Rational Gauss-type quadrature formulas with a prescribed node anywhere on the real line

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1 Preliminaries
   - Orthonormal rational functions
   - Rational interpolatory quadrature formulas

2 Rational Gauss-type quadrature with one fixed node
   - First kind
   - Second kind
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Orthonormal rational functions (ORFs)

**Notation**

$I = [-1, 1]$, $X \subseteq \mathbb{C}$: $\overline{X} = X \cup \{\infty\}$ and $X_I = \{x \in X : x \notin I\}$

**Rational functions**

$b_k(x) = \prod_{j=1}^{k} \frac{x}{1-x/\alpha_j}, \quad \alpha_j \in \overline{C}_I$

$L_n = \mathcal{L}\{\alpha_1, \ldots, \alpha_n\}$

$= \text{span}\{1, b_1(x), \ldots, b_n(x)\}$

**Polynomials**

$\forall j : \alpha_j = \infty \Rightarrow b_k(x) \equiv x^k$

$L_n \equiv \mathcal{P}_n = \text{span}\{1, x, \ldots, x^n\}$

**Orthonormal rational functions (ORFs)**

Given a positive measure $\mu$ on $I$ and inner product

$\langle f, g \rangle_\mu = \int_I f(x)g(x)\overline{d\mu(x)}$

$\rightarrow$ ORFs: $\varphi_k(x)$

$\varphi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$, $\varphi_k \perp \mu \mathcal{L}_{k-1}$, and $\|\varphi_k\|_\mu = \sqrt{\langle \varphi_k, \varphi_k \rangle_\mu} = 1$. 

Karl Deckers, Adhemar Bultheel, and Joris Van Deun
## Preliminaries

**Rational Gauss-type quadrature with one fixed node**

**Orthonormal rational functions (ORFs)**

### Notation

\[
I = [-1, 1], \quad X \subseteq \mathbb{C} : \overline{X} = X \cup \{\infty\} \text{ and } X_I = \{x \in X : x \notin I\}
\]

### Rational functions

\[
b_k(x) = \prod_{j=1}^{k} \frac{x}{1-\frac{x}{\alpha_j}}, \quad \alpha_j \in \overline{C}_I
\]

\[
\mathcal{L}_n = \mathcal{L}\{\alpha_1, \ldots, \alpha_n\} = \text{span}\{1, b_1(x), \ldots, b_n(x)\}
\]

### Polynomials

\[
\forall j : \alpha_j = \infty \Rightarrow b_k(x) \equiv x^k
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### Orthonormal rational functions (ORFs)

Given a positive measure \(\mu\) on \(I\) and inner product

\[
\langle f, g \rangle_\mu = \int_I f(x)\overline{g(x)}d\mu(x)
\]

\(\rightarrow\) **ORFs:** \(\varphi_k(x)\)

\[
\varphi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}, \varphi_k \perp_\mu \mathcal{L}_{k-1}, \text{ and } \|\varphi_k\|_\mu = \sqrt{\langle \varphi_k, \varphi_k \rangle_\mu} = 1.
\]
Orthonormal rational functions (ORFs)

Notation

\[ I = [-1, 1], \quad X \subseteq \mathbb{C} : \overline{X} = X \cup \{\infty\} \text{ and } X_I = \{x \in X : x \notin I\} \]

Rational functions

\[ b_k(x) = \prod_{j=1}^{k} \frac{x}{1-x/\alpha_j}, \quad \alpha_j \in \overline{\mathbb{C}} \]
\[ \mathcal{L}_n = \mathcal{L}\{\alpha_1, \ldots, \alpha_n\} \]
\[ = \text{span}\{1, b_1(x), \ldots, b_n(x)\} \]

Polynomials

\[ \forall j : \alpha_j = \infty \Rightarrow b_k(x) \equiv x^k \]
\[ \mathcal{L}_n \equiv \mathcal{P}_n = \text{span}\{1, x, \ldots, x^n\} \]

Orthonormal rational functions (ORFs)

Given a positive measure \( \mu \) on \( I \) and inner product

\[ \langle f, g \rangle_\mu = \int_I f(x)g(\overline{x})d\mu(x) \]

\[ \rightarrow \text{ ORFs: } \varphi_k(x) \]
\[ \varphi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}, \varphi_k \perp_\mu \mathcal{L}_{k-1}, \text{ and } \|\varphi_k\|_\mu = \sqrt{\langle \varphi_k, \varphi_k \rangle_\mu} = 1. \]
Orthonormal rational functions (ORFs)

Theorem (Recurrence)

The ORFs $\varphi_k$, $k > 0$, satisfy a three-term recurrence relation:

$$\varphi_k(x) = E_k \cdot F(\varphi_{k-2}(x), \varphi_{k-1}(x), \alpha_{k-2}, \alpha_{k-1}, \alpha_k, D_k)$$

with initial conditions $\varphi_{-1}(x) \equiv 0$, $\varphi_0(x) \equiv \eta_0 \|1\|^{-1}_{\mu}$, $|\eta_0| = 1$, and $\alpha_{-1} = \alpha_0 = \infty$.


Corollary (Generalized eigenvalue problem (GEP))

The zeros $\{x_{nk}\}_{k=1}^n$ of the ORF $\varphi_n(x)$ are eigenvalues of a GEP of the form $J_n v_{nk} = x_{nk} B_n v_{nk}$, where $J_n$ and $B_n$ are tridiagonal matrices, and $v_{nk}$ is the corresponding normalized eigenvector.

[J. Van Deun, Numer. Algorithms, 2007]
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[J. Van Deun, Numer. Algorithms, 2007]
Orthonormal rational functions (ORFs)

**Theorem**

*The ORF* \( \varphi_n \) *has* \( n \) *real distinct zeros in* \( I \) *iff* \( \alpha_n \in \mathbb{R}_I \).

[K. Deckers et al., Numer. Math., 2010 (submitted)]

**Theorem (Favard Theorem)**

Suppose the sequence of rational functions \( \{\hat{\phi}_k\}_{k=0}^\infty \), with \( \hat{\phi}_k \in \hat{\mathcal{L}}_k \setminus \hat{\mathcal{L}}_{k-1} \) and \( \hat{\mathcal{L}}_k = \mathcal{L}\{\hat{\alpha}_1, \ldots, \hat{\alpha}_k\} \), are generated by the three-term recurrence relation

\[
\hat{\phi}_k(x) = \hat{E}_k \cdot F(\hat{\phi}_{k-2}(x), \hat{\phi}_{k-1}(x), \hat{\alpha}_{k-2}, \hat{\alpha}_{k-1}, \hat{\alpha}_k, \hat{D}_k).
\]

Then - under certain conditions on \( \hat{E}_k \) and \( \hat{D}_k \) - the \( \hat{\phi}_k \) form an orthonormal system w.r.t. a positive measure \( \hat{\mu} \) on \( \hat{S} \subset \mathbb{R} \) and inner product \( \langle f, g \rangle_{\hat{\mu}} = \int_{\hat{S}} f(x)g(\overline{x})d\hat{\mu}(x) \).

Orthonormal rational functions (ORFs)

Theorem

The ORF $\varphi_n$ has $n$ real distinct zeros in $I$ iff $\alpha_n \in \overline{\mathbb{R}}_I$.

[K. Deckers et al., Numer. Math., 2010 (submitted)]

Theorem (Favard Theorem)

Suppose the sequence of rational functions $\{\hat{\varphi}_k\}_{k=0}^{\infty}$, with $\hat{\varphi}_k \in \hat{L}_k \setminus \hat{L}_{k-1}$ and $\hat{L}_k = \mathcal{L}\{\hat{\alpha}_1, \ldots, \hat{\alpha}_k\}$, are generated by the three-term recurrence relation

$$\hat{\varphi}_k(x) = \hat{E}_k \cdot F(\hat{\varphi}_{k-2}(x), \hat{\varphi}_{k-1}(x), \hat{\alpha}_{k-2}, \hat{\alpha}_{k-1}, \hat{\alpha}_k, \hat{D}_k).$$

Then - under certain conditions on $\hat{E}_k$ and $\hat{D}_k$ - the $\hat{\varphi}_k$ form an orthonormal system w.r.t. a positive measure $\hat{\mu}$ on $\hat{S} \subset \mathbb{R}$ and inner product $\langle f, g \rangle_{\hat{\mu}} = \int_{\hat{S}} f(x) \overline{g(x)} d\hat{\mu}(x)$.

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Rational interpolatory quadrature rule (RIQ)

Given the set of $n$ distinct nodes $\{x_{nk}\}_{k=1}^{n} \subset I$, there exists a unique set of weights $\{\lambda_{nk}\}_{k=1}^{n} \subset \mathbb{C}$ so that

$$\int_{I} f(x) d\mu(x) \approx \sum_{k=1}^{n} \lambda_{nk} f(x_{nk})$$

is exact for every $f \in \tilde{\mathcal{L}}_m$, with

$$\mathcal{L}_{n-1} \subseteq \tilde{\mathcal{L}}_m \subseteq \mathcal{L}\{\alpha_1, \ldots, \alpha_n, \overline{\alpha}_1, \ldots, \overline{\alpha}_n\} =: \mathcal{L}_n \cdot \mathcal{L}_n^c.$$

The space $\tilde{\mathcal{L}}_m$ is called the domain of validity.

Positive weights

- The weights $\{\lambda_{nk}\}_{k=1}^{n} \subset \mathbb{R}$ iff $\tilde{\mathcal{L}}_m = \tilde{\mathcal{L}}_m^c$;
- Positive RIQ = RIQ with weights $\{\lambda_{nk}\}_{k=1}^{n} \subset \mathbb{R}_0^+$. 
Rational interpolatory quadrature rule (RIQ)

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$$

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$$
\mathcal{L}_{n-1} \subseteq \tilde{L}_{m} \subseteq \mathcal{L}\{\alpha_{1}, \ldots, \alpha_{n}, \overline{\alpha}_{1}, \ldots, \overline{\alpha}_{n}\} =: \mathcal{L}_{n} \cdot \mathcal{L}_{n}^{c}.
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- The weights $\{\lambda_{nk}\}_{k=1}^{n} \subset \mathbb{R}$ iff $\tilde{L}_{m} \equiv \tilde{L}_{m}^{c}$;
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Rational interpolatory quadrature rule

Rational Gaussian quadrature

- \( = \) positive RIQ
- has maximal domain of validity; i.e., \( \tilde{\mathcal{L}}_m = \mathcal{L}_n \cdot \mathcal{L}_{n-1}^c \)
- exists iff \( \alpha_n \in \overline{\mathbb{R}}_I \)
- the nodes \( \{ x_{nk}^{(\mu,\alpha_n)} \}_{k=1}^n \) are zeros of the ORF \( \varphi_n \in \mathcal{L}_n \); hence, can be computed by means of the GEP

the corresponding weights \( \{ \lambda_{nk}^{(\mu,\alpha_n)} \}_{k=1}^n \) can be computed by means of the GEP too:

\[
\lambda_{nk}^{(\mu,\alpha_n)} = |v_{nk}|^2 \|1\|_\mu^2,
\]

where \( v_{nk} \) represents the first component of the eigenvector \( \underline{v}_{nk} \).
Rational Gauss-type quadrature

**Definition**

Rational Gauss-type quadrature = positive RIQ with \( j \) fixed nodes in \( I \), where \( 0 < j < n \), and the remaining \( n - j \) nodes are such that the domain of validity is as large as possible.

Rational Gauss-type quadrature with one fixed node

- generally, for each node that is fixed in advance, the dimension of the maximal possible domain of validity decrease with one
- maximal possible domain of validity for the case of one fixed node:
  - (first kind) \( \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^c \Rightarrow \) existence of the quadrature rule only depends on the node that has been fixed
  - (second kind) \( \mathcal{L}_n \cdot \mathcal{L}_{n-2}^c \Rightarrow \) existence also depends on the last two poles \( \alpha_{n-1} \) and \( \alpha_n \).
- for the second kind we will assume that \( \alpha_n \neq \overline{\alpha}_{n-1} \)
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Theorem

Suppose the quadrature

\[
\int_I f(x) d\mu(x) \approx \lambda_{n_1}^{(1)} f(a) + \sum_{k=2}^{n} \lambda_{nk}^{(1)} f(x_{nk}^{(1)}), \quad a \in I
\]

is exact for every \( f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^c \). Then the nodes \( \{x_{nk}^{(1)}\}_{k=1}^{n} \), with \( x_{n_1}^{(1)} = a \), are zeros of the quasi-orthogonal rational function

\[
Q_{n}^{(1)}(x) = \varphi_{n}(x) + \tau_{n} \left( \frac{1 - x/\alpha_{n-1}}{1 - x/\alpha_{n}} \right) \varphi_{n-1}(x),
\]

\[
\tau_{n} = -\frac{(1 - a/\alpha_{n})\varphi_{n}(a)}{(1 - a/\alpha_{n-1})\varphi_{n-1}(a)} \in \mathbb{C}.
\]

[K. Deckers et al., Numer. Math., 2010 (submitted)]
Rational Gauss-type quadrature: First kind

**Theorem**

Suppose \( c_n^{(1)} := \frac{\tau_n}{E_n} \left( \frac{1}{\alpha_{n-1}} - \frac{1}{\alpha_{n-1}} \right) \neq -1 \). Then there exist poles \( \alpha \in \mathbb{C} \) (not unique), and constants \( \hat{E}_n \neq 0 \) and

\[
\hat{D}_n = \frac{D_n + \tau_n/E_n}{1 + c_n^{(1)}} \quad \text{satisfying condition Favard theorem,}
\]

so that

\[
\left( \frac{1 - x/\alpha_n}{1 - x/\alpha} \right) Q_n^{(1)}(x) = \hat{E}_n \cdot F(\varphi_{n-2}(x), \varphi_{n-1}(x), \alpha_{n-2}, \alpha_{n-1}, \alpha, \hat{D}_n);
\]

hence, there exist positive measures \( \hat{\mu} \) on \( \hat{S} \subset \mathbb{R} \) (not unique), and inner product \( \langle f, g \rangle_{\hat{\mu}} = \int_{\hat{S}} f(x)g(\overline{x})d\hat{\mu}(x) \), so that

\[
\left( \frac{1 - x/\alpha_n}{1 - x/\alpha} \right) Q_n^{(1)}(x) \perp \hat{\mu} \mathcal{L}_{n-1}.
\]
Theorem

Suppose \( c_n^{(1)} := \frac{\tau_n}{E_n} \left( \frac{1}{\alpha_{n-1}} - \frac{1}{\alpha_{n-1}^2} \right) \neq -1. \) Then there exist poles \( \alpha \in \overline{C} \) (not unique), and constants \( \hat{E}_n \neq 0 \) and

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\hat{D}_n = \frac{D_n + \tau_n/E_n}{1 + c_n^{(1)}} \rightarrow \text{satisfying condition Favard theorem,}
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\[
\left( \frac{1 - x/\alpha_n}{1 - x/\alpha} \right) Q_n^{(1)}(x) \perp \hat{\mu} \mathcal{L}_{n-1}.
\]
**Corollary**

The zeros of $Q_n^{(1)}(x)$ are all real distinct iff

$$\Im\{\hat{D}_n\} = \frac{\Im\{1/\alpha_{n-2}\}}{|E_{n-1}|^2}, \quad \alpha_{n-1} \in \mathbb{R},$$

respectively

$$\Re\{\hat{D}_n\}^2 + \left(\Im\{\hat{D}_n\} - \frac{1}{\Im\{1/\alpha_{n-1}\}}\right)^2 = \left(\frac{1}{\Im\{1/\alpha_{n-1}\}}\right)^2 \frac{1}{1 + r_n}, \quad \alpha_{n-1} \notin \mathbb{R},$$

where

$$r_n = \frac{4\Im\{1/\alpha_{k-1}\} \cdot \Im\{1/\alpha_{k-2}\}}{|E_{n-1}|^2} > -1.$$
Lemma

Suppose the zeros of $Q_n^{(1)}(x)$ are all real distinct. Then at least $n - 1$ of them are in $(-1, 1)$, and there exist $\tau_n$ (and hence, $\hat{D}_n$) so that all the zeros are in $[-1, 1]$.


Boundary values $\hat{D}_n^\pm$ for $\hat{D}_n$, in order to have that $
abla_{nk}^{(1)} \subset I$, are obtained by setting

$$\tau_n^\pm := -\frac{(1 \pm 1/\alpha_n)\varphi_n(\pm 1)}{(1 \pm 1/\alpha_{n-1})\varphi_{n-1}(\pm 1)}$$
the nodes \( \{x_{nk}^{(1)}\}_{k=1}^n \) are rational Gaussian nodes:
\[
x_{nk}^{(1)} = x^{(\hat{\mu},\alpha)}_{nk}, \quad k = 1, \ldots, n \text{ and } \alpha \in \mathbb{R}
\]
\( \{\varphi_0, \ldots, \varphi_{n-1}\} \) forms an orthonormal system for the inner product \( \langle \cdot, \cdot \rangle_{\hat{\mu}} \Rightarrow \forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^c \supset \mathcal{L}_{n-1} : \int_I f(x) d\mu(x) = \int_{\hat{\mathcal{S}}} f(x) d\hat{\mu}(x); \)
hence, the weights \( \{\lambda_{nk}^{(1)}\}_{k=1}^n \) are rational Gaussian weights:
\[
\lambda_{nk}^{(1)} = \lambda^{(\hat{\mu},\alpha)}_{nk}, \quad k = 1, \ldots, n \text{ and } \alpha \in \mathbb{R}
\]
the nodes and weights can be computed by means of a modified GEP:
\[
J_n^{(1)} v_{nk}^{(1)} = x_{nk}^{(1)} B_n^{(1)} v_{nk}^{(1)},
\]
where
\[
J_n^{(1)} = \begin{pmatrix}
J_{n-1} & 0_{n-2} \\
0_{n-2} & \ast
\end{pmatrix}, \quad B_n^{(1)} = \begin{pmatrix}
B_{n-1} & 0_{n-2} \\
0_{n-2} & \ast
\end{pmatrix}
\]
and \( v_{nk}^{(1)} \) is the corresponding normalized eigenvector.
the nodes \( \{x_{nk}^{(1)}\}_{k=1}^n \) are rational Gaussian nodes:
\[
x_{nk}^{(1)} = x_{nk}^{(\hat{\mu},\alpha)}, \ k = 1, \ldots, n \quad \text{and} \quad \alpha \in \mathbb{R}_I
\]
\( \{\varphi_0, \ldots, \varphi_{n-1}\} \) forms an orthonormal system for the inner product \( \langle \cdot, \cdot \rangle_{\hat{\mu}} \Rightarrow \)
\[
\forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^c \supset \mathcal{L}_{n-1} : \int_I f(x) d\mu(x) = \int_{\hat{\Sigma}} f(x) d\hat{\mu}(x);
\]

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J_n^{(1)} = 
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\end{pmatrix}, \quad B_n^{(1)} = 
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0_{n-2} \times & 0_{n-2} \times
\end{pmatrix}
\]

and \( v_{nk}^{(1)} \) is the corresponding normalized eigenvector.
Rational Gauss-type quadrature: First kind

- the nodes \( \{x_{nk}^{(1)}\}_{k=1}^n \) are rational Gaussian nodes:
  \[
x_{nk}^{(1)} = x_{nk}^{(\hat{\mu},\alpha)}, \quad k = 1, \ldots, n \] and \( \alpha \in \overline{\mathbb{R}} \)

- \( \{\varphi_0, \ldots, \varphi_{n-1}\} \) forms an orthonormal system for the inner product \( \langle \cdot, \cdot \rangle_{\hat{\mu}} \Rightarrow \)
  \[
  \forall f \in L_{n-1} \cdot L_{n-1}^c \supset L_{n-1} : \int_I f(x) d\mu(x) = \int_{\hat{\mathcal{S}}} f(x) d\hat{\mu}(x);
  \]
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  \[
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- the nodes and weights can be computed by means of a modified GEP: \( J_n^{(1)} v_{nk}^{(1)} = x_{nk}^{(1)} B_n^{(1)} v_{nk}^{(1)} \), where
  \[
  J_n^{(1)} = \begin{pmatrix}
  J_{n-1} & 0_{n-2} \\
  0_{n-2}^T & \ast
  \end{pmatrix}, \quad B_n^{(1)} = \begin{pmatrix}
  B_{n-1} & 0_{n-2} \\
  0_{n-2}^T & \ast
  \end{pmatrix}
  \]
  and \( v_{nk}^{(1)} \) is the corresponding normalized eigenvector.
the nodes \( \{x_{nk}^{(1)}\}_{k=1}^n \) are rational Gaussian nodes:
\[
x_{nk}^{(1)} = x_{nk}^{(\hat{\mu},\alpha)}, \quad k = 1, \ldots, n \text{ and } \alpha \in \overline{\mathbb{R}}
\]
\( \{\varphi_0, \ldots, \varphi_{n-1}\} \) forms an orthonormal system for the inner product \( \langle \cdot, \cdot \rangle_{\hat{\mu}} \Rightarrow \)
\[
\forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^c \supset \mathcal{L}_{n-1} : \int_I f(x) d\mu(x) = \int_{\hat{\mu}} f(x) d\hat{\mu}(x);
\]
hence, the weights \( \{\lambda_{nk}^{(1)}\}_{k=1}^n \) are rational Gaussian weights:
\[
\lambda_{nk}^{(1)} = \lambda_{nk}^{(\hat{\mu},\alpha)}, \quad k = 1, \ldots, n \text{ and } \alpha \in \overline{\mathbb{R}}
\]
the nodes and weights can be computed by means of a modified GEP: 
\[
J_n^{(1)} \phantom{\nu_{nk}} = x_{nk}^{(1)} B_n^{(1)} \nu_{nk}^{(1)}, \quad \text{where}
\]
\[
J_n^{(1)} = \begin{pmatrix}
J_{n-1} & 0_{n-2} \\
0_{n-2}^T & x
\end{pmatrix}, \quad B_n^{(1)} = \begin{pmatrix}
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and \( \nu_{nk}^{(1)} \) is the corresponding normalized eigenvector.
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Rational Gauss-type quadrature: Second kind

[K. Deckers and A. Bultheel (In progress)]

**Theorem**

Suppose the quadrature

\[ \int_I f(x) d\mu(x) \approx \lambda_{n1}^{(2)} f(a) + \sum_{k=2}^{n} \lambda_{nk}^{(2)} f(x_{nk}^{(2)}), \quad a \in I \]

is exact for every \( f \in \mathcal{L}_n \cdot \mathcal{L}_{n-2}^c \). Then the nodes \( \{x_{nk}^{(2)}\}_{k=1}^{n} \), with \( x_{n1}^{(2)} = a \), are zeros of the quasi-orthogonal rational function

\[ Q_n^{(2)}(x) = \varphi_n(x) + \tau_n \varphi_{n-1}(x), \quad \tau_n = -\frac{\varphi_n(a)}{\varphi_{n-1}(a)} \in \mathbb{C}. \]
Rational Gauss-type quadrature: Second kind

\[ \alpha_{n-1} \notin \overline{\mathbb{R}} \]

- \( \alpha_n \notin \overline{\mathbb{R}} \cup \{ \overline{\alpha}_{n-1} \} \Rightarrow \{ \lambda^{(2)}_{nk} \}_{k=1}^n \not\subseteq \mathbb{R}_0^+ \)
- \( \alpha_n \in \overline{\mathbb{R}}_I \Rightarrow \{ \lambda^{(2)}_{nk} \}_{k=1}^n \subseteq \mathbb{R}_0^+ \) iff \( \tau_n = 0 \)

Rational Gaussian quadrature: \( x^{(2)}_{nk} = x^{(\mu,\alpha_n)}_{nk} \) and \( \lambda^{(2)}_{nk} = \lambda^{(\mu,\alpha_n)}_{nk} \), exact in \( \mathcal{L}_n \cdot \mathcal{L}^c_{n-1} \supset \mathcal{L}_n \cdot \mathcal{L}^c_{n-2} \)
Theorem

Let $\alpha_{n-1} \in \overline{\mathbb{R}}_I$ and suppose $c_n^{(2)} := \frac{\tau_n}{E_n} \left( \frac{1}{\alpha_{n-1}} - \frac{1}{\alpha_n} \right) \in (-1, \infty)$. Then there exist poles $\alpha \in \overline{C}_I$ (not unique), and constants $\hat{E}_n \neq 0$, $|k_n|^2 = 1 + c_n^{(2)}$, and

$$\hat{D}_n = \frac{D_n + \tau_n/E_n}{1 + c_n^{(2)}} \rightarrow \text{satisfying condition Favard theorem},$$

so that

$$\left( \frac{1-x/\alpha_n}{1-x/\alpha} \right) Q_n^{(2)}(x) = \hat{E}_n \cdot F(\varphi_{n-2}(x), k_n \varphi_{n-1}(x), \alpha_{n-2}, \alpha_{n-1}, \alpha, \hat{D}_n);$$

hence, there exist positive measures $\hat{\mu}$ on $\hat{S} \subsetneq \mathbb{R}$ (not unique), and inner product $\langle f, g \rangle_{\hat{\mu}} = \int_{\hat{S}} f(x)g(\overline{x})d\hat{\mu}(x)$, so that

$$\left( \frac{1-x/\alpha_n}{1-x/\alpha} \right) Q_n^{(2)}(x) \perp_{\hat{\mu}} \mathcal{L}_{n-1}.$$
Lemma

Suppose $c_n^{(2)} \notin (-1, \infty)$. Then there cannot exist distinct nodes $\{x_{nk}^{(2)}\}_{k=1}^n \subset I$ and weights $\{\lambda_{nk}^{(2)}\}_{k=1}^n \subset \mathbb{R}_0^+$ so that

$$\int_I f(x) d\mu(x) = \sum_{k=1}^n \lambda_{nk}^{(2)} f(x_{nk}^{(2)}), \quad \forall f \in \mathcal{L}_n \cdot \mathcal{L}_n^{c}.$$
Corollary

The zeros of $Q^{(2)}_n(x)$ are all real distinct iff

$$\Im\{\hat{D}_n\} = \frac{\Im\{1/\alpha_{n-2}\}}{|k_n|^2|E_{n-1}|^2}; \text{ i.e., iff } \Im\{\tau_nE_n\} - \Im\{1/\alpha_n\} = 0. \quad (1)$$

- $\alpha_n \in \overline{\mathbb{R}}$: (1) holds true for every $\tau_n$ for which $c^{(2)}_n \in (-1, \infty)$
  Boundary values $\hat{D}^\pm_n$ for $\hat{D}_n$, in order to have that
  $\{x^{(2)}_{nk}\}_{k=1}^n \subset I$, are obtained by setting $\tau_n^\pm := -\frac{\varphi_n(\pm 1)}{\varphi_{n-1}(\pm 1)}$.
- $\alpha_n \notin \overline{\mathbb{R}}$: (1) holds true iff $\tau_nE_{n-1} = \frac{1}{\alpha_{n-1}} - \frac{1}{\alpha_n}$
  Rational Gaussian quadrature (when changing the order of the last two poles), exact in $\mathcal{L}_n \cdot \mathcal{L}\{\overline{\alpha}_1, \ldots, \overline{\alpha}_{n-2}, \overline{\alpha}_n\} \supset \mathcal{L}_n \cdot \mathcal{L}^c_{n-2}$
Corollary

The zeros of $Q_n^{(2)}(x)$ are all real distinct iff

$$\Im\{\hat{D}_n\} = \frac{\Im\{1/\alpha_{n-2}\}}{|k_n|^2 |E_{n-1}|^2}; \text{ i.e., iff } \Im\{\tau_n E_n\} - \Im\{1/\alpha_n\} = 0. \quad (1)$$

- $\alpha_n \in \overline{\mathbb{R}}_I$: (1) holds true for every $\tau_n$ for which $c_n^{(2)} \in (-1, \infty)$
  - Boundary values $\hat{D}_n^\pm$ for $\hat{D}_n$, in order to have that $\{\chi_{nk}^{(2)}\}_{k=1}^n \subset I$, are obtained by setting $\tau_n^\pm := -\frac{\varphi_n(\pm1)}{\varphi_{n-1}(\pm1)}$.

- $\alpha_n \notin \overline{\mathbb{R}}$: (1) holds true iff $\tau_n \overline{E}_{n-1} = \frac{1}{\alpha_{n-1}} - \frac{1}{\alpha_n}$
  - Rational Gaussian quadrature (when changing the order of the last two poles), exact in $\mathcal{L}_n \cdot \mathcal{L}\{\overline{\alpha}_1, \ldots, \overline{\alpha}_{n-2}, \overline{\alpha}_n\} \supset \mathcal{L}_n \cdot \mathcal{L}_c^{1-n}$.
Corollary

The zeros of $Q_n^{(2)}(x)$ are all real distinct iff

$$\mathbb{S}\{\hat{D}_n\} = \mathbb{S}\{1/\alpha_{n-2}\} / |k_n|^2 |E_{n-1}|^2; \ i.e., \ iff \ \mathbb{S}\{\tau_n E_n\} - \mathbb{S}\{1/\alpha_n\} = 0. \ (1)$$

- $\alpha_n \in \overline{\mathbb{R}}_I$: (1) holds true for every $\tau_n$ for which $c_n^{(2)} \in (-1, \infty)$

  Boundary values $\hat{D}_n^\pm$ for $\hat{D}_n$, in order to have that

  $\{x_{nk}^{(2)}\}_{k=1}^n \subset I$, are obtained by setting $\tau_n^\pm := -\varphi_n(\pm 1) / \varphi_{n-1}(\pm 1)$.

- $\alpha_n \notin \overline{\mathbb{R}}$: (1) holds true iff $\tau_n E_{n-1} = 1/\alpha_{n-1} - 1/\alpha_n$

Rational Gaussian quadrature (when changing the order of the last two poles), exact in $L_n \cdot L\{\overline{\alpha}_1, \ldots, \overline{\alpha}_{n-2}, \overline{\alpha}_n\} \supset L_n \cdot L^c_{n-2}$
Rational Gauss-type quadrature: Second kind

- the nodes \( \{x_{nk}^{(2)}\}_{k=1}^n \) are rational Gaussian nodes:
  \[ x_{nk}^{(2)} = x_{nk}^{(\hat{\mu}, \alpha)}, \quad k = 1, \ldots, n \text{ and } \alpha \in \mathbb{R} \]
- \( \{\varphi_0, \ldots, \varphi_{n-2}, k_n \varphi_{n-1}\} \) forms an orthonormal system for the inner product \( \langle \cdot, \cdot \rangle_{\hat{\mu}} \Rightarrow \)
  \[ \forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-2}^c \supset \mathcal{L}_{n-1} : \int_I f(x) d\mu(x) = \int_{\hat{S}} f(x) d\hat{\mu}(x); \]
  hence, the weights \( \{\lambda_{nk}^{(2)}\}_{k=1}^n \) are rational Gaussian weights:
  \[ \lambda_{nk}^{(2)} = \lambda_{nk}^{(\hat{\mu}, \alpha)}, \quad k = 1, \ldots, n \text{ and } \alpha \in \mathbb{R} \]
- the nodes and weights can be computed by means of a modified GEP:
  \[ J_n^{(2)} \check{\nu}_{nk}^{(2)} = x_{nk}^{(2)} B_n^{(2)} \check{\nu}_{nk}^{(2)}, \]
  where
  \[ J_n^{(2)} = \begin{pmatrix} J_{n-1} & 0_{n-2} \\ 0_{n-2}^T & X \end{pmatrix}, \quad B_n^{(2)} = \begin{pmatrix} B_{n-1} & 0_{n-2} \\ 0_{n-2}^T & X \end{pmatrix} \]
  and \( \check{\nu}_{nk}^{(2)} \) is the corresponding normalized eigenvector.
Rational Gauss-type quadrature: Second kind

1. The nodes \( \{x_{nk}^{(2)}\}_{k=1}^n \) are rational Gaussian nodes:
   \[
x_{nk}^{(2)} = x_{nk}^{(\hat{\mu},\alpha)}, \quad k = 1, \ldots, n \text{ and } \alpha \in \overline{\mathbb{R}}_I
   \]

2. The set \( \{\varphi_0, \ldots, \varphi_{n-2}, k_n \varphi_{n-1}\} \) forms an orthonormal system for the inner product \( \langle \cdot, \cdot \rangle_{\hat{\mu}} \Rightarrow \)
   \[
   \forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-2}^c \supseteq \mathcal{L}_{n-1} : \int_I f(x) d\mu(x) = \int_{\hat{\Sigma}} f(x) d\hat{\mu}(x);
   \]
   hence, the weights \( \{\lambda_{nk}^{(2)}\}_{k=1}^n \) are rational Gaussian weights:
   \[
   \lambda_{nk}^{(2)} = \lambda_{nk}^{(\hat{\mu},\alpha)}, \quad k = 1, \ldots, n \text{ and } \alpha \in \overline{\mathbb{R}}_I
   \]

3. The nodes and weights can be computed by means of a modified GEP:
   \[
   J_{\left(2\right)}^{n} \nu_{\left(2\right)}^{\left(n\right)} = x_{nk}^{(2)} B_{\left(2\right)}^{n} \nu_{\left(2\right)}^{\left(nk\right)}, \text{ where}
   \]
   \[
   J_{\left(2\right)}^{n} = \begin{pmatrix}
   J_{n-1} & 0_{n-2} \\
   0_{n-2}^T & \times
   \end{pmatrix}, \quad B_{\left(2\right)}^{n} = \begin{pmatrix}
   B_{n-1} & 0_{n-2} \\
   0_{n-2}^T & \times
   \end{pmatrix}
   \]
   and \( \nu_{\left(2\right)}^{\left(nk\right)} \) is the corresponding normalized eigenvector.
Rational Gauss-type quadrature: Second kind

- The nodes \( \{x^{(2)}_{nk}\}_{k=1}^{n} \) are rational Gaussian nodes:
  \[
x^{(2)}_{nk} = x^{(\hat{\mu}, \alpha)}_{nk}, \quad k = 1, \ldots, n \text{ and } \alpha \in \mathbb{R}_I
\]

- The set \( \{\varphi_0, \ldots, \varphi_{n-2}, k_n \varphi_{n-1}\} \) forms an orthonormal system for the inner product \( \langle \cdot, \cdot \rangle_{\hat{\mu}} \Rightarrow \)
  \[
  \forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-2}^c \supset \mathcal{L}_{n-1} : \int_{I} f(x) d\mu(x) = \int_{\hat{S}} f(x) d\hat{\mu}(x);
  \]
  hence, the weights \( \{\lambda^{(2)}_{nk}\}_{k=1}^{n} \) are rational Gaussian weights:
  \[
  \lambda^{(2)}_{nk} = \lambda^{(\hat{\mu}, \alpha)}_{nk}, \quad k = 1, \ldots, n \text{ and } \alpha \in \mathbb{R}_I
\]

- The nodes and weights can be computed by means of a modified GEP: 
  \[
  J^{(2)}_n \mathbf{v}^{(2)}_{nk} = x^{(2)}_{nk} \mathbf{B}^{(2)}_n \mathbf{v}^{(2)}_{nk}, \text{ where}
\]

\[
J^{(2)}_n = \begin{pmatrix}
J_{n-1} & 0_{n-2} \\
0_{n-2} & X \\
0_{n-2}^T & X
\end{pmatrix}, \quad \mathbf{B}^{(2)}_n = \begin{pmatrix}
\mathbf{B}_{n-1} & 0_{n-2} \\
0_{n-2}^T & X \\
0_{n-2} & X
\end{pmatrix}
\]

and \( \mathbf{v}^{(2)}_{nk} \) is the corresponding normalized eigenvector.
the nodes \( \{x_{nk}^{(2)}\}_{k=1}^{n} \) are rational Gaussian nodes:
\[
x_{nk}^{(2)} = x_{nk}^{(\hat{\mu},\alpha)}, \quad k = 1, \ldots, n \quad \text{and} \quad \alpha \in \mathbb{R}
\]
\{\varphi_0, \ldots, \varphi_{n-2}, k_n \varphi_{n-1}\} forms an orthonormal system for the inner product \( \langle \cdot, \cdot \rangle_{\hat{\mu}} \Rightarrow \)
\[
\forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-2}^c \supset \mathcal{L}_{n-1} : \int_I f(x) d\mu(x) = \int_{\hat{S}} f(x) d\hat{\mu}(x);
\]
hence, the weights \( \{\lambda_{nk}^{(2)}\}_{k=1}^{n} \) are rational Gaussian weights:
\[
\lambda_{nk}^{(2)} = \lambda_{nk}^{(\hat{\mu},\alpha)}, \quad k = 1, \ldots, n \quad \text{and} \quad \alpha \in \mathbb{R}
\]
the nodes and weights can be computed by means of a modified GEP:
\[
\mathbf{J}_{n}^{(2)} \mathbf{v}_{nk}^{(2)} = x_{nk}^{(2)} \mathbf{B}_{n}^{(2)} \mathbf{v}_{nk}^{(2)} , \quad \text{where}
\]

\[
\mathbf{J}_{n}^{(2)} = \begin{pmatrix}
\mathbf{J}_{n-1} & 0_{n-2}^T \\
0_{n-2} & \mathbf{X}
\end{pmatrix}, \quad \mathbf{B}_{n}^{(2)} = \begin{pmatrix}
\mathbf{B}_{n-1} & 0_{n-2}^T \\
0_{n-2} & \mathbf{X}
\end{pmatrix}
\]

and \( \mathbf{v}_{nk}^{(2)} \) is the corresponding normalized eigenvector.
References

Thank you ...