$, then n will be such $n(k+1) < m+1 \leqslant (n+1)(k+1)$, that is $\frac{m}{k+1} - \frac{k}{k+1} \leqslant n < \frac{m}{k+1} + \frac{1}{k+1}$, hence $n = \lfloor \frac{m}{k+1} \rfloor$. Finally $\operatorname{secat}(\mathcal{A}_{\leqslant k}) = \operatorname{secat}(\mathcal{A}_k)$ by Corollary 1.11. □

As a particular case, when $\mathcal{A}$ is made of only one map ι_X, then $\mathcal{A}_k$ is made of the single map $g_k(\iota_X)$. Then:

Corollary 1.17. *For any map $\iota_X \colon A \to X$, $\operatorname{secat}(g_k(\iota_X)) = \lfloor \frac{\operatorname{secat}(\iota_X)}{k+1} \rfloor$.*

2. The Whitehead point of view

Definition 2.1. For any finite sequence $\mathcal{T} = (\tau_0 \colon B_0 \to X_0, \ldots, \tau_n \colon B_n \to X_n)$ of maps of $\mathbb{T}$, the *Whitehead construction* of $\mathcal{T}$ is the following sequence of homotopy commutative diagrams ($0 \leqslant i < n$):

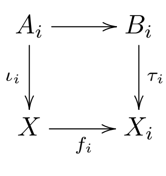


where the outside square is a homotopy pullback, the inside square is a homotopy pushout and the map $t_{i+1} : T_{i+1} \to \prod_0^{i+1} X_j$ is the whisker map induced by this homotopy pushout. The induction starts with $t_0 = \tau_0 : B_0 \to X_0$.

We denote T_n by $T(\mathcal{T})$ and t_n by $t(\mathcal{T})$.

Remark 2.2. The product symbol $\times$ means here the homotopy pullback over the terminal object e; it is the true pullback when the objects are e-fibrant. In the category **Top** or **Top***, all objects are e-fibrant, hence these are true pullbacks.

Theorem 2.3. *For $0 \leqslant i \leqslant n$, let*



be homotopy pullbacks in which $\mathcal{T} = (\tau_0, \ldots, \tau_n)$ are sequences of maps in $\mathbb{T}$. Then denoting $\mathcal{S} = (\iota_0, \ldots, \iota_n)$, the map $g(\mathcal{S})\colon G(\mathcal{S}) \to X$ has a homotopy section if and only if the induced map $\hat{f} = (f_0, \ldots, f_n)\colon X \to \prod_0^n X_j$ factors through $t(\mathcal{T})\colon T(\mathcal{T}) \to \prod_0^n X_j$ up to homotopy.

Keep in mind the important particular case in which $f_i = \mathrm{id}_X$, so that $\iota_i = \tau_i$ and $\hat{f}$ is the diagonal map $\Delta\colon X \to X^{n+1}$.

Proof. It is a standard argument (following the lines of [**7**, Theorem 8]) to prove that there is a homotopy pullback:

$$\begin{array}{ccc} G(\mathcal{S}) & \longrightarrow & T(\mathcal{T}) \\ {\scriptstyle g(\mathcal{S})}\downarrow & & \downarrow{\scriptstyle t(\mathcal{T})} \\ X & \xrightarrow[\hat{f}]{} & \prod_0^n X_j \end{array}$$

and the result follows. □

We extend the notion of 'category of a map' by the following definition:

Definition 2.4. Let $\mathbb{T}$ be pointed and let be a class of maps $\mathcal{X}$ with same source X. The *category* of $\mathcal{X}$ is the least integer n such that there exists a sequence $f_0\colon X \to X_0, \ldots, f_n\colon X \to X_n$ of $n+1$ maps in $\mathcal{X}$ such that the induced map $\hat{f} = (f_0, \ldots, f_n)\colon X \to \prod_0^n X_j$ factors through $t(\mathcal{T})\colon T(\mathcal{T}) \to \prod_0^n X_j$ up to homotopy, where $\mathcal{T} = (* \to X_0, \ldots, * \to X_n)$.

We denote this integer by $\mathrm{cat}\, \mathcal{X}$.

As a particular case, when there is only one map $f\colon X \to X_0$ in $\mathcal{X}$, we recover the usual definition of $\mathrm{cat}\, f$, and when this map f is the identity on X (so that $\hat{f} = \Delta$), we recover $\mathrm{cat}\, X$.

Observe that Theorem 2.3 shows that the category of a class is nothing but a particular case of sectional category of (another) class:

Corollary 2.5. *Let $\mathbb{T}$ be pointed and let be a class of maps $\mathcal{X}$ with same source. Then*

$$\mathrm{cat}\, \mathcal{X} = \mathrm{secat}\, \mathcal{A}$$

where $\mathcal{A}$ is the class consisting of the homotopy fibers of the maps of $\mathcal{X}$.

Example 2.6. Consider any $A \not\simeq *$ and $B \not\simeq *$ in $\mathbb{T}$ and let $\mathcal{X} = \{\mathrm{pr}_1\colon A \times B \to A, \mathrm{pr}_2\colon A \times B \to B\}$ the set of the two projections. The set of homotopy fibers of $\mathcal{X}$ is $\mathcal{A} = \{\mathrm{in}_2\colon B \hookrightarrow A \times B, \mathrm{in}_1\colon A \hookrightarrow A \times B\}$. By Corollary 2.5 $\mathrm{cat}\, \mathcal{X} = \mathrm{secat}\, \mathcal{A}$. Indeed, in this case $g(\mathrm{in}_2, \mathrm{in}_1) \simeq t(* \to A, * \to B)\colon A \vee B \hookrightarrow A \times B$ and $\hat{f} = (\mathrm{pr}_1, \mathrm{pr}_2) \simeq \mathrm{id}_{A \times B}$.

Example 2.7. Consider any $A \not\simeq *$ and $B \not\simeq *$ in $\mathbb{T}$ and let $\mathcal{X} = \{\mathrm{pr}_1\colon A \vee B \to A, \mathrm{pr}_2\colon A \vee B \to B\}$ the set of the two projections. Consider the set of homotopy fibers of $\mathcal{X}$: $\mathcal{A} = \{\iota_1\colon F_1 \to A \vee B, \iota_2\colon F_2 \to A \vee B\}$. Hence by Corollary 2.5, $\mathrm{cat}\, \mathcal{X} = \mathrm{secat}\, \mathcal{A}$. In this case $t(* \to A, * \to B) \simeq \hat{f} = (\mathrm{pr}_1, \mathrm{pr}_2)\colon A \vee B \hookrightarrow A \times B$ and of course $\hat{f}$ factors through $t(* \to A, * \to B)$ up to homotopy. Hence, $\mathrm{cat}\, \mathcal{X} = 1$.

EXAMPLE 2.8. Let A be a connected, CW H-space, and let $D\colon A \times A \to A$ the map such that $\mathrm{pr}_1 \cdot D \simeq \mathrm{pr}_2$. The diagonal map $\Delta\colon A \to A \times A$ is the homotopy fibre of D ([**10**, Proposition 3.7]). Thus $\mathrm{secat}\,(\Delta) = \mathrm{cat}\,(D)$ and $\mathrm{secat}\,(\{\mathrm{in}_1, \mathrm{in}_2, \Delta\}) = \mathrm{cat}\,(\{\mathrm{pr}_2, \mathrm{pr}_1, D\})$. Note that in this case $\mathrm{secat}\,(\Delta) = \mathrm{cat}\,(A)$ by Proposition 1.6.

3. The open covering point of view

In this section, we work in the category $\mathbf{Top}^*$, even if some things can be done in a wider context of a category $\mathbb{T}$.

PROPOSITION 3.1. *Let be any sequence* $\mathcal{T} = (\tau_0\colon B_0 \hookrightarrow X_0, \ldots, \tau_n\colon B_n \hookrightarrow X_n)$ *of closed cofibrations in* $\mathbf{Top}^*$*. Then:*

$$T(\mathcal{T}) = \{(x_0, \ldots, x_n) \in \prod_{j=0}^{n} X_j \mid x_k \in B_k \text{ for some } k\}$$

and $T(\mathcal{T}) \to \prod_{j=0}^{n} X_j$ *is a closed cofibration.*

PROOF. We have the following commutative diagram where all squares are pullbacks, and, since the projections are fibrations, homotopy pullbacks as well:

$$\begin{array}{ccccc}
T_i \times B_{i+1} & \rightarrowtail & \prod_0^i X_j \times B_{i+1} & \twoheadrightarrow & B_{i+1} \\
\downarrow & & \downarrow & & \downarrow \\
T_i \times X_{i+1} & \rightarrowtail & \prod_0^i X_j \times X_{i+1} & \twoheadrightarrow & X_{i+1} \\
\downarrow & & \downarrow & & \downarrow \\
T_i & \rightarrowtail & \prod_0^i X_j & \twoheadrightarrow & *.
\end{array}$$

Since $B_{i+1} \to X_{i+1}$ is a closed cofibration and $T_i \times X_{i+1} \to X_{i+1}$ is a fibration, $T_i \times B_{i+1} \to T_i \times X_{i+1}$ is also a closed cofibration, by [**13**, Theorem 12]. And similarly, assuming that $T_i \to \prod_0^i X_j$ is a closed cofibration by induction hypothesis, $T_i \times B_{i+1} \to \prod_0^i X_j \times B_{i+1}$ is also a closed cofibration. But then, the homotopy pushout T_{i+1} is the true pushout. Moreover, the map $T_{i+1} \to \prod_0^{i+1} X_j$ is closed, by [**9**, Proposition 2.46], and it is a cofibration by [**13**, Theorem 6]. □

DEFINITION 3.2. Let $\tau\colon B \to Y$ and $f\colon X \to Y$ be maps in $\mathbf{Top}^*$. A subspace U of X is said (τ, f)*-categorical* if there is a map $s\colon U \to B$ so that the restriction of f to U is homotopic to $\tau \circ s$. If the context makes it clear what τ and f are, we say also that U is B*-categorical.*

Saying that $\tau\colon B \to Y$ is a closed cofibration means that τ is an embedding and (Y, B) is a NDR-pair; in particular there is an open subset N of Y such that $B \subset N \subset Y$ and N is (τ, id_Y)-categorical.

We have the following characterization of $\mathrm{secat}\,\mathcal{A}$ or $\mathrm{cat}\,\mathcal{X}$ in terms of open categorical covering:

THEOREM 3.3. *Let* $\mathcal{A}$ *be a class of maps with the same target* X*, a well-pointed normal space. Then* $\mathrm{secat}\,\mathcal{A}$ *is the least integer* n*, such that there exists a sequence* $\mathcal{S} = (\iota_0\colon A_0 \to X, \ldots, \iota_n\colon A_n \to X)$ *of* $n+1$ *maps of* $\mathcal{A}$ *and there is an open covering* $(U_i)_{0 \leqslant i \leqslant n}$ *of* X*, each* U_i *being* A_i*-categorical.*

Theorem 3.4. *Let $\mathcal{X}$ be a class of maps with the same source X, a well-pointed normal space, and whose targets are path-connected spaces. Then $\operatorname{cat}\mathcal{X}$ is the least integer n, such that there exists a sequence $(f_0, \ldots, f_n)$ of $n+1$ maps of $\mathcal{X}$ and there is an open covering $(U_i)_{0\leqslant i\leqslant n}$ of X, each $f_i|_{U_i}$ being nullhomotopic.*

These theorems are consequences of the following proposition:

Proposition 3.5. *Let $\mathcal{T} = (\tau_0 \colon B_0 \hookrightarrow X_0, \ldots, \tau_n \colon B_n \hookrightarrow X_n)$ be any sequence of closed cofibrations and let $f_0 \colon X \to X_0, \ldots, f_n \colon X \to X_n$ be a sequence of maps in which X is a normal space. Then the induced map $\hat{f} = (f_0, \ldots, f_n)$ factors through $t(\mathcal{T}) \colon T(\mathcal{T}) \to \prod_0^n X_j$ up to homotopy if and only if there is an open covering $(U_i)_{0\leqslant i\leqslant n}$ of X, each U_i being B_i-categorical.*

Proof. ($\Leftarrow$.) By hypothesis, there is a covering $(U_0, \ldots, U_n)$ of X by open sets, and deformations $H_i \colon U_i \times I \to X_i$ of $f_i|_{U_i}$ into a map with values in B_i, for $0 \leqslant i \leqslant n$. As X is normal, there exists a covering of X by open sets, $(W_0, \ldots, W_n)$, such that $\overline{W}_i \subset U_i$, for $0 \leqslant i \leqslant n$. For any i, we choose a Urysohn function $\varphi_i \colon X \to I$ such that $\varphi_i(x) = 1$ if $x \in \overline{W}_i$ and $\varphi_i(x) = 0$ if $x \notin U_i$. We define now a continuous map $\hat{H}_i \colon X \times I \to X_i$ by:

$$\hat{H}_i(x,t) = \begin{cases} H_i(x, \varphi_i(x)t) & \text{if } x \in U_i, \\ f_i(x) & \text{otherwise.} \end{cases}$$

We collect these maps in a continuous map $H \colon X \times I \to \prod_0^n X_j$ defined by $H(x,t) = (\hat{H}_0(x,t) \ldots, \hat{H}_n(x,t))$. Observe that $H(x,0) = (f_0(x), \ldots, f_n(x)) = \hat{f}(x)$.

Set $r(x) = H(x,1)$. Since the $W_i's$ are a covering of X, for any point $x \in X$, there is a W_k with $x \in W_k$. By definition of $\hat{H}_k$, $\hat{H}_k(x,1) = H_k(x,1) \in B_k$. As the maps ι_i are closed cofibrations, $T(\mathcal{T}) = \{(x_0, \ldots, x_n) \in \prod_0^n X_j | x_k \in B_k \text{ for some } k\}$, and we deduce $r(X) \subset T(\mathcal{T})$ and r is a lifting up to homotopy (by the homotopy H) of $\hat{f}$.

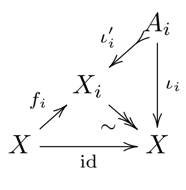
($\Rightarrow$.) By hypothesis, there is a map $r \colon X \to T(\mathcal{T})$ and a homotopy $H \colon X \times I \to \prod_0^n X_i$ between $\hat{f}$ and the composite $t(\mathcal{T}) \circ r$.

For any $0 \leqslant i \leqslant n$, as (X_i, B_i) is a NDR-pair, there exists also an open set N_i, $B_i \subset N_i \subset X_i$, and a deformation $G_i \colon N_i \times I \to X_i$ of $N_i \hookrightarrow X_i$ into a map with values in B_i. Let $p_i \colon \prod_0^n X_j \to X_i$ be the i-th projection. We set $h_i = p_i \circ t(\mathcal{T}) \circ r$ and $U_i = h_i^{-1}(N_i)$. Then, since $r(X) \subset T(\mathcal{T}) = \bigcup_{i=0}^n p_i^{-1}(B_i)$, $X = \bigcup_{i=0}^n U_i$. Hence the $U_i's$ are a covering of X. Define $H_i \colon U_i \times I \to X_i$ by:

$$H_i(u,t) = \begin{cases} p_i H(u, 2t) & \text{if } 0 \leqslant t \leqslant 1/2, \\ G_i(h_i(u), 2t-1) & \text{if } 1/2 \leqslant t \leqslant 1. \end{cases}$$

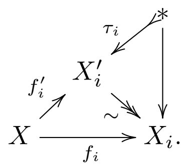
This is well defined since $p_i H(u,1) = h_i(u)$ an H_i is a homotopy between $f_i|_{U_i}$ and a map with values in B_i. □

Proof of Theorem 3.3. We can use Theorem 2.3 (where $f_i = \mathrm{id}_X$) and Proposition 3.5 directly if the maps in $\mathcal{A}$ are closed cofibrations. If they are not, we can replace them as follows:



decomposing first ι_i in $\mathcal{A}$ into a closed cofibration ι_i' followed by a fibration which is a homotopy equivalence (by [**14**, Proposition 2]), and then choosing a section f_i of it; this section exists since X is well-pointed and thus cofibrant. For an open set U_i of X, being (ι_i', f_i)-categorical is equivalent to be (ι_i, id_X)-categorical, and we can apply Proposition 3.5. □

Proof of Theorem 3.4. As in the previous proof, we make the following change of targets:



For an open set U_i of X, $f_i|_{U_i}$ nullhomotopic $\iff$ f_i factors through $* \to X_i$ (since X_i is path-connected) $\iff$ U_i is (τ_i, f_i')-categorical, and we can apply Proposition 3.5. □

Remark 3.6. Following the lines of [**8**, 1.3, 7.1 and 8.3], we can obtain lower bounds for secat and cat of a class of maps from cohomology. Consider the singular cohomology theory H^*, with any coefficient ring, and the corresponding reduced theory $\tilde{H}^*$. Let nil$R$ denote the nilpotency index of the ring R (this is the least integer n such that $R^n = 0$).

If $\mathcal{A}$ is a finite set of maps $\iota_i \colon A_i \to X$ with same target:

$$\operatorname{secat} \mathcal{A} + 1 \geqslant \operatorname{nil} (\cap_{\iota_i \in \mathcal{A}} \ker \iota_i^*)$$

where $\iota_i^* : \tilde{H}^*(X) \to \tilde{H}^*(A_i)$ denotes the induced homomorphism.

If $\mathcal{X}$ is a finite set of maps $f_i \colon X \to X_i$ with same source:

$$\operatorname{cat} \mathcal{X} + 1 \geqslant \operatorname{nil} (\cap_{f_i \in \mathcal{X}} \ \mathrm{im}\ f_i^*)$$

where $f_i^* : \tilde{H}^*(X_i) \to \tilde{H}^*(X)$ denotes the induced homomorphism.

Example 3.7. Let A and B be two spaces, and consider the inclusions $\mathrm{in}_1 \colon A \hookrightarrow A \times B$ and $\mathrm{in}_2 \colon B \hookrightarrow A \times B$. Assume A and B are 'reasonable' spaces, so that the inclusions are closed cofibrations and $A \times B$ is normal. Then $\operatorname{secat}(\mathcal{A}) = \operatorname{infcat}(\mathcal{A}) = \min\{\operatorname{cat} A, \operatorname{cat} B\}$.

Indeed, first recall that $\operatorname{secat} \mathrm{in}_1 = \operatorname{cat} B$ and $\operatorname{secat} \mathrm{in}_2 = \operatorname{cat} A$; hence by Remark 1.3, $\operatorname{secat} \mathcal{A} \leqslant \min\{\operatorname{cat} A, \operatorname{cat} B\}$.

Conversely, assume that $\operatorname{secat} \mathcal{A} = p + q - 1$ and that we have a covering of $A \times B$ formed by p open sets U_i $(i = 1, 2, \dots, p)$ with deformations of U_i into A and q open sets V_j $(j = 1, \dots, q)$ with deformations of V_j into B.

If $p = 0$, then fix any point $b_0 \in B$. The sets $\mathrm{pr}_1(A \times \ \{b_0\} \cap V_j)$ (pr_1 being the first projection) are contractible in A and cover A. Hence $\operatorname{cat} A \leqslant q - 1$. We can do the same reasoning if $q = 0$ and conclude that $\operatorname{cat} B \leqslant p - 1$.

We now suppose that $p \neq 0$ and $q \neq 0$. As above fix a point $b_0 \in B$. and consider the sets $A \times \ \{b_0\} \cap V_j$ and their projections $O_j = \mathrm{pr}_1(A \times \ \{b_0\} \cap V_j)$ which are contractible in A. If they cover A, we are done: $\operatorname{cat} A \leqslant q - 1$. But in general there may be points of A which are not in these projections. These points should lie in the projections $\mathrm{pr}_1(A \times \ \{b_0\} \cap U_i)$.

If the projections O_i do not cover A, then for each integer i $(1 \leqslant i \leqslant p)$, there is a point b_i and an integer j_i $(1 \leqslant j_i \leqslant q)$ such that

$$\mathrm{pr}_1(A \times \ \{b_0\} \cap U_i) \subseteq \mathrm{pr}_1(A \times \ \{b_i\} \cap V_{j_i}).$$

For if this is not the case then there is $a_0 \in A$ with $(a_0, b_0) \in U_i$ for some i such that $\{a_0\} \times B \cap V_j$ is empty for all j. It follows that $\{a_0\} \times B$ can be covered by the open sets U_i and the projections $\mathrm{pr}_2(\{a_0\} \times B \cap U_i)$ form a covering of B by p contractible open sets. Hence $\operatorname{cat} B \leqslant p - 1$, and since $\operatorname{secat} \mathcal{A} \leqslant \operatorname{cat} B$, we should have $q = 0$.

From this fact, we conclude that if $p \neq 0$ and $q \neq 0$, then the parts of A which are not possibly covered by the projections O_j $(1 \leqslant j \leqslant q)$, are covered by the projections $\mathrm{pr}_1(A \times \ \{b_i\} \cap V_{j_i})$ $(1 \leqslant j_i \leqslant q)$. Thus we obtain a covering of A by $p + q$ contractible sets. Hence $\operatorname{cat} A \leqslant p + q - 1 = \operatorname{secat} \mathcal{A}$.

The same holds for B; hence $\operatorname{secat} \{\mathrm{in}_1, \mathrm{in}_2\} = \min\{\operatorname{cat} A, \operatorname{cat} B\}$.

Note that incidentally we proved that $\operatorname{cat} A = \operatorname{cat} B$ whenever we need $p \neq 0$ and $q \neq 0$ in order to realize the sectional category with respect to the inclusions in_1 and in_2.

4. Topological complexity

The topological complexity of a space A, as defined in [**6**], is the sectional category of the diagonal map $\Delta\colon A \to A \times A$, i.e. $\mathrm{TC}\,(A) = \operatorname{secat}\,(\Delta)$. It is known that $\operatorname{cat}\,(A) \leqslant \mathrm{TC}\,(A) \leqslant \operatorname{cat}\,(A \times A)$.

The natural question we may ask in relation with the previous sections is: what about $\operatorname{secat}\,(\mathcal{A})$ where $\mathcal{A}$ is a class of maps with target $A \times A$ that contains the diagonal $\Delta\colon A \to A \times A$? Of course $\operatorname{secat}\,(\mathcal{A}) \leqslant \mathrm{TC}\,(A)$; but, for instance, what is $\operatorname{secat}\,(\mathcal{A})$ when $\mathcal{A}$ is the set of *all* maps from A to $A \times A$?

Remark 4.1. For any class $\mathcal{A}$ of maps with target $A \times A$, if $\mathcal{A}$ contains either in_1 or $\mathrm{in}_2\colon A \to A \times A$, then $\operatorname{secat}\,(\mathcal{A}) \leqslant \operatorname{cat}\,(A)$. Indeed $\operatorname{secat}\,(\mathcal{A}) \leqslant \operatorname{infcat}\,(\mathcal{A}) \leqslant \operatorname{secat}\,(\mathrm{in}_i) = \operatorname{cat}\,(A)$ $(i = 1$ or $2)$.

Proposition 4.2. *For any A in $\mathbb{T}$, consider the maps* $\mathrm{in}_1, \mathrm{in}_2, \Delta\colon A \to A \times A$. *Then:*

$$\operatorname{secat}\,(\{\Delta, \mathrm{in}_1\}) = \operatorname{secat}\,(\{\Delta, \mathrm{in}_2\}) = \operatorname{secat}\,(\{\mathrm{in}_1, \mathrm{in}_2\}) = \operatorname{cat}\,(A).$$

Proof. Consider the following homotopy commutative diagram:

$$\begin{array}{ccccc} * & \longrightarrow & A & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle \mathrm{in}_2} & & \downarrow \\ A & \underset{h}{\longrightarrow} & A \times A & \underset{\mathrm{pr}_1}{\longrightarrow} & A. \end{array}$$

The right square is a (homotopy) pullback. If h is either in_1 or Δ, which are both sections of pr_1, then the outer rectangle is a (homotopy) pullback as well; hence the left one is also a homotopy pullback. By Proposition 1.6, with $f = \mathrm{in}_2$, we get $\operatorname{secat}\,(\{\Delta, \mathrm{in}_1\}) \geqslant \operatorname{secat}\,(\{* \to A\}) = \operatorname{cat}\,(A)$, and with $f = \Delta$, we get $\operatorname{secat}\,(\{\mathrm{in}_1, \mathrm{in}_2\}) \geqslant \operatorname{secat}\,(\{* \to A\}) = \operatorname{cat}\,(A)$. Use Remark 4.1 to get equalities. □

Remark 4.3. If A is a surface, or any space with $\operatorname{cat}\,(A) \leqslant 2$, then also $\operatorname{secat}\,(\{\mathrm{in}_1, \mathrm{in}_2, \Delta\}) = \operatorname{cat}\,(A)$. This is clear for $\operatorname{cat} A \leqslant 1$. If the sectional category was strictly less than $\operatorname{cat} A = 2$, we would have a homotopy section for the join of only 2 of the three maps $\mathrm{in}_1, \mathrm{in}_2, \Delta$, and this is in contradiction with Proposition 4.2.

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