Asymptotics for Christoffel functions based on orthogonal rational functions

Karl Deckers*

*Department of Computer Science, K.U. Leuven, Heverlee, Belgium.

joint work with Doron Lubinsky, School of Mathematics, Georgia Tech, Atlanta, GA, USA.

Outline

Outline

1. Orthogonal rational functions
2. Rational Christoffel functions
3. Convergence
4. Influence of poles

Outline

Outline

Orthogonal rational functions (ORFs)

$$f_n(t) = \frac{c_n t^n + c_{n-1} t^{n-1} + \ldots + c_0}{(1 - t/\alpha_1)(1 - t/\alpha_2) \cdot \ldots \cdot (1 - t/\alpha_n)}, \; n = 1, 2, \ldots$$

Poles

- $\alpha_1, \alpha_2, \alpha_3, \ldots$
- Arbitrary complex or infinite, but outside $I := [-1, 1]$
- Fixed in advance

Orthogonal rational functions (ORFs)

Function spaces

Rational basis:

$$b_k(t) = \prod_{j=1}^{k} \frac{t}{1 - t/\alpha_j}, \qquad \alpha_j \in \overline{\mathbb{C}}_I := \{t \in \mathbb{C} \cup \{\infty\} \ : \ t \notin I\}$$

Space of rational functions:

$$\mathcal{L}_n := \mathcal{L}\{\alpha_1, \ldots, \alpha_n\} = \operatorname{span}\{1, b_1(t), \ldots, b_n(t)\}$$

Note that...

$\forall j : \alpha_j = \infty \Rightarrow b_k(t) \equiv t^k$ and $\mathcal{L}_n \equiv \mathcal{P}_n = \operatorname{span}\{1, t, \ldots, t^n\}$

Orthogonal rational functions (ORFs)

Function spaces

Rational basis:

$$b_k(t) = \prod_{j=1}^{k} \frac{t}{1 - t/\alpha_j}, \qquad \alpha_j \in \overline{\mathbb{C}}_I := \{t \in \mathbb{C} \cup \{\infty\} \ : \ t \notin I\}$$

Space of rational functions:

$$\mathcal{L}_n := \mathcal{L}\{\alpha_1, \ldots, \alpha_n\} = \mathrm{span}\{1, b_1(t), \ldots, b_n(t)\}$$

Note that...

$\forall j : \alpha_j = \infty \Rightarrow b_k(t) \equiv t^k$ and $\mathcal{L}_n \equiv \mathcal{P}_n = \mathrm{span}\{1, t, \ldots, t^n\}$

Orthogonal rational functions (ORFs)

Orthonormal rational functions (ORFs) on I

Given a positive measure μ on I and inner product

$$\langle f\,, g\rangle_\mu = \int_I f(t)\overline{g(\overline{t})}d\mu(t)$$

$\rightarrow$ ORFs $\varphi_k(t)$:

$$\varphi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$$
$$\varphi_k \perp_\mu \mathcal{L}_{k-1}$$
$$\|\varphi_k\|_\mu = \sqrt{\langle \varphi_k\,, \varphi_k\rangle_\mu} = 1$$

Rational Christoffel functions

Rational Christoffel functions

- rational reproducing kernels

$$K_n(x,y) := K_n(x,y;d\mu) = \sum_{j=0}^{n-1} \varphi_j(x)\overline{\varphi_j(y)},$$

have the property that $\langle f(\cdot), K_n(\cdot,y;d\mu)\rangle_\mu = f(y)$ for every $f \in \mathcal{L}_{n-1}$

- rational Christoffel functions

$$\begin{aligned} \lambda_n(x) := \lambda_n(x;d\mu) &= K_n^{-1}(x,x;d\mu) \\ &= \inf_{R\in\mathcal{L}_{n-1}} \frac{\|R\|_\mu^2}{|R(x)|^2}. \end{aligned}$$

Rational Christoffel functions

Theorem (Joris Van Deun, 2004)

Suppose

- μ is absolutely continuous and $\mu' > 0$ a.e. in I,
- $\{\alpha_j\}_{j>0}$ is bounded away from I
- asymptotic distribution of the poles is given by measure ν
- $\lim_{n\to\infty} n\lambda_n(x) = k(x)$ uniformly on I

Then

$$k(x) = \mu'(x)\pi\sqrt{1-x^2}\left[\int_{\overline{\mathbb{C}}_I} \Re\left\{\frac{\sqrt{u^2-1}}{u-x}\right\} d\nu(u)\right]^{-1}, \quad x \text{ a.e. in } I,$$

where the square root is positive for $u > 1$ and the branch cut is I.

Rational Christoffel functions

Aim is to prove - under certain conditions on the measure μ - uniform convergence of $\{n\lambda_n(x)\}_{n>0}$.

Definition

μ is regular on I in the sense of Stahl and Totik iff

$$\lim_{n\to\infty}\left[\sup_{p\in\mathcal{P}_{n-1}}\left(\frac{\|p\|_I}{\|p\|_\mu}\right)^{1/n}\right]=1, \qquad \|p\|_I := \max_{t\in I}|p(t)|.$$

Note: if $\mu' > 0$ a.e. in I, then μ is regular on I.

Rational Christoffel functions

Aim is to prove - under certain conditions on the measure μ - uniform convergence of $\{n\lambda_n(x)\}_{n>0}$.

Definition

μ is regular on I in the sense of Stahl and Totik iff

$$\lim_{n\to\infty}\left[\sup_{p\in\mathcal{P}_{n-1}}\left(\frac{\|p\|_I}{\|p\|_\mu}\right)^{1/n}\right]=1, \qquad \|p\|_I := \max_{t\in I}|p(t)|\,.$$

Note: if $\mu' > 0$ a.e. in I, then μ is regular on I.

Convergence

Lemma 1

Consider the Chebyshev measure of the second kind $d\omega(t) = \sqrt{1-t^2}dt$, $t \in I$, and suppose that $\{\alpha_j\}_{j>0}$ is bounded away from I, with asymptotic distribution ν. Then uniformly for x in compact subsets of $(-1,1)$,

$$\lim_{n\to\infty} n\lambda_n(x; d\omega) = \omega'(x)\pi\sqrt{1-x^2}\left[\int_{\overline{\mathbb{C}}_I} \Re\left\{\frac{\sqrt{u^2-1}}{u-x}\right\} d\nu(u)\right]^{-1}.$$

Convergence

The proof of Lemma 1 is based on:

- classical representation for the Christoffel function $\lambda_n^{[P]}(x; d\omega_n)$ for Bernstein-Szegő weights:

$$d\omega_n(t) = \frac{\sqrt{1-t^2}}{\left|\prod_{j=1}^{n-1}(1-t/\alpha_j)\right|^2}, \quad t \in I;$$

- $\lambda_n(x; d\omega) = \lambda_n^{[P]}(x; d\omega_n) \left|\prod_{j=1}^{n-1}(1-x/\alpha_j)\right|^2, x \in \overline{\mathbb{R}}.$

Convergence

The proof of Lemma 1 is based on:

- classical representation for the Christoffel function $\lambda_n^{[P]}(x; d\omega_n)$ for Bernstein-Szegő weights:

$$d\omega_n(t) = \frac{\sqrt{1-t^2}}{\left|\prod_{j=1}^{n-1}(1-t/\alpha_j)\right|^2}, \quad t \in I;$$

- $\lambda_n(x; d\omega) = \lambda_n^{[P]}(x; d\omega_n)\left|\prod_{j=1}^{n-1}(1-x/\alpha_j)\right|^2$, $x \in \overline{\mathbb{R}}$.

Convergence

Lemma 2

Suppose

- μ, ω are regular on I, and μ is absolutely continuous w.r.t. ω on $(a, b) \subset I$,
- $\frac{d\mu}{d\omega}$ is positive and continuous at $x \in (a, b)$
- $\{\alpha_j\}_{j>0}$ is bounded away from I, with asymptotic distribution ν,
- uniformly in some neighborhood of x,

$$\lim_{\epsilon \to 0+} \left(\limsup_{n\to\infty} \left| \frac{\lambda_n(x; d\omega)}{\lambda_{n \pm [\epsilon n]}(x; d\omega)} - 1 \right| \right) = 0.$$

Then uniformly $\lim_{n\to\infty} \frac{\lambda_n\left(x + \frac{s}{n}; d\mu\right)}{\lambda_n\left(x + \frac{s}{n}; d\omega\right)} = \frac{d\mu}{d\omega}(x)$, $s \in [-r, r]$, $r > 0$.

Convergence

The proof of Lemma 2 is based on:

- characterization of rational Christoffel functions $\lambda_n(x; d\mu) = \inf_{R \in \mathcal{L}_{n-1}} \frac{\|R\|_\mu^2}{|R(x)|^2}$;
- (under the given conditions on the sequence of poles) equivalent characterization of regular measures:

$$\lim_{n\to\infty} \left[\sup_{R \in \mathcal{L}_{n-1}} \left(\frac{\|R\|_I}{\|R\|_\mu} \right)^{1/n} \right] = 1, \qquad \|R\|_I := \max_{t \in I} |R(t)|;$$

Convergence

The proof of Lemma 2 is based on:

- characterization of rational Christoffel functions $\lambda_n(x; d\mu) = \inf_{R \in \mathcal{L}_{n-1}} \frac{\|R\|_\mu^2}{|R(x)|^2}$;
- (under the given conditions on the sequence of poles) equivalent characterization of regular measures:

$$\lim_{n\to\infty} \left[\sup_{R \in \mathcal{L}_{n-1}} \left(\frac{\|R\|_I}{\|R\|_\mu} \right)^{1/n} \right] = 1, \qquad \|R\|_I := \max_{t \in I} |R(t)|;$$

Main result

Theorem 1

Suppose

- μ is regular on I and absolutely continuous on $(a,b) \subset I$,
- μ' is positive and continuous at $x \in (a,b)$,
- $\{\alpha_j\}_{j>0}$ is bounded away from I, with asymptotic distribution ν.

Then uniformly for $s \in [-r, r]$, $r > 0$,

$$\lim_{n\to\infty} n\lambda_n\left(x + \frac{s}{n}\right) = \mu'(x)\pi\sqrt{1-x^2}\left[\int_{\overline{\mathbb{C}}_I} \Re\left\{\frac{\sqrt{u^2-1}}{u-x}\right\} d\nu(u)\right]^{-1}.$$

Universality

Corollary 1

Suppose

- μ is regular on I and absolutely continuous on $(a, b) \subset I$,
- μ' is positive and continuous in (a, b),
- $\{\alpha_j\}_{j>0}$ is bounded away from I, with asymptotic distribution ν.

Then for $x \in (a, b)$, uniformly for r and s in compact subsets of the real line,

$$\lim_{n\to\infty} \frac{K_n\left(x + \frac{r}{\tilde{K}_n(x,x)}, x + \frac{s}{\tilde{K}_n(x,x)}\right)}{K_n(x,x)} \rho_n(x, r, s) = \frac{\sin \pi(r-s)}{\pi(r-s)},$$

where $\tilde{K}_n(x,x) = \mu'(x)^{1/2}\mu'(y)^{1/2}K_n(x, y; d\mu)$, and $|\rho_n(x, r, s)| = 1$.

How poles of ORFs affect their Christoffel functions

In some results on asymptotics of orthogonal rational functions, the restriction on the poles is replaced by a Blaschke type assumption: $\sum_{j=1}^{\infty}\left(1-\left|\alpha_j-\sqrt{\alpha_j^2-1}\right|\right)=\infty$; hence, $\{\alpha_j\}_{j>0}$ does not necessarily have to be bounded away from I.

Q: Can we do the same for rational Christoffel functions?

A: No, we cannot.

However, we can ease the restrictions on the poles.

How poles of ORFs affect their Christoffel functions

In some results on asymptotics of orthogonal rational functions, the restriction on the poles is replaced by a Blaschke type assumption: $\sum_{j=1}^{\infty}\left(1-\left|\alpha_j-\sqrt{\alpha_j^2-1}\right|\right)=\infty$; hence, $\{\alpha_j\}_{j>0}$ does not necessarily have to be bounded away from I.

Q: Can we do the same for rational Christoffel functions?

A: No, we cannot.

However, we can ease the restrictions on the poles.

How poles of ORFs affect their Christoffel functions

In some results on asymptotics of orthogonal rational functions, the restriction on the poles is replaced by a Blaschke type assumption: $\sum_{j=1}^{\infty}\left(1-\left|\alpha_j-\sqrt{\alpha_j^2-1}\right|\right)=\infty$; hence, $\{\alpha_j\}_{j>0}$ does not necessarily have to be bounded away from I.

Q: Can we do the same for rational Christoffel functions?

A: No, we cannot.

However, we can ease the restrictions on the poles.

How poles of ORFs affect their Christoffel functions

In some results on asymptotics of orthogonal rational functions, the restriction on the poles is replaced by a Blaschke type assumption: $\sum_{j=1}^{\infty}\left(1-\left|\alpha_j-\sqrt{\alpha_j^2-1}\right|\right)=\infty$; hence, $\{\alpha_j\}_{j>0}$ does not necessarily have to be bounded away from I.

Q: Can we do the same for rational Christoffel functions?

A: No, we cannot.

However, we can ease the restrictions on the poles.

How poles of ORFs affect their Christoffel functions

Theorem 2

Suppose

- ω is the Chebyshev measure of the second kind,
- the poles $\{\alpha_j\}_{j>0}$ have asymptotic distribution ν with support in $\overline{\mathbb{C}}$,
- $x \in (-1, 1)$ is fixed,
- for every $\epsilon > 0$ there exists $\delta > 0$ such that $\limsup_{n\to\infty} \frac{1}{n} \sum_{j\leqslant n: |\alpha_j - x| \leqslant \delta} \frac{1}{\Im\{\alpha_j\}} < \epsilon$.

Then at x,

$$\lim_{n\to\infty} n\lambda_n(x; d\omega) = \omega'(x)\pi\sqrt{1-x^2}\left[\int_{\overline{\mathbb{C}}_I} \Re\left\{\frac{\sqrt{u^2-1}}{u-x}\right\} d\nu(u)\right]^{-1}.$$

Example

Consider the sequence of poles
$\{\alpha_j\}_{j=1}^{N} = \{\tilde{\alpha}_k\}_{k=1}^{n} \cup \{\hat{\alpha}_l\}_{l=1}^{[n^{1/3}]} \cup \{\infty\}$, with $\tilde{\alpha}_k = \tilde{\alpha} = \frac{7}{24}\mathbf{i}$, and

$$\hat{\alpha}_l = \frac{1}{2}(\beta_l + \beta_l^{-1}), \quad \beta_l = \left(1 - \frac{3}{4(l+1)^p}\right) e^{\frac{3\pi}{4}}, \; p > 0.$$

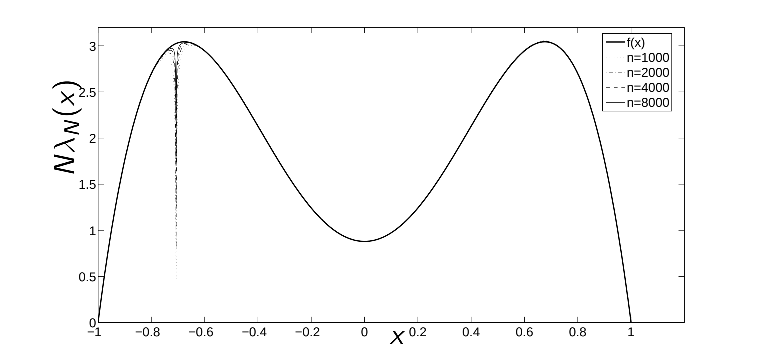
Then

- the Blaschke type assumption $\sum_{j=1}^{\infty} \left(1 - \left|\alpha_j - \sqrt{\alpha_j^2 - 1}\right|\right) = \infty$ is satisfied for every $p > 0$;
- the sequence of poles is not bounded away from I due to $\lim_{l\to\infty} \hat{\alpha}_l = -\frac{\sqrt{2}}{2} \in I$;
- the asymptotic distribution is given by $\nu = \delta_{\tilde{\alpha}}$.

Example (cont.)

Graphs of $N\lambda_N(x)$ and $f(x) := \pi(1-x^2)\left[\Re\left\{\frac{\sqrt{\tilde{\alpha}^2-1}}{\tilde{\alpha}-x}\right\}\right]^{-1}$

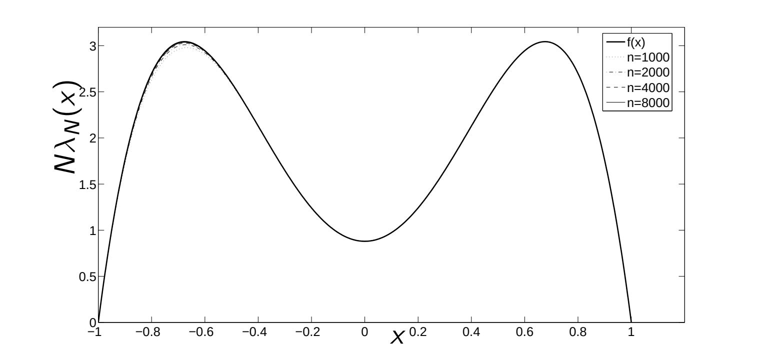
$p = 3 \Rightarrow$ condition w.r.t. $\sum \frac{1}{\Im\{\alpha_j\}}$ is not satisfied in $x = -\sqrt{2}/2$.



Example (cont.)

Graphs of $N\lambda_N(x)$ and $f(x) := \pi(1-x^2)\left[\Re\left\{\frac{\sqrt{\tilde{\alpha}^2-1}}{\tilde{\alpha}-x}\right\}\right]^{-1}$

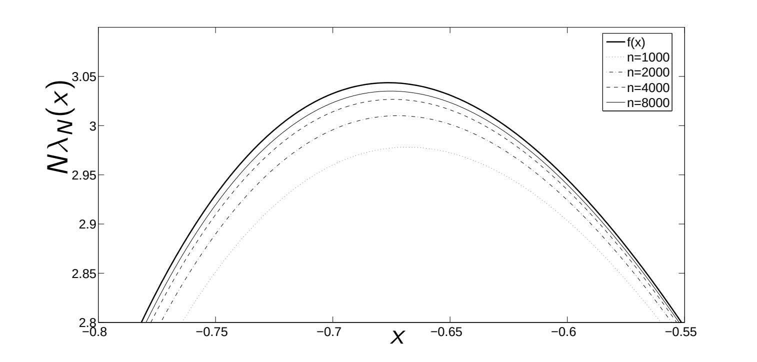
$p = 0.5 \Rightarrow$ condition w.r.t. $\sum \frac{1}{\Im\{\alpha_j\}}$ is satisfied for every $x \in (-1,1)$.



Example (cont.)

Graphs of $N\lambda_N(x)$ and $f(x) := \pi(1-x^2)\left[\Re\left\{\frac{\sqrt{\tilde{\alpha}^2-1}}{\tilde{\alpha}-x}\right\}\right]^{-1}$

$p = 0.5 \Rightarrow$ condition w.r.t. $\sum \frac{1}{\Im\{\alpha_j\}}$ is satisfied for every $x \in (-1,1)$.



References

- A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, *"Orthogonal Rational Functions"*, volume 5 of Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, 1999
- J. Van Deun, *"Orthogonal Rational Functions: Asymptotic Behaviour and Computational Aspects"*, PhD Thesis, K.U.Leuven, Dept. of Computer Science, May 2004.
- H. Stahl and V. Totik, *"General Orthogonal Polynomials"*, Encyclopedia of Mathematics and its Applications 43, Cambridge University Press, Cambridge, 1992.
- K. Deckers and A. Bultheel, *"Christoffel Functions and Universality Limits for Orthogonal Rational Functions"*, Analysis and Applications, 2011. (Accepted)

Thank you ...