Rational Gauss-type quadrature formulas with a prescribed node anywhere on the real line

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Outline

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Orthonormal rational functions (ORFs)

Notation

$I = [-1, 1], \quad X \subseteq \mathbb{C} : \ \overline{X} = X \cup \{\infty\}$ and $X_I = \{x \in X : x \notin I\}$

Rational functions

$b_k(x) = \prod_{j=1}^{k} \frac{x}{1-x/\alpha_j}, \quad \alpha_j \in \overline{\mathbb{C}}_I$

$\mathcal{L}_n = \mathcal{L}\{\alpha_1, \ldots, \alpha_n\}$
$\quad = \mathrm{span}\{1, b_1(x), \ldots, b_n(x)\}$

Polynomials

$\forall j : \alpha_j = \infty \Rightarrow b_k(x) \equiv x^k$

$\mathcal{L}_n \equiv \mathcal{P}_n = \mathrm{span}\{1, x, \ldots, x^n\}$

Orthonormal rational functions (ORFs)

Given a positive measure μ on I and inner product

$$\langle f, g \rangle_\mu = \int_I f(x)\overline{g(\overline{x})}d\mu(x)$$

$\rightarrow$ ORFs: $\varphi_k(x)$

$$\varphi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}, \ \varphi_k \perp_\mu \mathcal{L}_{k-1}, \text{ and } \|\varphi_k\|_\mu = \sqrt{\langle \varphi_k, \varphi_k \rangle_\mu} = 1.$$

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Orthonormal rational functions (ORFs)

Theorem (Recurrence)

The ORFs φ_k, $k > 0$, satisfy a three-term recurrence relation:

$$\varphi_k(x) = E_k \cdot F(\varphi_{k-2}(x), \varphi_{k-1}(x), \alpha_{k-2}, \alpha_{k-1}, \alpha_k, D_k)$$

with initial conditions $\varphi_{-1}(x) \equiv 0$, $\varphi_0(x) \equiv \eta_0 \|1\|_\mu^{-1}$, $|\eta_0| = 1$, and $\alpha_{-1} = \alpha_0 = \infty$.

[K. Deckers and A. Bultheel, IMA J. Numer. Anal., 2009]

Corollary (Generalized eigenvalue problem (GEP))

The zeros $\{x_{nk}\}_{k=1}^n$ of the ORF $\varphi_n(x)$ are eigenvalues of a GEP of the form $\mathbf{J}_n \underline{v}_{nk} = x_{nk} \mathbf{B}_n \underline{v}_{nk}$, where $\mathbf{J}_n$ and $\mathbf{B}_n$ are tridiagonal matrices, and $\underline{v}_{nk}$ is the corresponding normalized eigenvector.

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Theorem

The ORF φ_n has n real distinct zeros in I iff $\alpha_n \in \overline{\mathbb{R}}_I$.
[K. Deckers et al., Numer. Math., 2010 (submitted)]

Theorem (Favard Theorem)

Suppose the sequence of rational functions $\{\hat{\varphi}_k\}_{k=0}^{\infty}$, with $\hat{\varphi}_k \in \hat{\mathcal{L}}_k \setminus \hat{\mathcal{L}}_{k-1}$ and $\hat{\mathcal{L}}_k = \mathcal{L}\{\hat{\alpha}_1, \ldots, \hat{\alpha}_k\}$, are generated by the three-term recurrence relation

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Then - under certain conditions on $\hat{E}_k$ and $\hat{D}_k$ - the $\hat{\varphi}_k$ form an orthonormal system w.r.t. a positive measure $\hat{\mu}$ on $\hat{S} \subsetneq \mathbb{R}$ and inner product $\langle f, g \rangle_{\hat{\mu}} = \int_{\hat{S}} f(x)\overline{g(\overline{x})} d\hat{\mu}(x)$.
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Outline

Rational interpolatory quadrature rule

Rational interpolatory quadrature rule (RIQ)

Given the set of n distinct nodes $\{x_{nk}\}_{k=1}^{n} \subset I$, there exists a unique set of weights $\{\lambda_{nk}\}_{k=1}^{n} \subset \mathbb{C}$ so that

$$\int_I f(x)d\mu(x) \approx \sum_{k=1}^{n} \lambda_{nk} f(x_{nk})$$

is exact for every $f \in \tilde{\mathcal{L}}_m$, with
$\mathcal{L}_{n-1} \subseteq \tilde{\mathcal{L}}_m \subset \mathcal{L}\{\alpha_1, \ldots, \alpha_n, \overline{\alpha}_1, \ldots, \overline{\alpha}_n\} =: \mathcal{L}_n \cdot \mathcal{L}_n^c$.
The space $\tilde{\mathcal{L}}_m$ is called the domain of validity.

Positive weights

- The weights $\{\lambda_{nk}\}_{k=1}^{n} \subset \mathbb{R}$ iff $\tilde{\mathcal{L}}_m \equiv \tilde{\mathcal{L}}_m^c$;
- Positive RIQ = RIQ with weights $\{\lambda_{nk}\}_{k=1}^{n} \subset \mathbb{R}_0^+$.

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Rational interpolatory quadrature rule

Rational Gaussian quadrature

- $=$ positive RIQ
- has maximal domain of validity; i.e., $\tilde{\mathcal{L}}_m = \mathcal{L}_n \cdot \mathcal{L}_{n-1}^c$
- exists iff $\alpha_n \in \overline{\mathbb{R}}_I$
- the nodes $\{x_{nk}^{(\mu,\alpha_n)}\}_{k=1}^n$ are zeros of the ORF $\varphi_n \in \mathcal{L}_n$; hence, can be computed by means of the GEP
- the corresponding weights $\{\lambda_{nk}^{(\mu,\alpha_n)}\}_{k=1}^n$ can be computed by means of the GEP too:

$$\lambda_{nk}^{(\mu,\alpha_n)} = |v_{nk}|^2 \, \|1\|_\mu^2 \,,$$

where v_{nk} represents the first component of the eigenvector $\underline{v}_{nk}$.

Rational Gauss-type quadrature

Definition

Rational Gauss-type quadrature = positive RIQ with j fixed nodes in I, where $0 < j < n$, and the remaining $n - j$ nodes are such that the domain of validity is as large as possible.

Rational Gauss-type quadrature with one fixed node

- generally, for each node that is fixed in advance, the dimension of the maximal possible domain of validity decrease with one
- maximal possible domain of validity for the case of one fixed node:
  - (first kind) $\mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^c \Rightarrow$ existence of the quadrature rule only depends on the node that has been fixed
  - (second kind) $\mathcal{L}_n \cdot \mathcal{L}_{n-2}^c \Rightarrow$ existence also depends on the last two poles α_{n-1} and α_n.
- for the second kind we will assume that $\alpha_n \neq \overline{\alpha}_{n-1}$

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Outline

Rational Gauss-type quadrature: First kind

[K. Deckers et al., Numer. Math., 2010 (submitted)]

Theorem

Suppose the quadrature

$$\int_I f(x)d\mu(x) \approx \lambda_{n1}^{(1)} f(a) + \sum_{k=2}^{n} \lambda_{nk}^{(1)} f(x_{nk}^{(1)}), \quad a \in I$$

is exact for every $f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^{c}$. *Then the nodes* $\{x_{nk}^{(1)}\}_{k=1}^{n}$, *with* $x_{n1}^{(1)} = a$, *are zeros of the quasi-orthogonal rational function*

$$Q_n^{(1)}(x) = \varphi_n(x) + \tau_n \left(\frac{1 - x/\overline{\alpha}_{n-1}}{1 - x/\alpha_n} \right) \varphi_{n-1}(x),$$

$$\tau_n = -\frac{(1 - a/\alpha_n)\varphi_n(a)}{(1 - a/\overline{\alpha}_{n-1})\varphi_{n-1}(a)} \in \mathbb{C}.$$

Rational Gauss-type quadrature: First kind

Theorem

Suppose $c_n^{(1)} := \frac{\tau_n}{E_n}\left(\frac{1}{\alpha_{n-1}} - \frac{1}{\overline{\alpha}_{n-1}}\right) \neq -1$. *Then there exist poles* $\alpha \in \overline{\mathbb{C}}_I$ *(not unique), and constants* $\hat{E}_n \neq 0$ *and*

$$\hat{D}_n = \frac{D_n + \tau_n/E_n}{1 + c_n^{(1)}} \rightarrow \textit{satisfying condition Favard theorem},$$

so that

$$\left(\frac{1 - x/\alpha_n}{1 - x/\alpha}\right) Q_n^{(1)}(x) = \hat{E}_n \cdot F(\varphi_{n-2}(x), \varphi_{n-1}(x), \alpha_{n-2}, \alpha_{n-1}, \alpha, \hat{D}_n);$$

hence, there exist positive measures $\hat{\mu}$ *on* $\hat{S} \subsetneq \mathbb{R}$ *(not unique), and inner product* $\langle f, g\rangle_{\hat{\mu}} = \int_{\hat{S}} f(x)\overline{g(\overline{x})}d\hat{\mu}(x)$, *so that* $\left(\frac{1-x/\alpha_n}{1-x/\alpha}\right) Q_n^{(1)}(x) \perp_{\hat{\mu}} \mathcal{L}_{n-1}$.

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Rational Gauss-type quadrature: First kind

Corollary

The zeros of $Q_n^{(1)}(x)$ *are all real distinct iff*

$$\Im\{\hat{D}_n\} = \frac{\Im\{1/\alpha_{n-2}\}}{|E_{n-1}|^2}, \quad \alpha_{n-1} \in \overline{\mathbb{R}}_I,$$

respectively

$$\Re\{\hat{D}_n\}^2 + \left\{\Im\{\hat{D}_n\} - \frac{1}{\Im\{1/\overline{\alpha}_{n-1}\}}\right\}^2 = \frac{\left(\frac{1}{\Im\{1/\overline{\alpha}_{n-1}\}}\right)^2}{1+r_n}, \quad \alpha_{n-1} \notin \overline{\mathbb{R}},$$

where

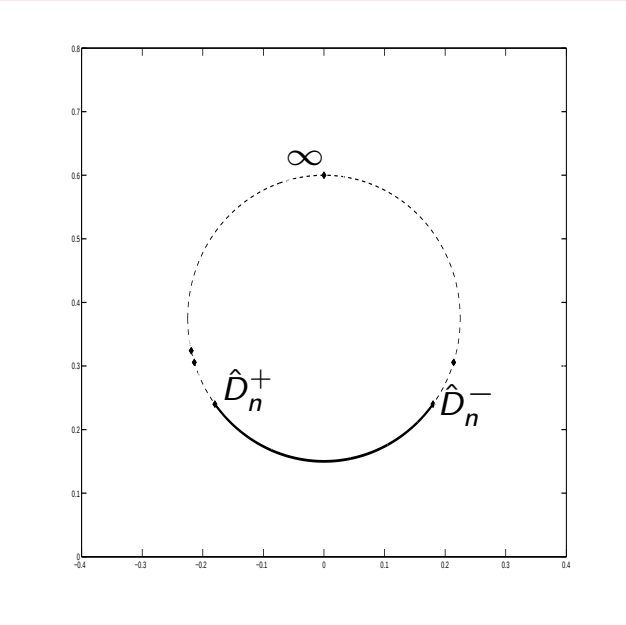
$$r_n = \frac{4\Im\{1/\overline{\alpha}_{k-1}\} \cdot \Im\{1/\alpha_{k-2}\}}{|E_{n-1}|^2} > -1.$$

Rational Gauss-type quadrature: First kind

Lemma

Suppose the zeros of $Q_n^{(1)}(x)$ are all real distinct. Then at least $n-1$ of them are in $(-1,1)$, and there exist τ_n (and hence, $\hat{D}_n$) so that all the zeros are in $[-1,1]$.

[J. Van Deun and A. Bultheel, J. Comput. Appl. Math., 2003]



Boundary values $\hat{D}_n^{\pm}$ for $\hat{D}_n$, in order to have that $\{x_{nk}^{(1)}\}_{k=1}^n \subset I$, are obtained by setting

$$\tau_n^{\pm} := -\frac{(1\mp 1/\alpha_n)\varphi_n(\pm 1)}{(1\mp 1/\overline{\alpha}_{n-1})\varphi_{n-1}(\pm 1)}$$

Rational Gauss-type quadrature: First kind

- the nodes $\{x_{nk}^{(1)}\}_{k=1}^{n}$ are rational Gaussian nodes: $x_{nk}^{(1)} = x_{nk}^{(\hat{\mu},\alpha)}$, $k = 1, \ldots, n$ and $\alpha \in \overline{\mathbb{R}}_I$
- $\{\varphi_0, \ldots, \varphi_{n-1}\}$ forms an orthonormal system for the inner product $\langle \cdot\,, \cdot \rangle_{\hat{\mu}} \Rightarrow$ $\forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^{c} \supset \mathcal{L}_{n-1} : \int_I f(x) d\mu(x) = \int_{\hat{S}} f(x) d\hat{\mu}(x)$; hence, the weights $\{\lambda_{nk}^{(1)}\}_{k=1}^{n}$ are rational Gaussian weights: $\lambda_{nk}^{(1)} = \lambda_{nk}^{(\hat{\mu},\alpha)}$, $k = 1, \ldots, n$ and $\alpha \in \overline{\mathbb{R}}_I$
- the nodes and weights can be computed by means of a modified GEP: $\mathbf{J}_n^{(1)} \underline{v}_{nk}^{(1)} = x_{nk}^{(1)} \mathbf{B}_n^{(1)} \underline{v}_{nk}^{(1)}$, where

$$\mathbf{J}_n^{(1)} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathbf{J}_{n-1}} & \begin{matrix} \underline{0}_{n-2} \\ * \end{matrix} \\ \hline \underline{0}_{n-2}^T & \times & \times \end{array}\right), \quad \mathbf{B}_n^{(1)} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathbf{B}_{n-1}} & \begin{matrix} \underline{0}_{n-2} \\ * \end{matrix} \\ \hline \underline{0}_{n-2}^T & \times & \times \end{array}\right)$$

and $\underline{v}_{nk}^{(1)}$ is the corresponding normalized eigenvector.

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Rational Gauss-type quadrature: Second kind

[K. Deckers and A. Bultheel (In progress)]

Theorem

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is exact for every $f \in \mathcal{L}_n \cdot \mathcal{L}_{n-2}^c$. *Then the nodes* $\{x_{nk}^{(2)}\}_{k=1}^n$, *with* $x_{n1}^{(2)} = a$, *are zeros of the quasi-orthogonal rational function*

$$Q_n^{(2)}(x) = \varphi_n(x) + \tau_n \varphi_{n-1}(x), \qquad \tau_n = -\frac{\varphi_n(a)}{\varphi_{n-1}(a)} \in \mathbb{C}.$$

Rational Gauss-type quadrature: Second kind

$\alpha_{n-1} \notin \overline{\mathbb{R}}$

- $\alpha_n \notin \overline{\mathbb{R}} \cup \{\overline{\alpha}_{n-1}\} \Rightarrow \{\lambda_{nk}^{(2)}\}_{k=1}^{n} \not\subseteq \mathbb{R}_0^+$
- $\alpha_n \in \overline{\mathbb{R}}_I \Rightarrow \{\lambda_{nk}^{(2)}\}_{k=1}^{n} \subseteq \mathbb{R}_0^+$ iff $\tau_n = 0$
  Rational Gaussian quadrature: $x_{nk}^{(2)} = x_{nk}^{(\mu,\alpha_n)}$ and $\lambda_{nk}^{(2)} = \lambda_{nk}^{(\mu,\alpha_n)}$, exact in $\mathcal{L}_n \cdot \mathcal{L}_{n-1}^c \supset \mathcal{L}_n \cdot \mathcal{L}_{n-2}^c$

Rational Gauss-type quadrature: Second kind

Theorem

Let $\alpha_{n-1} \in \overline{\mathbb{R}}_I$ *and suppose* $c_n^{(2)} := \frac{\tau_n}{E_n}\left(\frac{1}{\alpha_{n-1}} - \frac{1}{\alpha_n}\right) \in (-1, \infty)$.
Then there exist poles $\alpha \in \overline{\mathbb{C}}_I$ *(not unique), and constants* $\hat{E}_n \neq 0$, $|k_n|^2 = 1 + c_n^{(2)}$, *and*

$$\hat{D}_n = \frac{D_n + \tau_n/E_n}{1 + c_n^{(2)}} \rightarrow \textit{satisfying condition Favard theorem},$$

so that
$\left(\frac{1-x/\alpha_n}{1-x/\alpha}\right) Q_n^{(2)}(x) = \hat{E}_n \cdot F(\varphi_{n-2}(x), k_n\varphi_{n-1}(x), \alpha_{n-2}, \alpha_{n-1}, \alpha, \hat{D}_n)$;
hence, there exist positive measures $\hat{\mu}$ *on* $\hat{S} \subsetneq \mathbb{R}$ *(not unique), and inner product* $\langle f, g\rangle_{\hat{\mu}} = \int_{\hat{S}} f(x)\overline{g(\overline{x})}d\hat{\mu}(x)$, *so that*
$\left(\frac{1-x/\alpha_n}{1-x/\alpha}\right) Q_n^{(2)}(x) \perp_{\hat{\mu}} \mathcal{L}_{n-1}$.

Rational Gauss-type quadrature: Second kind

Lemma

Suppose $c_n^{(2)} \notin (-1, \infty)$. *Then there cannot exist distinct nodes* $\{x_{nk}^{(2)}\}_{k=1}^n \subset I$ *and weights* $\{\lambda_{nk}^{(2)}\}_{k=1}^n \subset \mathbb{R}_0^+$ *so that*

$$\int_I f(x) d\mu(x) = \sum_{k=1}^n \lambda_{nk}^{(2)} f(x_{nk}^{(2)}), \quad \forall f \in \mathcal{L}_n \cdot \mathcal{L}_{n-2}^c.$$

Rational Gauss-type quadrature: Second kind

Corollary

The zeros of $Q_n^{(2)}(x)$ are all real distinct iff

$$\Im\{\hat{D}_n\} = \frac{\Im\{1/\alpha_{n-2}\}}{|k_n|^2\,|E_{n-1}|^2};\ \textit{i.e., iff}\ \Im\{\tau_n\overline{E_n}\} - \Im\{1/\alpha_n\} = 0. \qquad (1)$$

- $\alpha_n \in \overline{\mathbb{R}}_I$: (1) holds true for every τ_n for which $c_n^{(2)} \in (-1,\infty)$
  Boundary values $\hat{D}_n^{\pm}$ for $\hat{D}_n$, in order to have that $\{x_{nk}^{(2)}\}_{k=1}^n \subset I$, are obtained by setting $\tau_n^{\pm} := -\frac{\varphi_n(\pm 1)}{\varphi_{n-1}(\pm 1)}$.
- $\alpha_n \notin \overline{\mathbb{R}}$: (1) holds true iff $\tau_n\overline{E}_{n-1} = \frac{1}{\alpha_{n-1}} - \frac{1}{\overline{\alpha}_n}$
  Rational Gaussian quadrature (when changing the order of the last two poles), exact in $\mathcal{L}_n \cdot \mathcal{L}\{\overline{\alpha}_1,\ldots,\overline{\alpha}_{n-2},\overline{\alpha}_n\} \supset \mathcal{L}_n \cdot \mathcal{L}_{n-2}^c$

Rational Gauss-type quadrature: Second kind

Corollary

The zeros of $Q_n^{(2)}(x)$ are all real distinct iff

$$\Im\{\hat{D}_n\} = \frac{\Im\{1/\alpha_{n-2}\}}{|k_n|^2\,|E_{n-1}|^2};\ \textit{i.e., iff}\ \Im\{\tau_n\overline{E_n}\} - \Im\{1/\alpha_n\} = 0. \qquad (1)$$

- $\alpha_n \in \overline{\mathbb{R}}_I$: (1) holds true for every τ_n for which $c_n^{(2)} \in (-1,\infty)$ Boundary values $\hat{D}_n^{\pm}$ for $\hat{D}_n$, in order to have that $\{x_{nk}^{(2)}\}_{k=1}^n \subset I$, are obtained by setting $\tau_n^{\pm} := -\frac{\varphi_n(\pm 1)}{\varphi_{n-1}(\pm 1)}$.
- $\alpha_n \notin \overline{\mathbb{R}}$: (1) holds true iff $\tau_n\overline{E}_{n-1} = \frac{1}{\alpha_{n-1}} - \frac{1}{\overline{\alpha}_n}$ Rational Gaussian quadrature (when changing the order of the last two poles), exact in $\mathcal{L}_n \cdot \mathcal{L}\{\overline{\alpha}_1,\ldots,\overline{\alpha}_{n-2},\overline{\alpha}_n\} \supset \mathcal{L}_n \cdot \mathcal{L}_{n-2}^c$

Rational Gauss-type quadrature: Second kind

Corollary

The zeros of $Q_n^{(2)}(x)$ are all real distinct iff

$$\Im\{\hat{D}_n\} = \frac{\Im\{1/\alpha_{n-2}\}}{|k_n|^2\,|E_{n-1}|^2};\ \textit{i.e., iff}\ \Im\{\tau_n\overline{E_n}\} - \Im\{1/\alpha_n\} = 0. \qquad (1)$$

- $\alpha_n \in \overline{\mathbb{R}}_I$: (1) holds true for every τ_n for which $c_n^{(2)} \in (-1,\infty)$
  Boundary values $\hat{D}_n^{\pm}$ for $\hat{D}_n$, in order to have that $\{x_{nk}^{(2)}\}_{k=1}^n \subset I$, are obtained by setting $\tau_n^{\pm} := -\frac{\varphi_n(\pm 1)}{\varphi_{n-1}(\pm 1)}$.
- $\alpha_n \notin \overline{\mathbb{R}}$: (1) holds true iff $\tau_n\overline{E}_{n-1} = \frac{1}{\alpha_{n-1}} - \frac{1}{\overline{\alpha}_n}$
  Rational Gaussian quadrature (when changing the order of the last two poles), exact in $\mathcal{L}_n \cdot \mathcal{L}\{\overline{\alpha}_1,\ldots,\overline{\alpha}_{n-2},\overline{\alpha}_n\} \supset \mathcal{L}_n \cdot \mathcal{L}_{n-2}^c$

Rational Gauss-type quadrature: Second kind

- the nodes $\{x_{nk}^{(2)}\}_{k=1}^{n}$ are rational Gaussian nodes: $x_{nk}^{(2)} = x_{nk}^{(\hat{\mu},\alpha)}$, $k = 1, \ldots, n$ and $\alpha \in \overline{\mathbb{R}}_I$
- $\{\varphi_0, \ldots, \varphi_{n-2}, k_n\varphi_{n-1}\}$ forms an orthonormal system for the inner product $\langle \cdot\,, \cdot \rangle_{\hat{\mu}} \Rightarrow$ $\forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-2}^{c} \supset \mathcal{L}_{n-1} : \int_I f(x)d\mu(x) = \int_{\hat{S}} f(x)d\hat{\mu}(x)$; hence, the weights $\{\lambda_{nk}^{(2)}\}_{k=1}^{n}$ are rational Gaussian weights: $\lambda_{nk}^{(2)} = \lambda_{nk}^{(\hat{\mu},\alpha)}$, $k = 1, \ldots, n$ and $\alpha \in \overline{\mathbb{R}}_I$
- the nodes and weights can be computed by means of a modified GEP: $\mathbf{J}_n^{(2)} \underline{v}_{nk}^{(2)} = x_{nk}^{(2)} \mathbf{B}_n^{(2)} \underline{v}_{nk}^{(2)}$, where

$$\mathbf{J}_n^{(2)} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathbf{J}_{n-1}} & \begin{matrix} \underline{0}_{n-2} \\ \times \end{matrix} \\ \hline \underline{0}_{n-2}^T & \times & \times \end{array}\right), \quad \mathbf{B}_n^{(2)} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathbf{B}_{n-1}} & \begin{matrix} \underline{0}_{n-2} \\ \times \end{matrix} \\ \hline \underline{0}_{n-2}^T & \times & \times \end{array}\right)$$

and $\underline{v}_{nk}^{(2)}$ is the corresponding normalized eigenvector.

Rational Gauss-type quadrature: Second kind

- the nodes $\{x_{nk}^{(2)}\}_{k=1}^{n}$ are rational Gaussian nodes: $x_{nk}^{(2)} = x_{nk}^{(\hat{\mu},\alpha)}$, $k = 1, \ldots, n$ and $\alpha \in \overline{\mathbb{R}}_I$
- $\{\varphi_0, \ldots, \varphi_{n-2}, k_n\varphi_{n-1}\}$ forms an orthonormal system for the inner product $\langle \cdot\,, \cdot \rangle_{\hat{\mu}} \Rightarrow$ $\forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-2}^{c} \supset \mathcal{L}_{n-1} : \int_I f(x)d\mu(x) = \int_{\hat{S}} f(x)d\hat{\mu}(x)$; hence, the weights $\{\lambda_{nk}^{(2)}\}_{k=1}^{n}$ are rational Gaussian weights: $\lambda_{nk}^{(2)} = \lambda_{nk}^{(\hat{\mu},\alpha)}$, $k = 1, \ldots, n$ and $\alpha \in \overline{\mathbb{R}}_I$
- the nodes and weights can be computed by means of a modified GEP: $\mathbf{J}_n^{(2)} \underline{v}_{nk}^{(2)} = x_{nk}^{(2)} \mathbf{B}_n^{(2)} \underline{v}_{nk}^{(2)}$, where

$$\mathbf{J}_n^{(2)} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathbf{J}_{n-1}} & \underline{0}_{n-2} \\ & & \times \\ \hline \underline{0}_{n-2}^T & \times & \times \end{array}\right), \quad \mathbf{B}_n^{(2)} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathbf{B}_{n-1}} & \underline{0}_{n-2} \\ & & \times \\ \hline \underline{0}_{n-2}^T & \times & \times \end{array}\right)$$

and $\underline{v}_{nk}^{(2)}$ is the corresponding normalized eigenvector.

Rational Gauss-type quadrature: Second kind

- the nodes $\{x_{nk}^{(2)}\}_{k=1}^{n}$ are rational Gaussian nodes: $x_{nk}^{(2)} = x_{nk}^{(\hat{\mu},\alpha)}$, $k = 1, \ldots, n$ and $\alpha \in \overline{\mathbb{R}}_I$
- $\{\varphi_0, \ldots, \varphi_{n-2}, k_n\varphi_{n-1}\}$ forms an orthonormal system for the inner product $\langle \cdot\,, \cdot \rangle_{\hat{\mu}} \Rightarrow$ $\forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-2}^{c} \supset \mathcal{L}_{n-1} : \int_I f(x)d\mu(x) = \int_{\hat{S}} f(x)d\hat{\mu}(x)$; hence, the weights $\{\lambda_{nk}^{(2)}\}_{k=1}^{n}$ are rational Gaussian weights: $\lambda_{nk}^{(2)} = \lambda_{nk}^{(\hat{\mu},\alpha)}$, $k = 1, \ldots, n$ and $\alpha \in \overline{\mathbb{R}}_I$
- the nodes and weights can be computed by means of a modified GEP: $\mathbf{J}_n^{(2)} \underline{v}_{nk}^{(2)} = x_{nk}^{(2)} \mathbf{B}_n^{(2)} \underline{v}_{nk}^{(2)}$, where

$$\mathbf{J}_n^{(2)} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathbf{J}_{n-1}} & \underline{0}_{n-2} \\ & & \times \\ \hline \underline{0}_{n-2}^T & \times & \times \end{array}\right), \quad \mathbf{B}_n^{(2)} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathbf{B}_{n-1}} & \underline{0}_{n-2} \\ & & \times \\ \hline \underline{0}_{n-2}^T & \times & \times \end{array}\right)$$

and $\underline{v}_{nk}^{(2)}$ is the corresponding normalized eigenvector.

Rational Gauss-type quadrature: Second kind

- the nodes $\{x_{nk}^{(2)}\}_{k=1}^{n}$ are rational Gaussian nodes: $x_{nk}^{(2)} = x_{nk}^{(\hat{\mu},\alpha)}$, $k = 1, \ldots, n$ and $\alpha \in \overline{\mathbb{R}}_I$
- $\{\varphi_0, \ldots, \varphi_{n-2}, k_n\varphi_{n-1}\}$ forms an orthonormal system for the inner product $\langle \cdot\,, \cdot \rangle_{\hat{\mu}} \Rightarrow$
  $\forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-2}^{c} \supset \mathcal{L}_{n-1} : \int_I f(x)d\mu(x) = \int_{\hat{S}} f(x)d\hat{\mu}(x)$;
  hence, the weights $\{\lambda_{nk}^{(2)}\}_{k=1}^{n}$ are rational Gaussian weights: $\lambda_{nk}^{(2)} = \lambda_{nk}^{(\hat{\mu},\alpha)}$, $k = 1, \ldots, n$ and $\alpha \in \overline{\mathbb{R}}_I$
- the nodes and weights can be computed by means of a modified GEP: $\mathbf{J}_n^{(2)} \underline{v}_{nk}^{(2)} = x_{nk}^{(2)} \mathbf{B}_n^{(2)} \underline{v}_{nk}^{(2)}$, where

$$\mathbf{J}_n^{(2)} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathbf{J}_{n-1}} & \underline{0}_{n-2} \\ & & \times \\ \hline \underline{0}_{n-2}^T & \times & \times \end{array}\right), \quad \mathbf{B}_n^{(2)} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathbf{B}_{n-1}} & \underline{0}_{n-2} \\ & & \times \\ \hline \underline{0}_{n-2}^T & \times & \times \end{array}\right)$$

and $\underline{v}_{nk}^{(2)}$ is the corresponding normalized eigenvector.

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Thank you ...