Polynomial functors on hermitian spaces

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(joint work with Christine Vespa)

In the early fifties, Eilenberg and Mac Lane [3] introduced the notion of *polynomial functors* between categories of modules to study the homology of topological spaces which have now their name. The interest of this notion has remained strong, because of other connections with algebraic topology (Henn-Lannes-Schwartz), representation theory, algebraic K-theory and stable homology of linear groups (Betley, Suslin, Scorichenko). This classical notion of polynomial functor can be defined in the same way for functors from a (small) symmetric monoidal category $(\mathcal{C}, +, 0)$ whose unit 0 is an *zero object* to a (nice) abelian category.

But for some purposes, this setting is not enough. For example, many recent works deal with FI-modules (functors from the category of finite sets with injections to abelian groups, see [1]); finitely generated FI-modules carry polynomial properties (for dimension functions, when the values are in finite-dimensional vector spaces over a field). Another example comes from the quotients of the lower central serie of the automorphism group of a free group — see recent works by Satoh and Bartholdi. But our main motivation to introduce a generalized notion of polynomial functors in the study of *stable homology of congruence groups*. To be more precise, let I a ring *without unit*, n a non-negative integer and

$$GL_n(I) := Ker\,(GL_n(I \oplus \mathbb{Z}) \to GL_n(\mathbb{Z}))$$

the corresponding general linear group, which is congruence group ($I \oplus \mathbb{Z}$ is the ring obtained by adding formally a unit to I). The study of the homology of these groups is known to be extremely hard and related to the problem of excision in algebraic K-theory. For a qualitative approach of this problem, let us remark that the stabilization maps $H_*(GL_n(I)) \to H_*(GL_{n+1}(I))$ and the natural action of $GL_n(\mathbb{Z})$ on $H_*(GL_n(I))$ assemble to give a functor

$$\mathbf{S}(\mathbb{Z}) \to \mathbf{Ab} \qquad \mathbb{Z}^n \mapsto H_d(GL_n(I))$$

for each $d \in \mathbb{N}$ (which will be denoted by $H_d(GL(I))$). Here, we denote by $\mathbf{S}(R)$, for any ring R, the category of finitely generated left free R-modules with split R-linear injections, the splitting being given in the structure.

Conjecture. *For any ring without unit I and any $d \in \mathbb{N}$, the functor $H_d(GL(I))$: $\mathbf{S}(\mathbb{Z}) \to \mathbf{Ab}$ is weakly polynomial of degree $\leq 2d$.*

This conjecture is inspired by the beautiful work of Suslin [4].

We will now explain the meaning of *weakly polynomial* and how wich kind of classification result we can get for this kind of polynomial functors (following [2]).

The category $\mathbf{S}(R)$ can be seen as a particular case of the category $\mathbf{H}(A)$ of hermitian spaces over a ring with involution A (the objects are the finitely generated free A-modules endowed with a non-degenerate hermitian form, the morphisms are A-linear maps which preserve the hermitian forms): $\mathbf{S}(R)$ is equivalent to $\mathbf{H}(R^{op} \times R)$, where $R^{op} \times R$ is endowed with the canonical involution. We deal with this general hermitian setting, which is not harder than $\mathbf{S}(\mathbb{Z})$.

Strongly polynomial functors

In the sequel, $(\mathcal{C}, +, 0)$ denotes a (small) symmetric monoidal category whose unit 0 is an *initial object* (as FI with the disjoint union or $\mathbf{H}(A)$ with the hermitian sum) and $\mathcal{A}$ a Grothendieck category. The category $\mathbf{Fct}(\mathcal{C}, \mathcal{A})$ of functors from $\mathcal{C}$ to $\mathcal{A}$ is also a Grothendieck category.

Definition. For any object x of $\mathcal{C}$, let $\tau_x : \mathbf{Fct}(\mathcal{C}, \mathcal{A}) \to \mathbf{Fct}(\mathcal{C}, \mathcal{A})$ denote the precomposition by the functor $- + x : \mathcal{C} \to \mathcal{C}$. We denote also by δ_x (respectively κ_x) the cokernel (resp. kernel) of the natural transformation $\mathrm{Id} = \tau_0 \to \tau_x$ induced by the unique map $0 \to x$.

A functor $F : \mathcal{C} \to \mathcal{A}$ is said *strongly polynomial of degree* $\leq d$ if $\delta_{a_0}\delta_{a_1} \dots \delta_{a_d}(F) = 0$ for any $(d+1)$-tuple $(a_0, \dots, a_d)$ of objects of $\mathcal{C}$.

This notion is not so well-behaved, because is it not stable under subfunctors.

Weakly polynomial functors

To avoid this problem, we change the definition by working in a suitable quotient category:

Proposition and definition. The full subcategory $\mathcal{SN}(\mathcal{C}, \mathcal{A})$ of $\mathbf{Fct}(\mathcal{C}, \mathcal{A})$ of functors F such that $F = \sum_{x \in \mathrm{Ob}\,\mathcal{C}} \kappa_x(F)$ is localizing. We denote by $\mathbf{St}(\mathcal{C}, \mathcal{A})$ the quotient category $\mathbf{Fct}(\mathcal{C}, \mathcal{A})/\mathcal{SN}(\mathcal{C}, \mathcal{A})$.

For any object x of $\mathcal{C}$, τ_x induces an exact functor (always denoted in the same way) of $\mathbf{St}(\mathcal{C}, \mathcal{A})$; in this category, the natural transformation $\mathrm{Id} \to \tau_x$ is monic. So its cokernel δ_x is exact.

An object X of $\mathbf{St}(\mathcal{C}, \mathcal{A})$ is said polynomial of degree $\leq d$ if $\delta_{a_0}\delta_{a_1} \dots \delta_{a_d}(X) = 0$ for any $(d+1)$-tuple $(a_0, \dots, a_d)$ of objects of $\mathcal{C}$.

The full subcategory $\mathcal{P}ol_d(\mathcal{C}, \mathcal{A})$ of $\mathbf{St}(\mathcal{C}, \mathcal{A})$ of these objects is bilocalizing.

A functor $F : \mathcal{C} \to \mathcal{A}$ is said *weakly polynomial of degree* $\leq d$ if its image in $\mathbf{St}(\mathcal{C}, \mathcal{A})$ belongs to $\mathcal{P}ol_d(\mathcal{C}, \mathcal{A})$.

Main result

Theorem ([2]). Let A be a ring with involution and $\mathcal{A}$ a Grothendieck category. For any $d \in \mathbb{N}$, the forgetful functor $\mathbf{H}(A) \to \mathbf{F}(A)$ (category of finitely generated free A-modules, with usual morphisms) induces an equivalence of categories:

$$\mathcal{P}ol_d(\mathbf{F}(A), \mathcal{A})/\mathcal{P}ol_{d-1}(\mathbf{F}(A), \mathcal{A}) \to \mathcal{P}ol_d(\mathbf{H}(A), \mathcal{A})/\mathcal{P}ol_{d-1}(\mathbf{H}(A), \mathcal{A})$$

(the source category can be described from the wreath product of A and $\mathfrak{S}_d$).

References

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[4] A. Suslin, *Excision in integer algebraic K-theory*, Trudy Mat. Inst. Steklov **208** (1995), 290–317.