# Self-similar solutions of the one-dimensional Landau–Lifshitz–Gilbert equation

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**Abstract**

We consider the one-dimensional Landau–Lifshitz–Gilbert (LLG) equation, a model describing the dynamics for the spin in ferromagnetic materials. Our main aim is the analytical study of the bi-parametric family of self-similar solutions of this model. In the presence of damping, our construction provides a family of global solutions of the LLG equation which are associated to a discontinuous initial data of infinite (total) energy, and which are smooth and have finite energy for all positive times. Special emphasis will be given to the behaviour of this family of solutions with respect to the Gilbert damping parameter.

We would like to emphasize that our analysis also includes the study of self-similar solutions of the Schrödinger map and the heat flow for harmonic maps into the 2-sphere as special cases. In particular, the results presented here recover some of the previously known results in the setting of the 1d-Schrödinger map equation.
*Keywords and phrases:* Landau–Lifshitz–Gilbert equation, Landau–Lifshitz equation, ferromagnetic spin chain, Schrödinger maps, heat-flow for harmonic maps, self-similar solutions, asymptotics.

## Contents



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## 1 Introduction and statement of results

In this work we consider the one-dimensional Landau–Lifshitz–Gilbert equation (LLG)

$$\partial_t \vec{\mathrm{m}} = \beta \vec{\mathrm{m}} \times \vec{m}_{ss} - \alpha \vec{\mathrm{m}} \times (\vec{\mathrm{m}} \times \vec{\mathrm{m}}_{ss}), \quad s \in \mathbb{R}, \quad t > 0, \tag{LLG}$$

where $\vec{\mathrm{m}} = (\mathrm{m}_1, \mathrm{m}_2, \mathrm{m}_3) : \mathbb{R} \times (0, \infty) \longrightarrow \mathbb{S}^2$ is the spin vector, $\beta \geq 0$, $\alpha \geq 0$, $\times$ denotes the usual cross-product in $\mathbb{R}^3$, and $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$.

Here we have not included the effects of anisotropy or an external magnetic field. The first term on the right in (LLG) represents the exchange interaction, while the second one corresponds to the Gilbert damping term and may be considered as a dissipative term in the equation of motion. The parameters $\beta \geq 0$ and $\alpha \geq 0$ are the so-called exchange constant and Gilbert damping coefficient, and take into account the exchange of energy in the system and the effect of damping on the spin chain respectively. Note that, by considering the time-scaling $\vec{\mathrm{m}}(s,t) \to \vec{\mathrm{m}}(s, (\alpha^2+\beta^2)^{1/2} t)$, in what follows we will assume w.l.o.g. that

$$\alpha, \ \beta \in [0,1] \qquad \text{and} \qquad \alpha^2 + \beta^2 = 1. \tag{1.1}$$

The Landau–Lifshitz–Gilbert equation was first derived on phenomenological grounds by L. Landau and E. Lifshitz to describe the dynamics for the magnetization or spin $\vec{\mathrm{m}}(s,t)$ in ferromagnetic materials [24, 11]. The nonlinear evolution equation (LLG) is related to several physical and mathematical problems and it has been seen to be a physically relevant model for several magnetic materials [19, 20]. In the setting of the LLG equation, of particular importance is to consider the effect of dissipation on the spin [27, 7, 6].

The Landau–Lifshitz family of equations includes as special cases the well-known heat-flow for harmonic maps and the Schrödinger map equation onto the 2-sphere. Precisely, when $\beta = 0$ (and therefore $\alpha = 1$) the LLG equation reduces to the one-dimensional *heat-flow equation for harmonic maps*

$$\partial_t \vec{\mathrm{m}} = -\vec{\mathrm{m}} \times (\vec{\mathrm{m}} \times \vec{\mathrm{m}}_{ss}) = \vec{\mathrm{m}}_{ss} + |\vec{\mathrm{m}}_s|^2 \vec{\mathrm{m}} \tag{HFHM}$$

(notice that $|\vec{\mathrm{m}}|^2 = 1$, and in particular $\vec{\mathrm{m}} \cdot \vec{\mathrm{m}}_{ss} = -|\vec{\mathrm{m}}_s|^2$). The opposite limiting case of the LLG equation (that is $\alpha = 0$, i.e. no dissipation/damping and therefore $\beta = 1$) corresponds to the *Schrödinger map equation onto the sphere*

$$\partial_t \vec{\mathrm{m}} = \vec{\mathrm{m}} \times \vec{\mathrm{m}}_{ss}. \tag{SM}$$

Both special cases have been objects of intense research and we refer the interested reader to [21, 14, 25, 13] for surveys.

Of special relevance is the connection of the LLG equation with certain non-linear Schrödinger equations. This connection is established as follows: Let us suppose that $\vec{\mathrm{m}}$ is the tangent vector of a curve in $\mathbb{R}^3$, that is $\vec{\mathrm{m}} = \vec{X}_s$, for some curve $\vec{X}(s,t) \in \mathbb{R}^3$ parametrized by the arc-length. It can be shown [7] that if $\vec{\mathrm{m}}$ evolves under (LLG) and we define the so-called filament function $u$ associated to $\vec{X}(s,t)$ by

$$u(s,t) = \mathrm{c}(s,t) e^{i \int_0^s \tau(\sigma,t)\, d\sigma}, \tag{1.2}$$

in terms of the curvature c and torsion $\tau$ associated to the curve, then $u$ solves the following non-local non-linear Schrödinger equation with damping

$$iu_t + (\beta - i\alpha) u_{ss} + \frac{u}{2}\left( \beta |u|^2 + 2\alpha \int_0^s \operatorname{Im}(\bar{u} u_s) - A(t) \right) = 0, \tag{1.3}$$

where $A(t) \in \mathbb{R}$ is a time-dependent function defined in terms of the curvature and torsion and their derivatives at the point $s = 0$. The transformation (1.2) was first considered in the undamped case by Hasimoto in [18]. Notice that if $\alpha = 0$, equation (1.3) can be transformed into the well-known completely integrable cubic Schrödinger equation.

The main purpose of this paper is the analytical study of self-similar solutions of the LLG equation of the form

$$\vec{\mathrm{m}}(s,t) = \vec{m}\left(\frac{s}{\sqrt{t}}\right), \tag{1.4}$$

for some profile $\vec{m} : \mathbb{R} \to \mathbb{S}^2$, with emphasis on the behaviour of these solutions with respect to the Gilbert damping parameter $\alpha \in [0,1]$.

For $\alpha = 0$, self-similar solutions have generated considerable interest [22, 21, 4, 15, 9]. We are not aware of any other study of such solutions for $\alpha > 0$ in the one dimensional case (see [10] for a study of self-similar solutions of the harmonic map flow in higher dimensions). However, Lipniacki [26] has studied self-similar solutions for a related model with nonconstant arc-length. On the other hand, little is known analytically about the effect of damping on the evolution of a one-dimensional spin chain. In particular, Lakshmanan and Daniel obtained an explicit solitary wave solution in [7, 6] and demonstrated the damping of the solution in the presence of dissipation in the system. In this setting, we would like to understand how the dynamics of self-similar solutions to this model is affected by the introduction of damping in the equations governing the motion of these curves.

As will be shown in Section 2 self-similar solutions of (LLG) of the type (1.4) constitute a bi-parametric family of solutions $\{\vec{m}_{c_0,\alpha}\}_{c_0,\alpha}$ given by

$$\vec{\mathrm{m}}_{c_0,\alpha}(s,t) = \vec{m}_{c_0,\alpha}\left(\frac{s}{\sqrt{t}}\right), \qquad c_0 > 0, \qquad \alpha \in [0,1], \tag{1.5}$$

where $\vec{m}_{c_0,\alpha}$ is the solution of the Serret–Frenet equations

$$\begin{cases} \vec{m}' = c\vec{n}, \\ \vec{n}' = -c\vec{m} + \tau\vec{b}, \\ \vec{b}' = -\tau\vec{n}, \end{cases} \tag{1.6}$$

with curvature and torsion given respectively by

$$c_{c_0,\alpha}(s) = c_0 e^{-\frac{\alpha s^2}{4}}, \quad \tau_{c_0,\alpha}(s) = \frac{\beta s}{2}, \tag{1.7}$$

and initial conditions

$$\vec{m}_{c_0,\alpha}(0) = (1,0,0), \quad \vec{n}_{c_0,\alpha}(0) = (0,1,0), \quad \vec{b}_{c_0,\alpha}(0) = (0,0,1). \tag{1.8}$$

The first result of this paper is the following:

**Theorem 1.1.** *Let* $\alpha \in [0,1]$*,* $c_0 > 0$[1] *and* $\vec{m}_{c_0,\alpha}$ *be the solution of the Serret–Frenet system* (1.6) *with curvature and torsion given by* (1.7) *and initial conditions* (1.8)*. Define* $\vec{\mathrm{m}}_{c_0,\alpha}(s,t)$ *by*

$$\vec{\mathrm{m}}_{c_0,\alpha}(s,t) = \vec{m}_{c_0,\alpha}\left(\frac{s}{\sqrt{t}}\right), \qquad t > 0.$$

*Then,*

[1]The case $c_0 = 0$ corresponds to the constant solution for (LLG), that is

$$\vec{\mathrm{m}}_{c_0,\alpha}(s,t) = \vec{m}\left(\frac{s}{\sqrt{t}}\right) = (1,0,0), \qquad \forall \alpha \in [0,1].$$

*(i) The function* $\vec{\mathrm{m}}_{c_0,\alpha}(\cdot,t)$ *is a regular* $\mathcal{C}^\infty(\mathbb{R};\mathbb{S}^2)$*-solution of* (LLG) *for* $t>0$*.*

*(ii) There exist unitary vectors* $\vec{A}^{\pm}_{c_0,\alpha} = (A^{\pm}_{j,c_0,\alpha})_{j=1}^3 \in \mathbb{S}^2$ *such that the following pointwise convergence holds when t goes to zero:*

$$\lim_{t\to 0^+} \vec{\mathrm{m}}_{c_0,\alpha}(s,t) = \begin{cases} \vec{A}^{+}_{c_0,\alpha}, & \text{if } s>0, \\ \vec{A}^{-}_{c_0,\alpha}, & \text{if } s<0, \end{cases} \tag{1.9}$$

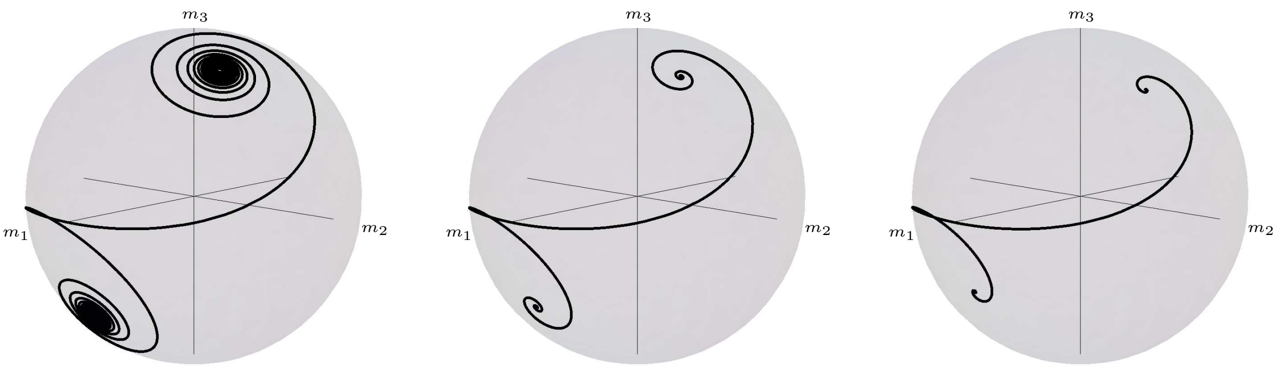
*where* $\vec{A}^{-}_{c_0,\alpha} = (A^{+}_{1,c_0,\alpha}, -A^{+}_{2,c_0,\alpha}, -A^{+}_{3,c_0,\alpha})$*.*

*(iii) Moreover, there exists a constant* $C(c_0,\alpha,p)$ *such that for all* $t>0$

$$\|\vec{\mathrm{m}}_{c_0,\alpha}(\cdot,t) - \vec{A}^{+}_{c_0,\alpha}\chi_{(0,\infty)}(\cdot) - \vec{A}^{-}_{c_0,\alpha}\chi_{(-\infty,0)}(\cdot)\|_{L^p(\mathbb{R})} \leq C(c_0,\alpha,p)t^{\frac{1}{2p}}, \tag{1.10}$$

*for all* $p\in(1,\infty)$*. In addition, if* $\alpha>0$*,* (1.10) *also holds for* $p=1$*. Here,* $\chi_E$ *denotes the characteristic function of a set* $E$*.*

The graphics in Figure 1 depict the profile $\vec{m}_{c_0,\alpha}(s)$ for fixed $c_0 = 0.8$ and the values of $\alpha = 0.01$, $\alpha = 0.2$, and $\alpha = 0.4$. In particular it can be observed how the convergence of $\vec{m}_{c_0,\alpha}$ to $\vec{A}^{\pm}_{c_0,\alpha}$ is accelerated by the diffusion $\alpha$.




(a) $\alpha = 0.01$ (b) $\alpha = 0.2$ (c) $\alpha = 0.4$

Figure 1: The profile $\vec{m}_{c_0,\alpha}$ for $c_0 = 0.8$ and different values of $\alpha$.

Notice that the initial condition

$$\vec{\mathrm{m}}_{c_0,\alpha}(s,0) = \vec{A}^{+}_{c_0,\alpha}\chi_{(0,\infty)}(s) + \vec{A}^{-}_{c_0,\alpha}\chi_{(-\infty,0)}(s), \tag{1.11}$$

has a jump singularity at the point $s=0$ whenever the vectors $\vec{A}^{+}_{c_0,\alpha}$ and $\vec{A}^{-}_{c_0,\alpha}$ satisfy

$$\vec{A}^{+}_{c_0,\alpha} \neq \vec{A}^{-}_{c_0,\alpha}.$$

In this situation (and we will be able to prove analytically this is the case at least for certain ranges of the parameters $\alpha$ and $c_0$, see Proposition 1.5 below), Theorem 1.1 provides a bi-parametric family of global smooth solutions of (LLG) associated to a discontinuous singular initial data (jump-singularity).

As has been already mentioned, in the absence of damping ($\alpha = 0$), singular self-similar solutions of the Schrödinger map equation were previously obtained in [15], [22] and [4]. In this framework, Theorem 1.1 establishes the persistence of a jump singularity for self-similar solutions in the presence of dissipation.

Some further remarks on the results stated in Theorem 1.1 are in order. Firstly, from the self-similar nature of the solutions $\vec{m}_{c_0,\alpha}(s,t)$ and the Serret–Frenet equations (1.6), it follows that the curvature and torsion associated to these solutions are of the self-similar form and given by

$$\mathrm{c}_{c_0,\alpha}(s,t) = \frac{c_0}{\sqrt{t}} e^{-\frac{\alpha s^2}{4t}} \qquad \text{and} \qquad \tau_{c_0,\alpha}(s,t) = \frac{\beta s}{2\sqrt{t}}. \tag{1.12}$$

As a consequence, the total energy $E(t)$ of the spin $\vec{m}_{c_0,\alpha}(s,t)$ found in Theorem 1.1 is expressed as

$$\begin{aligned} E(t) &= \frac{1}{2}\int_{-\infty}^{\infty} |\vec{m}_s(s,t)|^2 \, ds = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{c}^2_{c_0,\alpha}(s,t)\, ds \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{c_0}{\sqrt{t}} e^{-\frac{\alpha s^2}{4t}}\right)^2 ds = c_0^2\sqrt{\frac{\pi}{\alpha t}}, \qquad \alpha > 0, \quad t > 0. \end{aligned} \tag{1.13}$$

It is evident from (1.13) that the total energy of the spin chain at the initial time $t = 0$ is infinite, while the total energy of the spin becomes finite for all positive times, showing the dissipation of energy in the system in the presence of damping.

Secondly, it is also important to remark that in the setting of Schrödinger equations, for fixed $\alpha \in [0,1]$ and $c_0 > 0$, the solution $\vec{m}_{c_0,\alpha}(s,t)$ of (LLG) established in Theorem 1.1 is associated through the Hasimoto transformation (1.2) to the filament function

$$u_{c_0,\alpha}(s,t) = \frac{c_0}{\sqrt{t}} e^{(-\alpha + i\beta)\frac{s^2}{4t}}, \tag{1.14}$$

which solves

$$iu_t + (\beta - i\alpha)u_{ss} + \frac{u}{2}\left(\beta|u|^2 + 2\alpha \int_0^s \mathrm{Im}(\bar{u}u_s) - A(t)\right) = 0, \quad \text{with} \quad A(t) = \frac{\beta c_0^2}{t} \tag{1.15}$$

and is such that at initial time $t = 0$

$$u_{c_0,\alpha}(s,0) = 2c_0\sqrt{\pi(\alpha + i\beta)}\delta_0.$$

Here $\delta_0$ denotes the delta distribution at the point $s = 0$ and $\sqrt{z}$ denotes the square root of a complex number $z$ such that $\mathrm{Im}(\sqrt{z}) > 0$.

Notice that the solution $u_{c_0,\alpha}(s,t)$ is very rough at initial time, and in particular $u_{c_0,\alpha}(s,0)$ does not belong to the Sobolev class $H^s$ for any $s \geq 0$. Therefore, the standard arguments (that is a Picard iteration scheme based on Strichartz estimates and Sobolev-Bourgain spaces) cannot be applied at least not in a straightforward way to study the local well-posedness of the initial value problem for the Schrödinger equations (1.15). The existence of solutions of the Scrödinger equations (1.15) associated to an initial data proportional to a Dirac delta opens the question of developing a well-posedness theory for Schrödinger equations of the type considered here to include initial data of infinite energy. This question was addressed by A. Vargas and L. Vega in [29] and A. Grünrock in [12] in the case $\alpha = 0$ and when $A(t) = 0$ (see also [2] for a related problem), but we are not aware of any results in this setting when $\alpha > 0$ (see [14] for related well-posedness results in the case $\alpha > 0$ for initial data in Sobolev spaces of positive index). Notice that when $\alpha > 0$ , the solution (1.14) has infinite energy at the initial time, however the energy becomes finite for any $t > 0$. Moreover, as a consequence of the exponential decay in the space variable when $\alpha > 0$, $u_{c_0,\alpha}(t) \in H^m(\mathbb{R})$, for all $t > 0$ and $m \in \mathbb{N}$. Hence these solutions do not fit into the usual functional framework for solutions of the Schrödinger equations (1.15).

As already mentioned, one of the main goals of this paper is to study both the qualitative and quantitative effect of the damping parameter $\alpha$ and the parameter $c_0$ on the dynamical behaviour of the family $\{\vec{\mathrm{m}}_{c_0,\alpha}\}_{c_0,\alpha}$ of self-similar solutions of (LLG) found in Theorem 1.1. Precisely, in an attempt to fully understand the regularization of the solution at positive times close to the initial time $t = 0$, and to understand how the presence of damping affects the dynamical behaviour of these self-similar solutions, we aim to give answers to the following questions:

- Q1: Can we obtain a more precise behaviour of the solutions $\vec{m}_{c_0,\alpha}(s,t)$ at positive times $t$ close to zero?
- Q2: Can we understand the limiting vectors $\vec{A}^{\pm}_{c_0,\alpha}$ in terms of the parameters $c_0$ and $\alpha$?

In order to address our first question, we observe that, due to the self-similar nature of these solutions (see (1.5)), the behaviour of the family of solutions $\vec{\mathrm{m}}_{c_0,\alpha}(s,t)$ at positive times close to the initial time $t = 0$ is directly related to the study of the asymptotics of the associated profile $\vec{m}_{c_0,\alpha}(s)$ for large values of $s$. In addition, the symmetries of $\vec{m}_{c_0,\alpha}(s)$ (see Theorem 1.2 below) allow to reduce ourselves to obtain the behaviour of the profile $\vec{m}_{c_0,\alpha}(s)$ for large positive values of the space variable. The precise asymptotics of the profile is given in the following theorem.

**Theorem 1.2** (Asymptotics). *Let $\alpha \in [0,1]$, $c_0 > 0$ and $\{\vec{m}_{c_0,\alpha}, \vec{n}_{c_0,\alpha}, \vec{b}_{c_0,\alpha}\}$ be the solution of the Serret–Frenet system* (1.6) *with curvature and torsion given by* (1.7) *and initial conditions* (1.8). *Then,*

*(i) (Symmetries). The components of $\vec{m}_{c_0,\alpha}(s)$, $\vec{n}_{c_0,\alpha}(s)$ and $\vec{b}_{c_0,\alpha}(s)$ satisfy respectively that*

- *$m_{1,c_0,\alpha}(s)$ is an even function, and $m_{j,c_0,\alpha}(s)$ is an odd function for $j \in \{2,3\}$.*
- *$n_{1,c_0,\alpha}(s)$ and $b_{1,c_0,\alpha}(s)$ are odd functions, while $n_{j,c_0,\alpha}(s)$ and $b_{j,c_0,\alpha}(s)$ are even functions for $j \in \{2,3\}$.*

*(ii) (Asymptotics). There exist an unit vector $\vec{A}^+_{c_0,\alpha} \in \mathbb{S}^2$ and $\vec{B}^+_{c_0,\alpha} \in \mathbb{R}^3$ such that the following asymptotics hold for all $s \geq s_0 = 4\sqrt{8 + c_0^2}$:*

$$\begin{aligned}\vec{m}_{c_0,\alpha}(s) =& \vec{A}^+_{c_0,\alpha} - \frac{2c_0}{s}\vec{B}^+_{c_0,\alpha}e^{-\alpha s^2/4}(\alpha\sin(\vec{\phi}(s)) + \beta\cos(\vec{\phi}(s))) \\ & - \frac{2c_0^2}{s^2}\vec{A}^+_{c_0,\alpha}e^{-\alpha s^2/2} + O\left(\frac{e^{-\alpha s^2/4}}{s^3}\right),\end{aligned} \tag{1.16}$$

$$\vec{n}_{c_0,\alpha}(s) = \vec{B}^+_{c_0,\alpha}\sin(\vec{\phi}(s)) + \frac{2c_0}{s}\vec{A}^+_{c_0,\alpha}\alpha e^{-\alpha s^2/4} + O\left(\frac{e^{-\alpha s^2/4}}{s^2}\right), \tag{1.17}$$

$$\vec{b}_{c_0,\alpha}(s) = \vec{B}^+_{c_0,\alpha}\cos(\vec{\phi}(s)) + \frac{2c_0}{s}\vec{A}^+_{c_0,\alpha}\beta e^{-\alpha s^2/4} + O\left(\frac{e^{-\alpha s^2/4}}{s^2}\right). \tag{1.18}$$

*Here, $\sin(\vec{\phi})$ and $\cos(\vec{\phi})$ are understood acting on each of the components of $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$, with*

$$\phi_j(s) = a_j + \beta\int_{s_0^2/4}^{s^2/4}\sqrt{1 + c_0^2\frac{e^{-2\alpha\sigma}}{\sigma}}\,d\sigma, \quad j \in \{1,2,3\}, \tag{1.19}$$

*for some constants* $a_1, a_2, a_3 \in [0, 2\pi)$, *and the vector* $\vec{B}^+_{c_0,\alpha}$ *is given in terms of* $\vec{A}^+_{c_0,\alpha} = (A^+_{j,c_0,\alpha})_{j=1}^3$ *by*

$$\vec{B}^+_{c_0,\alpha} = ((1-(A^+_{1,c_0,\alpha})^2)^{1/2}, (1-(A^+_{2,c_0,\alpha})^2)^{1/2}, (1-(A^+_{3,c_0,\alpha})^2)^{1/2}).$$

As we will see in Section 2, the convergence and rate of convergence of the solutions $\vec{\mathrm{m}}_{c_0,\alpha}(s,t)$ of the LLG equation established in parts *(ii)* and *(iii)* of Theorem 1.1 will be obtained as a consequence of the more refined asymptotic analysis of the associated profile given in Theorem 1.2.

With regard to the asymptotics of the profile established in part *(ii)* of Theorem 1.2, it is important to mention the following:

(a) The errors in the asymptotics in Theorem 1.2-*(ii)* depend only on $c_0$. In other words, the bounds for the errors terms are independent of $\alpha \in [0,1]$. More precisely, we use the notation $O(f(s))$ to denote a function for which exists a constant $C(c_0) > 0$ depending on $c_0$, but independent on $\alpha$, such that

$$|O(f(s))| \le C(c_0)|f(s)|, \quad \text{for all } s \ge s_0. \tag{1.20}$$

(b) The terms $\vec{A}^+_{c_0,\alpha}$, $\vec{B}^+_{c_0,\alpha}$, $B^+_j \sin(a_j)$ and $B^+_j \cos(a_j)$, $j \in \{1,2,3\}$, and the error terms in Theorem 1.2-*(ii)* depend continuously on $\alpha \in [0,1]$ (see Subsection 3.3 and Corollary 3.14). Therefore, the asymptotics (1.16)–(1.18) show how the profile $\vec{m}_{c_0,\alpha}$ converges to $\vec{m}_{c_0,0}$ as $\alpha \to 0^+$ and to $\vec{m}_{c_0,1}$ as $\alpha \to 1^-$. In particular, we recover the asymptotics for $\vec{m}_{c_0,0}$ given in [15].

(c) We also remark that using the Serret–Frenet formulae and the asymptotics in Theorem 1.2-*(ii)*, it is straightforward to obtain the asymptotics for the derivatives of $\vec{\mathrm{m}}_{c_0,\alpha}(s,t)$.

(d) When $\alpha = 0$ and for fixed $j \in \{1,2,3\}$, we can write $\phi_j$ in (1.19) as

$$\phi_j(s) = a_j + \frac{s^2}{4} + c_0^2 \ln(s) + C(c_0) + O\left(\frac{1}{s^2}\right),$$

and we recover the logarithmic contribution in the oscillation previously found in [15]. Moreover, in this case the asymptotics in part *(ii)* represents an improvement of the one established in Theorem 1 in [15].

When $\alpha > 0$, $\phi_j$ behaves like

$$\phi_j(s) = a_j + \frac{\beta s^2}{4} + C(\alpha, c_0) + O\left(\frac{e^{-\alpha s^2/2}}{\alpha s^2}\right), \tag{1.21}$$

and there is no logarithmic correction in the oscillations in the presence of damping.

Consequently, the phase function $\vec{\phi}$ defined in (1.19) captures the different nature of the oscillatory character of the solutions in both the absence and the presence of damping in the system of equations.

(e) When $\alpha = 1$, there exists an explicit formula for $\vec{m}_{c_0,1}$, $\vec{n}_{c_0,1}$ and $\vec{b}_{c_0,1}$, and in particular we have explicit expressions for the vectors $\vec{A}^\pm_{c_0,1}$ in terms of the parameter $c_0 > 0$ in the asymptotics given in part *(ii)*. See Appendix.

(f) At first glance, one might think that the term $-2c_0^2\vec{A}^+_{c_0,\alpha}e^{-\alpha s^2/2}/s^2$ in (1.16) could be included in the error term $O(e^{-\alpha s^2/4}/s^3)$. However, we cannot do this because

$$\frac{e^{-\alpha s^2/2}}{s^2} > \frac{e^{-\alpha s^2/4}}{s^3}, \qquad \text{for all } 2 \le s \le \left(\frac{2}{3\alpha}\right)^{1/2}, \ \alpha \in (0, 1/8], \tag{1.22}$$

and in our notation the big-$O$ must be independent of $\alpha$. (The exact interval where the inequality in (1.22) holds can be determined using the so-called Lambert $W$ function.)

(g) Let $\vec{B}^+_{c_0,\alpha,\sin} = (B_j \sin(a_j))_{j=1}^3$, $\vec{B}^+_{c_0,\alpha,\cos} = (B_j \cos(a_j))_{j=1}^3$. Then the orthogonality of $\vec{m}_{c_0,\alpha}$, $\vec{n}_{c_0,\alpha}$ and $\vec{b}_{c_0,\alpha}$ together with the asymptotics (1.16)–(1.18) yield

$$\vec{A}^+_{c_0,\alpha} \cdot \vec{B}^+_{c_0,\alpha,\sin} = \vec{A}^+_{c_0,\alpha} \cdot \vec{B}^+_{c_0,\alpha,\cos} = \vec{B}^+_{c_0,\alpha,\sin} \cdot \vec{B}^+_{c_0,\alpha,\cos} = 0,$$

which gives relations between the phases.

(h) Finally, the amplitude of the leading order term controlling the wave-like behaviour of the solution $\vec{m}_{c_0,\alpha}(s)$ around $\vec{A}^\pm_{c_0,\alpha}$ for values of $s$ sufficiently large is of the order $c_0\, e^{-\alpha s^2/4}/s$, from which one observes how the convergence of the solution to its limiting values $\vec{A}^\pm_{c_0,\alpha}$ is accelerated in the presence of damping in the system. See Figure 1.

We conclude the introduction by stating the results answering the second of our questions. Precisely, Theorems 1.3 and 1.4 below establish the dependence of the vectors $\vec{A}^\pm_{c_0,\alpha}$ in Theorem 1.1 with respect to the parameters $\alpha$ and $c_0$. Theorem 1.3 provides the behaviour of the limiting vector $\vec{A}^+_{c_0,\alpha}$ for a fixed value of $\alpha \in (0,1)$ and "small" values of $c_0 > 0$, while Theorem 1.4 states the behaviour of $\vec{A}^+_{c_0,\alpha}$ for fixed $c_0 > 0$ and $\alpha$ close to the limiting values $\alpha = 0$ and $\alpha = 1$. Recall that $\vec{A}^-_{c_0,\alpha}$ is expressed in terms of the coordinates of $\vec{A}^+_{c_0,\alpha}$ as

$$\vec{A}^-_{c_0,\alpha} = (A^+_{1,c_0,\alpha}, -A^+_{2,c_0,\alpha}, -A^+_{3,c_0,\alpha}) \tag{1.23}$$

(see part *(ii)* of Theorem 1.1).

**Theorem 1.3.** *Let $\alpha \in [0,1]$, $c_0 > 0$, and $\vec{A}^+_{c_0,\alpha} = (A^+_{j,c_0,\alpha})_{j=1}^3$ be the unit vector given in Theorem 1.2. Then $\vec{A}^+_{c_0,\alpha}$ is a continuous function of $c_0 > 0$. Moreover, if $\alpha \in (0,1]$ the following inequalities hold true:*

$$|A^+_{1,c_0,\alpha} - 1| \le \frac{c_0^2\pi}{\alpha}\left(1 + \frac{c_0^2\pi}{8\alpha}\right), \tag{1.24}$$

$$\left|A^+_{2,c_0,\alpha} - c_0\frac{\sqrt{\pi(1+\alpha)}}{\sqrt{2}}\right| \le \frac{c_0^2\pi}{4} + \frac{c_0^2\pi}{\alpha\sqrt{2}}\left(1 + \frac{c_0^2\pi}{8} + c_0\frac{\sqrt{\pi(1+\alpha)}}{2\sqrt{2}}\right) + \left(\frac{c_0^2\pi}{2\sqrt{2}\alpha}\right)^2, \tag{1.25}$$

$$\left|A^+_{3,c_0,\alpha} - c_0\frac{\sqrt{\pi(1-\alpha)}}{\sqrt{2}}\right| \le \frac{c_0^2\pi}{4} + \frac{c_0^2\pi}{\alpha\sqrt{2}}\left(1 + \frac{c_0^2\pi}{8} + c_0\frac{\sqrt{\pi(1-\alpha)}}{2\sqrt{2}}\right) + \left(\frac{c_0^2\pi}{2\sqrt{2}\alpha}\right)^2. \tag{1.26}$$

The following result provides an approximation of the behaviour of $\vec{A}^+_{c_0,\alpha}$ for fixed $c_0 > 0$ and values of the Gilbert parameter close to 0 and 1.

**Theorem 1.4.** *Let $c_0 > 0$, $\alpha \in [0,1]$ and $\vec{A}^+_{c_0,\alpha}$ be the unit vector given in Theorem 1.2. Then $\vec{A}^+_{c_0,\alpha}$ is a continuous function of $\alpha$ in $[0,1]$, and the following inequalities hold true:*

$$|\vec{A}^+_{c_0,\alpha} - \vec{A}^+_{c_0,0}| \le C(c_0)\sqrt{\alpha}|\ln(\alpha)|, \quad \text{for all } \alpha \in (0,1/2], \tag{1.27}$$

$$|\vec{A}^+_{c_0,\alpha} - \vec{A}^+_{c_0,1}| \le C(c_0)\sqrt{1-\alpha}, \quad \text{for all } \alpha \in [1/2,1]. \tag{1.28}$$

*Here, $C(c_0)$ is a positive constant depending on $c_0$ but otherwise independent of $\alpha$.*

As a by-product of Theorems 1.3 and 1.4, we obtain the following proposition which asserts that the solutions $\vec{\mathrm{m}}_{c_0,\alpha}(s,t)$ of the LLG equation found in Theorem 1.1 are indeed associated to a discontinuous initial data at least for certain ranges of $\alpha$ and $c_0$.

**Proposition 1.5.** *With the same notation as in Theorems 1.1 and 1.2, the following statements hold:*

*(i) For fixed $\alpha \in (0,1)$ there exists $c_0^* > 0$ depending on $\alpha$ such that*

$$\vec{A}^+_{c_0,\alpha} \neq \vec{A}^-_{c_0,\alpha} \qquad \text{for all } \ c_0 \in (0, c_0^*).$$

*(ii) For fixed $c_0 > 0$, there exists $\alpha_0^* > 0$ small enough such that*

$$\vec{A}^+_{c_0,\alpha} \neq \vec{A}^-_{c_0,\alpha} \qquad \text{for all } \ \alpha \in (0, \alpha_0^*).$$

*(iii) For fixed $0 < c_0 \neq k\sqrt{\pi}$ with $k \in \mathbb{N}$, there exists $\alpha_1^* > 0$ with $1 - \alpha_1^* > 0$ small enough such that*

$$\vec{A}^+_{c_0,\alpha} \neq \vec{A}^-_{c_0,\alpha} \qquad \text{for all } \ \alpha \in (\alpha_1^*, 1).$$

**Remark 1.6.** *Based on the numerical results in Section 5, we conjecture that $\vec{A}^+_{c_0,\alpha} \neq \vec{A}^-_{c_0,\alpha}$ for all $\alpha \in [0,1)$ and $c_0 > 0$.*

We would like to point out that some of our results and their proofs combine and extend several ideas previously introduced in [15] and [16]. The approach we use in the proof of the main results in this paper is based on the integration of the Serret–Frenet system of equations via a Riccati equation, which in turn can be reduced to the study of a second order ordinary differential equation given by

$$f''(s) + \frac{s}{2}(\alpha + i\beta)f'(s) + \frac{c_0^2}{4}e^{-\frac{\alpha s^2}{2}} f(s) = 0 \tag{1.29}$$

when the curvature and torsion are given by (1.7).

Unlike in the undamped case, in the presence of damping no explicit solutions are known for equation (1.29) and the term containing the exponential in the equation (1.29) makes it difficult to use Fourier analysis methods to study analytically the behaviour of the solutions to this equation. The fundamental step in the analysis of the behaviour of the solutions of (1.29) consists in introducing new auxiliary variables $z$, $h$ and $y$ defined by

$$z = |f|^2, \qquad y = \mathrm{Re}(\bar{f}f') \qquad \text{and} \qquad h = \mathrm{Im}(\bar{f}f')$$

in terms of solutions $f$ of (1.29), and studying the system of equations satisfied by these key quantities. As we will see later on, these variables are the "natural" ones in our problem, in the sense that the components of the tangent, normal and binormal vectors can be written in terms of these quantities. It is important to emphasize that, in order to obtain error bounds in the asymptotic analysis independent of the damping parameter $\alpha$ (and hence recover the asymptotics when $\alpha = 0$ and $\alpha = 1$ as particular cases), it will be fundamental to exploit the cancellations due to the oscillatory character of $z$, $y$ and $h$.

The outline of this paper is the following. Section 2 is devoted to the construction of the family of self-similar solutions $\{\vec{\mathrm{m}}_{c_0,\alpha}\}_{c_0,\alpha}$ of the LLG equation. In Section 3 we reduce the study of the properties of this family of self-similar solutions to that of the properties of the solutions of the complex second order complex ODE (1.29). This analysis is of independent interest. Section 4 contains the proofs of the main results of this paper as a consequence of those established in

Section 3. In Section 5 we give provide some numerical results for $\vec{A}^{+}_{c_0,\alpha}$, as a function of $\alpha \in [0,1]$ and $c_0 > 0$, which give some inside for the scattering problem and justify Remark 1.6. Finally, we have included the study of the self-similar solutions of the LLG equation in the case $\alpha = 1$ in Appendix.

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## 2 Self-similar solutions of the LLG equation

First we derive what we will refer to as the geometric representation of the LLG equation. To this end, let us assume that $\vec{\mathrm{m}}(s,t) = \vec{X}_s(s,t)$ for some curve $\vec{X}(s,t)$ in $\mathbb{R}^3$ parametrized with respect to the arc-length with curvature $\mathrm{c}(s,t)$ and torsion $\uptau(s,t)$. Then, using the Serret–Frenet system of equations (1.6), we have

$$\vec{\mathrm{m}}_{ss} = \mathrm{c}_s \vec{\mathrm{n}} + \mathrm{c}(-\mathrm{c}\vec{\mathrm{n}} + \uptau\vec{\mathrm{b}}),$$

and thus we can rewrite (LLG) as

$$\partial_t \vec{\mathrm{m}} = \beta(\mathrm{c}_s \vec{\mathrm{b}} - \mathrm{c}\uptau\vec{\mathrm{n}}) + \alpha(\mathrm{c}\uptau\vec{\mathrm{b}} + \mathrm{c}_s\vec{\mathrm{n}}), \tag{2.1}$$

in terms of intrinsic quantities $\mathrm{c}$, $\uptau$ and the Serret–Frenet trihedron $\{\vec{\mathrm{m}}, \vec{\mathrm{n}}, \vec{\mathrm{b}}\}$.

We are interested in self-similar solutions of (LLG) of the form

$$\vec{\mathrm{m}}(s,t) = \vec{m}\left(\frac{s}{\sqrt{t}}\right) \tag{2.2}$$

for some profile $\vec{m} : \mathbb{R} \longrightarrow \mathbb{S}^2$. First, notice that due to the self-similar nature of $\vec{\mathrm{m}}(s,t)$ in (2.2), from the Serret–Frenet equations (1.6) it follows that the unitary normal and binormal vectors and the associated curvature and torsion are self-similar and given by

$$\vec{\mathrm{n}}(s,t) = \vec{n}\left(\frac{s}{\sqrt{t}}\right), \qquad \vec{\mathrm{b}}(s,t) = \vec{b}\left(\frac{s}{\sqrt{t}}\right), \tag{2.3}$$

$$\mathrm{c}(s,t) = \frac{1}{\sqrt{t}}\, c\left(\frac{s}{\sqrt{t}}\right) \quad \text{and} \quad \uptau(s,t) = \frac{1}{\sqrt{t}}\, \tau\left(\frac{s}{\sqrt{t}}\right). \tag{2.4}$$

Assume that $\vec{\mathrm{m}}(s,t)$ is a solution of the LLG equation, or equivalently of its geometric version (2.1). Then, from (2.2)–(2.4) it follows that the Serret–Frenet trihedron $\{\vec{m}(\cdot), \vec{n}(\cdot), \vec{b}(\cdot)\}$ solves

$$-\frac{s}{2} c\vec{n} = \beta(c'\vec{b} - c\tau\vec{n}) + \alpha(c\tau\vec{b} + c'\vec{n}), \tag{2.5}$$

As a consequence,

$$-\frac{s}{2} c = \alpha c' - \beta c\tau \quad \text{and} \quad \beta c' + \alpha c\tau = 0.$$

Thus, we obtain

$$c(s) = c_0\, e^{-\frac{\alpha s^2}{4}} \quad \text{and} \quad \tau(s) = \frac{\beta s}{2}, \tag{2.6}$$

for some positive constant $c_0$ (recall that we are assuming w.l.o.g. that $\alpha^2+\beta^2=1$). Therefore, in view of (2.4), the curvature and torsion associated to a self-similar solution of (LLG) of the form (2.2) are given respectively by

$$\mathrm{c}(s,t)=\frac{c_0}{\sqrt{t}}e^{-\frac{\alpha s^2}{4t}} \quad \text{and} \quad \tau(s,t)=\frac{\beta s}{2t}, \qquad c_0>0. \tag{2.7}$$

Notice that given $(\mathrm{c},\tau)$ as above, for fixed time $t>0$ one can solve the Serret–Frenet system of equations to obtain the solution up to a rigid motion in the space which in general may depend on $t$. As a consequence, and in order to determine the dynamics of the spin chain, we need to find the time evolution of the trihedron $\{\vec{\mathrm{m}}(s,t),\vec{\mathrm{n}}(s,t),\vec{\mathrm{b}}(s,t)\}$ at some fixed point $s^*\in\mathbb{R}$. To this end, from the above expressions of the curvature and torsion associated to $\vec{\mathrm{m}}(s,t)$ and evaluating the equation (2.1) at the point $s^*=0$, we obtain that $\vec{\mathrm{m}}_t(0,t)=\vec{0}$. On the other hand, differentiating the geometric equation (2.1) with respect to $s$, and using the Serret–Frenet equations (1.6) together with the compatibility condition $\vec{\mathrm{m}}_{st}=\vec{\mathrm{m}}_{ts}$, we get the following relation for the time evolution of the normal vector

$$\mathrm{c}\vec{\mathrm{n}}_t=\beta(\mathrm{c}_{ss}\vec{\mathrm{b}}+c^2\tau\vec{\mathrm{m}}-\mathrm{c}\tau^2\vec{\mathrm{b}})+\alpha((\mathrm{c}\tau)_s\vec{\mathrm{b}}-\mathrm{c}\mathrm{c}_s\vec{\mathrm{m}}+\mathrm{c}_s\tau\vec{\mathrm{b}}).$$

The evaluation of the above identity at $s^*=0$ together with the expressions for the curvature and torsion in (2.7) yield $\vec{\mathrm{n}}_t(0,t)=\vec{0}$. The above argument shows that

$$\vec{\mathrm{m}}_t(0,t)=\vec{0}, \qquad \vec{\mathrm{n}}_t(0,t)=\vec{0} \quad \text{and} \quad \vec{\mathrm{b}}_t(0,t)=(\vec{\mathrm{m}}\times\vec{\mathrm{n}})_t(0,t)=\vec{0}.$$

Therefore we can assume w.l.o.g. that

$$\vec{\mathrm{m}}(0,t)=(1,0,0), \qquad \vec{\mathrm{n}}(0,t)=(0,1,0) \quad \text{and} \quad \vec{\mathrm{b}}(0,t)=(0,0,1),$$

and in particular

$$\vec{m}(0)=\vec{\mathrm{m}}(0,1)=(1,0,0), \quad \vec{n}(0)=\vec{\mathrm{n}}(0,1)=(0,1,0), \quad \text{and} \quad \vec{b}(0)=\vec{\mathrm{b}}(0,1)=(0,0,1). \tag{2.8}$$

Given $\alpha\in[0,1]$ and $c_0>0$, from the theory of ODE's, it follows that there exists a unique $\{\vec{m}_{c_0,\alpha}(\cdot),\vec{n}_{c_0,\alpha}(\cdot),\vec{b}_{c_0,\alpha}(\cdot)\}\in\left(\mathcal{C}^\infty(\mathbb{R};\mathbb{S}^2)\right)^3$ solution of the Serret–Frenet equations (1.6) with curvature and torsion (2.6) and initial conditions (2.8) such that

$$\vec{m}_{c_0,\alpha}\perp\vec{n}_{c_0,\alpha}, \quad \vec{m}_{c_0,\alpha}\perp\vec{b}_{c_0,\alpha}, \quad \vec{n}_{c_0,\alpha}\perp\vec{b}_{c_0,\alpha}$$

and

$$|\vec{m}_{c_0,\alpha}|^2=|\vec{n}_{c_0,\alpha}|^2=|\vec{b}_{c_0,\alpha}|^2=1.$$

Define $\vec{\mathrm{m}}_{c_0,\alpha}(s,t)$ as

$$\vec{\mathrm{m}}_{c_0,\alpha}(s,t)=\vec{m}_{c_0,\alpha}\left(\frac{s}{\sqrt{t}}\right). \tag{2.9}$$

Then, $\vec{m}_{c_0,\alpha}(\cdot,t)\in\mathcal{C}^\infty\left(\mathbb{R};\mathbb{S}^2\right)$ for all $t>0$, and bearing in mind both the relations in (2.3)–(2.4) and the fact that the vectors $\{\vec{m}_{c_0,\alpha}(\cdot),\vec{n}_{c_0,\alpha}(\cdot),\vec{b}_{c_0,\alpha}(\cdot)\}$ satisfy the identity (2.5), a straightforward calculation shows that $\vec{m}_{c_0,\alpha}(\cdot,t)$ is a regular $\mathcal{C}^\infty(\mathbb{R};\mathbb{S}^2)$-solution of the LLG equation for all $t>0$. Notice that the case $c_0=0$ yields the constant solution $\vec{m}_{0,\alpha}(s,t)=(1,0,0)$. Therefore in what follows we will assume that $c_0>0$.

The rest of the paper is devoted to establish analytical properties of the solutions $\{\vec{\mathrm{m}}_{c_0,\alpha}(s,t)\}_{c_0,\alpha}$ defined by (2.9) for fixed $\alpha\in[0,1]$ and $c_0>0$. As already mentioned, due to the self-similar nature of these solutions, it suffices to study the properties of the associated profile $\vec{m}_{c_0,\alpha}(\cdot)$ or, equivalently, of the solution $\{\vec{m}_{c_0,\alpha},\vec{n}_{c_0,\alpha},\vec{b}_{c_0,\alpha}\}$ of the Serret–Frenet system (1.6) with curvature and torsion given by (2.6) and initial conditions (2.8). As we will continue to see, the analysis of the profile solution $\{\vec{m}_{c_0,\alpha},\vec{n}_{c_0,\alpha},\vec{b}_{c_0,\alpha}\}$ can be reduced to the study of the properties of the solutions of a certain second order complex differential equation.

## 3 Integration of the Serret–Frenet system

### 3.1 Reduction to the study of a second order ODE

Classical changes of variables from the differential geometry of curves allow us to reduce the nine equations in the Serret–Frenet system into three complex-valued second order equations (see [8, 28, 23]). Theses changes of variables are related to stereographic projection and this approach was also used in [15]. However, their choice of stereographic projection has a singularity at the origin, which leads to an indetermination of the initial conditions of some of the new variables. For this reason, we consider in the following lemma a stereographic projection that is compatible with the initial conditions (2.8). Although the proof of the lemma below is a slight modification of that in [23, Subsections 2.12 and 7.3], we have included its proof here both for the sake of completeness and to clarify to the unfamiliar reader how the integration of the Frenet equations can be reduced to the study of a second order differential equation.

**Lemma 3.1.** *Let $\vec{m} = (m_j(s))_{j=1}^3$, $\vec{n} = (n_j(s))_{j=1}^3$ and $\vec{b} = (b_j(s))_{j=1}^3$ be a solution of the Serret–Frenet equations* (1.6) *with positive curvature $c$ and torsion $\tau$. Then, for each $j \in \{1, 2, 3\}$ the function*

$$f_j(s) = e^{\frac{1}{2}\int_0^s c(\sigma)\eta_j(\sigma)\,d\sigma}, \qquad \text{with} \qquad \eta_j(s) = \frac{(n_j(s) + ib_j(s))}{1 + m_j(s)},$$

*solves the equation*

$$f_j''(s) + \left(i\tau(s) - \frac{c'(s)}{c(s)}\right) f_j'(s) + \frac{c^2(s)}{4} f_j(s) = 0, \tag{3.1}$$

*with initial conditions*

$$f_j(0) = 1, \quad f_j'(0) = \frac{c(0)(n_j(0) + ib_j(0))}{2(1 + m_j(0))}.$$

*Moreover, the coordinates of $\vec{m}$, $\vec{n}$ and $\vec{b}$ are given in terms of $f_j$ and $f_j'$ by*

$$m_j(s) = 2\left(1 + \frac{4}{c(s)^2}\left|\frac{f_j'(s)}{f_j(s)}\right|^2\right)^{-1} - 1, \quad n_j(s) + ib_j(s) = \frac{4f_j'(s)}{c(s)f_j(s)}\left(1 + \frac{4}{c(s)^2}\left|\frac{f_j'(s)}{f_j(s)}\right|^2\right)^{-1}. \tag{3.2}$$

*The above relations are valid at least as long as $m_j > -1$ and $|f_j| > 0$.*

*Proof.* For simplicity, we omit the index $j$. The proof relies on several transformations that are rather standard in the study of curves. First we define the complex function

$$N = (n + ib)e^{i\int_0^s \tau(\sigma)\,d\sigma}. \tag{3.3}$$

Then $N' = i\tau N + (n' + ib')e^{i\int_0^s \tau(\sigma)\,d\sigma}$. On the other hand, the Serret–Frenet equations imply that

$$n' + ib' = -cm - i\tau N e^{-i\int_0^s \tau(\sigma)\,d\sigma}.$$

Therefore, setting

$$\psi = ce^{i\int_0^s \tau(\sigma)\,d\sigma},$$

we get

$$N' = -\psi m. \tag{3.4}$$

Using again the Serret–Frenet equations, we also obtain

$$m' = \frac{1}{2}(\overline{\psi}N + \psi\overline{N}). \tag{3.5}$$

Let us consider now the auxiliary function

$$\varphi = \frac{N}{1+m}. \tag{3.6}$$

Differentiating and using (3.4), (3.5) and (3.6)

$$\begin{aligned}\varphi' &= \frac{N'}{1+m} - \frac{Nm'}{(1+m)^2} \\ &= \frac{N'}{1+m} - \frac{\varphi m'}{1+m} \\ &= -\frac{\varphi^2\overline{\psi}}{2} - \frac{\psi}{2(1+m)}(2m+\varphi\overline{N}).\end{aligned}$$

Noticing that we can recast the relation $m^2+n^2+b^2=1$ as $N\overline{N}=(1-m)(1+m)$ and recalling the definition of $\varphi$ in (3.6), we have $\varphi\overline{N}=1-m$, so that

$$\varphi' + \frac{\varphi^2\overline{\psi}}{2} + \frac{\psi}{2} = 0. \tag{3.7}$$

Finally, define the stereographic projection of $(m,n,b)$ by

$$\eta = \frac{n+ib}{1+m}. \tag{3.8}$$

Observe that from the definitions of $N$ and $\varphi$, respectively in (3.3) and (3.6), we can rewrite $\eta$ as

$$\eta = \varphi e^{-i\int_0^s \tau(\sigma)\,d\sigma},$$

and from (3.7) it follows that $\eta$ solves the Riccati equation

$$\eta' + i\tau\eta + \frac{c}{2}(\eta^2+1) = 0, \tag{3.9}$$

(recall that $\psi = ce^{i\int_0^s \tau(\sigma)\,d\sigma}$). Finally, setting

$$f(s) = e^{\frac{1}{2}\int_0^s c(\sigma)\eta(\sigma)\,d\sigma}, \tag{3.10}$$

we get

$$\eta = \frac{2f'}{cf} \tag{3.11}$$

and equation (3.1) follows from (3.9). The initial conditions are an immediate consequence of the definition of $\eta$ and $f$ in (3.8) and (3.10).

A straightforward calculation shows that the inverse transformation of the stereographic projection is

$$m = \frac{1-|\eta|^2}{1+|\eta|^2}, \quad n = \frac{2\,\mathrm{Re}\,\eta}{1+|\eta|^2}, \quad b = \frac{2\,\mathrm{Im}\,\eta}{1+|\eta|^2},$$

so that we obtain (3.2) using (3.11) and the above identities. □

Going back to our problem, Lemma 3.1 reduces the analysis of the solution $\{\vec{m},\vec{n},\vec{b}\}$ of the Serret–Frenet system (1.6) with curvature and torsion given by (2.6) and initial conditions (2.8) to the study of the second order differential equation

$$f''(s) + \frac{s}{2}(\alpha+i\beta)f'(s) + \frac{c_0^2}{4}e^{-\alpha s^2/2}f(s) = 0, \tag{3.12}$$

with three initial conditions: For $(m_1, n_1, b_1) = (1, 0, 0)$ the associated initial condition for $f_1$ is

$$f_1(0) = 1, \quad f_1'(0) = 0, \tag{3.13}$$

for $(m_2, n_2, b_2) = (0, 1, 0)$ is

$$f_2(0) = 1, \quad f_2'(0) = \frac{c_0}{2}, \tag{3.14}$$

and for $(m_3, n_3, b_3) = (0, 0, 1)$ is

$$f_3(0) = 1, \quad f_3'(0) = \frac{ic_0}{2}. \tag{3.15}$$

It is important to notice that, by multiplying (3.12) by $\bar{f}'$ and taking the real part, it is easy to see that

$$\frac{d}{ds}\left[\frac{1}{2}\left(e^{\frac{\alpha s^2}{2}}|f'|^2 + \frac{c_0^2}{4}|f|^2\right)\right] = 0.$$

Thus,

$$E(s) := \frac{1}{2}\left(e^{\frac{\alpha s^2}{2}}|f'|^2 + \frac{c_0^2}{4}|f|^2\right) = E_0, \qquad \forall\, s \in \mathbb{R}, \tag{3.16}$$

with $E_0$ a constant defined by the value of $E(s)$ at some point $s_0 \in \mathbb{R}$. The conservation of the energy $E(s)$ allows us to simplify the expressions of $m_j$, $n_j$ and $b_j$ for $j \in \{1, 2, 3\}$ in the formulae (3.2) in terms of the solution $f_j$ to (3.12) associated to the initial conditions (3.13)–(3.15).

Indeed, on the one hand notice that the energies associated to the initial conditions (3.13)–(3.15) are respectively

$$E_{0,1} = \frac{c_0^2}{8}, \qquad E_{0,2} = \frac{c_0^2}{4} \qquad \text{and} \qquad E_{0,3} = \frac{c_0^2}{4}. \tag{3.17}$$

On the other hand, from (3.16), it follows that

$$\left(1 + \frac{4}{c_0^2 e^{-\frac{\alpha s^2}{2}}} \frac{|f_j'|^2(s)}{|f_j|^2(s)}\right)^{-1} = \frac{c_0^2}{8E_{0,j}}|f_j|^2(s), \qquad j \in \{1, 2, 3\}.$$

Therefore, from (3.17), the above identity and formulae (3.2) in Lemma 3.1, we conclude that

$$m_1(s) = 2|f_1(s)|^2 - 1, \qquad n_1(s) + ib_1(s) = \frac{4}{c_0} e^{\alpha s^2/4} \bar{f}_1(s) f_1'(s), \tag{3.18}$$

$$m_j(s) = |f_j(s)|^2 - 1, \qquad n_j(s) + ib_j(s) = \frac{2}{c_0} e^{\alpha s^2/4} \bar{f}_j(s) f_j'(s), \quad j \in \{2, 3\}. \tag{3.19}$$

The above identities give the expressions of the tangent, normal and binormal vectors in terms of the solutions $\{f_j\}_{j=1}^3$ of the second order differential equation (3.12) associated to the initial conditions (3.13)–(3.15).

By Lemma 3.1, the formulae (3.18) and (3.19) are valid as long as $m_j > -1$, which is equivalent to the condition $|f_j| \neq 0$. As shown in Appendix, for $\alpha = 1$ there is $\tilde{s} > 0$ such that $m_j(\tilde{s}) = -1$ and then (3.18) and (3.19) are (a priori) valid just in a bounded interval. However, the trihedron $\{\vec{m}, \vec{n}, \vec{b}\}$ is defined globally and $f_j$ can also be extended globally as the solution of the linear equation (3.12). Then, it is simple to verify that the functions given by the l.h.s. of formulae (3.18) and (3.19) satisfy the Serret–Frenet system and hence, by the uniqueness of the solution, the formulae (3.18) and (3.19) are valid for all $s \in \mathbb{R}$.

### 3.2 The second-order equation. Asymptotics

In this section we study the properties of the complex-valued equation

$$f''(s) + \frac{s}{2}(\alpha + i\beta)f'(s) + \frac{c_0^2}{4}f(s)e^{-\alpha s^2/2} = 0, \tag{3.20}$$

for fixed $c_0 > 0$, $\alpha \in [0,1)$, $\beta > 0$ such that $\alpha^2 + \beta^2 = 1$. We begin noticing that in the case $\alpha = 0$, the solution can be written explicitly in terms of parabolic cylinder functions or confluent hypergeometric functions (see [1]). Another analytical approach using Fourier analysis techniques has been taken in [15], leading to the asymptotics

$$f(s) = C_1 e^{i(c_0^2/2)\ln(s)} + C_2 \frac{e^{-is^2/4}}{s} e^{-i(c_0^2/2)\ln(s)} + O(1/s^2), \tag{3.21}$$

as $s \to \infty$, where the constants $C_1$, $C_2$ and $O(1/s^2)$ depend on the initial conditions and $c_0$.

For $\alpha = 1$, equation (3.20) can be also solved explicitly and the solution is given by

$$f(s) = \frac{2f'(0)}{c_0}\sin\left(\frac{c_0}{2}\int_0^s e^{-\sigma^2/4}\, d\sigma\right) + f(0)\cos\left(\frac{c_0}{2}\int_0^s e^{-\sigma^2/4}\, d\sigma\right).$$

In the case $\alpha \in (0,1)$, one cannot compute the solutions of (3.20) in terms of known functions and we will follow a more analytical analysis. In contrast with the situation when $\alpha = 0$, it is far from evident to use Fourier analysis to study (3.20) when $\alpha > 0$.

For the rest of this section we will assume that $\alpha \in [0,1)$. In addition, we will also assume that $s > 0$ and we will develop the asymptotic analysis necessary to establish part *(ii)* of Theorem 1.2. At this point, it is important to recall the expressions given in (3.18)–(3.19) for the coordinates of the tangent, normal and binormal vectors associated to our family of solutions of the LLG equation in terms $f$. Bearing this in mind, we observe that the study of the asymptotic behaviour of these vectors are dictated by the asymptotic behaviour of the variables

$$z = |f|^2, \quad y = \mathrm{Re}(\bar{f}f'), \quad \text{and} \quad h = \mathrm{Im}(\bar{f}f') \tag{3.22}$$

associated to the solution $f$ of (3.20).

As explained in the remark (a) after Theorem 1.2, we need to work with remainder terms that are independent of $\alpha$. To this aim, we proceed in two steps: first we found uniform estimates for $\alpha \in [0,1/2]$ in Propositions 3.2 and 3.3, then we treat the case $\alpha \in [1/2,1)$ in Lemma 3.6. In Subsection 3.3 we provide some continuity results that allows us to take $\alpha \to 1^-$ and give the full statement in Corollary 3.14. Finally, notice that these asymptotics lead to the asymptotics for the original equation (3.20) (see Remark 3.9).

We begin our analysis by establishing the following:

**Proposition 3.2.** *Let $c_0 > 0$, $\alpha \in [0,1)$, $\beta > 0$ such that $\alpha^2 + \beta^2 = 1$, and $f$ be a solution of (3.20). Define $z$, $y$ and $h$ as $z = |f|^2$ and $y + ih = \bar{f}f'$. Then*

*(i) There exists $E_0 \geq 0$ such that the identity*

$$\frac{1}{2}\left(e^{\alpha\frac{s^2}{2}}|f'|^2 + \frac{c_0^2}{4}|f|^2\right) = E_0$$

*holds true for all $s \in \mathbb{R}$. In particular, $f$, $f'$, $z$, $y$ and $h$ are bounded functions. Moreover, for all $s \in \mathbb{R}$*

$$|f(s)| \leq \frac{\sqrt{8E_0}}{c_0}, \quad |f'(s)| \leq \sqrt{2E_0}\, e^{-\alpha s^2/4}, \tag{3.23}$$

$$|z(s)| \leq \frac{8E_0}{c_0^2} \quad \text{and} \quad |h(s)| + |y(s)| \leq \frac{8E_0}{c_0}e^{-\alpha s^2/4}. \tag{3.24}$$

*(ii) The limit*

$$z_\infty := \lim_{s\to\infty} z(s)$$

*exists.*

*(iii) Let* $\gamma := 2E_0 - c_0^2 z_\infty/2$ *and* $s_0 = 4\sqrt{8+c_0^2}$. *For all* $s \geq s_0$, *we have*

$$z(s) - z_\infty = -\frac{4}{s}(\alpha y + \beta h) - \frac{4\gamma}{s^2} e^{-\alpha s^2/2} + R_0(s), \tag{3.25}$$

*where*

$$|R_0(s)| \leq C(E_0, c_0) \frac{e^{-\alpha s^2/4}}{s^3}. \tag{3.26}$$

*Proof.* Part *(i)* is just the conservation of energy proved in (3.16). Next, using the conservation law in part *(i)*, we obtain that the variables $\{z, y, h\}$ solve the first-order real system

$$z' = 2y, \tag{3.27}$$

$$y' = \beta \frac{s}{2} h - \alpha \frac{s}{2} y + e^{-\alpha s^2/2}\left(2E_0 - \frac{c_0^2}{2} z\right), \tag{3.28}$$

$$h' = -\beta \frac{s}{2} y - \alpha \frac{s}{2} h. \tag{3.29}$$

To show *(ii)*, plugging (3.27) into (3.29) and integrating from 0 to some $s > 0$ we obtain

$$z(s) - \frac{1}{s}\int_0^s z(\sigma)\, d\sigma = -\frac{4}{\beta s}\left(h(s) - h(0) + \frac{\alpha}{2}\int_0^s \sigma h(\sigma)\, d\sigma\right). \tag{3.30}$$

Also, using the above identity,

$$\frac{d}{ds}\left(\frac{1}{s}\int_0^s z(\sigma)\, d\sigma\right) = -\frac{4}{\beta s^2}\left(h(s) - h(0) + \frac{\alpha}{2}\int_0^s \sigma h(\sigma)\, d\sigma\right). \tag{3.31}$$

Now, since from part *(i)* $|h(s)| \leq \frac{8E_0}{c_0} e^{-\alpha s^2/4}$, both $h$ and $\alpha \int_0^s \sigma h(\sigma)\, d\sigma$ are bounded functions, thus from (3.31) it follows that the limit of $\frac{1}{s}\int_0^s z$ exists, as $s \to \infty$. Hence (3.30) and previous observations conclude that the limit $z_\infty := \lim_{s\to\infty} z(s)$ exists and furthermore

$$z_\infty := \lim_{s\to\infty} z(s) = \lim_{s\to\infty} \frac{1}{s}\int_0^s z(\sigma). \tag{3.32}$$

We continue to prove *(iii)*. Integrating (3.31) between $s > 0$ and $+\infty$ and using integration by parts, we obtain

$$z_\infty - \frac{1}{s}\int_0^s z(\sigma)\, d\sigma = -\frac{4}{\beta}\int_s^\infty \frac{h(\sigma)}{\sigma^2}\, d\sigma + \frac{4}{\beta}\frac{h(0)}{s} - \frac{2\alpha}{\beta}\left[\frac{1}{s}\int_0^s \sigma h(\sigma)\, d\sigma + \int_s^\infty h(\sigma)\, d\sigma\right]. \tag{3.33}$$

From (3.30) and (3.33), we get

$$z(s) - z_\infty = -\frac{4}{\beta}\frac{h(s)}{s} + \frac{2\alpha}{\beta}\int_s^\infty h(\sigma)\, d\sigma + \frac{4}{\beta}\int_s^\infty \frac{h(\sigma)}{\sigma^2}. \tag{3.34}$$

In order to compute the integrals in (3.34), using (3.27) and (3.28), we write

$$h = \frac{2}{\beta}\left(\frac{y'}{s} + \frac{\alpha}{4} z' - \frac{2E_0}{s} e^{-\alpha s^2/2} + \frac{c_0^2}{2s} z e^{-\alpha s^2/2}\right).$$

Then, integrating by parts and using the bound for $y$ in (3.24),

$$\int_s^\infty h(\sigma) = \frac{2}{\beta}\left(-\frac{y}{s} + \int_s^\infty \frac{y}{\sigma^2} + \frac{\alpha}{4}(z_\infty - z) - 2E_0 \int_s^\infty \frac{e^{-\alpha\sigma^2/2}}{\sigma} + \frac{c_0^2}{2}\int_s^\infty \frac{z}{\sigma} e^{-\alpha\sigma^2/2}\right). \quad (3.35)$$

Also, from (3.27) and (3.34), we obtain

$$\int_s^\infty \frac{h(\sigma)}{\sigma^2} = \frac{2}{\beta}\left(\int_s^\infty \frac{y'}{\sigma^3} + \frac{\alpha}{2}\int_s^\infty \frac{y}{\sigma^2} - 2E_0\int_s^\infty \frac{e^{-\alpha\sigma^2/2}}{\sigma^3} + \frac{c_0^2}{2}\int_s^\infty \frac{z}{\sigma^3}e^{-\alpha\sigma^2/2}\right). \quad (3.36)$$

Multiplying (3.34) by $\beta^2$, using (3.35), (3.36) and the identity

$$\alpha\int_s^\infty \frac{e^{-\alpha\sigma^2/2}}{\sigma^n} = \frac{e^{-\alpha s^2/2}}{s^{n+1}} - (n+1)\int_s^\infty \frac{e^{-\alpha\sigma^2/2}}{\sigma^{n+2}}, \quad \text{for all } \alpha \geq 0,\ n \geq 1,$$

we conclude that

$$\begin{aligned}(\alpha^2+\beta^2)(z-z_\infty) = &-\frac{4}{s}(\alpha y + \beta h) - \frac{8E_0}{s^2}e^{-\alpha s^2/2} \\ &+ 8\alpha\int_s^\infty \frac{y}{\sigma^2} + 8\int_s^\infty \frac{y'}{\sigma^3} + 2c_0^2\int_s^\infty e^{-\alpha\sigma^2/2} z\left(\frac{\alpha}{\sigma} + \frac{2}{\sigma^3}\right).\end{aligned} \quad (3.37)$$

Finally, using (3.27) and the boundedness of $z$ and $y$, an integration by parts argument shows that

$$8\alpha\int_s^\infty \frac{y}{\sigma^2} + 8\int_s^\infty \frac{y'}{\sigma^3} = -4\alpha\frac{z}{s^2} - 8\frac{y}{s^3} - 12\frac{z}{s^4} + 8\int_s^\infty z\left(\frac{\alpha}{\sigma^3} - \frac{6}{\sigma^5}\right). \quad (3.38)$$

Bearing in mind that $\alpha^2+\beta^2 = 1$, from (3.37) and (3.38), we obtain the following identity

$$\begin{aligned}z - z_\infty = &-\frac{4}{s}(\alpha y + \beta h) - \frac{8E_0}{s^2}e^{-\alpha s^2/2} - 4\alpha\frac{z}{s^2} - 8\frac{y}{s^3} - 12\frac{z}{s^4} + 8\int_s^\infty z\left(\frac{\alpha}{\sigma^3} + \frac{6}{\sigma^5}\right)d\sigma \\ &+ 2c_0^2\int_s^\infty e^{-\alpha\sigma^2/2} z\left(\frac{\alpha}{\sigma} + \frac{2}{\sigma^3}\right)d\sigma,\end{aligned} \quad (3.39)$$

for all $s > 0$. In order to prove *(iii)*, we first write $z = z - z_\infty + z_\infty$ and observe that

$$8\alpha\int_s^\infty \frac{z}{\sigma^3} = 8\alpha\int_s^\infty \frac{z - z_\infty}{\sigma^3} + \frac{4\alpha z_\infty}{s^2},$$

$$\int_s^\infty \frac{z}{\sigma^5} = \int_s^\infty \frac{z-z_\infty}{\sigma^5} + \frac{z_\infty}{4s^4} \qquad \text{and}$$

$$\int_s^\infty e^{-\alpha\sigma^2/2} z\left(\frac{\alpha}{\sigma} + \frac{2}{\sigma^3}\right) = \int_s^\infty e^{-\alpha\sigma^2/2}(z - z_\infty)\left(\frac{\alpha}{\sigma} + \frac{2}{\sigma^3}\right) + \frac{z_\infty}{s^2}e^{-\alpha s^2/2}.$$

Therefore, we can recast (3.39) as (3.25) with

$$\begin{aligned}R_0(s) = &-\frac{4\alpha(z - z_\infty)}{s^2} - \frac{8y}{s^3} - \frac{12(z - z_\infty)}{s^4} + 8\int_s^\infty (z - z_\infty)\left(\frac{\alpha}{\sigma^3} + \frac{6}{\sigma^5}\right)d\sigma \\ &+ 2c_0^2\int_s^\infty e^{-\alpha\sigma^2/2}(z - z_\infty)\left(\frac{\alpha}{\sigma} + \frac{2}{\sigma^3}\right)d\sigma.\end{aligned} \quad (3.40)$$

Let us take $s_0 \geq 1$ to be fixed in what follows. For $t \geq s_0$, we denote $\|\cdot\|_t$ the norm of $L^\infty([t,\infty))$. From the definition of $R_0$ in (3.40) and the elementary inequalities

$$\alpha\int_s^\infty \frac{e^{-\alpha\sigma^2/2}}{\sigma^n} \leq \frac{e^{-\alpha s^2/2}}{s^{n+1}}, \quad \text{for all } \alpha \geq 0, \quad n \geq 1, \quad (3.41)$$

and

$$\int_s^\infty \frac{e^{-\alpha\sigma^2/2}}{\sigma^n} \leq \frac{e^{-\alpha s^2/2}}{(n-1)s^{n-1}}, \qquad \text{for all } \alpha \geq 0, \quad n > 1, \tag{3.42}$$

we obtain

$$\|R_0\|_t \leq \frac{8\|y\|_t}{t^3} + \frac{4}{t^2}\left(8 + c_0^2 e^{-\alpha t^2/2}\right)\|z - z_\infty\|_t.$$

Hence, choosing $s_0 = 4\sqrt{8 + c_0^2}$, so that $\frac{4}{t^2}\left(8 + c_0^2 e^{-\alpha t^2/2}\right) \leq 1/2$, from (3.24) and (3.25) we conclude that there exists a constant $C(E_0, c_0) > 0$ such that

$$\|z - z_\infty\|_t \leq \frac{C(E_0, c_0)}{t} e^{-\alpha t^2/4}, \qquad \text{for all } \alpha \in [0, 1) \quad \text{and} \quad t \geq s_0,$$

which implies that

$$|z(s) - z_\infty| \leq \frac{C(E_0, c_0)}{s} e^{-\alpha s^2/4}, \qquad \text{for all } \alpha \in [0, 1), \quad s \geq s_0. \tag{3.43}$$

Finally, plugging (3.24) and (3.43) into (3.40) and bearing in mind the inequalities (3.41) and (3.42), we deduce that

$$|R_0(s)| \leq C(E_0, c_0)\, \frac{e^{-\alpha s^2/4}}{s^3}, \qquad \forall\, s \geq s_0 = 4\sqrt{8 + c_0^2}, \tag{3.44}$$

and the proof of *(iii)* is completed. □

Formula (3.25) in Proposition 3.2 gives $z$ in terms of $y$ and $h$. Therefore, we can reduce our analysis to that of the variables $y$ and $h$ or, in other words, to that of the system (3.27)–(3.29). In fact, a first attempt could be to define $w = y + ih$, so that from (3.28) and (3.29), we have that $w$ solves

$$\left(we^{(\alpha+i\beta)s^2/4}\right)' = e^{(-\alpha+i\beta)s^2/4}\left(\gamma - \frac{c_0^2}{2}(z - z_\infty)\right). \tag{3.45}$$

From (3.43) in Proposition 3.2 and (3.45), we see that the limit $w_* = \lim_{s\to\infty} w(s)e^{(\alpha+i\beta)s^2/4}$ exists (at least when $\alpha \neq 0$), and integrating (3.45) from some $s > 0$ to $\infty$ we find that

$$w(s) = e^{-(\alpha+i\beta)s^2/4}\left(w_* - \int_s^\infty e^{(-\alpha+i\beta)\sigma^2/4}\left(\gamma - \frac{c_0^2}{2}(z - z_\infty)\right) d\sigma\right).$$

In order to obtain an asymptotic expansion, we need to estimate $\int_s^\infty e^{(-\alpha+i\beta)\sigma^2/4}(z - z_\infty)$, for $s$ large. This can be achieved using (3.43),

$$\left|\int_s^\infty e^{(-\alpha+i\beta)\sigma^2/4}(z - z_\infty)\, d\sigma\right| \leq C(E_0, c_0) \int_s^\infty \frac{e^{-\alpha\sigma^2/2}}{\sigma}\, d\sigma \tag{3.46}$$

and the asymptotic expansion

$$\int_s^\infty \frac{e^{-\alpha\sigma^2/2}}{\sigma}\, d\sigma = e^{-\alpha s^2/2}\left(\frac{1}{\alpha s^2} - \frac{2}{\alpha^2 s^4} + \frac{8}{\alpha^3 s^6} + \cdots\right).$$

However this estimate diverges as $\alpha \to 0$. The problem is that the bound used in obtaining (3.46) does not take into account the cancellations due to the oscillations. Therefore, and in order to obtain the asymptotic behaviour of $z$, $y$ and $h$ valid for all $\alpha \in [0, 1)$, we need a more refined analysis. In the next proposition we study the system (3.27)–(3.29), where we consider the cancellations due the oscillations (see Lemma 3.5 below). The following result provides estimates that are valid for $s \geq s_1$, for some $s_1$ independent of $\alpha$, if $\alpha$ is small.

**Proposition 3.3.** *With the same notation and terminology as in Proposition 3.2, let*

$$s_1 = \max\left\{4\sqrt{8+c_0^2}, 2c_0\left(\frac{1}{\beta}-1\right)^{1/2}\right\}.$$

*Then for all* $s \geq s_1$,

$$y(s) = be^{-\alpha s^2/4}\sin(\phi(s_1;s)) - \frac{2\alpha\gamma}{s}e^{-\alpha s^2/2} + O\left(\frac{e^{-\alpha s^2/2}}{\beta^2 s^2}\right), \tag{3.47}$$

$$h(s) = be^{-\alpha s^2/4}\cos(\phi(s_1;s)) - \frac{2\beta\gamma}{s}e^{-\alpha s^2/2} + O\left(\frac{e^{-\alpha s^2/2}}{\beta^2 s^2}\right), \tag{3.48}$$

*where*

$$\phi(s_1;s) = a + \beta\int_{s_1^2/4}^{s^2/4}\sqrt{1+c_0^2\frac{e^{-2\alpha t}}{t}}\,dt,$$

$a \in [0,2\pi)$ *is a real constant, and* $b$ *is a positive constant given by*

$$b^2 = \left(2E_0 - \frac{c_0^2}{4}z_\infty\right)z_\infty. \tag{3.49}$$

*Proof.* First, notice that plugging the expression for $z(s) - z_\infty$ in (3.25) into (3.28), the system (3.28)–(3.29) for the variables $y$ and $h$ rewrites equivalently as

$$y' = \frac{s}{2}(\beta h - \alpha y) + \frac{2c_0^2}{s}e^{-\alpha s^2/2}(\beta h + \alpha y) + \gamma e^{-\alpha s^2/2} + R_1(s), \tag{3.50}$$

$$h' = -\frac{s}{2}(\beta y + \alpha h), \tag{3.51}$$

where

$$R_1(s) = -\frac{c_0^2}{2}e^{-\alpha s^2/2}R_0(s) + \frac{2c_0^2\gamma e^{-\alpha s^2}}{s^2}, \tag{3.52}$$

and $R_0$ is given by (3.40).

Introducing the new variables,

$$u(t) = e^{\alpha t}y(2\sqrt{t}), \quad v(t) = e^{\alpha t}h(2\sqrt{t}), \tag{3.53}$$

we recast (3.50)–(3.51) as

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} \alpha K & \beta(1+K) \\ -\beta & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F \\ 0 \end{pmatrix}, \tag{3.54}$$

with

$$K = \frac{c_0^2 e^{-2\alpha t}}{t}, \quad F = \gamma\frac{e^{-\alpha t}}{\sqrt{t}} + \frac{e^{-\alpha t}}{\sqrt{t}}R_1(2\sqrt{t}),$$

where $R_1$ is the function defined in (3.52). In this way, we can regard (3.54) as a non-autonomous system. It is straightforward to check that the matrix

$$A = \begin{pmatrix} \alpha K & \beta(1+K) \\ -\beta & 0 \end{pmatrix}$$

is diagonalizable, i.e. $A = PDP^{-1}$, with

$$D = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}, \quad P = \begin{pmatrix} -\frac{\alpha K}{2\beta} - i\Delta^{1/2} & -\frac{\alpha K}{2\beta} + i\Delta^{1/2} \\ 1 & 1 \end{pmatrix},$$

$$\lambda_\pm = \frac{\alpha K}{2} \pm i\beta\Delta^{1/2}, \qquad \text{and} \qquad \Delta = 1 + K - \frac{\alpha^2 K^2}{4\beta^2}. \tag{3.55}$$

At this point we remark that the condition $t \geq t_1$, with $t_1 := s_1^2/4$ and $s_1 \geq 2c_0(\frac{1}{\beta}-1)^{1/2}$, implies that

$$0 < K\left(\frac{1}{\beta} - 1\right) \leq 1, \quad \forall t \geq t_1, \tag{3.56}$$

so that

$$\Delta = 1 + K - \frac{(1-\beta^2)}{4\beta^2}K^2 = \left(1 + \frac{K}{2} + \frac{K}{2\beta}\right)\left(1 + \frac{K}{2}\left(1 - \frac{1}{\beta}\right)\right) \geq \frac{1}{2}, \quad \forall t \geq t_1. \tag{3.57}$$

Thus, defining

$$w = (w_1, w_2) = P^{-1}(u, v), \tag{3.58}$$

we get

$$\left(e^{-\int_{t_1}^t D} w\right)' = e^{-\int_{t_1}^t D}\left((P^{-1})'Pw + P^{-1}\tilde{F}\right), \tag{3.59}$$

with $\tilde{F} = (F, 0)$. From the definition of $w$ and taking into account that $u$ and $v$ are real functions, we have that $w_1 = \bar{w}_2$ and therefore the study of (3.59) reduces to the analysis of the equation:

$$\left(e^{-\int_{t_1}^t \lambda_+} w_1\right)' = e^{-\int_{t_1}^t \lambda_+} G(t), \tag{3.60}$$

with

$$G(t) = i\frac{\alpha K'}{4\beta\Delta^{1/2}}(w_1 + \bar{w}_1) - \frac{\Delta'}{4\Delta}(w_1 - \bar{w}_1) + i\frac{F}{2\Delta^{1/2}}.$$

From (3.60) we have

$$w_1(t) = e^{\int_{t_1}^t \lambda_+}\left(w_1(t_1) + w_\infty - \int_t^\infty e^{-\int_{t_1}^\tau \lambda_+} G(\tau)\, d\tau\right), \tag{3.61}$$

with

$$w_\infty = \int_{t_1}^\infty e^{-\int_{t_1}^\tau \lambda_+} G(\tau).$$

Since

$$w_1 = \frac{iu}{2\Delta^{1/2}} + \frac{v}{2} + \frac{i\alpha K v}{4\beta\Delta^{1/2}}, \tag{3.62}$$

we recast $G$ as $G = i(G_1 + G_2 + G_3)$ with

$$G_1 = \frac{\alpha K' v}{4\beta\Delta^{1/2}} - \frac{\Delta'}{4\Delta^{3/2}}\left(u + \frac{\alpha K v}{2\beta}\right), \quad G_2 = \frac{\gamma e^{-\alpha t}}{2t^{1/2}\Delta^{1/2}} \quad \text{and} \quad G_3 = \frac{e^{-\alpha t}}{2t^{1/2}\Delta^{1/2}} R_1(2t^{1/2}).$$

Now, from the definition of $K$ and $\Delta$, we have

$$K' = -K\left(2\alpha + \frac{1}{t}\right), \qquad K'' = K\left(\left(2\alpha + \frac{1}{t}\right)^2 + \frac{1}{t^2}\right),$$
$$\Delta' = K'\left(1 - \frac{\alpha^2 K}{2\beta^2}\right) \qquad \text{and} \quad \Delta'' = K\left(\left(2\alpha + \frac{1}{t}\right)^2 + \frac{1}{t^2}\right)\left(1 - \frac{\alpha^2 K}{2\beta^2}\right) - \frac{\alpha^2}{2\beta^2}K^2\left(2\alpha + \frac{1}{t}\right)^2.$$

Also, since $s_1 = \max\{4\sqrt{8 + c_0^2}, 2c_0(1/\beta - 1)^{1/2}\}$, for all $t \geq t_1 = s_1^2/4$, we have in particular that $t \geq 8 + c_0^2$ and $t \geq c_0^2(1/\beta - 1)$, hence

$$\frac{c_0^2}{t\beta} = \frac{c_0^2}{t}\left(\frac{1}{\beta} - 1\right) + \frac{c_0^2}{t} \leq 2 \tag{3.63}$$

and

$$\left|1 - \frac{\alpha^2 K}{4\beta^2}\right| \le 1 + \frac{1}{4\beta}\left(\frac{c_0^2}{t\beta}\right) \le \frac{2}{\beta}. \tag{3.64}$$

Therefore

$$|K'| \le c_0^2 e^{-2\alpha t}\left(\frac{2\alpha}{t} + \frac{1}{t^2}\right), \qquad |\Delta'| \le \frac{2c_0^2 e^{-2\alpha t}}{\beta}\left(\frac{2\alpha}{t} + \frac{1}{t^2}\right) \tag{3.65}$$

and

$$|\Delta''| \le \frac{24 c_0^2}{\beta} e^{-2\alpha t}\left(\frac{\alpha}{t} + \frac{1}{t^2}\right). \tag{3.66}$$

From Proposition 3.2, $u$ and $v$ are bounded in terms of the energy. Thus, from the definition of $G_1$ and the estimates (3.56), (3.57) and (3.65), we obtain

$$|G_1(t)| \le \frac{C(E_0, c_0)e^{-2\alpha t}}{\beta^2}\left(\frac{\alpha}{t} + \frac{1}{t^2}\right).$$

Since

$$\left|e^{\pm \int_{t_1}^{\tau} \lambda_+}\right| \le 2, \tag{3.67}$$

we conclude that

$$\left|\int_t^\infty e^{-\int_{t_1}^{\tau} \lambda_+} G_1(\tau)\, d\tau\right| \le \frac{C(E_0, c_0)}{\beta^2}\int_t^\infty e^{-2\alpha\tau}\left(\frac{\alpha}{\tau} + \frac{1}{\tau^2}\right) \le \frac{C(E_0, c_0)e^{-2\alpha t}}{\beta^2 t}. \tag{3.68}$$

Here we have used the inequality

$$\alpha\int_t^\infty \frac{e^{-2\alpha\sigma}}{\sigma^n}\, d\sigma \le \frac{e^{-2\alpha t}}{2t^n}, \quad n \ge 1, \tag{3.69}$$

which follows by integrating by parts.

In order to handle the terms involving $G_2$ and $G_3$, we need to take advantage of the oscillatory character of the involved integrals, which is exploited in Lemma 3.5. From (3.57), (3.65) and (3.66), straightforward calculations show that the function defined by $f = \gamma/(2t^{1/2}\Delta^{1/2})$ satisfies the hypothesis in part *(ii)* of Lemma 3.5 with $a = 1/2$ and $L = C(E_0, c_0)/\beta$. Thus invoking this lemma with $f = \gamma/(2t^{1/2}\Delta^{1/2})$ and noticing that

$$\frac{1}{\Delta^{1/2}} = 1 + \left(\frac{1}{\Delta^{1/2}} - 1\right)$$

and that

$$\left|\frac{1}{\Delta^{1/2}} - 1\right| = \left|\frac{1-\Delta}{\Delta^{1/2}(\Delta^{1/2}+1)}\right| \le \frac{|K|\left|1 - \frac{\alpha^2 K}{4\beta^2}\right|}{|\Delta^{1/2}(\Delta^{1/2}+1)|} \le \frac{2\sqrt{2}c_0^2}{\beta t},$$

where we have used (3.57) and (3.64), we conclude that

$$\int_t^\infty e^{-\int_{t_1}^{\tau}\lambda_+} G_2(\tau)\, d\tau = \frac{\gamma}{2(\alpha + i\beta)t^{1/2}} e^{-\int_{t_1}^{t}\lambda_+} e^{-\alpha t} + R_2(t), \tag{3.70}$$

with

$$|R_2(t)| \le \frac{C(E_0, c_0)e^{-\alpha t}}{\beta^2 t^{3/2}}.$$

For $G_3$, we first write explicitly (recall the definition of $R_1$ in (3.52))

$$G_3(t) = -\frac{c_0^2 R_0(2\sqrt{t})e^{-3\alpha t}}{4t^{1/2}\Delta^{1/2}} + \frac{c_0^2\gamma e^{-5\alpha t}}{4t^{3/2}\Delta^{1/2}} := G_{3,1}(t) + G_{3,2}(t). \tag{3.71}$$

Using (3.44) and (3.57), we see that $|G_{3,1}(t)| \leq C(E_0, c_0)e^{-4\alpha t}/t^2$, so that we can treat this term as we did for $G_1$ to obtain

$$\left| \int_t^\infty e^{-\int_{t_1}^\tau \lambda_+} G_{3,1}(\tau)\, d\tau \right| \leq \frac{C(E_0, c_0)e^{-4\alpha t}}{t}. \tag{3.72}$$

For the second term, using (3.57), (3.65) and (3.63), it is easy to see that the function $f$ defined by $f = (c_0^2\gamma)/(4t^{3/2}\Delta^{1/2})$ satisfies

$$|f(t)| \leq \frac{C(E_0, c_0)}{t^{3/2}} \quad \text{and} \quad |f'(t)| \leq C(E_0, c_0)\left(\frac{\alpha}{t^{3/2}} + \frac{1}{t^{5/2}}\right),$$

as a consequence, invoking part *(i)* of Lemma 3.5, we obtain

$$\left| \int_t^\infty e^{-\int_{t_1}^\tau \lambda_+} G_{3,2}(\tau)\, d\tau \right| \leq \frac{C(E_0, c_0)e^{-5\alpha t}}{\beta t^{3/2}}. \tag{3.73}$$

From (3.61), (3.67), (3.68), (3.72) and (3.73), we deduce that

$$w_1(t) = e^{\int_{t_1}^t \lambda_+}(w_1(t_1) + w_\infty) - \frac{\gamma(\beta + i\alpha)}{2t^{1/2}}e^{-\alpha t} + R_3(t) \quad \text{with} \quad |R_3(t)| \leq \frac{C(E_0, c_0)e^{-\alpha t}}{\beta^2 t}. \tag{3.74}$$

Now we *claim* that

$$e^{\int_{t_1}^t \lambda_+} = C_{\alpha,c_0}e^{i\beta I(t)} + H(t), \quad \text{with} \quad I(t) = \int_{t_1}^t \sqrt{1 + K(\sigma)}\, d\sigma, \quad |H(t)| \leq \frac{3c_0^2 e^{-2\alpha t}}{t}, \tag{3.75}$$

and

$$C_{\alpha,c_0} = \exp\left(\frac{\alpha}{2}\int_{t_1}^\infty K\, d\sigma\right) \exp\left(-i\frac{\alpha^2}{4\beta}\int_{t_1}^\infty \frac{K^2}{\Delta^{1/2} + (1+K)^{1/2}}\, d\sigma\right).$$

Indeed, recall that $\lambda_+ = \frac{\alpha K}{2} + i\beta\Delta^{1/2}$ so that

$$e^{\int_{t_1}^t \lambda_+} = e^{\alpha\int_{t_1}^t \frac{K}{2}} e^{i\beta\int_{t_1}^t \Delta^{1/2}}. \tag{3.76}$$

First, we notice that

$$\alpha\int_{t_1}^t \frac{K}{2} = c_0^2\alpha\int_{t_1}^\infty \frac{e^{-2\alpha\sigma}}{2\sigma} - c_0^2\alpha\int_t^\infty \frac{e^{-2\alpha\sigma}}{2\sigma},$$

where both integrals are finite in view of (3.69). Moreover, by combining with the fact that $|1 - e^{-x}| \leq x$, for $x \geq 0$, we can write

$$\exp\left(-c_0^2\alpha\int_t^\infty \frac{e^{-2\alpha\sigma}}{2\sigma}\right) = 1 + H_1(t),$$

with

$$|H_1(t)| \leq \frac{c_0^2 e^{-2\alpha t}}{4t}, \quad \text{for all } t \geq c_0^2/4. \tag{3.77}$$

The above argument shows that

$$e^{\alpha\int_{t_1}^t \frac{K}{2}} = e^{\alpha\int_{t_1}^\infty \frac{K}{2}}(1 + H_1(t)), \tag{3.78}$$

with $H_1(t)$ satisfying (3.77).

For the second term of the eigenvalue, using the definition of $\Delta$ in (3.55), we write

$$
\begin{aligned}
i\beta \int_{t_1}^{t} \Delta^{1/2} &= i\beta \int_{t_1}^{t} \left(\Delta^{1/2} - \sqrt{1+K}\right) + i\beta \int_{t_1}^{t} \sqrt{1+K} \\
&= -i\frac{\alpha^2}{4\beta} \int_{t_1}^{t} \frac{K^2}{\Delta^{1/2} + (1+K)^{1/2}} + i\beta \int_{t_1}^{t} \sqrt{1+K}.
\end{aligned}
$$

Proceeding as before and using that $|1 - e^{ix}| \le |x|$, for $x \in \mathbb{R}$, and that

$$
\alpha \int_{t}^{\infty} \frac{K^2}{\Delta^{1/2} + \sqrt{1+K}} \le \alpha \int_{t}^{\infty} K^2(\sigma)\, d\sigma = \alpha c_0^4 \int_{t}^{\infty} \frac{e^{-4\alpha\tau}}{\tau^2} \le \frac{c_0^4 e^{-4\alpha t}}{4t^2},
$$

we conclude that

$$
e^{i\beta \int_{t_1}^{t} \Delta^{1/2}} = e^{i\beta I(t)} e^{-i\frac{\alpha^2}{4\beta}\int_{t_1}^{\infty} \frac{K^2}{\Delta^{1/2}+(1+K)^{1/2}}} (1 + H_2), \tag{3.79}
$$

with

$$
|H_2(t)| \le \frac{c_0^4 e^{-4\alpha t}}{16\beta t^2} \le \frac{c_0^2 e^{-4\alpha t}}{8t},
$$

bearing in mind (3.63). Therefore, from (3.76), (3.78) and (3.79),

$$
e^{\int_{t_1}^{t} \lambda_+} = C_{\alpha,c_0} e^{i\beta I(t)} (1 + H_1(t))(1 + H_2(t)).
$$

The claim follows from the above identity, the bounds for $H_1$ and $H_2$, and the fact that $C_{\alpha,c_0}$ satisfies that $|C_{\alpha,c_0}| = |e^{\int_{t_1}^{\infty} \lambda_+}| \le 2$ (see (3.67)). From (3.74), the claim and writing

$$
C_{\alpha,c_0}(w_1(t_1) + w_\infty) = (be^{ia})/2 \tag{3.80}
$$

for some real constants $a$ and $b$ such that $b \ge 0$ and $a \in [0, 2\pi)$, it follows that

$$
w_1(t) = \frac{b}{2} e^{i(\beta I(t) + a)} - \frac{\gamma(\beta + i\alpha)}{2t^{1/2}} + R_{w_1}(t) \quad \text{with} \quad |R_{w_1}(t)| \le \frac{C(E_0, c_0)e^{-\alpha t}}{\beta^2 t}. \tag{3.81}
$$

The above bound for $R_{w_1}(t)$ easily follows from the bounds for $R_3(t)$ and $H(t)$ in (3.74) and (3.75) respectively, and the fact that

$$
|w_1(t)| \le C(E_0, c_0), \qquad \forall\, t \ge t_1. \tag{3.82}
$$

This last inequality is a consequence of (3.53), (3.57), (3.62), (3.63) and the bounds for $y$ and $h$ established in (3.24) in Proposition 3.2.

Going back to the definition of $w$ in (3.58), we have $(u, v) = P(w_1, w_2)$, that is

$$
\begin{aligned}
u &= -\frac{\alpha K}{2\beta}(w_1 + \bar{w}_1) - i\Delta^{1/2}(w_1 - \bar{w}_1) = 2\,\mathrm{Im}(w_1) + R_4(t), \\
v &= (w_1 + \bar{w}_1) = 2\,\mathrm{Re}(w_1),
\end{aligned} \tag{3.83}
$$

with

$$
\begin{aligned}
|R_4(t)| &= \left| \frac{-\alpha K}{\beta} \mathrm{Re}(w_1) + 2(\Delta^{1/2} - 1)\,\mathrm{Im}(w_1) \right| \\
&\le \frac{K}{\beta} |\mathrm{Re}(w_1)| + 2\frac{|\Delta - 1|}{\Delta^{1/2} + 1} |\mathrm{Im}(w_1)| \\
&\le \frac{2c_0^2 e^{-2\alpha t}}{\beta t} (|\mathrm{Re}(w_1)| + |\mathrm{Im}(w_1)|) \le \frac{C(E_0, c_0)e^{-2\alpha t}}{\beta t},
\end{aligned}
$$

where we have used (3.57), (3.64), and (3.82). From (3.81) and (3.83), we obtain

$$u(t) = b\sin(\beta I(t) + a) - \frac{\alpha\gamma}{t^{1/2}}e^{-\alpha t} + R_5(t),$$
$$v(t) = b\cos(\beta I(t) + a) - \frac{\beta\gamma}{t^{1/2}}e^{-\alpha t} + R_6(t),$$

with

$$|R_5(t)| + |R_6(t)| \le C(E_0, c_0)e^{-\alpha t}/(\beta^2 t).$$

The asymptotics for $y$ and $h$ given in (3.47) and (3.48) are a direct consequence of (3.53) and the above identities and bounds.

Finally, we compute the value of $b$. In fact, from (3.47) and (3.48)

$$\lim_{s\to\infty}(y^2(s) + h^2(s))e^{\alpha s^2/2} = b^2.$$

On the other hand, since $y + ih = \bar{f}f'$ and using the conservation of energy (3.16)

$$\left(y^2(s) + h^2(s)\right)e^{\alpha s^2/2} = |y + ih|^2(s)e^{\alpha s^2/2} = |f'|^2|f|^2 e^{\alpha s^2/2} = (2E_0 - \frac{c_0^2}{4}|f|^2)|f|^2,$$

so that, taking the limit as $s \to \infty$ and recalling that $z = |f|^2$, (3.49) follows. □

**Remark 3.4.** *From the definitions of $b$ in* (3.49)*, and $be^{ia}$ in* (3.80) *(in terms of $C_{\alpha,c_0}$, $w_1(t_1)$ and $w_\infty$ in* (3.80)*), it is simple to verify that $b$ and $be^{ia}$ depend continuously on $\alpha \in [0,1)$, provided that $z_\infty$ is a continuous function of $\alpha$. In Subsection 3.3 we will prove that $z_\infty$ depends continuously on $\alpha$, for $\alpha \in [0,1]$, and establish the continuous dependence of the constants $b$ and $be^{ia}$ with respect to the parameter $\alpha$ in Lemma 3.13 above.*

In the proof of Proposition 3.3, we have used the following key lemma that establishes the control of certain integrals by exploiting their oscillatory character.

**Lemma 3.5.** *With the same notation as in the proof of Proposition 3.2.*

*(i) Let $f \in C^1((t_1,\infty))$ such that*

$$|f(t)| \le L/t^a \quad \text{and} \quad |f'(t)| \le L\left(\frac{\alpha}{t^a} + \frac{1}{t^{a+1}}\right),$$

*for some constants $L$, $a > 0$. Then, for all $t \ge t_1$ and $l \ge 1$*

$$\int_t^\infty e^{-\int_{t_1}^\tau \lambda_+} e^{-l\alpha\tau} f(\tau)\,d\tau = \frac{1}{(\alpha + i\beta)} e^{-\int_{t_1}^t \lambda_+} e^{-l\alpha t} f(t) + F(t),$$

*with*

$$|F(t)| \le \frac{C(l, a, c_0)Le^{-l\alpha t}}{\beta t^a}. \tag{3.84}$$

*(ii) If in addition $f \in C^2((t_1,\infty))$,*

$$|f'(t)| \le L/t^{a+1} \quad \text{and} \quad |f''(t)| \le L\left(\frac{\alpha}{t^{a+1}} + \frac{1}{t^{a+2}}\right), \tag{3.85}$$

*then*

$$|F(t)| \le \frac{C(l, a, c_0)Le^{-l\alpha t}}{\beta t^{a+1}}. \tag{3.86}$$

*Here $C(l,a,c_0)$ is a positive constant depending only on $l$, $a$ and $c_0$.*

*Proof.* Define $\lambda = \lambda_+$. Recall (see proof of Proposition 3.2) that

$$\lambda_+ = \frac{\alpha K}{2} + i\beta\Delta^{1/2} \quad \text{and} \quad \Delta = 1 + K - \frac{\alpha^2 K^2}{4\beta^2}, \quad \text{with} \quad K = c_0^2 \frac{e^{-2\alpha t}}{t}.$$

Setting $R_\lambda = 1/\lambda - 1/(i\beta)$ and integrating by parts, we obtain

$$\left(1 + \frac{l\alpha}{i\beta}\right) \int_t^\infty e^{-\int_{t_1}^\tau \lambda} e^{-l\alpha\tau} f(\tau)\, d\tau = e^{-\int_{t_1}^t \lambda} e^{-l\alpha t} f(t) \left(\frac{1}{i\beta} + R_\lambda\right) + \int_t^\infty e^{-\int_{t_1}^\tau \lambda} e^{-l\alpha\tau} \left(-l\alpha f R_\lambda + \frac{f'}{\lambda} - \frac{f\lambda'}{\lambda^2}\right) d\tau,$$

or, equivalently,

$$\int_t^\infty e^{-\int_{t_1}^\tau \lambda} e^{-\alpha\tau} f(\tau)\, d\tau = \frac{1}{l\alpha + i\beta} e^{-\int_{t_1}^t \lambda} e^{-\alpha t} f(t) + F(t),$$

with

$$F(t) = \frac{i\beta}{l\alpha + i\beta} \left( e^{-\int_{t_1}^t \lambda} e^{-l\alpha t} R_\lambda f + \int_t^\infty e^{-\int_{t_1}^\tau \lambda} e^{-l\alpha\tau} \left(-l\alpha f R_\lambda + \frac{f'}{\lambda} - \frac{f\lambda'}{\lambda^2}\right) d\tau \right).$$

Using (3.57), (3.63) and (3.65), it is easy to check that for all $t \geq t_1$

$$|\lambda| \geq \frac{\beta}{\sqrt{2}} \quad \text{and} \quad |\lambda'| \leq 3c_0^2 \left(\frac{2\alpha}{t} + \frac{1}{t^2}\right). \tag{3.87}$$

On the other hand,

$$|R_\lambda| = \left|\frac{i\beta - \lambda}{i\beta\lambda}\right| \leq \frac{\sqrt{2}}{\beta^2} \left(\beta|1 - \Delta^{1/2}| + \frac{\alpha K}{2}\right),$$

with, using the definition of $\Delta$ in (3.57) and (3.63),

$$\frac{\alpha K}{2} \leq \frac{c_0^2}{2t} \quad \text{and} \quad |1 - \Delta^{1/2}| = \frac{|1-\Delta|}{1 + \Delta^{1/2}} \leq |1 - \Delta| \leq \frac{c_0^2}{t} + \frac{c_0^2}{4\beta t}\left(\frac{c_0^2}{\beta t}\right) \leq \frac{2c_0^2}{\beta t}.$$

Previous lines show that

$$|R_\lambda| \leq \frac{10c_0^2}{\beta^2 t}. \tag{3.88}$$

The estimate (3.84) easily follows from the bounds (3.67), (3.69), (3.87), (3.88) and the hypotheses on $f$. To obtain part *(ii)* we only need to improve the estimate for the term

$$\int_t^\infty e^{-\int_{t_1}^\tau \lambda} e^{-l\alpha\tau} \frac{f'}{\lambda}\, d\tau$$

in the above argument. In particular, it suffices to prove that

$$\left| \int_t^\infty e^{-\int_{t_1}^\tau \lambda} e^{-l\alpha\tau} \frac{f'}{\lambda} \right| \leq C(l, c_0, a) \frac{L e^{-l\alpha t}}{\beta^2 t^{a+1}}.$$

Now, consider the function $g = f'/\lambda$. Notice that from (3.63), (3.87) and the hypotheses on $f$ in (3.85), we have

$$|g(t)| \leq \frac{\sqrt{2}L}{\beta t^{a+1}}$$

and

$$
\begin{aligned}
|g'(t)| \leq & \frac{\sqrt{2}}{\beta} L \left( \frac{\alpha}{t^{a+1}} + \frac{1}{t^{a+2}} \right) + \frac{6L}{\beta} \left( \frac{c_0^2}{\beta t} \right) \left( \frac{2\alpha}{t^{a+1}} + \frac{1}{t^{a+2}} \right) \\
\leq & \frac{14L}{\beta} \left( \frac{2\alpha}{t^{a+1}} + \frac{1}{t^{a+2}} \right).
\end{aligned}
$$

Therefore, from part *(i)*, we obtain

$$
\left| \int_t^\infty e^{-\int_{t_1}^\tau \lambda} e^{-l\alpha\tau} \frac{f'}{\lambda} \right| \leq C(l, c_0, a) L e^{-l\alpha t} \left( \frac{1}{\beta t^{a+1}} + \frac{1}{\beta^2 t^{a+1}} \right) \leq \frac{C(l, c_0, a) L e^{-l\alpha t}}{\beta^2 t^{a+1}},
$$

as desired. ☐

We remark that if $\alpha \in [0, 1/2]$, the asymptotics in Proposition 3.3 are uniform in $\alpha$. Indeed,

$$
\max_{\alpha \in [0,1/2]} \left\{ 4\sqrt{8 + c_0^2}, 2c_0 \left( \frac{1}{\beta} - 1 \right)^{1/2} \right\} = 4\sqrt{8 + c_0^2} = s_0.
$$

Therefore in this situation we can omit the dependence on $s_1$ in the function $\phi(s_1; s)$, because the asymptotics are valid with

$$
\phi(s) := \phi(s_0; s) = a + \beta \int_{s_0^2/4}^{s^2/4} \sqrt{1 + c_0^2 \frac{e^{-2\alpha t}}{t}}\, dt. \tag{3.89}
$$

We continue to show that the factor $1/\beta^2$ in the big-$O$ in formulae (3.47) and (3.48) are due to the method used and this factor can be avoided if $\alpha$ is far from zero. More precisely, we have the following:

**Lemma 3.6.** *Let $\alpha \in [1/2, 1)$. With the same notation as in Propositions 3.2 and 3.3, we have the following asymptotics: for all $s \geq s_0$,*

$$
y(s) = b e^{-\alpha s^2/4} \sin(\phi(s)) - \frac{2\alpha\gamma}{s} e^{-\alpha s^2/2} + O\left( \frac{e^{-\alpha s^2/2}}{s^2} \right), \tag{3.90}
$$

$$
h(s) = b e^{-\alpha s^2/4} \cos(\phi(s)) - \frac{2\beta\gamma}{s} e^{-\alpha s^2/2} + O\left( \frac{e^{-\alpha s^2/2}}{s^2} \right). \tag{3.91}
$$

*Here, the function $\phi$ is defined by* (3.89) *and the bounds controlling the error terms depend on $c_0$, and the energy $E_0$, and are independent of $\alpha \in [1/2, 1)$*

*Proof.* Let $\alpha \in [1/2, 1)$ and define $w = y + ih$. From Proposition 3.3 and (1.21), we have that for all $\alpha \in [1/2, 1)$

$$
\lim_{s \to \infty} w e^{(\alpha + i\beta)s^2/4} = bie^{-i\tilde{a}}, \tag{3.92}
$$

where $\tilde{a} := a + C(\alpha, c_0)$, $a$ and $b$ are the constants defined in Proposition 3.3 and $C(\alpha, c_0)$ is the constant in (1.21). Then, since $w$ satisfies

$$
\left( w e^{(\alpha + i\beta)s^2/4} \right)' = e^{(-\alpha + i\beta)s^2/4} \left( \gamma - \frac{c_0^2}{2}(z - z_\infty) \right), \tag{3.93}
$$

integrating the above identity between $s$ and infinity,

$$
w e^{(\alpha + i\beta)s^2/4} = ibe^{-i\tilde{a}} - \int_s^\infty e^{(-\alpha + i\beta)\sigma^2/4} \left( \gamma - \frac{c_0^2}{2}(z - z_\infty) \right) d\sigma.
$$

Now, integrating by parts and using (3.41) (recall that $1 \leq 2\alpha$), we see that

$$\int_s^\infty e^{(-\alpha+i\beta)\sigma^2/4}\, d\sigma = 2(\alpha+i\beta)\frac{e^{(-\alpha+i\beta)s^2/4}}{s} + O\left(\frac{e^{-\alpha s^2/4}}{s^3}\right), \qquad \forall\, s \geq s_0.$$

Next, notice that from (3.43) in Proposition 3.2, we also obtain

$$\int_s^\infty e^{(-\alpha+i\beta)\sigma^2/4}(z-z_\infty)\, d\sigma = O\left(\frac{e^{-\alpha s^2/2}}{s^2}\right), \quad \forall\, s \geq s_0.$$

The above argument shows that for all $s \geq s_0$

$$w(s) = ibe^{-\alpha s^2/4}e^{-i(\tilde{a}+\beta s^2/4)} - \frac{2(\alpha+i\beta)\gamma}{s}e^{-\alpha s^2/2} + O\left(\frac{e^{-\alpha s^2/2}}{s^2}\right). \tag{3.94}$$

The asymptotics for $y$ and $h$ in the statement of the lemma easily follow from (3.94) bearing in mind that $w = y + ih$ and recalling that the function $\phi$ behaves like (1.21) when $\alpha > 0$. □

In the following corollary we summarize the asymptotics for $z$, $y$ and $h$ obtained in this section. Precisely, as a consequence of Proposition 3.2-*(iii)*, Proposition 3.3 and Lemma 3.6, we have the following:

**Corollary 3.7.** *Let $\alpha \in [0,1)$. With the same notation as before, for all $s \geq s_0 = 4\sqrt{8+c_0^2}$,*

$$y(s) = be^{-\alpha s^2/4}\sin(\phi(s)) - \frac{2\alpha\gamma}{s}e^{-\alpha s^2/2} + O\left(\frac{e^{-\alpha s^2/2}}{s^2}\right), \tag{3.95}$$

$$h(s) = be^{-\alpha s^2/4}\cos(\phi(s)) - \frac{2\beta\gamma}{s}e^{-\alpha s^2/2} + O\left(\frac{e^{-\alpha s^2/2}}{s^2}\right), \tag{3.96}$$

$$z(s) = z_\infty - \frac{4b}{s}e^{-\alpha s^2/4}(\alpha\sin(\phi(s)) + \beta\cos(\phi(s))) + \frac{4\gamma e^{-\alpha s^2/2}}{s^2} + O\left(\frac{e^{-\alpha s^2/4}}{s^3}\right), \tag{3.97}$$

*where*

$$\phi(s) = a + \beta\int_{s_0^2/4}^{s^2/4}\sqrt{1 + c_0^2\frac{e^{-2\alpha t}}{t}}\, dt,$$

*for some constant $a \in [0, 2\pi)$,*

$$b = z_\infty^{1/2}\left(2E_0 - \frac{c_0^2}{4}z_\infty\right)^{1/2}, \qquad \gamma = 2E_0 - \frac{c_0^2}{2}z_\infty \qquad \text{and} \qquad z_\infty = \lim_{s\to\infty} z(s).$$

*Here, the bounds controlling the error terms depend on $c_0$ and the energy $E_0$, and are independent of $\alpha \in [0,1)$.*

**Remark 3.8.** *In the case when $s < 0$, the same arguments to the ones leading to the asymptotics in the above corollary will lead to an analogous asymptotic behaviour for the variables $z$, $h$ and $y$ for $s < 0$. As mentioned at the beginning of Subsection 3.2, here we have reduced ourselves to the case of $s > 0$ when establishing the asymptotic behaviour of the latter quantities due to the parity of the solution we will be applying these results to.*

**Remark 3.9.** *The asymptotics in Corollary 3.7 lead to the asymptotics for the solutions $f$ of the equation* (3.20)*, at least if $|f|_\infty := z_\infty^{1/2}$ is strictly positive. Indeed, this implies that there exists $s^* \geq s_0$ such that $f(s) \neq 0$ for all $s \geq s^*$. Then writing $f$ in its polar form $f = \rho \exp(i\theta)$, we have $\rho^2\theta' = \operatorname{Im}(\bar{f}f')$. Hence, using* (3.22)*, we obtain $\rho = z^{1/2}$ and $\theta' = h/z$. Therefore, for all $s \geq s^*$,*

$$\theta(s) - \theta(s^*) = \int_{s^*}^{s} \frac{h(\sigma)}{z(\sigma)}\, d\sigma. \tag{3.98}$$

*Hence, using the asymptotics for $z$ and $h$ in Corollary 3.7, we can obtain the asymptotics for $f$. In the case that $\alpha \in (0,1]$, we can also show that the phase converges. Indeed, the asymptotics in Corollary 3.7 yield that the integral in* (3.98) *converges as $s \to \infty$ for $\alpha > 0$, and we conclude that there exists a constant $\theta_\infty \in \mathbb{R}$ such that*

$$f(s) = z(s)^{1/2} \exp\left( i\theta_\infty - i\int_s^\infty \frac{h(\sigma)}{z(\sigma)}\, d\sigma \right), \quad \text{for all } \ s \geq s^*.$$

*The asymptotics for $f$ is obtained by plugging the asymptotics in Corollary 3.7 into the above expression.*

## 3.3 The second-order equation. Dependence on the parameters

The aim of this subsection is to study the dependence of the $f$, $z$, $y$ and $h$ on the parameters $c_0 > 0$ and $\alpha \in [0,1]$. This will allow us to pass to the limit $\alpha \to 1^-$ in the asymptotics in Corollary 3.7 and will give us the elements for the proofs of Theorems 1.3 and 1.4.

### 3.3.1 Dependence on $\alpha$

We will denote by $f(s,\alpha)$ the solution of (3.20) with some initial conditions $f(0,\alpha)$, $f'(0,\alpha)$ that are independent of $\alpha$. Indeed, we are interested in initial conditions that depend only on $c_0$ (see (3.13)–(3.15)). Moreover, in view of (3.17), we assume that the energy $E_0$ in (3.16) is a function of $c_0$. In order to simplify the notation, we denote with a subindex $\alpha$ the derivative with respect to $\alpha$ and by $'$ the derivative with respect to $s$. Analogously to Subsection 3.2, we define

$$z(s,\alpha) = |f(s,\alpha)|^2, \quad y(s,\alpha) = \operatorname{Re}(\bar{f}(s,\alpha)f'(s,\alpha)), \quad h(s,\alpha) = \operatorname{Im}(\bar{f}(s,\alpha)f'(s,\alpha)) \tag{3.99}$$

and

$$z_\infty(\alpha) = \lim_{s\to\infty} |f(s,\alpha)|^2.$$

Observe that in Proposition 3.2-(*ii*), we proved the existence of $z_\infty(\alpha)$, for $\alpha \in [0,1)$. For $\alpha \in (0,1]$, the estimates in (3.24) hold true and hence $z(s,\alpha)$ is a bounded function whose derivative decays exponentially. Therefore, it admits a limit at infinity for all $\alpha \in [0,1]$ and $z_\infty(1)$ is well-defined.

The next lemma provides estimates for $z_\alpha$, $h_\alpha$ and $y_\alpha$.

**Lemma 3.10.** *Let $\alpha \in (0,1)$. There exists a constant $C(c_0)$, depending on $c_0$ but not on $\alpha$, such that for all $s \geq 0$,*

$$|z_\alpha(s,\alpha)| \leq C(c_0) \min\left\{ \frac{s^2}{\sqrt{1-\alpha}} + s^3, \frac{s^2}{\sqrt{\alpha(1-\alpha)}}, \frac{1}{\alpha^2\sqrt{1-\alpha}} \right\}, \tag{3.100}$$

$$|y_\alpha(s,\alpha)| + |h_\alpha(s,\alpha)| \leq C(c_0)e^{-\alpha s^2/4} \min\left\{ \frac{s^2}{\sqrt{1-\alpha}} + s^3, \frac{s^2}{\sqrt{\alpha(1-\alpha)}} \right\}. \tag{3.101}$$

*Proof.* Differentiating (3.12) with respect to $\alpha$,

$$f_\alpha'' + \frac{s}{2}(\alpha + i\beta) f_\alpha' + \frac{c_0^2}{4} f_\alpha e^{-\alpha s^2/2} = g, \tag{3.102}$$

where

$$g(s,\alpha) = -\left(1 - i\frac{\alpha}{\beta}\right)\frac{s}{2} f' + \frac{c_0^2 s^2}{8} f e^{-\alpha s^2/2}.$$

Also, since the initial conditions do not depend on $\alpha$,

$$f_\alpha(0,\alpha) = f_\alpha'(0,\alpha) = 0. \tag{3.103}$$

Using the estimates in (3.23) and that $\alpha^2 + \beta^2 = 1$, we obtain

$$|g| \leq C(c_0)\left(\frac{s}{\beta} e^{-\alpha s^2/4} + s^2 e^{-\alpha s^2/2}\right), \quad \text{for all } s \geq 0. \tag{3.104}$$

Multiplying (3.102) by $\bar{f}_\alpha'$ and taking real part, we have

$$\frac{1}{2}\left(|f_\alpha'|^2\right)' + \frac{\alpha s}{2}|f_\alpha'|^2 + \frac{c_0^2}{8}\left(|f_\alpha|^2\right)' e^{-\alpha s^2/2} = \mathrm{Re}(g \bar{f}_\alpha'). \tag{3.105}$$

Multiplying (3.105) by $2e^{\alpha s^2/2}$ and integrating, taking into account (3.103),

$$|f_\alpha'|^2 e^{\alpha s^2/2} + \frac{c_0^2}{4}|f_\alpha|^2 = 2\int_0^s e^{\alpha\sigma^2/2}\,\mathrm{Re}(g\bar{f}_\alpha')\, d\sigma. \tag{3.106}$$

Let us define the real-valued function $\eta = |f_\alpha'| e^{\alpha s^2/4}$. Then (3.106) yields

$$\eta^2(s) \leq 2\int_0^s e^{\alpha\sigma^2/4}|g|\eta\, d\sigma, \quad \text{for all } s \geq 0.$$

Thus, by the Gronwall inequality (see e.g. [3, Lemma A.5]),

$$\eta(s) \leq \int_0^s e^{\alpha\sigma^2/4}|g|, d\sigma, \quad \text{for all } s \geq 0. \tag{3.107}$$

From (3.104), (3.106) and (3.107), we conclude that

$$\begin{aligned}
(|f_\alpha'| e^{\alpha s^2/4} + \frac{c_0}{2}|f_\alpha|)^2 &\leq 2(|f_\alpha|^2 e^{\alpha s^2/2} + \frac{c_0^2}{4}|f_\alpha|^2) \\
&\leq 4\int_0^s e^{\alpha\sigma^2/4}|g|\eta\, d\sigma \leq 4\left(\sup_{\sigma\in[0,s]}\eta(\sigma)\right)\left(\int_0^s e^{\alpha\sigma^2/4}|g|\, d\sigma\right) \\
&\leq \left(\int_0^s e^{\alpha\sigma^2/4}|g|\, d\sigma\right)^2.
\end{aligned}$$

Thus, using (3.104), from the above inequality it follows

$$|f_\alpha'| e^{\alpha s^2/4} + \frac{c_0}{2}|f_\alpha| \leq C(c_0)\int_0^s \left(\frac{\sigma}{\beta} + \sigma^2 e^{-\alpha\sigma^2/4}\right) d\sigma, \quad \text{for all } s \geq 0. \tag{3.108}$$

In particular, for all $s \geq 0$,

$$\begin{aligned}
|f_\alpha(s)| &\leq C(c_0)\min\left\{\frac{s^2}{\sqrt{1-\alpha}} + s^3, \frac{s^2}{\sqrt{\alpha(1-\alpha)}}\right\}, \\
|f_\alpha'(s)| &\leq C(c_0) e^{-\alpha s^2/4}\min\left\{\frac{s^2}{\sqrt{1-\alpha}} + s^3, \frac{s^2}{\sqrt{\alpha(1-\alpha)}}\right\},
\end{aligned} \tag{3.109}$$

where we have used that

$$\int_0^s \sigma^2 e^{-\alpha\sigma^2/4}\, d\sigma \le s^2 \int_0^s e^{-\alpha\sigma^2/4}\, d\sigma \le s^2\sqrt{\pi/\alpha}.$$

Notice that from (3.103) and (3.109),

$$|f_\alpha(s)| \le \int_0^s |f'_\alpha|\, d\sigma \le \frac{C(c_0)}{\sqrt{\alpha(1-\alpha)}} \int_0^s \sigma^2 e^{-\alpha\sigma^2/4}\, d\sigma,$$

and

$$\int_0^\infty \sigma^2 e^{-\alpha\sigma^2/4}\, d\sigma = \frac{2\sqrt{\pi}}{\alpha^{3/2}}, \tag{3.110}$$

so that

$$|f_\alpha(s)| \le \frac{C(c_0)}{\alpha^2\sqrt{1-\alpha}}. \tag{3.111}$$

On the other hand, differentiating the relations in (3.99) with respect to $\alpha$,

$$|z_\alpha| \le 2|f_\alpha||f|, \quad |y_\alpha + ih_\alpha| \le |f_\alpha||f'| + |f||f'_\alpha|. \tag{3.112}$$

By putting together (3.23), (3.109), (3.111) and (3.112), we obtain (3.100) and (3.101). □

**Lemma 3.11.** *The function $z_\infty$ is continuous in $(0,1]$. More precisely, there exists a constant $C(c_0)$ depending on $c_0$ but not on $\alpha$, such that*

$$|z_\infty(\alpha_2) - z_\infty(\alpha_1)| \le \frac{C(c_0)}{L(\alpha_2,\alpha_1)}|\alpha_2 - \alpha_1|, \quad \text{for all } \alpha_1, \alpha_2 \in (0,1], \tag{3.113}$$

*where*

$$L(\alpha_2,\alpha_1) := \alpha_1^2\alpha_2^{3/2}\left(\alpha_1^{3/2}\sqrt{1-\alpha_2} + \alpha_2^{3/2}\sqrt{1-\alpha_1}\right).$$

*In particular,*

$$|z_\infty(1) - z_\infty(\alpha)| \le C(c_0)\sqrt{1-\alpha}, \quad \text{for all } \alpha \in [1/2, 1]. \tag{3.114}$$

*Proof.* Let $\alpha_1, \alpha_2 \in (0,1]$, $\alpha_1 < \alpha_2$. By classical results from the ODE theory, the functions $y(s,\alpha)$, $h(s,\alpha)$ and $z(s,\alpha)$ are smooth in $\mathbb{R}\times[0,1)$ and continuous in $\mathbb{R}\times[0,1]$ (see e.g. [5, 17]). Hence, integrating (3.27) with respect to $s$, we deduce that

$$z_\infty(\alpha_2) - z_\infty(\alpha_1) = 2\int_0^\infty (y(s,\alpha_2) - y(s,\alpha_1))\, ds = 2\int_0^\infty \int_{\alpha_1}^{\alpha_2} \frac{dy}{d\mu}(s,\mu)\, d\mu\, ds. \tag{3.115}$$

To estimate the last integral, we use (3.101)

$$\int_{\alpha_1}^{\alpha_2} |\frac{dy}{d\mu}(s,\mu)|\, d\mu \le C(c_0)\frac{s^2}{\sqrt{\alpha_1}} \int_{\alpha_1}^{\alpha_2} \frac{e^{-\mu s^2/4}}{\sqrt{1-\mu}}\, d\mu. \tag{3.116}$$

Now, integrating by parts,

$$\int_{\alpha_1}^{\alpha_2} \frac{e^{-\mu s^2/4}}{\sqrt{1-\mu}}\, d\mu = 2\left(\sqrt{1-\alpha_1}e^{-\alpha_1 s^2/4} - \sqrt{1-\alpha_2}e^{-\alpha_2 s^2/4}\right) - \frac{s^2}{2}\int_{\alpha_1}^{\alpha_2} \sqrt{1-\mu}e^{-\mu s^2/4}\, d\mu.$$

Therefore, by combining with (3.115) and (3.116),

$$|z_\infty(\alpha_2) - z_\infty(\alpha_1)| \le \frac{C(c_0)}{\sqrt{\alpha_1}}\left(\sqrt{1-\alpha_1}\int_0^\infty s^2 e^{-\alpha_1 s^2/4}\, ds - \sqrt{1-\alpha_2}\int_0^\infty s^2 e^{-\alpha_2 s^2/4}\, ds\right),$$

and bearing in mind (3.110), we conclude that

$$|z_\infty(\alpha_2) - z_\infty(\alpha_1)| \leq \frac{C(c_0)}{\sqrt{\alpha_1}} \left( \frac{\sqrt{1-\alpha_1}}{\alpha_1^{3/2}} - \frac{\sqrt{1-\alpha_2}}{\alpha_2^{3/2}} \right),$$

which, after some algebraic manipulations and using that $\alpha_1$, $\alpha_2 \in (0,1]$, leads to (3.113). □

The estimate for $z_\infty$ near zero is more involved and it is based in an improvement of the estimate for the derivative of $z_\infty$.

**Lemma 3.12.** *The function $z_\infty$ is continuous in $[0,1]$. Moreover, there exists a constant $C(c_0) > 0$, depending on $c_0$ but not on $\alpha$ such that for all $\alpha \in (0,1/2]$,*

$$|z_\infty(\alpha) - z_\infty(0)| \leq C(c_0)\sqrt{\alpha}|\ln(\alpha)|. \tag{3.117}$$

*Proof.* As in the proof of Lemma 3.11, we recall that the functions $y(s,\alpha)$, $h(s,\alpha)$ and $z(s,\alpha)$ are smooth in any compact subset of $\mathbb{R} \times [0,1)$. From now on we will use the identity (3.39) fixing $s = 1$. We can verify that the two integral terms in (3.39) are continuous functions at $\alpha = 0$, which proves that $z_\infty$ is continuous in 0. In view of Lemma 3.11, we conclude that $z_\infty$ is continuous in $[0,1]$.

Now we claim that

$$\left|\frac{dz_\infty}{d\alpha}(\alpha)\right| \leq C(c_0)\frac{|\ln(\alpha)|}{\sqrt{\alpha}}, \quad \text{for all } \alpha \in (0,1/2]. \tag{3.118}$$

In fact, once (3.118) is proved, we can compute

$$|z_\infty(\alpha) - z_\infty(0)| = \left|\int_0^\alpha \frac{dz_\infty}{d\mu}(\mu)d\mu\right| \leq C(c_0)\int_0^\alpha \frac{|\ln(\mu)|}{\sqrt{\mu}}\,d\mu = 2C(c_0)\sqrt{\alpha}(|\ln(\alpha)| + 2),$$

which implies (3.117).

It remains to prove the claim. Differentiating (3.39) (recall that $s = 1$) with respect to $\alpha$, and using that $y(1,\cdot)$, $h(1,\cdot)$ and $z(1,\cdot)$ are continuous differentiable in $[0,1/2]$, we deduce that there exists a constant $C(c_0) > 0$ such that

$$\left|\frac{dz_\infty}{d\alpha}(\alpha)\right| \leq C(c_0) + 8|I_1(\alpha)| + 2c_0^2|I_2(\alpha)|, \tag{3.119}$$

with

$$I_1(\alpha) = \int_1^\infty \frac{z}{\sigma^3} + \alpha\int_1^\infty \frac{z_\alpha}{\sigma^3} + 6\int_1^\infty \frac{z_\alpha}{\sigma^5} \tag{3.120}$$

and

$$I_2(\alpha) = -\frac{\alpha}{2}\int_1^\infty e^{-\alpha\sigma^2/2}z\sigma + \alpha\int_1^\infty e^{-\alpha\sigma^2/2}\frac{z_\alpha}{\sigma} + 2\int_1^\infty e^{-\alpha\sigma^2/2}\frac{z_\alpha}{\sigma^3}. \tag{3.121}$$

By (3.24) and (3.100), $z$ is uniformly bounded and $z_\alpha$ grows at most as a cubic polynomial, so that the first and the last integral in the r.h.s. of (3.120) are bounded independently of $\alpha \in [0,1/2]$. In addition, (3.100) also implies that

$$|z_\alpha| = |z_\alpha|^{1/2}|z_\alpha|^{1/2} \leq C(c_0)(s^3)^{1/2}\left(\frac{1}{\alpha^2}\right)^{1/2} = C(c_0)\frac{s^{3/2}}{\alpha}, \tag{3.122}$$

which shows that the remaining integral in (3.120) is bounded.

Thus, the above argument shows that

$$|I_1(\alpha)| \le C(c_0) \qquad \text{for all} \quad \alpha \in [0,1/2]. \tag{3.123}$$

The same arguments also yield that the first two integrals in the r.h.s. of (3.121) are bounded by $C(c_0)\alpha^{-1/2}$. Using once more that $|z_\alpha| \le C(c_0)s^2\alpha^{-1/2}$, we obtain the following bounds for the remaining two integrals in (3.121)

$$\left|\alpha \int_1^s e^{-\alpha\sigma^2/2}\frac{z_\alpha}{\sigma}\right| \le \frac{C(c_0)}{\sqrt{\alpha}}\int_1^\infty \alpha\sigma e^{-\alpha\sigma^2/2}\,d\sigma = \frac{C(c_0)}{\sqrt{\alpha}}e^{-\alpha/2} \le \frac{C(c_0)}{\sqrt{\alpha}}$$

and

$$\left|2\int_1^\infty e^{-\alpha\sigma/2}\frac{z_\alpha}{\sigma^3}\,d\sigma\right| \le \frac{C(c_0)}{\sqrt{\alpha}}\int_1^\infty \frac{e^{-\alpha\sigma^2/2}}{\sigma}\,d\sigma \le C(c_0)\frac{|\ln(\alpha)|}{\sqrt{\alpha}}.$$

In conclusion, we have proved that

$$|I_2(\alpha)| \le C(c_0)\frac{|\ln(\alpha)|}{\sqrt{\alpha}},$$

which combined with (3.119) and (3.123), completes the proof of claim. □

We end this section showing that the previous continuity results allow us to "pass to the limit" $\alpha \to 1^-$ in Corollary 3.7. Using the notation $b(\alpha) = b$ and $a(\alpha) = a$ for the constants defined for $\alpha \in [0,1)$ in Proposition 3.3 in Subsection 3.2, we have

**Lemma 3.13.** *The value $b(\alpha)$ is a continuous function of $\alpha \in [0,1]$ and the value $b(\alpha)e^{ia(\alpha)}$ is continuous function of $\alpha \in [0,1)$ that can be continuously extended to $[0,1]$. The function $a(\alpha)$ has a (possible discontinuous) extension for $\alpha \in [0,1]$ such that $a(\alpha) \in [0,2\pi)$.*

*Proof.* By Lemma 3.12, we have the continuity of $z_\infty$ in [0,1]. Therefore, in view of Remark 3.4, the function $be^{ia}$ is a continuous function of $\alpha \in [0,1)$ and by (3.49) $b$ is actually well-defined and continuous in $\alpha \in [0,1]$.

It only remains to prove that the limit

$$L := \lim_{\alpha \to 1^-} b(\alpha)e^{ia(\alpha)} \tag{3.124}$$

exists. If $b(1) = 0$, it is immediate that $L = 0$ and we can give any arbitrary value in $[0,2\pi)$ to $a(1)$. Let us suppose that $b(1) > 0$. Integrating (3.93), we get

$$w(s)e^{(\alpha+i\beta)s^2/4} = w(s_0)e^{(\alpha+i\beta)s_0^2/4} + \int_{s_0}^s e^{(-\alpha+i\beta)\sigma^2/4}\left(\gamma - \frac{c_0^2}{2}(z - z_\infty)\right)d\sigma,$$

and this relation is valid for any $\alpha \in (0,1]$. Let $\alpha \in (0,1)$. In view of (3.92), letting $s \to \infty$, we have

$$ibe^{i(a+C(\alpha,c_0))} = w(s_0)e^{(\alpha+i\beta)s_0^2/4} + \int_{s_0}^\infty e^{(-\alpha+i\beta)\sigma^2/4}\left(\gamma - \frac{c_0^2}{2}(z - z_\infty)\right)d\sigma, \tag{3.125}$$

where $C(\alpha, c_0)$ is the constant in (1.21). Notice that the r.h.s. of (3.125) is well-defined for any $\alpha \in (0,1]$ and by the arguments given in the proof of Lemma 3.11 and the dominated convergence theorem, the r.h.s. is also continuous for any $\alpha \in (0,1]$. Therefore, the limit $L$ in (3.124) exists and is given by the r.h.s. of (3.125) evaluated in $\alpha = 1$ and divided by $ie^{iC(1,c_0)}$. Moreover,

$$\lim_{\alpha\to 1^-} e^{ia(\alpha)} = \frac{L}{b(1)},$$

so that by the compactness of the the unit circle in $\mathbb{C}$, there exists $\theta \in [0,2\pi)$ such that $e^{i\theta} = L/b(1)$ and we can extend $a$ by defining $a(1) = \theta$. □

The following result summarizes an improvement of Corollary 3.7 to include the case $\alpha = 1$ and the continuous dependence of the constants appearing in the asymptotics on $\alpha$. Precisely, we have the following:

**Corollary 3.14.** *Let $\alpha \in [0,1]$, $\beta \geq 0$ with $\alpha^2 + \beta^2 = 1$ and $c_0 > 0$. Then,*

*(i) The asymptotics in Corollary 3.7 holds true for all $\alpha \in [0,1]$.*

*(ii) Moreover, the values $b$ and $be^{ia}$ are continuous functions of $\alpha \in [0,1]$ and each term in the asymptotics for $z$, $y$ and $h$ in Corollary 3.7 depends continuously on $\alpha \in [0,1]$.*

*(iii) In addition, the bounds controlling the error terms depend on $c_0$ and are independent of $\alpha \in [0,1]$.*

*Proof.* Let $s \geq s_0$ fixed. As noticed in the proof of Lemma 3.11, the functions $y(s,\alpha)$, $h(s,\alpha)$, $z(s,\alpha)$ are continuous in $\alpha = 1$. In addition, by Lemma 3.13 $be^{ia}$ is continuous in $\alpha = 1$, using the definition of $\phi$, it is immediate that $b\sin(\phi(s))$ and $b\cos(\phi(s))$ are continuous in $\alpha = 1$. Therefore the big-$O$ terms in (3.95), (3.96) and (3.97) are also are continuous in $\alpha = 1$. The proof of the corollary follows by letting $\alpha \to 1^-$ in (3.95), (3.96) and (3.97). □

### 3.3.2 Dependence on $c_0$

In this subsection, we study the dependence of $z_\infty$ as a function of $c_0$, for a fixed value of $\alpha$. To this aim, we need to take into account the initial conditions given in (3.13)–(3.15). More generally, let us assume that $f$ is a solution of (3.20) with initial conditions $f(0)$ and $f'(0)$ that depend smoothly on $c_0$, for any $c_0 > 0$, and that $E_0 > 0$ is the associated energy defined in (3.16). To keep our notation simple, we omit the parameter $c_0$ in the functions $f$ and $z_\infty$. Under these assumptions, we have

**Proposition 3.15.** *Let $\alpha \in [0,1]$ and $c_0 > 0$. Then $z_\infty$ is a continuous function of $c_0 \in (0,\infty)$. Moreover if $\alpha \in (0,1]$, the following estimate hold*

$$\left| z_\infty - \left| f(0) + \frac{f'(0)\sqrt{\pi}}{\sqrt{\alpha + i\beta}} \right|^2 \right| \leq \frac{\sqrt{2E_0}c_0\pi}{\alpha} \left| f(0) + \frac{f'(0)\sqrt{\pi}}{\sqrt{\alpha + i\beta}} \right| + \left( \frac{\sqrt{2E_0}c_0\pi}{2\alpha} \right)^2. \tag{3.126}$$

*Proof.* Since we are assuming that the initial conditions $f(0)$ and $f'(0)$ depend smoothly on $c_0$, by classical results from the ODE theory, the functions $f$, $y$, $h$ and $z$ are smooth with respect to $s$ and $c_0$. From (3.39) with $s = 1$, we have that $z_\infty$ can be written in terms of continuous functions of $c_0$ (the continuity of the integral terms follows from the dominated convergence theorem), so that $z_\infty$ depends continuously on $c_0$.

To prove (3.126), we multiply (3.20) by $e^{(\alpha+i\beta)s^2/4}$, so that

$$(f' e^{(\alpha+i\beta)s^2/4})' = -\frac{c_0^2}{4} f(s) e^{(-\alpha+i\beta)s^2/4}.$$

Hence, integrating twice, we have

$$f(s) = f(0) + G(s) + F(s), \tag{3.127}$$

with

$$G(s) = f'(0) \int_0^s e^{-(\alpha+i\beta)\sigma^2/4}\, d\sigma \quad \text{and} \quad F(s) = -\frac{c_0^2}{4} \int_0^s e^{-(\alpha+i\beta)\sigma^2/4} \int_0^\sigma e^{(-\alpha+i\beta)\tau^2/4} f(\tau)\, d\tau\, d\sigma.$$

Since by Proposition 3.2 $|f(s)| \leq \frac{2\sqrt{2E_0}}{c_0}$, we obtain

$$|F(s)| \leq \frac{\sqrt{2E_0}c_0}{2} \int_0^s e^{-\alpha\sigma^2/4} \int_0^\sigma e^{-\alpha\tau^2/4}\, d\tau\, d\sigma \leq \frac{\sqrt{2E_0}c_0}{2} \cdot \frac{\pi}{\alpha}. \tag{3.128}$$

Using (3.127) and the identity,

$$|z_1 + z_2|^2 = |z_1|^2 + 2\,\mathrm{Re}(\bar{z}_1 z_2) + |z_2|^2, \quad z_1, z_2 \in \mathbb{C},$$

we conclude that $z(s) = |f(s)|^2$ satisfies

$$z(s) = |f(0) + G(s)|^2 + 2\,\mathrm{Re}(\bar{F}(s)(f(0) + G(s))) + |F(s)|^2.$$

Therefore, for all $s \geq 0$,

$$|z(s) - |f(0) + G(s)|^2| \leq 2|F(s)||f(0) + G(s)| + |F(s)|^2.$$

Hence we can use the bound (3.128) and then let $s \to \infty$. Noticing that

$$\lim_{s\to\infty} G(s) = f'(0) \int_0^\infty e^{-(\alpha+i\beta)\sigma^2/4}\, d\sigma = f'(0) \frac{\sqrt{\pi}}{\sqrt{\alpha + i\beta}},$$

the estimate (3.126) follows. □

# 4 Proof of the main results

In Section 3 we have performed a careful analysis of the equation (3.12), taking also into consideration the initial conditions (3.13)–(3.15). Therefore, the proofs of our main theorem consist mainly in coming back to the original variables using the identities (3.18) and (3.19). For the sake of completeness, we provide the details in the following proofs.

***Proof of Theorem 1.2.*** Let $\alpha \in [0, 1]$, $c_0 > 0$ and $\{\vec{m}_{c_0,\alpha}(\cdot), \vec{n}_{c_0,\alpha}(\cdot), \vec{b}_{c_0,\alpha}(\cdot)\}$ be the unique $\mathcal{C}^\infty(\mathbb{R}; \mathbb{S}^2)$-solution of the Serret–Frenet equations (1.6) with curvature and torsion (2.6) and initial conditions (2.8). In order to simplify the notation, in the rest of the proof we drop the subindexes $c_0$ and $\alpha$ and simply write $\{\vec{m}(\cdot), \vec{n}(\cdot), \vec{b}(\cdot)\}$ for $\{\vec{m}_{c_0,\alpha}(\cdot), \vec{n}_{c_0,\alpha}(\cdot), \vec{b}_{c_0,\alpha}(\cdot)\}$.

First observe that if we define $\{\vec{M}, \vec{N}, \vec{B}\}$ in terms of $\{\vec{m}, \vec{n}, \vec{b}\}$ by

$$\begin{aligned}
\vec{M}(s) &= (m(-s), -m(-s), -m(-s)), \\
\vec{N}(s) &= (-n(-s), n(-s), n(-s)), \\
\vec{B}(s) &= (-b(-s), b(-s), b(-s)), \qquad s \in \mathbb{R},
\end{aligned}$$

then $\{\vec{M}, \vec{N}, \vec{B}\}$ is also a solution of the Serret system (1.6) with curvature and torsion (2.6). Notice also that

$$\{\vec{M}(0), \vec{N}(0), \vec{B}(0)\} = \{\vec{m}(0), \vec{n}(0), \vec{b}(0)\}.$$

Therefore, from the uniqueness of the solution we conclude that

$$\vec{M}(s) = \vec{m}(s), \qquad \vec{N}(s) = \vec{n}(s) \qquad \text{and} \qquad \vec{B}(s) = \vec{b}(s), \qquad \forall\, s \in \mathbb{R}.$$

This proves part *(i)* of Theorem 1.2.

Second, in Section 3 we have seen that one can write the components of the Frenet trihedron $\{\vec{m}, \vec{n}, \vec{b}\}$ as

$$m_1(s) = 2|f_1(s)|^2 - 1, \qquad n_1(s) + ib_1(s) = \frac{4}{c_0} e^{\alpha s^2/4} \bar{f}_1(s) f_1'(s), \tag{4.1}$$

$$m_j(s) = |f_j(s)|^2 - 1, \qquad n_j(s) + ib_j(s) = \frac{2}{c_0} e^{\alpha s^2/4} \bar{f}_j(s) f_j'(s), \quad j \in \{2,3\}, \tag{4.2}$$

with $f_j$ solution of the second order ODE (3.12) with initial conditions (3.13)-(3.15) respectively, and associated initial energies (see (3.17))

$$E_{0,1} = \frac{c_0^2}{8} \qquad \text{and} \qquad E_{j,1} = \frac{c_0^2}{8}, \qquad \text{for} \qquad j \in \{2,3\}. \tag{4.3}$$

Notice that the identities (4.1)–(4.2) rewrite equivalently as

$$\begin{cases} m_{1,c_0,\alpha} = 2z_1 - 1, \quad n_{1,c_0,\alpha} = \dfrac{4}{c_0} e^{\alpha s^2/4}\, y_1, \quad b_{1,c_0,\alpha} = \dfrac{4}{c_0} e^{\alpha s^2/4}\, h_1, \\ m_{j,c_0,\alpha} = z_j - 1, \quad n_{j,c_0,\alpha} = \dfrac{2}{c_0} e^{\alpha s^2/4}\, y_j, \quad b_{j,c_0,\alpha} = \dfrac{2}{c_0} e^{\alpha s^2/4}\, h_j, \quad j \in \{2,3\}, \end{cases} \tag{4.4}$$

in terms of the quantities $\{z_j, y_j, h_j\}$ defined by

$$z_j = |f_j|^2, \quad y_j = \operatorname{Re}(\bar{f}_j f_j') \quad \text{and} \quad h_j = \operatorname{Im}(\bar{f}_j f_j').$$

Denote by $z_{j,\infty}$, $a_j$, $b_j$, $\gamma_j$ and $\phi_j$ the constants and function appearing in the asymptotics of $\{y_j, h_j, z_j\}$ proved in Section 3 in Corollary 3.14.

Taking the limit as $s \to +\infty$ in (4.1)–(4.2), and since $|\vec{m}(s)| = 1$, we obtain that there exists $\vec{A}^+ = (A_j^+)_{j=1}^3 \in \mathbb{S}^2$ with

$$A_1^+ = 2z_{1,\infty} - 1, \qquad A_j^+ = z_{j,\infty} - 1, \qquad \text{for } j \in \{2,3\}. \tag{4.5}$$

The asymptotics stated in part *(ii)* of Theorem 1.2 easily follows from formulae (4.1)–(4.2) and the asymptotics for $\{z_j, y_j, h_j\}$ established in Corollary 3.14. Indeed, it suffices to observe that from the formulae for $b_j$ and $\gamma_j$ in terms of the initial energies $E_{0,j}$ and $z_{j,\infty}$ given in Corollary 3.14, (4.3) and (4.5) we obtain

$$b_1^2 = \frac{c_0^2}{16}(1 - (A_1^+)^2), \quad b_2^2 = \frac{c_0^2}{4}(1 - (A_2^+)^2), \quad b_3^2 = \frac{c_0^2}{4}(1 - (A_3^+)^2), \tag{4.6}$$

$$\gamma_1 = -\frac{c_0^2}{4} A_1^+, \quad \gamma_2 = -\frac{c_0^2}{2} A_2^+, \quad \gamma_3 = -\frac{c_0^2}{2} A_3^+. \tag{4.7}$$

Substituting these constants in (3.95), (3.96) and (3.97) in Corollary 3.14, we obtain (1.16), (1.17) and (1.18). This completes the proof of Theorem 1.2-*(ii)*. □

***Proof of Theorem 1.1***. Let $\alpha \in [0,1]$, and $c_0 > 0$. As before, dropping the subindexes, we will denote by $\{\vec{m}, \vec{n}, \vec{b}\}$ the unique solution of the Serret–Frenet equations (1.6) with curvature and torsion (2.6) and initial conditions (2.8). Define

$$\vec{\mathrm{m}}(s,t) = \vec{m}\left(\frac{s}{\sqrt{t}}\right). \tag{4.8}$$

As has been already mentioned (see Section 2), part *(i)* of Theorem 1.1 follows from the fact that the triplet $\{\vec{m},\vec{n},\vec{b}\}$ is a regular-$(\mathcal{C}^\infty(\mathbb{R};\mathbb{S}^2))^3$ solution of (1.6)-(2.6)-(2.8) and satisfies the equation

$$-\frac{s}{2}c\vec{n} = \beta(c'\vec{b} - c\tau\vec{n}) + \alpha(c\tau\vec{b} + c'\vec{n}).$$

Next, from the parity of the components of the profile $\vec{m}(\cdot)$ and the asymptotics established in parts *(i)* and *(ii)* in Theorem 1.2, it is immediate to prove the pointwise convergence (1.9). In addition, $\vec{A}^- = (A_1^+, -A_2^+, -A_3^+)$ in terms of the components of the vector $\vec{A}^+ = (A_j^+)_{j=1}^3$.

Now, using the symmetries of $\vec{m}(\cdot)$, the change of variables $\eta = s/\sqrt{t}$ gives us

$$\|\vec{\mathrm{m}}(\cdot,t) - \vec{A}^+\chi_{(0,\infty)}(\cdot) - \vec{A}^-\chi_{(-\infty,0)}(\cdot)\|_{L^p(\mathbb{R})} = \sum_{j=1}^{3}\left(2t^{1/2}\int_0^\infty |m_j(\eta) - A_j^+|^p\, d\eta\right)^{1/p}. \tag{4.9}$$

Therefore, it only remains to prove that the last integral is finite. To this end, let $s_0 = 4\sqrt{8 + c_0^2}$. On the one hand, notice that since $\vec{m}$ and $\vec{A}^+$ are unitary vectors,

$$\int_0^{s_0} |m_j(s) - A_j|^p\, ds \le 2^p s_0. \tag{4.10}$$

On the other hand, from the asymptotics for $\vec{m}(\cdot)$ in (1.16), (1.20), and the fact that the vectors $\vec{A}^+$ and $\vec{B}^+$ satisfy $|\vec{A}^+|^2 = 1$ and $|\vec{B}^+|^2 = 2$, we obtain

$$\begin{aligned}\left(\int_{s_0}^\infty |m_j(s) - A_j^+|^p\, ds\right)^{1/p} \le& 2\sqrt{2}c_0(\alpha+\beta)\left(\int_{s_0}^\infty \frac{e^{-\alpha s^2 p/4}}{s^p}\right)^{1/p} + 2c_0^2\left(\int_{s_0}^\infty \frac{e^{-\alpha s^2 p/2}}{s^{2p}}\right)^{1/p}\\ &+ C(c_0)\left(\int_{s_0}^\infty \frac{e^{-\alpha s^2 p/4}}{s^{3p}}\right)^{1/p}.\end{aligned} \tag{4.11}$$

Since the r.h.s. of (4.11) is finite for all $p \in (1,\infty)$ if $\alpha \in [0,1]$, and for all $p \in [1,\infty)$ if $\alpha \in (0,1]$, inequality (1.10) follows from (4.9), (4.10) and (4.11). This completes the proof of Theorem 1.1. □

***Proof of Theorem 1.3***. The proof is a consequence of Proposition 3.15. In fact, recall the relations (4.5) and (3.17), that is

$$A_1^+ = 2z_{1,\infty} - 1, \qquad \text{and} \qquad A_j^+ = z_{j,\infty} - 1, \qquad \text{for } j \in \{2,3\},$$

and

$$E_{0,1} = \frac{c_0^2}{8}, \qquad E_{0,j} = \frac{c_0^2}{4}, \qquad \text{for } j \in \{2,3\},$$

Thus the continuity of $\vec{A}^+_{c_0,\alpha}$ with respect to $c_0$, follows from the continuity of $z_\infty$ in Proposition 3.15.

Using the initial conditions (3.13)–(3.15), the values for the energies $E_{0,j}$ for $j \in \{1,2,3\}$, and the identity

$$\frac{\sqrt{\pi}}{\sqrt{\alpha + i\beta}} = \frac{\sqrt{\pi}}{\sqrt{2}}\left(\sqrt{1+\alpha} - i\sqrt{1-\alpha}\right),$$

we now compute

$$\left| f_j(0) + \frac{f_j'(0)\sqrt{\pi}}{\sqrt{\alpha+i\beta}}\right|^2 = \begin{cases} 1, & \text{if } j = 1,\\ 1 + \frac{c_0^2\pi}{4} + \frac{c_0\sqrt{\pi}}{\sqrt{2}}\sqrt{1+\alpha}, & \text{if } j = 2,\\ 1 + \frac{c_0^2\pi}{4} + \frac{c_0\sqrt{\pi}}{\sqrt{2}}\sqrt{1-\alpha}, & \text{if } j = 3.\end{cases} \tag{4.12}$$

Then, substituting the values (4.12) in (3.126) and using the above relations together with the inequality $\sqrt{1+x} \leq 1 + x/2$ for $x \geq 0$, we obtain the estimates (1.24)–(1.26). $\square$

***Proof of Theorem 1.4.*** Recall that the components of $\vec{A}^+_{c_0,\alpha}$ are given explicitly in (4.5) in terms of the functions $z_{j,\infty}$, for $j \in \{1,2,3\}$. The continuity on $[0,1]$ of $A^+_{j,c_0,\alpha}$ as a function of $\alpha$ for $j \in \{1,2,3\}$ follows from that of $z_{j,\infty}$ established in Lemma 3.12. Notice also that the estimates (1.27) and (1.28) are an immediate consequence of (3.117) in Lemma 3.12 and (3.114) in Lemma 3.11, respectively. $\square$

Before giving the proof of Proposition 1.5, we recall that when $\alpha = 0$ or $\alpha = 1$, the vector $\vec{A}^+_{c_0,\alpha} = (A_{j,c_0,\alpha})^3_{j=1}$ is determined explicitly in terms of the parameter $c_0$ (see [15] for the case $\alpha = 0$ and Appendix for the case $\alpha = 1$). Precisely,

$$A_{1,c_0,0} = e^{-\frac{\pi c_0^2}{2}}, \tag{4.13}$$

$$A_{2,c_0,0} = 1 - \frac{e^{-\frac{\pi c_0^2}{4}}}{8\pi} \sinh(\pi c_0^2/2)|c_0\Gamma(ic_0^2/4) + 2e^{i\pi/4}\Gamma(1/2 + ic_0^2/4)|^2, \tag{4.14}$$

$$A_{3,c_0,0} = 1 - \frac{e^{-\frac{\pi c_0^2}{4}}}{8\pi} \sinh(\pi c_0^2/2)|c_0\Gamma(ic_0^2/4) - 2e^{-i\pi/4}\Gamma(1/2 + ic_0^2/4)|^2 \tag{4.15}$$

and

$$\vec{A}^+_{c_0,1} = (\cos(c_0\sqrt{\pi}), \sin(c_0\sqrt{\pi}), 0). \tag{4.16}$$

***Proof of Proposition 1.5.*** Recall that (see Theorem 1.1)

$$\vec{A}^-_{c_0,\alpha} = (A^+_{1,c_0,\alpha}, -A^+_{2,c_0,\alpha}, -A^+_{3,c_0,\alpha}), \tag{4.17}$$

with $A^+_{j,c_0,\alpha}$ the components of $\vec{A}^+_{c_0,\alpha}$. Therefore $\vec{A}^+_{c_0,\alpha} \neq \vec{A}^-_{c_0,\alpha}$ iff $A^+_{1,c_0,\alpha} \neq 1$ or $-1$.

Parts *(ii)* and *(iii)* follow from the continuity of $A^+_{1,c_0,\alpha}$ in $[0,1]$ established in Theorem 1.4 bearing in mind that, from the expressions for $A^+_{1,c_0,0}$ in (4.13) and $A^+_{1,c_0,1}$ in (4.16), we have that $A^+_{1,c_0,0} \neq \pm 1$ for all $c_0 > 0$ and $A^+_{1,c_0,1} \neq \pm 1$ if $c_0 \neq k\sqrt{\pi}$ with $k \in \mathbb{N}$.

In order to proof part *(i)*, we will argue by contradiction. Assume that for some $\alpha \in (0,1)$, there exists a sequence $\{c_{0,n}\}_{n\in\mathbb{N}}$ such that $c_{0,n} > 0$, $c_{0,n} \longrightarrow 0$ as $n \to \infty$ and $\vec{A}^+_{c_{0,n},\alpha} = \vec{A}^-_{c_{0,n}\alpha}$. Hence from (4.17) the second and third component of $\vec{A}^+_{c_{0,n},\alpha}$ are zero. Thus the estimate (1.25) in Theorem 1.3 yields

$$c_{0,n}\frac{\sqrt{\pi(1+\alpha)}}{\sqrt{2}} \leq \frac{c_{0,n}^2\pi}{4} + \frac{c_{0,n}^2\pi}{\alpha\sqrt{2}}\left(1 + \frac{c_{0,n}^2\pi}{8} + c_{0,n}\frac{\sqrt{\pi(1+\alpha)}}{2\sqrt{2}}\right) + \left(\frac{c_{0,n}^2\pi}{2\sqrt{2}\alpha}\right)^2.$$

Dividing by $c_{0,n} > 0$ and letting $c_{0,n} \to 0$ as $n \to \infty$, the contradiction follows.

$\square$

## 5 Some numerical results

As has been already pointed out, only in the cases $\alpha = 0$ and $\alpha = 1$ we have an explicit formula for $\vec{A}^+_{c_0,\alpha}$ (see (4.13)–(4.16)). Theorems 1.3 and 1.4 give information about the behaviour of $\vec{A}^+_{c_0,\alpha}$ for small values of $c_0$ for a fixed valued of $\alpha$, and for values of $\alpha$ near to 0 or 1 for a fixed valued of $c_0$. The aim of this section is to give some numerical results that allow us to understand the map

$(\alpha, c_0) \in [0,1] \times (0,\infty) \mapsto \vec{A}^{\pm}_{c_0,\alpha} \in \mathbb{S}^2$. For a fixed value of $\alpha$, we will discuss first the injectivity and surjectivity (in some appropriate sense) of the map $c_0 \mapsto \vec{A}^{\pm}_{c_0,\alpha}$ and second the behaviour of $\vec{A}^{+}_{c_0,\alpha}$ as $c_0 \to \infty$.

For fixed $\alpha$, define $\theta_{c_0,\alpha}$ to be the angle between the unit vectors $\vec{A}^{+}_{c_0,\alpha}$ and $-\vec{A}^{-}_{c_0,\alpha}$ associated to the family of solutions $\vec{\mathrm{m}}_{c_0,\alpha}(s,t)$ established in Theorem 1.1, that is $\theta_{c_0,\alpha}$ such that

$$\cos(\theta_{c_0,\alpha}) = 1 - 2(A^{+}_{1,c_0,\alpha})^2. \tag{5.1}$$

It is pertinent to ask whether $\theta_{c_0,\alpha}$ may attain any value in the interval $[0,\pi]$ by varying the parameter $c_0 > 0$.

In Figure 2 we plot the function $\theta_{c_0,\alpha}$ associated to the family of solutions $\vec{\mathrm{m}}_{c_0,\alpha}(s,t)$ established in Theorem 1.1 for $\alpha = 0$, $\alpha = 0.4$ and $\alpha = 1$, as a function of $c_0 > 0$. The curves $\theta_{c_0,0}$ and $\theta_{c_0,1}$ are exact since we have explicit formulae for $A^{+}_{1,c_0,\alpha}$ when $\alpha = 0$ and $\alpha = 1$ (see (4.13) and (4.16)). We deduce that in the case $\alpha = 0$, there is a bijective relation between $c_0 > 0$ and the angles in $(0,\pi)$. In the case $\alpha = 1$, there are infinite values of $c_0 > 0$ that allow to reach any angle in $[0,\pi]$. If $\alpha \in (0,1)$, numerical simulations show that there exists $\theta^*_\alpha \in (0,\pi)$ such that the angles in $(\theta^*_\alpha, \pi)$ are reached by a unique value of $c_0$, but for angles in $[0, \theta^*_\alpha]$ there are at least two values of $c_0 > 0$ that produce them (See $\theta_{c_0,0.4}$ in Figure 2).

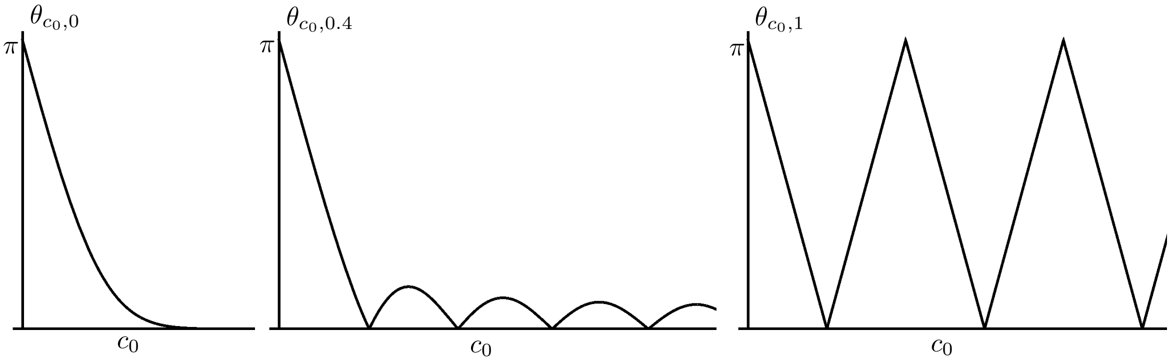



Figure 2: The angles $\theta_{c_0,\alpha}$ as a function of $c_0$ for $\alpha = 0$, $\alpha = 0.4$ and $\alpha = 1$.

These numerical results suggest that, due to the invariance of (LLG) under rotations[2], for a fixed $\alpha \in [0,1)$ one can solve the following inverse problem: Given any distinct vectors $\vec{A}^+, \vec{A}^- \in \mathbb{S}^2$ there exists $c_0 > 0$ such that the associated solution $\vec{\mathrm{m}}_{c_0,\alpha}(s,t)$ given by Theorem 1.1 (possibly multiplied by a rotation matrix) provides a solution of (LLG) with initial condition

$$\vec{\mathrm{m}}(\cdot,0) = \vec{A}^+ \chi_{(0,\infty)}(\cdot) + \vec{A}^- \chi_{(-\infty,0)}(\cdot). \tag{5.2}$$

Note that in the case $\alpha = 1$ the restriction $\vec{A}^+ \neq \vec{A}^-$ can be dropped.

In addition, Figure 2 suggests that $\vec{A}^{+}_{c_0,\alpha} \neq \vec{A}^{-}_{c_0,\alpha}$ for fixed $\alpha \in [0,1)$ and $c_0 > 0$. Indeed, notice that $\vec{A}^{+}_{c_0,\alpha} \neq \vec{A}^{-}_{c_0,\alpha}$ if and only if $A_1 \neq \pm 1$ or equivalently $\cos\theta_{c_0,\alpha} \neq -1$, that is $\theta_{c_0,\alpha} \neq \pi$, which is true if $\alpha \in [0,1)$ for any $c_0 > 0$ (See Figure 2). Notice also that when $\alpha = 1$, then the value $\pi$ is attained by different values of $c_0$.

The next natural question is the injectivity of the application $c_0 \longrightarrow \theta_{c_0,\alpha}$, for fixed $\alpha$. Precisely, can we generate the same angle using different values of $c_0$? In the case $\alpha = 0$, the

[2] In fact, using that

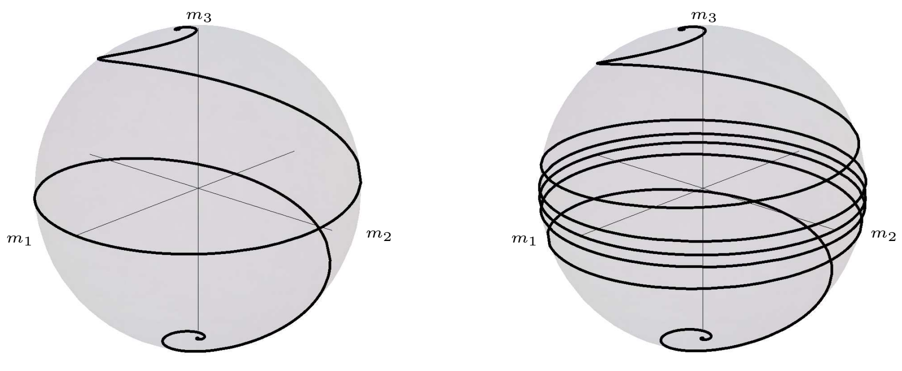
$$(M\vec{a}) \times (M\vec{b}) = (\det M) M^{-T} (\vec{a} \times \vec{b}), \qquad \text{for all } M \in \mathcal{M}_{3,3}(\mathbb{R}),\ \vec{a}, \vec{b} \in \mathbb{R}^3,$$

it is easy to verify that if $\vec{\mathrm{m}}(s,t)$ is a solution of (LLG) with initial condition $\vec{\mathrm{m}}^0$, then $\vec{\mathrm{m}}_R := R\vec{\mathrm{m}}$ is a solution of (LLG) with initial condition $\vec{\mathrm{m}}^0_R := R\vec{\mathrm{m}}^0$, for any $R \in SO(3)$.

plot of $\theta_{c_0,0}$ in Figure 2 shows that the value of $c_0$ is unique, in fact one has following formula $\sin(\theta_{c_0,0}/2) = A_{1,c_0,0} = e^{-\frac{c_0^2}{2}\pi}$ (see [15]). In the case $\alpha = 1$, we have $\sin(\theta_{c_0,1}/2) = A_{1,c_0,1} = \cos(c_0\sqrt{\pi})$, moreover

$$\vec{A}^+_{c_0,1} = \vec{A}^+_{c_0+2k\sqrt{\pi},1}, \quad \text{for any } k \in \mathbb{Z}. \tag{5.3}$$

As before, if $\alpha \in (0,1)$ we do not have an analytic answer and we have to rely on numerical simulations. However, it is difficult to test the uniqueness of $c_0$ numerically. Using the command `FindRoot` in Mathematica, we have found such values. For instance, for $\alpha = 0.4$, we obtain that $c_0 \approx 2.1749$ and $c_0 \approx 6.6263$ give the same value of $\vec{A}^+_{c_0,0.4}$. The respective profiles $\vec{m}_{c_0,0.4}(\cdot)$ are shown in Figure 3. This multiplicity of solutions suggests that the Cauchy problem for (LLG) with initial condition (5.2) is ill-posed, at least for certain values of $c_0$. This interesting problem will be studied in a forthcoming paper.




(a) $\vec{m}_{c_0,0.4}(\cdot)$, with $c_0 \approx 2.1749$ (b) $\vec{m}_{c_0,0.4}(\cdot)$, with $c_0 \approx 6.6263$

Figure 3: Two profiles $\vec{m}_{c_0,0.4}(\cdot)$, with the same limit vector $\vec{A}^+_{c_0,0.4}$.

The rest of this section is devoted to give some numerical results on the behaviour of the limiting vector $\vec{A}^+_{c_0,\alpha}$. In particular, the results below aim to complement those established in Theorem 1.3 on the behaviour of $\vec{A}^+_{c_0,\alpha}$ for small values of $c_0$, when $\alpha$ is fixed.

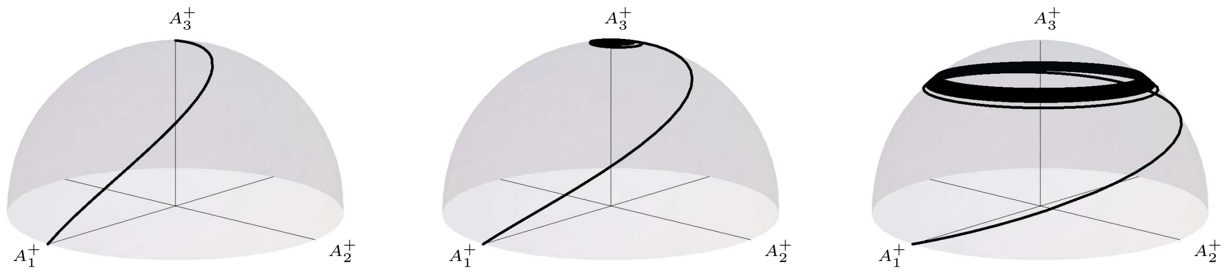
We start recalling what it is known in the extremes cases $\alpha = 0$ and $\alpha = 1$. Precisely, if $\alpha = 0$, the explicit formulae (4.13)–(4.15) for $\vec{A}^+_{c_0,0}$ allow us to prove that

$$\lim_{c_0\to 0^+} A^+_{3,c_0,0} = 0 \quad \text{and} \quad \lim_{c_0\to\infty} A^+_{3,c_0,1} = 1, \tag{5.4}$$

and also that $\{A^+_{3,c_0,0} : c_0 \in (0,\infty)\} = (0,1)$. When $\alpha = 1$ the picture is completely different. In fact $A^+_{3,c_0,1} = 0$ for all $c_0 > 0$, and the limit vectors remain in the equator plane $\mathbb{S}^1 \times \{0\}$. The natural question is what happens with $\vec{A}^+_{c_0,\alpha}$ when $\alpha \in (0,1)$ as a function of $c_0$.

Although we do not provide a rigorous answer to this question, in Figure 4 we show some numerical results. Precisely, Figure 4 depicts the curves $\vec{A}^+_{c_0,0.01}$, $\vec{A}^+_{c_0,0.4}$ and $\vec{A}^+_{c_0,0.8}$ as functions of $c_0$, for $c_0 \in [0,1000]$. We see that the behaviour of $\vec{A}^+_{c_0,\alpha}$ changes when $\alpha$ increases in the sense that the first and second coordinates start oscillating more and more as $\alpha$ goes to 1. In all the cases the third component remains monotonically increasing with $c_0$, but the value of $A^+_{3,1000,\alpha}$ seems to be decreasing with $\alpha$. At this point it is not clear what the limit value of $A^+_{3,c_0,\alpha}$ as

$c_0 \to \infty$ is. For this reason, we perform a more detailed analysis of $A^+_{3,c_0,\alpha}$ and we show the curves $A^+_{3,1,\alpha}$, $A^+_{3,10,\alpha}$, $A^+_{3,1000,\alpha}$ (for fixed $\alpha \in [0,1]$) in Figure 5. From these results we conjecture that $\{A^+_{3,c_0,\cdot}\}_{c_0>0}$ is a pointwise nondecreasing sequence of functions that converges to 1 for any $\alpha < 1$ as $c_0 \to \infty$. This would imply that, for $\alpha \in (0,1)$ fixed, $A_{1,c_0,\alpha} \to 0$ as $c_0 \to \infty$, and since $A_{1,c_0,\alpha} \to 1$ as $c_0 \to 0$ (see (1.24)), we could conclude by continuity (see Theorem 1.3) that for any angle $\theta \in (0,\pi)$ there exists $c_0 > 0$ such that $\theta$ is the angle between $\vec{A}^+_{c_0,\alpha}$ and $-\vec{A}^+_{c_0,\alpha}$ (see (5.1)). This provides an alternative way to justify the surjectivity of the map $c_0 \mapsto \vec{A}^+_{c_0,\alpha}$ (in the sense explained above).




(a) $\vec{A}^+_{c_0,0.01}$ (b) $\vec{A}^+_{c_0,0.4}$ (c) $\vec{A}^+_{c_0,0.8}$

Figure 4: The curves $\vec{A}^+_{c_0,0.01}$, $\vec{A}^+_{c_0,0.4}$ and $\vec{A}^+_{c_0,0.8}$ as functions of $c_0$, for $c_0 \in [0,1000]$.

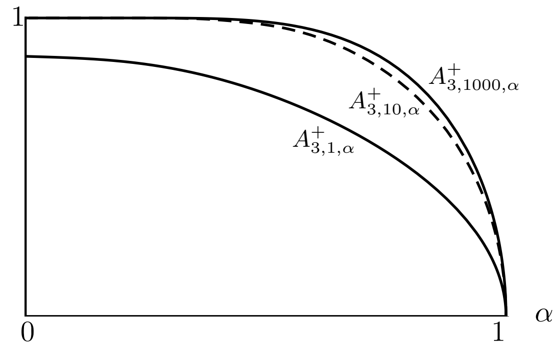



Figure 5: The curves $A^+_{3,1,\alpha}$ ,$A^+_{3,10,\alpha}$, $A^+_{3,1000,\alpha}$ as functions of $\alpha$, for $\alpha \in [0,1]$.

The curves in Figure 5 also allow us to discuss further the results in Theorem 1.4. In fact, when $\alpha$ is close to 1 the slope of the functions become unbounded and, roughly speaking, the behaviour of $A^+_{3,c_0,\alpha}$ is in agreement with the result in Theorem 1.4, that is

$$A^+_{3,c_0,\alpha} \sim C(c_0)\sqrt{1-\alpha}, \qquad \text{as } \alpha \to 1^-.$$

Numerically, the analysis is more difficult when $\alpha \sim 0$, because the number of computations needed to have an accurate profile of $A^+_{3,c_0,\alpha}$ increases drastically as $\alpha \to 0^+$. In any case, Figure 5 suggests that $A^+_{3,c_0,\alpha}$ converges to $A^+_{3,c_0,0}$ faster than $\sqrt{\alpha}|\ln(\alpha)|$. We think that this rate of convergence can be improved to $\alpha|\ln(\alpha)|$. In fact, in the proof of Lemma 3.10 we only used energy estimates. Probably, taking into account the oscillations in equation (3.102) (as did in Proposition 3.3), it would be possible to establish the necessary estimates to prove the following conjecture:

$$|\vec{A}^+_{c_0,\alpha} - \vec{A}^+_{c_0,0}| \le C(c_0)\alpha|\ln(\alpha)|, \qquad \text{for } \alpha \in (0,1/2].$$

## 6 Appendix

In this appendix we show how to compute explicitly the solution $\vec{m}_{c_0,\alpha}(s,t)$ of the LLG equation in the case $\alpha = 1$. As a consequence, we will obtain an explicit formula for the limiting vector $\vec{A}^+_{c_0,1}$ and the other constants appearing in the asymptotics of the associated profile established in Theorem 1.2 in terms of the parameter $c_0$ in the case when $\alpha = 1$.

We start by recalling that if $\alpha = 1$ then $\beta = 0$. We need to find the solution $\{\vec{m}, \vec{n}, \vec{b}\}$ of the Serret–Frenet system (1.6) with $c(s) = c_0 e^{-s^2/4}$, $\tau \equiv 0$ and the initial conditions (1.8). Hence, it is immediate that

$$m_3 = n_3 \equiv 0, \quad b_1 = b_2 \equiv 0 \quad \text{and} \quad b_3 \equiv 1.$$

To compute the other components, we use the Riccati equation (3.9) satisfied by the stereographic projection of $\{m_j, n_j, b_j\}$

$$\eta_j = \frac{n_j + i b_j}{1 + m_j}, \qquad \text{for} \qquad j \in \{1, 2\}, \tag{6.1}$$

found in the proof of Lemma 3.1. For the values of curvature and torsion $c(s) = c_0 e^{-s^2/4}$ and $\tau(s) = 0$ the Riccati equation (3.9) reads

$$\eta_j' + \frac{i\beta s}{2}\eta_j + \frac{c_0}{2} e^{-\alpha s^2/4}(\eta_j^2 + 1) = 0. \tag{6.2}$$

We see that when $\alpha = 1$, and thus $\beta = 0$, (6.2) is a separable equation that we write as:

$$\frac{d\eta_j}{\eta_j^2 + 1} = -\frac{c_0}{2} e^{-\alpha s^2/4},$$

so integrating, we get

$$\eta_j(s) = \tan\left(\arctan(\eta_j(0)) - \frac{c_0}{2}\operatorname{Erf}(s)\right), \tag{6.3}$$

where $\operatorname{Erf}(s)$ is the non-normalized error function

$$\operatorname{Erf}(s) = \int_0^s e^{-\sigma^2/4}\, d\sigma.$$

Also, using (1.8) and (6.1) we get the initial conditions $\eta_1(0) = 0$ and $\eta_2(0) = 1$. In particular, if $c_0$ is small (6.3) is the global solution of the Riccati equation, but it blows-up in finite time if $c_0$ is large. As long as $\eta_j$ is well-defined, by Lemma 3.1,

$$f_j(s) = e^{\frac{c_0}{2}\int_0^s e^{-\alpha\sigma^2/4}\eta_j(\sigma)\, d\sigma}.$$

The change of variables

$$\mu = \arctan(\eta_j(0)) - \frac{c_0}{2}\operatorname{Erf}(s)$$

yields

$$\int_0^s e^{-\alpha\sigma^2/4}\eta_j(\sigma)\, d\sigma = \frac{2}{c_0}\ln\left|\frac{\cos\left(\arctan(\eta_j(0)) - \frac{c_0}{2}\operatorname{Erf}(s)\right)}{\cos(\arctan(\eta_j(0)))}\right|,$$

and after some simplifications, we obtain

$$f_1(s) = \left|\cos\left(\frac{c_0}{2}\operatorname{Erf}(s)\right)\right| \quad \text{and} \quad f_2(s) = \left|\cos\left(\frac{c_0}{2}\operatorname{Erf}(s)\right) + \sin\left(\frac{c_0}{2}\operatorname{Erf}(s)\right)\right|.$$

In view of (3.18) and (3.19), we conclude that

$$m_1(s) = 2|f_1(s)|^2 - 1 = \cos\left(c_0\operatorname{Erf}(s)\right) \quad \text{and} \quad m_2(s) = |f_2(s)|^2 - 1 = \sin\left(c_0\operatorname{Erf}(s)\right). \tag{6.4}$$

A priori, the formulae in (6.4) are valid only as long as $\eta$ is well-defined, but a simple verification show that these are the global solutions of (1.6), with

$$n_1(s) = -\sin(c_0 \operatorname{Erf}(s)) \quad \text{and} \quad n_2(s) = \cos(c_0 \operatorname{Erf}(s)).$$

In conclusion, we have proved the following:

**Proposition 6.1.** *Let $\alpha = 1$, and thus $\beta = 0$. Then, the trihedron $\{\vec{m}_{c_0,1}, \vec{n}_{c_0,1}, \vec{b}_{c_0,1}\}$ solution of* (1.6)–(1.8) *is given by*

$$\begin{aligned}
\vec{m}_{c_0,1}(s) &= (\cos(c_0 \operatorname{Erf}(s)), \sin(c_0 \operatorname{Erf}(s)), 0),\\
\vec{n}_{c_0,1}(s) &= -(\sin(c_0 \operatorname{Erf}(s)), \cos(c_0 \operatorname{Erf}(s)), 0),\\
\vec{b}_{c_0,1}(s) &= (0,0,1),
\end{aligned}$$

*for all $s \in \mathbb{R}$. In particular, the limiting vectors $\vec{A}^+_{c_0,1}$ and $\vec{A}^-_{c_0,1}$ in Theorem 1.2 are given in terms of $c_0$ as follows:*

$$\vec{A}^{\pm}_{c_0,1} = (\cos(c_0\sqrt{\pi}), \pm\sin(c_0\sqrt{\pi}), 0).$$

Proposition 6.1 allows us to give an alternative explicit proof of Theorem 1.2 when $\alpha = 1$.

**Corollary 6.2.** *[Explicit asymptotics when $\alpha = 1$] With the same notation as in Proposition 6.1, the following asymptotics for $\{\vec{m}_{c_0,1}, \vec{n}_{c_0,1}, \vec{b}_{c_0,1}\}$ holds true:*

$$\begin{aligned}
\vec{m}_{c_0,1}(s) &= \vec{A}^+_{c_0,1} - \frac{2c_0}{s}\vec{B}^+_{c_0,1} e^{-s^2/4}\sin(\vec{a}) - \frac{2c_0^2}{s^2}\vec{A}^+_{c_0,1} e^{-s^2/2} + O\left(\frac{e^{-s^2/4}}{s^3}\right),\\
\vec{n}_{c_0,1}(s) &= \vec{B}^+_{c_0,1}\sin(\vec{a}) + \frac{2c_0}{s}\vec{A}^+_{c_0,1} e^{-s^2/4} - \frac{2c_0^2}{s^2}\vec{B}^+_{c_0,1} e^{-s^2/2}\sin(\vec{a}) + O\left(\frac{e^{-s^2/4}}{s^3}\right),\\
\vec{b}_{c_0,1}(s) &= \vec{B}^+_{c_0,1}\cos(\vec{a}),
\end{aligned}$$

*where the vectors $\vec{A}^+_{c_0,1}$, $\vec{B}^+_{c_0,1}$ and $\vec{a} = (a_j)_{j=1}^3$ are given explicitly in terms of $c_0$ by*

$$\vec{A}^+_{c_0,1} = (\cos(c_0\sqrt{\pi}), \sin(c_0\sqrt{\pi}), 0), \qquad \vec{B}^+_{c_0,1} = (|\sin(c_0\sqrt{\pi})|, |\cos(c_0\sqrt{\pi})|, 1),$$

$$a_1 = \begin{cases} \frac{3\pi}{2}, & \text{if } \sin(c_0\sqrt{\pi}) \geq 0,\\ \frac{\pi}{2}, & \text{if } \sin(c_0\sqrt{\pi}) < 0, \end{cases} \qquad a_2 = \begin{cases} \frac{\pi}{2}, & \text{if } \cos(c_0\sqrt{\pi}) \geq 0,\\ \frac{3\pi}{2}, & \text{if } \cos(c_0\sqrt{\pi}) < 0, \end{cases} \qquad \text{and} \qquad a_3 = 0.$$

*Here, the bounds controlling the error terms depend on $c_0$.*

*Proof.* By Proposition 6.1,

$$\begin{cases}
\vec{m}_{c_0,1}(s) = (\cos(c_0\sqrt{\pi} - c_0 \operatorname{Erfc}(s)), \sin(c_0\sqrt{\pi} - c_0 \operatorname{Erfc}(s)), 0),\\
\vec{n}_{c_0,1}(s) = -(\sin(c_0\sqrt{\pi} - c_0 \operatorname{Erfc}(s)), \cos(c_0\sqrt{\pi} - c_0 \operatorname{Erfc}(s)), 0),\\
\vec{b}_{c_0,1}(s) = (0,0,1),
\end{cases} \tag{6.5}$$

where the complementary error function is given by

$$\operatorname{Erfc}(s) = \int_s^\infty e^{-\sigma^2/4}\, d\sigma = \sqrt{\pi} - \operatorname{Erf}(s).$$

It is simple to check that

$$\begin{aligned}
\sin(c_0 \operatorname{Erfc}(s)) &= e^{-s^2/4}\left(\frac{2c_0}{s} - \frac{4c_0}{s^3} + \frac{24c_0}{s^5} + O\left(\frac{c_0}{s^7}\right)\right),\\
\cos(c_0 \operatorname{Erfc}(s)) &= 1 + e^{-s^2/2}\left(-\frac{2c_0^2}{s^2} + \frac{8c_0^2}{s^4} - \frac{56c_0^2}{s^6} + O\left(\frac{c_0^2}{s^8}\right)\right),
\end{aligned}$$

so that, using (6.5), we obtain that

$$m_1(s) = n_2(s) = \cos(c_0\sqrt{\pi}) + \frac{2c_0}{s}e^{-s^2/4}\sin(c_0\sqrt{\pi}) - \frac{2c_0^2}{s^2}e^{-s^2/2}\cos(c_0\sqrt{\pi}) + O\left(\frac{e^{-s^2/4}}{s^3}\right),$$

$$m_2(s) = -n_1(s) = \sin(c_0\sqrt{\pi}) - \frac{2c_0}{s}e^{-s^2/4}\cos(c_0\sqrt{\pi}) - \frac{2c_0^2}{s^2}e^{-s^2/2}\sin(c_0\sqrt{\pi}) + O\left(\frac{e^{-s^2/4}}{s^3}\right).$$

The conclusion follows from the definitions of $\vec{A}^+_{c_0,1}$, $\vec{B}^+_{c_0,1}$ and $\vec{a}$. □

**Remark 6.3.** *Notice that $\vec{a}$ is not a continuous function of $c_0$, but the vectors $(B_j^+\sin(a_j))_{j=1}^3$ and $(B_j^+\cos(a_j))_{j=1}^3$ are.*

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