Asymptotics for Christoffel functions based on orthogonal rational functions

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Outline

1. Orthogonal rational functions

2. Rational Christoffel functions

3. Convergence

4. Influence of poles
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3. Convergence
4. Influence of poles
Orthogonal rational functions (ORFs)

\[ f_n(t) = \frac{c_n t^n + c_{n-1} t^{n-1} + \ldots + c_0}{(1 - t/\alpha_1)(1 - t/\alpha_2) \cdots (1 - t/\alpha_n)}, \quad n = 1, 2, \ldots \]

Poles

- \(\alpha_1, \alpha_2, \alpha_3, \ldots\)
- Arbitrary complex or infinite, but outside \(I := [-1, 1]\)
- Fixed in advance
Orthogonal rational functions (ORFs)

**Function spaces**

Rational basis:

\[ b_k(t) = \prod_{j=1}^{k} \frac{t}{1 - t/\alpha_j}, \quad \alpha_j \in \overline{\mathbb{C}}_I := \{ t \in \mathbb{C} \cup \{\infty\} : t \notin I \} \]

Space of rational functions:

\[ \mathcal{L}_n := \mathcal{L}\{\alpha_1, \ldots, \alpha_n\} = \text{span}\{1, b_1(t), \ldots, b_n(t)\} \]

Note that...

\[ \forall j : \alpha_j = \infty \Rightarrow b_k(t) \equiv t^k \text{ and } \mathcal{L}_n \equiv \mathcal{P}_n = \text{span}\{1, t, \ldots, t^n\} \]
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Orthogonal rational functions (ORFs)

Given a positive measure $\mu$ on $I$ and inner product

$$\langle f, g \rangle_\mu = \int_I f(t)\overline{g(t)}d\mu(t)$$

$\rightarrow$ ORFs $\varphi_k(t)$:

$$\varphi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$$

$$\varphi_k \perp_\mu \mathcal{L}_{k-1}$$

$$\| \varphi_k \|_\mu = \sqrt{\langle \varphi_k, \varphi_k \rangle_\mu} = 1$$
Rational Christoffel functions

- Rational reproducing kernels

\[ K_n(x, y) := K_n(x, y; d\mu) = \sum_{j=0}^{n-1} \varphi_j(x) \varphi_j(y), \]

have the property that \(<f(\cdot), K_n(\cdot, y; d\mu)>_\mu = f(y)\) for every \(f \in \mathcal{L}_{n-1}\)

- Rational Christoffel functions

\[ \lambda_n(x) := \lambda_n(x; d\mu) = K_n^{-1}(x, x; d\mu) \]
\[ = \inf_{R \in \mathcal{L}_{n-1}} \frac{\|R\|_{\mu}^2}{|R(x)|^2}. \]
Theorem (Joris Van Deun, 2004)

Suppose

- \( \mu \) is absolutely continuous and \( \mu' > 0 \) a.e. in \( I \),
- \( \{\alpha_j\}_{j>0} \) is bounded away from \( I \)
- asymptotic distribution of the poles is given by measure \( \nu \)
- \( \lim_{n \to \infty} n \lambda_n(x) = k(x) \) uniformly on \( I \)

Then

\[
k(x) = \mu'(x) \pi \sqrt{1 - x^2} \left[ \int_{\mathbb{C}_I} \Re \left\{ \frac{\sqrt{u^2 - 1}}{u - x} \right\} \, d\nu(u) \right]^{-1}, \quad x \text{ a.e. in } I,
\]

where the square root is positive for \( u > 1 \) and the branch cut is \( I \).
Aim is to prove - under certain conditions on the measure $\mu$ - uniform convergence of $\{n\lambda_n(x)\}_{n>0}$.

**Definition**

$\mu$ is **regular** on $I$ in the sense of Stahl and Totik iff

$$\lim_{n \to \infty} \left[ \sup_{p \in \mathcal{P}_{n-1}} \left( \frac{\|p\|_I}{\|p\|_\mu} \right)^{1/n} \right] = 1, \quad \|p\|_I := \max_{t \in I} |p(t)|.$$

Note: if $\mu' > 0$ a.e. in $I$, then $\mu$ is regular on $I$. 
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$\Box$
Lemma 1

Consider the Chebyshev measure of the second kind
\[ d\omega(t) = \sqrt{1 - t^2} \, dt, \, t \in I, \] and suppose that \( \{\alpha_j\}_{j>0} \) is bounded away from \( I \), with asymptotic distribution \( \nu \). Then uniformly for \( x \) in compact subsets of \((-1, 1)\),

\[
\lim_{n \to \infty} n \lambda_n(x; d\omega) = \\
\omega'(x) \pi \sqrt{1 - x^2} \left[ \int_{\mathbb{C}_I} \Re \left\{ \frac{\sqrt{u^2 - 1}}{u - x} \right\} d\nu(u) \right]^{-1}.
\]
The proof of Lemma 1 is based on:

- classical representation for the Christoffel function

\[ \lambda_n^{[P]}(x; d\omega_n) \] for Bernstein-Szegő weights:

\[
d\omega_n(t) = \frac{\sqrt{1 - t^2}}{\left| \prod_{j=1}^{n-1} (1 - t/\alpha_j) \right|^2}, \quad t \in I;
\]

\[ \lambda_n(x; d\omega) = \lambda_n^{[P]}(x; d\omega_n) \left| \prod_{j=1}^{n-1} (1 - x/\alpha_j) \right|^2, \quad x \in \overline{\mathbb{R}}. \]
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Lemma 2

Suppose

- \( \mu, \omega \) are regular on \( I \), and \( \mu \) is absolutely continuous w.r.t. \( \omega \) on \((a, b) \subset I\),
- \( \frac{d\mu}{d\omega} \) is positive and continuous at \( x \in (a, b) \)
- \( \{\alpha_j\}_{j>0} \) is bounded away from \( I \), with asymptotic distribution \( \nu \),
- uniformly in some neighborhood of \( x \),

\[
\lim_{\epsilon \to 0^+} \left( \limsup_{n \to \infty} \left| \frac{\lambda_n(x; d\omega)}{\lambda_n[\epsilon n](x; d\omega)} - 1 \right| \right) = 0.
\]

Then uniformly \( \lim_{n \to \infty} \frac{\lambda_n(x + \frac{s}{n}; d\mu)}{\lambda_n(x + \frac{s}{n}; d\omega)} = \frac{d\mu}{d\omega}(x), \ s \in [-r, r], \ r > 0. \)
The proof of Lemma 2 is based on:

- characterization of rational Christoffel functions
  \[ \lambda_n(x; d\mu) = \inf_{R \in \mathcal{L}_{n-1}} \frac{\|R\|_\mu^2}{|R(x)|^2} ; \]

- (under the given conditions on the sequence of poles)
  equivalent characterization of regular measures:
  \[ \lim_{n \to \infty} \left[ \sup_{R \in \mathcal{L}_{n-1}} \left( \frac{\|R\|_I}{\|R\|_\mu} \right)^{1/n} \right] = 1, \quad \|R\|_I := \max_{t \in I} |R(t)| ; \]
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- characterization of rational Christoffel functions
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- (under the given conditions on the sequence of poles) equivalent characterization of regular measures:

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\]
Main result

**Theorem 1**

Suppose

- $\mu$ is regular on $I$ and absolutely continuous on $(a, b) \subset I$,
- $\mu'$ is positive and continuous at $x \in (a, b)$,
- $\{\alpha_j\}_{j > 0}$ is bounded away from $I$, with asymptotic distribution $\nu$.

Then uniformly for $s \in [-r, r]$, $r > 0$,

$$\lim_{n \to \infty} n\lambda_n \left( x + \frac{s}{n} \right) = \mu'(x) \pi \sqrt{1-x^2} \left[ \int_{\mathbb{C} \setminus I} \Re \left\{ \frac{\sqrt{u^2-1}}{u-x} \right\} d\nu(u) \right]^{-1}.$$
Corollary 1

Suppose

- $\mu$ is regular on $I$ and absolutely continuous on $(a, b) \subset I$,
- $\mu'$ is positive and continuous in $(a, b)$,
- $\{\alpha_j\}_{j \geq 0}$ is bounded away from $I$, with asymptotic distribution $\nu$.

Then for $x \in (a, b)$, uniformly for $r$ and $s$ in compact subsets of the real line,

$$\lim_{n \to \infty} \frac{K_n \left( x + \frac{r}{\tilde{K}_n(x,x)}, x + \frac{s}{\tilde{K}_n(x,x)} \right)}{K_n(x,x)} \rho_n(x, r, s) = \frac{\sin \pi (r - s)}{\pi (r - s)},$$

where $\tilde{K}_n(x,x) = \mu'(x)^{1/2} \mu'(y)^{1/2} K_n(x, y; d\mu)$, and $|\rho_n(x, r, s)| = 1$. 
How poles of ORFs affect their Christoffel functions

In some results on asymptotics of orthogonal rational functions, the restriction on the poles is replaced by a Blaschke type assumption: \( \sum_{j=1}^{\infty} \left( 1 - |\alpha_j - \sqrt{\alpha_j^2 - 1}| \right) = \infty \); hence, \( \{\alpha_j\}_{j>0} \) does not necessarily have to be bounded away from \( I \).

Q: Can we do the same for rational Christoffel functions?
A: No, we cannot.

However, we can ease the restrictions on the poles.
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How poles of ORFs affect their Christoffel functions

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However, we can ease the restrictions on the poles.
Theorem 2

Suppose

- $\omega$ is the Chebyshev measure of the second kind,
- the poles $\{\alpha_j\}_{j>0}$ have asymptotic distribution $\nu$ with support in $\overline{\mathbb{C}}$,
- $x \in (-1, 1)$ is fixed,
- for every $\epsilon > 0$ there exists $\delta > 0$ such that
  \[
  \limsup_{n \to \infty} \frac{1}{n} \sum_{j \leq n: |\alpha_j - x| \leq \delta} \frac{1}{\Im \{\alpha_j\}} < \epsilon.
  \]

Then at $x$,

\[
\lim_{n \to \infty} n \lambda_n(x; d\omega) = \omega'(x) \pi \sqrt{1 - x^2} \left[ \int_{\mathbb{C}_i} \Re \left\{ \frac{\sqrt{u^2 - 1}}{u - x} \right\} d\nu(u) \right]^{-1}.
\]
Example

Consider the sequence of poles
\[ \{\alpha_j\}_{j=1}^N = \{\tilde{\alpha}_k\}_{k=1}^n \cup \{\hat{\alpha}_l\}_{l=1}^{[n^{1/3}]} \cup \{\infty\}, \] with \( \tilde{\alpha}_k = \tilde{\alpha} = \frac{7}{24}i, \) and
\[ \hat{\alpha}_l = \frac{1}{2}(\beta_l + \beta_l^{-1}), \quad \beta_l = \left(1 - \frac{3}{4(l + 1)^p}\right) e^{\frac{3\pi}{4}}, \quad p > 0. \]

Then
- the Blaschke type assumption
  \[ \sum_{j=1}^{\infty} \left(1 - \left|\alpha_j - \sqrt{\alpha_j^2 - 1}\right|\right) = \infty \] is satisfied for every \( p > 0; \)
- the sequence of poles is not bounded away from \( l \) due to
  \[ \lim_{l \to \infty} \hat{\alpha}_l = -\frac{\sqrt{2}}{2} \in l; \]
- the asymptotic distribution is given by \( \nu = \delta_{\tilde{\alpha}}. \)
Graphs of $N\lambda_N(x)$ and $f(x) := \pi(1 - x^2) \left[ \Re \left\{ \frac{\sqrt{\alpha^2 - 1}}{\bar{\alpha} - x} \right\} \right]^{-1}$

$p = 3 \Rightarrow$ condition w.r.t. $\sum \frac{1}{\Im \{\alpha_j\}}$ is not satisfied in $x = -\sqrt{2}/2$. 
Example (cont.)

Graphs of $N\lambda_N(x)$ and $f(x):=\pi(1-x^2)\left[\Re\left\{\frac{\sqrt{\bar{\alpha}^2-1}}{\bar{\alpha}-x}\right\}\right]^{-1}$

$p = 0.5 \Rightarrow$ condition w.r.t. $\sum \frac{1}{\Im\{\alpha_j\}}$ is satisfied for every $x \in (-1, 1)$. 
Example (cont.)

Graphs of $N\lambda_N(x)$ and $f(x) := \pi(1 - x^2) \left[ \Re \left\{ \frac{\sqrt{\alpha^2 - 1}}{\alpha - x} \right\} \right]^{-1}$

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Thank you ...