Rational Gauss-Radau and rational Szegő-Lobatto quadrature on the interval and the unit circle respectively

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Aim of the talk

Present relation between Rational Gauss-Radau and Rational Szegő-Lobatto quadrature formulas.

Motivation

- interchange properties,
- analytic computations,
- reduces computational effort,
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Preliminaries

Interval $I = [-1, 1]$

$\overline{\mathbb{C}}_I = (\mathbb{C} \cup \{\infty\}) \setminus I$

$\{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}}_I$

$\mathcal{L}_n = \left\{ \frac{p_n(x)}{\prod_{k=1}^n (1 - x/\alpha_k)} : p_n \in \mathbb{P}_n \right\}, \quad n = 1, 2, \ldots$

Unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

$\{\beta_1, \beta_2, \ldots\} \subset \mathbb{D}$

$\mathring{\mathcal{L}}_n = \left\{ \frac{p_n(z)}{\prod_{k=1}^n (1 - \overline{\beta}_k z)} : p_n \in \mathbb{P}_n \right\}, \quad n = 1, 2, \ldots$

Quadrature rules

$$J_\mu(f) := \int_K f(t) d\mu(t) \approx \sum_{j=1}^n \lambda_j f(t_j) =: J_n(f), \quad \{t_j\}_{j=1}^n \subset K$$

where $K = I$ or $K = \mathbb{T}$, and $t = x$ or $t = z$.

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positive rational interpolatory quadrature rule on I

μ a positive bounded Borel measure on I,

$\{\lambda_j\}_{j=1}^n \subset \mathbb{R}_0^+$,

$J_\mu(f) = J_n(f)$ at least for all $f \in \mathcal{L}_{n-1}$.

Special cases

Rational Gaussian quadrature formula, which has maximal domain of exactness:
$\mathcal{L}_n \cdot \mathcal{L}_{n-1}^c = \{f(x) \cdot \overline{g(\overline{x})} : f \in \mathcal{L}_n, g \in \mathcal{L}_{n-1}\}$.

Rational Gauss-Radau quadrature formula (i.e, one node x_α fixed in advance), with domain of exactness: $\mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^c$.

Construction quadratures

Rational Gaussian quadrature

nodes x_k = zeros of $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, where $\varphi_n \perp_\mu \mathcal{L}_{n-1}$ w.r.t. the inner product $\langle f\,, g\rangle_\mu = \int_{-1}^{1} f(x)\overline{g(\overline{x})}d\mu(x)$.

$\lambda_k = \{\sum_{j=0}^{n-1} |\varphi_j(x_k)|^2\}^{-1}$.

Rational Gauss-Radau quadrature

nodes x_k = zeros of the quasi-orthogonal rational function (qORF)

$$Q_{n,\tau_n} = \varphi_n(x) + \tau_n \frac{1 - x/\overline{\alpha}_{n-1}}{1 - x/\alpha_n} \varphi_{n-1}(x), \quad \tau_n \in \mathbb{C}.$$

$\lambda_k = \{\sum_{j=0}^{n-1} |\varphi_j(x_k)|^2\}^{-1}$.

In remainder we assume $\alpha_n \in \mathbb{R}_I$.

positive rational interpolatory quadrature rule on $\mathbb{T}$

$\mathring{\mu}$ a positive bounded Borel measure on $\mathbb{T}$,
$\{\mathring{\lambda}_j\}_{j=1}^n \subset \mathbb{R}_0^+$,
$J_{\mathring{\mu}}(\mathring{f}) = J_n(\mathring{f})$ for all
$\mathring{f} \in \mathring{\mathcal{L}}_p \cdot \mathring{\mathcal{L}}_{p*} = \{f(z) \cdot \overline{g(1/\overline{z})} : f, g \in \mathring{\mathcal{L}}_p\}$,
$(n-1)/2 \leq p \leq n-1.$

Special cases

Rational Szegő quadrature formula, which has maximal domain of exactness: $\mathring{\mathcal{L}}_{n-1} \cdot \mathring{\mathcal{L}}_{(n-1)*}$.

Rational Szegő-Lobatto quadrature formula (i.e., two nodes z_α and z_β fixed in advance) with domain of exactness: $\mathring{\mathcal{L}}_{n-2} \cdot \mathring{\mathcal{L}}_{(n-2)*}$.

Construction quadratures

Rational Szegő quadrature

nodes z_k = zeros of a para-orthogonal rational function (pORF)

$$\mathring{Q}_{n,\xi_n}(z) = \frac{z - \beta_{n-1}}{1 - \overline{\beta}_n z}\phi_{n-1}(z) + \xi_n \frac{1 - \overline{\beta}_{n-1} z}{1 - \overline{\beta}_n z}\phi^*_{n-1}(z),$$

where $\phi_n \perp_{\mathring{\mu}} \mathring{\mathcal{L}}_{n-1}$ w.r.t. the inner product
$\left\langle \mathring{f}, \mathring{g} \right\rangle_{\mathring{\mu}} = \int_{-\pi}^{\pi} \mathring{f}(z)\overline{\mathring{g}(1/\overline{z})} d\mathring{\mu}(\theta)$,
$\phi^*_n(z) = \overline{\phi_n(1/\overline{z})} \prod_{j=1}^{n} \frac{z-\beta_k}{1-\overline{\beta}_k z}$, and $\xi_n \in \mathbb{T}$.
$\mathring{\lambda}_k = \{\sum_{j=0}^{n-1} |\phi_j(z_k)|^2\}^{-1}$.

Construction quadratures

Rational Szegő-Lobatto quadrature

nodes z_k = zeros of a pORF

$\tilde{\mathring{Q}}_{n,\tilde{\xi}_n}(z) = \frac{z-\beta_{n-1}}{1-\overline{\beta}_n z}\tilde{\phi}_{n-1}(z) + \tilde{\xi}_n \frac{1-\overline{\beta}_{n-1}z}{1-\overline{\beta}_n z}\tilde{\phi}^*_{n-1}(z)$, where

$\tilde{\phi}_{n-1}(z) = c_k \left(\frac{z-\beta_{n-2}}{1-\overline{\beta}_{n-1} z}\phi_{n-2}(z) + \tilde{\delta}_{n-1} \frac{1-\overline{\beta}_{n-2}z}{1-\overline{\beta}_{n-1} z}\phi^*_{n-2}(z) \right)$,

$\tilde{\delta}_{n-1} \in \mathbb{D}$, and $\tilde{\xi}_n \in \mathbb{T}$.

$\mathring{\lambda}_k = \{\sum_{j=0}^{n-1} |\tilde{\phi}_j(x_k)|^2\}^{-1}$.

In remainder we assume $\beta_n \in (-1,1)$.

Connection between I and $\mathbb{T}$

Joukowski transformation

$$x = J(z) = \frac{1}{2}(z + z^{-1}).$$

$$\{\alpha_1, \alpha_2, \ldots\} \subset \overline{\mathbb{C}}_I \rightsquigarrow \{\hat{\beta}_1, \hat{\beta}_2, \ldots\} \subset \mathbb{D}.$$

$$\hat{\beta}_{2k} = \overline{\hat{\beta}}_{2k-1} = \beta_k,\ \alpha_k = J(\beta_k).$$

The measures

μ on $I \rightsquigarrow \mathring{\mu}$ on $\mathbb{T}$ a symmetric measure

$$\mathring{\mu}(E) = \mu(\{\cos\theta, \theta \in E \cap [0,\pi)\}) + \mu(\{\cos\theta, \theta \in E \cap [-\pi, 0)\}),$$

The integrals

$$\int_I f(x)d\mu(x) = \frac{1}{2}\int_{\mathbb{T}} \mathring{f}(z)d\mathring{\mu}(z), \quad \mathring{f} = f \circ J.$$

Connection between I and $\mathbb{T}$

Theorem 'Sym'

$\{\beta_1, \ldots, \beta_{n-1}\}$ real or complex conjugate pairs, and $\mathring{\mu}$ positive symmetric Borel measure on $\mathbb{T}$. Then,

- zeros of $\mathring{Q}_{n,\xi_n}$ appear in complex conjugate pairs iff $\xi_n = \pm 1$.
- $z = 1$ is zero iff $\xi_n = -1$ and $z = -1$ iff $\xi_n = (-1)^{n+1}$.
- Let $J_n(\mathring{f})$ be an n-point rational Szegő-Lobatto quadrature formula based on zeros of $\mathring{Q}_{n,\pm 1}$. Then for $k = 1, \ldots, n$, the weight $\mathring{\lambda}_k = \mathring{\lambda}_j$ if $z_j = \overline{z}_k$.

Theorem 'Rel'

$\mathring{\mu}$ related with μ by Joukowski transform, then

$$\begin{aligned} \varphi_k(x) &= F\left(\overline{\phi_{2k}(\overline{z})}, \phi_{2k}^*(z)\right) \\ &= G\left(\overline{\phi_{2k-1}^*(\overline{z})}, \phi_{2k-1}(z)\right). \end{aligned}$$

Relating rational Szegő-Lobatto and Gauss-Radau rules

Theorem 'SL-GR'

μ and $\mathring{\mu}$ related by Joukowski transform. Let $J_{2n}(\mathring{f}) = \mathring{\lambda}_\alpha(z_\alpha)\mathring{f}(z_\alpha) + \mathring{\lambda}_\beta \mathring{f}(\overline{z}_\alpha) + \sum_{k=1}^{2n-2} \mathring{\lambda}_k \mathring{f}(z_k)$ be a $2n$-point rational Szegő-Lobatto quadrature formula with fixed nodes $\{z_\alpha, \overline{z}_\alpha\} \subset \mathbb{T} \setminus \{-1, +1\}$ for $J_{\mathring{\mu}}(\mathring{f})$. Set $x_\alpha = J(z_\alpha)$, $\lambda_\alpha = \mathring{\lambda}_\alpha$, $x_k = J(z_k)$ and $\lambda_k = \mathring{\lambda}_k$. Then the formula $J_n(f) = \lambda_\alpha f(x_\alpha) + \sum_{j=1}^{n-1} \lambda_j f(x_j)$ coincides with the n-point rational Gauss-Radau formula based on the zeros of the qORF

$$Q_{n,\tau_n}(x) = H(z)\tilde{\mathring{Q}}_{2n,1}(z)$$

where τ_n depends of x_α

Relating rational Szegő-Lobatto and Gauss-Radau rules

Proof

From Theorem 'Sym' it follows that $\tilde{\mathring{\hat{Q}}}_{2n,1}$ has $2n$ zeros (all different from 1 and -1), appearing in complex conjugate pairs, and $\mathring{\lambda}_\alpha = \mathring{\lambda}_\beta$, $\mathring{\lambda}_{n-1+k} = \mathring{\lambda}_k$.

If $f \in \mathcal{L}_{n-1}\mathcal{L}_{n-1}^c$ then $\mathring{f}(z) = (f \circ J)(z) \in \hat{\mathring{\mathcal{L}}}_{2n-2}\hat{\mathring{\mathcal{L}}}_{(2n-2)^*}$

$$\begin{aligned} J_n(f) &= \lambda_\alpha f(x_\alpha) + \sum_{k=1}^{n-1} \lambda_k f(x_k) \\ &= \frac{1}{2}\left\{ \mathring{\lambda}_\alpha(z_\alpha)\mathring{f}(z_\alpha) + \mathring{\lambda}_\beta \mathring{f}(\overline{z}_\alpha) + \sum_{k=1}^{2n-2} \mathring{\lambda}_k \mathring{f}(z_k) \right\} \\ &= \frac{1}{2} J_{2n}(\mathring{f}) = \frac{1}{2} J_{\mathring{\mu}}(\mathring{f}) = J_\mu(f). \end{aligned}$$

Relating rational Gauss-Radau and Szegő-Lobatto rules

Theorem 'GR-SL'

Let μ and $\mathring{\mu}$ related by the Joukowski transformation. Let $J_n(f) = \lambda_\alpha f(x_\alpha) + \sum_{j=1}^{n} \lambda_k f(x_k)$ be an n-point rational Gauss-Radau quadrature formula with fixed node x_α for $J_\mu(f)$, based on the zeros of the qORF Q_{n,τ_n}. Set $x_k = \cos\theta_k$, $x_\alpha = \cos\theta_\alpha$, and define $z_k = \overline{z}_{n-1+k} = e^{i\theta_k}$, $\mathring{\lambda}_k = \mathring{\lambda}_{n-1+k} = \lambda_k$, $k = 1, \ldots, n-1$. $z_\alpha = e^{i\theta_\alpha}$, $\mathring{\lambda}_\alpha = \lambda_\alpha$. Then

$$J_{2n}(\mathring{f}) = \mathring{\lambda}_\alpha \mathring{f}(z_\alpha) + \mathring{\lambda}_\alpha \mathring{f}(\overline{z}_\alpha) + \sum_{k=1}^{2n-2} \mathring{\lambda}_k \mathring{f}(z_k)$$

coincides with the $2n$-point rational Szegő-Lobatto quadrature formula with fixed nodes $\{z_\alpha, \overline{z}_\alpha\}$ for $J_{\mathring{\mu}}(\mathring{f})$ based on the zeros of the pORF $\tilde{\mathring{Q}}_{2n,\tilde{\xi}_{2n}}$ with $\tilde{\xi}_{2n} = 1$, and with $\tilde{\delta}_{2n-1}$ depending on τ_n.

Outline of the proof

It is known that the qORF is orthogonal to $\mathcal{L}_{n-1}$ w.r.t. a measure $\tilde{\mu}$ (not necessarily supported in I).

We can express $Q_{n,\tau_n}(x)$ in terms of $\hat{\phi}_{2n-2}(z)$ and $\hat{\phi}^*_{2n-2}(z)$ in two ways:

- by assuming that $Q_{n,\tau_n}(x) = H(z)\tilde{\mathring{\hat{Q}}}_{2n,1}(z)$, which is equivalent with assuming that the measure $\tilde{\mu}$ is supported on I;
- by making use of the definition of $Q_{n,\tau_n}(x)$ together with Theorem 'Rel'.

By comparison we then obtain a relation between $\tilde{\delta}_{2n-1}$ and τ_n.

Relating rational Gauss-Radau and Szegő-Lobatto rules

Next, we need the following lemma.

Lemma

$\{\beta_1, \ldots, \beta_{n-1}\}$ *real or complex conjugate pairs, and* $\mathring{\mu}$ *positive symmetric Borel measure on* $\mathbb{T}$.

$$\phi_{n-1}^c(z) = b_n \frac{z-\beta_{n-1}}{z-\overline{\beta}_{n-1}}\phi_{n-1}(z) + a_n \frac{1-\overline{\beta}_{n-1}z}{z-\overline{\beta}_{n-1}}\overline{\phi_{n-1}(1/\overline{z})}$$

a_n and b_n can be expressed in terms of δ_n. Since $\phi_{n-1}^c(z)$ is independent of δ_n, so are a_n and b_n. Hence, the expressions for a_n and b_n remains the same when replacing δ_n with $\tilde{\delta}_n$. As a result, we find that

$$\tilde{\delta}_n \in C \cap \mathbb{D},\ a_n \neq 0$$

$$\tilde{\delta}_n \in L \cap \mathbb{D},\ a_n = 0$$

with C a circle and L a line that depends of a_n and b_n.

Finally, we have the following lemma.

Lemma

Let τ_n be such that $Q_{n,\tau_n}(x)$ has all its zeros in $(-1,1)$. Then the corresponding $\tilde{\delta}_{2n-1}$ satisfies the condition given in the previous lemma.

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