Orthogonal Rational Functions and Rational Gauss-type Quadrature Rules

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Outline

1. ORFs
   - Notation
   - Orthonormal rational functions (ORFs)

2. RIQs
   - Rational interpolatory quadrature rules (RIQs)
   - Characterization

3. Gauss-type RIQs
   - Rational Gaussian quadrature
   - Rational Gauss-Radau quadrature
   - Rational Gauss-Lobatto quadrature

4. Conclusion and References
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Notation

\[ f_n(x) = \frac{c_n x^n + c_{n-1} x^{n+1} + \ldots + c_0}{(1-x/\alpha_1)(1-x/\alpha_2) \ldots (1-x/\alpha_n)}, \quad n = 1, 2, \ldots \]

Poles

- \( \alpha_1, \alpha_2, \alpha_3, \ldots \)
- Arbitrary complex or infinite, but outside \([-1, 1]\)
- Fixed in advance

Function spaces

- \( \mathcal{L}_{-1} = \{0\}, \mathcal{L}_0 = \mathbb{C} \)
- \( n > 0: \mathcal{L}_n = \mathcal{L}\{\alpha_1, \ldots, \alpha_n\} = \text{space of rational functions} \)
  with poles among \( \{\alpha_1, \ldots, \alpha_n\} \)
- \( \mathcal{L} = \text{closure of } \bigcup_{n=0}^{\infty} \mathcal{L}_n \)
- \( \mathcal{L}_{-1} \subset \mathcal{L}_0 \subset \mathcal{L}_1 \subset \ldots \subset \mathcal{L}_n \subset \ldots \subset \mathcal{L} \)
Notation

$l = [-1, 1]$, $X \subseteq \mathbb{C}$: $\overline{X} = X \cup \{\infty\}$ and $X_l = \{x \in X : x \notin l\}$

Rational basis

$b_k(x) = \prod_{j=1}^{k} \frac{x}{1-x/\alpha_j}$, $\alpha_j \in \overline{\mathbb{C}_l}$

$L_n = \text{span}\{1, b_1(x), \ldots, b_n(x)\}$

Polynomials

$\forall j : \alpha_j = \infty \Rightarrow b_k(x) \equiv x^k$

$L_n \equiv \mathcal{P}_n = \text{span}\{1, x, \ldots, x^n\}$

Orthonormal rational functions (ORFs)

Given a positive measure $\mu$ on $l$ and inner product

$\langle f, g \rangle_{\mu} = \int_{l} f(x)g^c(x)d\mu(x)$, $g^c(x) = g(\overline{x})$

$\rightarrow$ ORFs: $\varphi_k(x)$

$\varphi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$, $\varphi_k \perp_{\mu} \mathcal{L}_{k-1}$, and $\|\varphi_k\|_{\mu} = \sqrt{\langle \varphi_k, \varphi_k \rangle_{\mu}} = 1$. 
Orthonormal rational functions (ORFs)

Given a positive measure $\mu$ on $I$ and inner product

$$\langle f, g \rangle_\mu = \int_I f(x)g^c(x)d\mu(x)$$

$g^c(x) = \overline{g(x)}$

$\rightarrow$ ORFs: $\varphi_k(x)$

$$\varphi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}, \varphi_k \perp_\mu \mathcal{L}_{k-1}, \text{ and } \|\varphi_k\|_\mu = \sqrt{\langle \varphi_k, \varphi_k \rangle_\mu} = 1.$$
Notation

\[ I = [-1, 1], \quad X \subseteq \mathbb{C} : \overline{X} = X \cup \{ \infty \} \text{ and } X_I = \{ x \in X : x \notin I \} \]

Rational basis

\[ b_k(x) = \prod_{j=1}^{k} \frac{x}{1-x/\alpha_j}, \quad \alpha_j \in \overline{C}_I \]

\[ \mathcal{L}_n = \text{span}\{1, b_1(x), \ldots, b_n(x)\} \]

Polynomials

\[ \forall j : \alpha_j = \infty \Rightarrow b_k(x) \equiv x^k \]

\[ \mathcal{L}_n \equiv \mathcal{P}_n = \text{span}\{1, x, \ldots, x^n\} \]

Orthonormal rational functions (ORFs)

Given a positive measure \( \mu \) on \( I \) and inner product

\[ \langle f, g \rangle_\mu = \int_I f(x)g^c(x)d\mu(x) \quad g^c(x) = \overline{g(\overline{x})} \]

\[ \rightarrow \text{ORFs: } \varphi_k(x) \]

\[ \varphi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}, \varphi_k \perp_\mu \mathcal{L}_{k-1}, \text{ and } \| \varphi_k \|_\mu = \sqrt{\langle \varphi_k, \varphi_k \rangle_\mu} = 1. \]
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4. Conclusion and References
Theorem (Three-term Recurrence)

The ORFs $\varphi_k, k > 0$, satisfy a three-term recurrence relation:

$$\varphi_k(x) = E_k \cdot F(\varphi_{k-2}(x), \varphi_{k-1}(x), \alpha_{k-2}, \alpha_{k-1}, \alpha_k, D_k)$$

with initial conditions $\varphi_{-1}(x) \equiv 0$, $\varphi_0(x) \equiv \eta_0 \frac{1}{1 - \mu}$, $|\eta_0| = 1$,
and $\alpha_{-1} \in \mathbb{R}$ and $\alpha_0 = \overline{C}$.


Corollary (Generalized eigenvalue problem (GEP))

The zeros $\{x_{nk}\}_{k=1}^n$ of the ORF $\varphi_n(x)$ are eigenvalues of a GEP of the form $J_n \underline{v}_{nk} = x_{nk} B_n \underline{v}_{nk}$, where $J_n$ and $B_n$ are tridiagonal matrices, and $\underline{v}_{nk}$ is the corresponding normalized eigenvector.

[J. Van Deun, Numer. Algorithms, 2007]
Orthonormal rational functions (ORFs)

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with initial conditions $\varphi_{-1}(x) \equiv 0$, $\varphi_0(x) \equiv \eta_0 \|1\|_{\mu}^{-1}$, $|\eta_0| = 1$, and $\alpha_{-1} \in \bar{\mathbb{R}}$, and $\alpha_0 = \bar{\mathbb{C}}$.


Corollary (Generalized eigenvalue problem (GEP))

The zeros $\{x_{nk}\}_{k=1}^n$ of the ORF $\varphi_n(x)$ are eigenvalues of a GEP of the form $J_n v_{nk} = x_{nk} B_n v_{nk}$, where $J_n$ and $B_n$ are tridiagonal matrices, and $v_{nk}$ is the corresponding normalized eigenvector.

[J. Van Deun, Numer. Algorithms, 2007]
Orthonormal rational functions (ORFs)

Theorem

The ORF $\varphi_n$ has $n$ real distinct zeros in $I$ iff $\alpha_n \in \overline{R}_I$.


Theorem (Favard Theorem)

Suppose the sequence of rational functions $\{\hat{\varphi}_k\}_{k=0}^{\infty}$, with $\hat{\varphi}_k \in \hat{L}_k \setminus \hat{L}_{k-1}$ and $\hat{L}_k = \mathcal{L}\{\hat{\alpha}_1, \ldots, \hat{\alpha}_k\}$, are generated by the three-term recurrence relation

$$\hat{\varphi}_k(x) = \hat{E}_k \cdot F(\hat{\varphi}_{k-2}(x), \hat{\varphi}_{k-1}(x), \hat{\alpha}_{k-2}, \hat{\alpha}_{k-1}, \hat{\alpha}_k, \hat{D}_k).$$

Then - under certain conditions on $\hat{E}_k$ and $\hat{D}_k$ - the $\hat{\varphi}_k$ form an orthonormal system w.r.t. a positive measure $\hat{\mu}$ on $\hat{S} \subset \mathbb{R}$ and inner product $\langle f, g \rangle_{\hat{\mu}} = \int_{\hat{S}} f(x) g^c(x) d\hat{\mu}(x)$.

Orthonormal rational functions (ORFs)

Theorem

The ORF \( \varphi_n \) has \( n \) real distinct zeros in \( I \) iff \( \alpha_n \in \overline{\mathbb{R}}_I \).


Theorem (Favard Theorem)

Suppose the sequence of rational functions \( \{\hat{\varphi}_k\}_{k=0}^{\infty} \), with \( \hat{\varphi}_k \in \hat{L}_k \setminus \hat{L}_{k-1} \) and \( \hat{L}_k = \mathcal{L}\{\hat{\alpha}_1, \ldots, \hat{\alpha}_k\} \), are generated by the three-term recurrence relation

\[
\hat{\varphi}_k(x) = \hat{E}_k \cdot F(\hat{\varphi}_{k-2}(x), \hat{\varphi}_{k-1}(x), \hat{\alpha}_{k-2}, \hat{\alpha}_{k-1}, \hat{\alpha}_k, \hat{D}_k).
\]

Then - under certain conditions on \( \hat{E}_k \) and \( \hat{D}_k \) - the \( \hat{\varphi}_k \) form an orthonormal system w.r.t. a positive measure \( \hat{\mu} \) on \( \hat{S} \subset \mathbb{R} \) and inner product \( \langle f, g \rangle_{\hat{\mu}} = \int_{\hat{S}} f(x)g(x^c) \, d\hat{\mu}(x) \).

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4. Conclusion and References
Rational interpolatory quadrature rules (RIQs)

Given a (possible complex) measure $\sigma$ on $I$ and a set of $n$ distinct nodes $\{x_{nk}\}_{k=1}^{n} \subset I$. Then there exists a unique set of weights $\{\lambda_{nk}\}_{k=1}^{n} \subset \mathbb{C}$ so that

$$J_\sigma(f) := \int_{I} f(x) d\sigma(x) \approx \sum_{k=1}^{n} \lambda_{nk} f(x_{nk}) =: J_n(f) \quad (1)$$

is exact for (at least) every $f \in \mathcal{L}_{n-1}$.

Domain of validity

Assume that (1) is exact for every $f \in \tilde{\mathcal{L}}_m$, where $\mathcal{L}_{n-1} \subseteq \tilde{\mathcal{L}}_m \subseteq \mathcal{L}\{\alpha_1, \ldots, \alpha_n, \bar{\alpha}_1, \ldots, \bar{\alpha}_n\} =: \mathcal{L}_n \cdot \mathcal{L}_{n}^c$, and suppose there is no larger space of rational functions for which (1) is exact. Then $\tilde{\mathcal{L}}_m$ is called the domain of validity.
Rational interpolatory quadrature rules (RIQs)

Given a (possible complex) measure $\sigma$ on $I$ and a set of $n$ distinct nodes $\{x_{nk}\}_{k=1}^{n} \subset I$. Then there exists a unique set of weights $\{\lambda_{nk}\}_{k=1}^{n} \subset \mathbb{C}$ so that

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Domain of validity

Assume that (1) is exact for every $f \in \tilde{\mathcal{L}}_{m}$, where $\mathcal{L}_{n-1} \subseteq \tilde{\mathcal{L}}_{m} \subseteq \mathcal{L}\{\alpha_1, \ldots, \alpha_n, \overline{\alpha}_1, \ldots, \overline{\alpha}_n\} =: \mathcal{L}_{n} \cdot \mathcal{L}_{n}^{c}$, and suppose there is no larger space of rational functions for which (1) is exact. Then $\tilde{\mathcal{L}}_{m}$ is called the domain of validity.
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Characterization

In the remainder we only consider positive measures $\sigma = \mu$ (for complex measures, see [K. Deckers and A. Bultheel, Report TW 574, 2010])

RIQs can be characterized by means of quasi-orthogonal rational functions (qORFs)

→ two kinds of RIQs / qORFs:
  - RIQs of the first kind: $\tilde{\mathcal{L}}_m = \mathcal{L}_n \cdot \mathcal{L}_{k-1}^c$ with $1 \leq k \leq n$;
  - RIQs of the second kind: $\tilde{\mathcal{L}}_m = \mathcal{L}_{n-1} \cdot \mathcal{L}_k^c$ with $0 \leq k \leq n - 1$.

[K. Deckers and A. Bultheel, Report TW 573, 2010]
RIQs of the first kind

The nodes in a RIQ with domain of validity $\mathcal{L}_n \cdot \mathcal{L}_{k-1}^c$, where $1 \leq k \leq n$, are the zeros of a $P_{n,n-k} \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ satisfying $P_{n,n-k} \perp \mu \mathcal{L}_{k-1}$. i.e., $P_{n,n-k}$ is a qORF of the form

$$P_{n,n-k}(x) = \varphi_n(x) + \sum_{j=1}^{n-k} a_j \varphi_{n-j}(x), \quad a_{n-k} \neq 0. \quad (2)$$

Property

Suppose the qORF (2) has all real zeros. Then it has at least $k$ zeros inside $I$.
RIQs of the first kind

The nodes in a RIQ with domain of validity \( L_n \cdot L_{k-1}^c \), where \( 1 \leq k \leq n \), are the zeros of a \( P_{n,n-k} \in L_n \setminus L_{n-1} \) satisfying \( P_{n,n-k} \perp \mu L_{k-1} \). 

i.e., \( P_{n,n-k} \) is a qORF of the form

\[
P_{n,n-k}(x) = \varphi_n(x) + \sum_{j=1}^{n-k} a_j \varphi_{n-j}(x), \quad a_{n-k} \neq 0. \tag{2}
\]

Property

Suppose the qORF (2) has all real zeros. Then it has at least \( k \) zeros inside \( I \).
RIQs of the first kind

The nodes in a RIQ with domain of validity $\mathcal{L}_n \cdot \mathcal{L}_{k-1}^c$, where $1 \leq k \leq n$, are the zeros of a $P_{n,n-k} \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ satisfying $P_{n,n-k} \perp_\mu \mathcal{L}_{k-1}$.

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$$P_{n,n-k}(x) = \varphi_n(x) + \sum_{j=1}^{n-k} a_j \varphi_{n-j}(x), \quad a_{n-k} \neq 0.$$  \hfill (2)

Property

Suppose the qORF (2) has all real zeros. Then it has at least $k$ zeros inside $I$. 

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RIQs of the second kind

The nodes in a RIQ with domain of validity $L_{n-1} \cdot L^c_k$, where $0 \leq k \leq n - 1$, are the zeros of a $Q_{n,n-k} \in L_n \setminus L_{n-1}$ satisfying

$Q_{n,n-k} \perp_{\mu} L_k(\vec{\alpha}_n) := \{ \frac{p_k}{\pi_k} : p_k(\vec{\alpha}_n) = 0 \}$.

i.e., $Q_{n,n-k}$ is a qORF of the form

$$Q_{n,n-k}(x) = \varphi_n(x) + \sum_{j=1}^{n-k} a_j \cdot \left( \frac{1 - x/\alpha_{n-j}}{1 - x/\alpha_n} \right) \varphi_{n-j}(x), \quad a_{n-k} \neq 0. \quad (3)$$

Property

Suppose the qORF (3) has all real zeros. Then it has at least $k$ zeros inside $I$. 
RIQs of the second kind

The nodes in a RIQ with domain of validity $\mathcal{L}_{n-1} \cdot \mathcal{L}_k^c$, where $0 \leq k \leq n - 1$, are the zeros of a $Q_{n,n-k} \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ satisfying $Q_{n,n-k} \perp_{\mu} \mathcal{L}_k(\overline{\alpha}_n) := \{p_k: p_k(\overline{\alpha}_n) = 0\}$. i.e., $Q_{n,n-k}$ is a qORF of the form

$$Q_{n,n-k}(x) = \varphi_n(x) + \sum_{j=1}^{n-k} a_j \cdot \left( \frac{1 - x/\overline{\alpha}_{n-j}}{1 - x/\alpha_n} \right) \varphi_{n-j}(x), \quad a_{n-k} \neq 0.$$ (3)

Property

Suppose the qORF (3) has all real zeros. Then it has at least $k$ zeros inside $I$. 
RIQs of the second kind

The nodes in a RIQ with domain of validity $\mathcal{L}_{n-1} \cdot \mathcal{L}_c^k$, where $0 \leq k \leq n-1$, are the zeros of a $Q_{n,n-k} \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ satisfying

$$Q_{n,n-k} \perp \mu \mathcal{L}_k(\overline{\alpha}_n) := \{ \frac{p_k}{\pi_k} : p_k(\overline{\alpha}_n) = 0 \}.$$

i.e., $Q_{n,n-k}$ is a qORF of the form

$$Q_{n,n-k}(x) = \varphi_n(x) + \sum_{j=1}^{n-k} a_j \cdot \left( \frac{1 - x/\overline{\alpha}_{n-j}}{1 - x/\alpha_n} \right) \varphi_{n-j}(x), \quad a_{n-k} \neq 0. \quad (3)$$

Property

Suppose the qORF (3) has all real zeros. Then it has at least $k$ zeros inside $I$. 
Positive RIQs

Positive rational interpolatory quadrature rules

- Positive rational interpolatory quadrature $\equiv$ RIQ with weights
  $\{\lambda_{nk}\}_{k=1}^{n} \subset \mathbb{R}^{+}$.
- Domain of validity satisfies $\tilde{L}_{m} \equiv \tilde{L}_{m}^{c}$;

Rational Gauss-type quadrature rules

- Rational Gauss-type quadrature $\equiv$ positive RIQ with $j$ fixed
  nodes in $I$, where $0 \leq j \leq 2$, and the remaining $n - j$ nodes
  are such that the domain of validity has dimension $2n - j$ (or
  larger).
- Existence not only depends on the fixed node(s), but also on
  the sequence of poles.
Positive RIQs

Positive rational interpolatory quadrature rules

Positive rational interpolatory quadrature = RIQ with weights \( \{\lambda_{nk}\}_{k=1}^{n} \subset \mathbb{R}_{0}^{+} \).

Domain of validity satisfies \( \tilde{\mathcal{L}}_{m} \equiv \tilde{\mathcal{L}}_{m}^{c} \).

Rational Gauss-type quadrature rules

Rational Gauss-type quadrature = positive RIQ with \( j \) fixed nodes in \( I \), where \( 0 \leq j \leq 2 \), and the remaining \( n - j \) nodes are such that the domain of validity has dimension \( 2n - j \) (or larger).

Existence not only depends on the fixed node(s), but also on the sequence of poles.
Rational Gauss-type quadrature rules

- **Rational Gaussian quadrature:** \( j = 0 \) and \( \tilde{L}_m = L_n \cdot L_{n-1}^c \)
- **Rational Gauss-Radau quadrature:** \( j = 1 \) and
  - \( \tilde{L}_m \supseteq L_n \cdot L_{n-2}^c \) (first kind), or
  - \( \tilde{L}_m \supseteq L_{n-1} \cdot L_{n-1}^c \) (second kind)
- **Rational Gauss-Lobatto quadrature:** \( j = 2 \) and
  - \( \tilde{L}_m \supseteq L_n \cdot L_{n-3}^c \) (first kind), or
  - \( \tilde{L}_m \supseteq L_{n-1} \cdot L_{n-2}^c \) (second kind)

In what follows: existence and construction of these rational Gauss-type quadrature rules.
Rational Gauss-type quadrature rules

- **Rational Gaussian quadrature**: \( j = 0 \) and \( \tilde{L}_m = L_n \cdot L_{n-1}^c \)
- **Rational Gauss-Radau quadrature**: \( j = 1 \) and
  - \( \tilde{L}_m \supseteq L_n \cdot L_{n-2}^c \) (first kind), or
  - \( \tilde{L}_m \supseteq L_{n-1} \cdot L_{n-1}^c \) (second kind)
- **Rational Gauss-Lobatto quadrature**: \( j = 2 \) and
  - \( \tilde{L}_m \supseteq L_n \cdot L_{n-3}^c \) (first kind), or
  - \( \tilde{L}_m \supseteq L_{n-1} \cdot L_{n-2}^c \) (second kind)

In what follows: existence and construction of these rational Gauss-type quadrature rules.
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Rational Gaussian quadrature

- has maximal domain of validity; i.e., \( \tilde{L}_m = L_n \cdot L^c_{n-1} \)
- exists iff \( \alpha_n \in \mathbb{R}_l \)
- the nodes \( \{ x^{(\mu, \alpha_n)}_{nk} \}_{k=1}^n \) are zeros of the ORF \( \varphi_n \in L_n \); hence, can be computed by means of the GEP
- the corresponding weights \( \{ \lambda^{(\mu, \alpha_n)}_{nk} \}_{k=1}^n \) can be computed by means of the GEP too:

\[
\lambda^{(\mu, \alpha_n)}_{nk} = |v_{nk}|^2 \|1\|^2_\mu,
\]

where \( v_{nk} \) represents the first component of the eigenvector \( v_{nk} \).
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4. Conclusion and References
Consider the qORF $Q_{n,1}(x)$ with fixed zero in $x_{n1}$:

$$Q_{n,1}(x) = \varphi_n(x) + a_{n1} \left( \frac{1 - x/\alpha_{n-1}}{1 - x/\alpha_n} \right) \varphi_{n-1}(x),$$

$$a_{n1} = -\frac{(1 - x_{n1}/\alpha_n)\varphi_n(x_{n1})}{(1 - x_{n1}/\alpha_{n-1})\varphi_{n-1}(x_{n1})} \in \mathbb{C}.$$

Questions:

- are the zeros $\{x_{nk}\}_{k=1}^n$ all real distinct and in $I$?
- are the weights $\{\lambda_{nk}\}_{k=1}^n$ in the corresponding RIQ

$$\int_I f(x) d\mu(x) = \sum_{k=1}^n \lambda_{nk} f(x_{nk}), \quad \forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^c$$

all in $\mathbb{R}_0^+$?
Rational Gauss-Radau quadrature: Second kind


**Theorem**

Suppose $c_n := \frac{a_{n1}}{E_n} \left( \frac{1}{\alpha_{n-1}} - \frac{1}{\alpha_{n-1}} \right) \neq -1$. Then there exist poles $\alpha \in \overline{\mathbb{C}}_I$ (not unique), and constants $\hat{E}_n \neq 0$ and

$$
\hat{D}_n = \frac{D_n + a_{n1}/E_n}{1 + c_n} \to \text{satisfying condition Favard theorem,}
$$

so that

$$
\left( \frac{1 - x/\alpha_n}{1 - x/\alpha} \right) Q_{n,1}(x) = \hat{E}_n \cdot F(\varphi_{n-2}(x), \varphi_{n-1}(x), \alpha_{n-2}, \alpha_{n-1}, \alpha, \hat{D}_n);
$$

hence, $\left( \frac{1 - \cdot/\alpha_n}{1 - \cdot/\alpha} \right) Q_{n,1}(\cdot) \perp \hat{\mu} \mathcal{L}_{n-1}$, where $\hat{\mu}$ is a (not unique) positive measure on $\hat{S} \subset \mathbb{R}$.
Rational Gauss-Radau quadrature: Second kind


**Theorem**

Suppose \( c_n := \frac{a_{n1}}{E_n} \left( \frac{1}{\alpha_{n-1}} - \frac{1}{\alpha_{n-1}} \right) \neq -1 \). Then there exist poles \( \alpha \in \overline{\mathbb{C}} \) (not unique), and constants \( \hat{E}_n \neq 0 \) and

\[
\hat{D}_n = \frac{D_n + a_{n1}/E_n}{1 + c_n} \rightarrow \text{satisfying condition Favard theorem,}
\]

so that

\[
\left( \frac{1 - x/\alpha_n}{1 - x/\alpha} \right) Q_{n,1}(x) = \hat{E}_n \cdot F(\varphi_{n-2}(x), \varphi_{n-1}(x), \alpha_{n-2}, \alpha_{n-1}, \alpha, \hat{D}_n);
\]

hence, \( \left( \frac{1 - \cdot/\alpha_n}{1 - \cdot/\alpha} \right) Q_{n,1}(\cdot) \perp \mu \mathcal{L}_{n-1} \), where \( \mu \) is a (not unique) positive measure on \( \hat{S} \subset \mathbb{R} \).
If $c_n = -1$, then numerator polynomial of $Q_{n,1}(x)$ has zero in $x = \alpha_{n-1}$.

**Corollary**

The zeros of $Q_{n,1}(x)$ are all real distinct iff

$$\Im\{\hat{D}_n\} = \Im\{1/\alpha_{n-2}\}|E_{n-1}|^{-2}, \quad \alpha_{n-1} \in \mathbb{R}_I,$$

respectively

$$\Re\{\hat{D}_n\}^2 + \left(\Im\{\hat{D}_n\} - \frac{1}{\Im\{1/\alpha_{n-2}\}}\right)^2 = \left(\frac{1}{\Im\{1/\alpha_{n-1}\}}\right)^2 \left(\frac{1}{1 + r_n}\right), \quad \alpha_{n-1} \notin \mathbb{R},$$

where

$$r_n = 4\Im\{1/\alpha_{k-1}\} \cdot \Im\{1/\alpha_{k-2}\}|E_{n-1}|^{-2} > -1.$$
Rational Gauss-Radau quadrature: Second kind

- Under the same condition on $\hat{D}_n \rightarrow n - 1$ zeros are in $I$
- Boundary values $\hat{D}_n^{\pm}$ for $\hat{D}_n$, in order to have that
  \[ \{x_{nk}\}_{k=1}^n \subset I, \]
  are obtained by setting
  \[ a_{n1}^{\pm} := -\frac{(1\mp 1/\alpha_n)\varphi_n(\pm 1)}{(1\mp 1/\alpha_{n-1})\varphi_{n-1}(\pm 1)} \]
- Forbidden value $\hat{D}_{n\emptyset}$ for $\hat{D}_n$ is obtained by setting
  \[ a_{n1\emptyset} := -\lim_{x \rightarrow \infty} \frac{(1-x/\alpha_n)\varphi_n(x)}{(1-x/\alpha_{n-1})\varphi_{n-1}(x)} \]
Rational Gauss-Radau quadrature: Second kind

- The nodes \( \{x_{nk}\}_{k=1}^n \) are rational Gaussian nodes: 
  \[ x_{nk} = x_{nk}^{(\mu,\alpha)} \], 
  \( k = 1, \ldots, n \) and \( \alpha \in \mathbb{R}_I \)

- \( \{\varphi_0, \ldots, \varphi_{n-1}\} \) forms an orthonormal system for the inner product \( \langle \cdot, \cdot \rangle_{\hat{\mu}} \Rightarrow \)
  \( \forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}^c_{n-1} \supset \mathcal{L}_{n-1} : \int_I f(x) d\mu(x) = \int_{\hat{\mu}} f(x) d\hat{\mu}(x) \);
  hence, the weights \( \{\lambda_{nk}\}_{k=1}^n \) are rational Gaussian weights:
  \[ \lambda_{nk} = \lambda_{nk}^{(\hat{\mu},\alpha)} \], \( k = 1, \ldots, n \) and \( \alpha \in \mathbb{R}_I \)

- The nodes and weights can be computed by means of a modified GEP:
  \( \hat{J}_n \hat{\nu}_{nk} = x_{nk} \hat{B}_n \hat{\nu}_{nk} \), where

\[
\hat{J}_n = \begin{pmatrix}
J_{n-1} & 0_{n-2} \\
0_{n-2} & \ldots
\end{pmatrix},
\hat{B}_n = \begin{pmatrix}
B_{n-1} & 0_{n-2} \\
0_{n-2} & \ldots
\end{pmatrix}
\]

and \( \hat{\nu}_{nk} \) is the corresponding normalized eigenvector.
Rational Gauss-Radau quadrature: Second kind

- The nodes $\{x_{nk}\}_{k=1}^n$ are rational Gaussian nodes: $x_{nk} = x_{nk}^{(\hat{\mu},\alpha)}$, $k = 1, \ldots, n$ and $\alpha \in \overline{\mathbb{R}}_I$

- $\{\varphi_0, \ldots, \varphi_{n-1}\}$ forms an orthonormal system for the inner product $\langle \cdot, \cdot \rangle_{\hat{\mu}}$ \(
\forall f \in L_{n-1} \cdot L_{n-1}^c \supset L_{n-1} : \int_I f(x) d\mu(x) = \int_{\hat{\mathbb{S}}} f(x) d\hat{\mu}(x) \)

- Hence, the weights $\{\lambda_{nk}\}_{k=1}^n$ are rational Gaussian weights: $\lambda_{nk} = \lambda_{nk}^{(\hat{\mu},\alpha)}$, $k = 1, \ldots, n$ and $\alpha \in \overline{\mathbb{R}}_I$

- The nodes and weights can be computed by means of a modified GEP: $\hat{J}_n \hat{v}_{nk} = x_{nk} \hat{B}_n \hat{v}_{nk}$, where

$$
\hat{J}_n = \begin{pmatrix}
J_{n-1} & 0_{n-2} \\
0_I & \ast
\end{pmatrix}, \quad \hat{B}_n = \begin{pmatrix}
B_{n-1} & 0_{n-2} \\
0_I & \ast
\end{pmatrix}
$$

and $\hat{v}_{nk}$ is the corresponding normalized eigenvector.
Rational Gauss-Radau quadrature: Second kind

- The nodes \( \{x_{nk}\}_{k=1}^{n} \) are rational Gaussian nodes: \( x_{nk} = x_{nk}^{(\hat{\mu},\alpha)} \), \( k = 1, \ldots, n \) and \( \alpha \in \mathbb{R}_I \)

- \( \{\varphi_0, \ldots, \varphi_{n-1}\} \) forms an orthonormal system for the inner product \( \langle \cdot, \cdot \rangle_{\hat{\mu}} \) ⇒
  \[ \forall f \in \mathcal{L}_{n-1} \cap \mathcal{L}^c_{n-1} \supseteq \mathcal{L}_{n-1}: \int_I f(x) d\mu(x) = \int_{\hat{\mathcal{S}}} f(x) d\hat{\mu}(x); \]
  hence, the weights \( \{\lambda_{nk}\}_{k=1}^{n} \) are rational Gaussian weights:
  \( \lambda_{nk} = \lambda_{nk}^{(\hat{\mu},\alpha)} \), \( k = 1, \ldots, n \) and \( \alpha \in \mathbb{R}_I \)

- The nodes and weights can be computed by means of a modified GEP:
  \( \hat{J}_n \hat{\nu}_{nk} = x_{nk} \hat{B}_n \hat{\nu}_{nk} \), where

\[
\hat{J}_n = \begin{pmatrix}
J_{n-1} & 0_{n-2} \\
0_{n-2}^T & \ast
\end{pmatrix}, \quad \hat{B}_n = \begin{pmatrix}
B_{n-1} & 0_{n-2} \\
0_{n-2}^T & \ast
\end{pmatrix}
\]

and \( \hat{\nu}_{nk} \) is the corresponding normalized eigenvector.
Rational Gauss-Radau quadrature: Second kind

- The nodes \( \{x_{nk}\}_{k=1}^{n} \) are rational Gaussian nodes: \( x_{nk} = x_{nk}(\hat{\mu},\alpha) \), \( k = 1, \ldots, n \) and \( \alpha \in \mathbb{R}_I \)

- \( \{\varphi_0, \ldots, \varphi_{n-1}\} \) forms an orthonormal system for the inner product \( \langle \cdot, \cdot \rangle_{\hat{\mu}} \Rightarrow \)

\[ \forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}^c \supset \mathcal{L}_{n-1} : \int_I f(x) d\mu(x) = \int_{\hat{\mu}} f(x) d\hat{\mu}(x) ; \]

hence, the weights \( \{\lambda_{nk}\}_{k=1}^{n} \) are rational Gaussian weights:

\( \lambda_{nk} = \lambda_{nk}(\hat{\mu},\alpha) \), \( k = 1, \ldots, n \) and \( \alpha \in \mathbb{R}_I \)

- The nodes and weights can be computed by means of a modified GEP: \( \hat{J}_n \hat{\nu}_{nk} = x_{nk} \hat{B}_n \hat{\nu}_{nk} \), where

\[
\hat{J}_n = \begin{pmatrix}
J_{n-1} & 0_{n-2} \\
0_{n-2} & \ast
\end{pmatrix}, \quad \hat{B}_n = \begin{pmatrix}
B_{n-1} & 0_{n-2} \\
0_{n-2} & \ast
\end{pmatrix}
\]

and \( \hat{\nu}_{nk} \) is the corresponding normalized eigenvector.
Rational Gauss-Radau quadrature: First kind

Consider the qORF \( P_{n,1}(x) \) with fixed zero in \( x_{n1} \):

\[
P_{n,1}(x) = \varphi_n(x) + a_{n1}\varphi_{n-1}(x), \quad a_{n1} = -\frac{\varphi_n(x_{n1})}{\varphi_{n-1}(x_{n1})} \in \mathbb{C}.
\]

Questions:
- are the zeros \( \{x_{nk}\}_{k=1}^n \) all real distinct and in \( I \)?
- are the weights \( \{\lambda_{nk}\}_{k=1}^n \) in the corresponding RIQ

\[
\int_I f(x) d\mu(x) = \sum_{k=1}^n \lambda_{nk} f(x_{nk}), \quad \forall f \in \mathcal{L}_n \cdot \mathcal{L}_n^c \quad \text{all in } \mathbb{R}_0^+?
\]
Rational Gauss-Radau quadrature: First kind

Conditions on \( \{\alpha_{n-1}, \alpha_n\} \)

- If \( \alpha_n = \overline{\alpha}_{n-1} \), then First kind = Second kind.
  
  So, suppose \( \alpha_n \neq \overline{\alpha}_{n-1} \).

- If \( \alpha_{n-1} \notin \overline{\mathbb{R}} \), then
  
  \[
  \alpha_n \notin \overline{\mathbb{R}} \Rightarrow \{\lambda_{nk}\}_{k=1}^n \notin \mathbb{R}^+_0
  \]

- If \( \alpha_{n-1} \in \overline{\mathbb{R}}_I \), then
  
  \[
  \alpha_n \in \overline{\mathbb{R}}_I \Rightarrow \{\lambda_{nk}\}_{k=1}^n \subset \mathbb{R}^+_0 \iff a_{n1} = 0, \text{ i.e.,}
  \]

  Rational Gaussian quadrature: \( x_{nk} = x_{nk}^{(\mu,\alpha_n)} \) and
  \[
  \lambda_{nk} = \lambda_{nk}^{(\mu,\alpha_n)}, \text{ exact in } L_n \cdot L_{n-1}^c \supset L_n \cdot L_{n-2}^c
  \]

  So, suppose \( \alpha_{n-1} \in \overline{\mathbb{R}}_I \).
Theorem

Suppose \( c_n := \frac{a_{n1}}{E_n} \left( \frac{1}{\alpha_{n-1}} - \frac{1}{\alpha_n} \right) \in (-1, \infty) \). Then there exist poles \( \alpha \in \mathbb{C}_I \) (not unique), and constants \( \hat{E}_n \neq 0, |k_n|^2 = 1 + c_n \), and

\[
\hat{D}_n = \frac{D_n + a_{n1}/E_n}{1 + c_n} \rightarrow \text{satisfying condition Favard theorem},
\]

so that

\[
\left( \frac{1-x/\alpha_n}{1-x/\alpha} \right) P_{n,1}(x) = \hat{E}_n \cdot F(\varphi_{n-2}(x), k_n\varphi_{n-1}(x), \alpha_{n-2}, \alpha_{n-1}, \alpha, \hat{D}_n);
\]

hence, \( \left( \frac{1-\cdot/\alpha_n}{1-\cdot/\alpha} \right) P_{n,1}(\cdot) \perp \hat{\mu} \mathcal{L}_{n-1}, \) where \( \hat{\mu} \) is a (not unique) positive measure on \( \hat{S} \not\subset \mathbb{R} \).
Rational Gauss-Radau quadrature: First kind

Lemma

Suppose \( c_n \notin (-1, \infty) \). Then there cannot exist distinct nodes \( \{x_{nk}\}_{k=1}^n \subset I \) and weights \( \{\lambda_{nk}\}_{k=1}^n \subset \mathbb{R}_0^+ \) so that

\[
\int_I f(x) d\mu(x) = \sum_{k=1}^n \lambda_{nk} f(x_{nk}), \quad \forall f \in \mathcal{L}_n \cdot \mathcal{L}_c^{n-2}.
\]

Corollary

The zeros of \( P_{n,1}(x) \) are all real distinct iff

\[
\Im \{ \hat{D}_n \} = \frac{\Im \{1/\alpha_{n-2}\}}{|k_n|^2 |E_{n-1}|^2}; \quad \text{i.e., iff} \quad \Im \{ a_{n1} \overline{E}_n \} - \Im \{1/\alpha_n\} = 0. \tag{4}
\]
Rational Gauss-Radau quadrature: First kind

- $\alpha_n \in \overline{\mathbb{R}}_I$: (4) holds true for every $a_{n1}$ for which $c_n \in (-1, \infty)$
Boundary values $c_n^- < 0 < c_n^+$ for $c_n$, in order to have that
$\{x_{nk}\}_{k=1}^n \subset I$, are obtained by setting $a_{n1}^\pm := -\frac{\varphi_n(\pm 1)}{\varphi_{n-1}(\pm 1)}$.

- $\alpha_n \notin \overline{\mathbb{R}}$: (4) holds true iff $a_{n1} \overline{E}_{n-1} = \frac{1}{\alpha_{n-1}} - \frac{1}{\alpha_n}$
Rational Gaussian quadrature (when changing the order of the last two poles), exact in $\mathcal{L}_n \cdot \mathcal{L}\{\overline{\alpha}_1, \ldots, \overline{\alpha}_{n-2}, \overline{\alpha}_n\} \supset \mathcal{L}_n \cdot \mathcal{L}_n^c$
Rational Gauss-Radau quadrature: First kind

- \( \alpha_n \in \overline{\mathbb{R}}_I \): (4) holds true for every \( a_{n1} \) for which \( c_n \in (-1, \infty) \).

  Boundary values \( c_n^- < 0 < c_n^+ \) for \( c_n \), in order to have that \( \{x_{nk}\}_{k=1}^n \subset I \), are obtained by setting \( a_{n1}^\pm := -\frac{\varphi_n(\pm 1)}{\varphi_{n-1}(\pm 1)} \).

- \( \alpha_n \notin \overline{\mathbb{R}} \): (4) holds true iff \( a_{n1}E_{n-1} = \frac{1}{\alpha_{n-1}} - \frac{1}{\alpha_n} \).

  Rational Gaussian quadrature (when changing the order of the last two poles), exact in \( \mathcal{L}_n \cdot \mathcal{L}\{\overline{\alpha}_1, \ldots, \overline{\alpha}_{n-2}, \overline{\alpha}_n\} \supset \mathcal{L}_n \cdot \mathcal{L}_{n-2}^c \).
the nodes $\{x_{nk}\}_{k=1}^n$ are rational Gaussian nodes: $x_{nk} = x_{nk}^{(\hat{\mu},\alpha)}$, $k = 1, \ldots, n$ and $\alpha \in \mathbb{R}_I$

$\{\varphi_0, \ldots, \varphi_{n-2}, k_n\varphi_{n-1}\}$ forms an orthonormal system for the inner product $\langle \cdot, \cdot \rangle_{\hat{\mu}} \Rightarrow \\
\forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-2}^c \supset \mathcal{L}_{n-1} : \int_I f(x) d\mu(x) = \int_{\hat{\mathcal{S}}} f(x) d\hat{\mu}(x)$; hence, the weights $\{\lambda_{nk}\}_{k=1}^n$ are rational Gaussian weights: $\lambda_{nk} = \lambda_{nk}^{(\hat{\mu},\alpha)}$, $k = 1, \ldots, n$ and $\alpha \in \mathbb{R}_I$

the nodes and weights can be computed by means of a modified GEP: $\hat{J}_n\hat{v}_{nk} = x_{nk}\hat{B}_n\hat{v}_{nk}$, where

$$\hat{J}_n = \begin{pmatrix} J_{n-1} & 0_{n-2}^T \\ 0_{n-2} & \times \end{pmatrix}, \quad \hat{B}_n = \begin{pmatrix} B_{n-1} & 0_{n-2}^T \\ 0_{n-2} & \times \end{pmatrix}$$

and $\hat{v}_{nk}$ is the corresponding normalized eigenvector.
Rational Gauss-Radau quadrature: First kind

- the nodes \( \{x_{nk}\}_{k=1}^n \) are rational Gaussian nodes: \( x_{nk} = x_{nk}^{(\hat{\mu},\alpha)} \), \( k = 1, \ldots, n \) and \( \alpha \in \overline{\mathbb{R}}_I \)

- \( \{\varphi_0, \ldots, \varphi_{n-2}, k_n \varphi_{n-1}\} \) forms an orthonormal system for the inner product \( \langle \cdot, \cdot \rangle_{\hat{\mu}} \Rightarrow \forall f \in \mathcal{L}_{n-1}^c \supset \mathcal{L}_{n-1}: \int_I f(x) d\mu(x) = \int_{\hat{\mathcal{S}}} f(x) d\hat{\mu}(x); \) hence, the weights \( \{\lambda_{nk}\}_{k=1}^n \) are rational Gaussian weights: \( \lambda_{nk} = \lambda_{nk}^{(\hat{\mu},\alpha)} \), \( k = 1, \ldots, n \) and \( \alpha \in \overline{\mathbb{R}}_I \)

- the nodes and weights can be computed by means of a modified GEP: \( \hat{J}_n \hat{v}_{nk} = x_{nk} \hat{B}_n \hat{v}_{nk} \), where

\[
\hat{J}_n = \begin{pmatrix}
J_{n-1} & 0_{n-2} \\
0_{n-2} & x
\end{pmatrix}, \quad \hat{B}_n = \begin{pmatrix}
B_{n-1} & 0_{n-2} \\
0_{n-2} & x
\end{pmatrix}
\]

and \( \hat{v}_{nk} \) is the corresponding normalized eigenvector.
Rational Gauss-Radau quadrature: First kind

- the nodes \( \{ x_{nk} \}_{k=1}^n \) are rational Gaussian nodes: \( x_{nk} = x_{nk}^{(\mu, \alpha)} \), \( k = 1, \ldots, n \) and \( \alpha \in \overline{\mathbb{R}}_I \)

- \( \{ \varphi_0, \ldots, \varphi_{n-2}, k_n \varphi_{n-1} \} \) forms an orthonormal system for the inner product \( \langle \cdot, \cdot \rangle_{\mu} \implies \forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}^c_{n-2} \supset \mathcal{L}_{n-1} : \int_I f(x) d\mu(x) = \int_{\hat{\mathcal{S}}} f(x) d\hat{\mu}(x) ; \) hence, the weights \( \{ \lambda_{nk} \}_{k=1}^n \) are rational Gaussian weights: \( \lambda_{nk} = \lambda_{nk}^{(\mu, \alpha)} \), \( k = 1, \ldots, n \) and \( \alpha \in \overline{\mathbb{R}}_I \)

- the nodes and weights can be computed by means of a modified GEP: \( \hat{J}_n \hat{v}_{nk} = x_{nk} \hat{B}_n \hat{v}_{nk} \), where

\[
\hat{J}_n = \begin{pmatrix}
J_{n-1} & 0_{n-2} \\
0_{n-2}^T & X
\end{pmatrix}, \quad \hat{B}_n = \begin{pmatrix}
B_{n-1} & 0_{n-2} \\
0_{n-2}^T & X
\end{pmatrix}
\]

and \( \hat{v}_{nk} \) is the corresponding normalized eigenvector.
Rational Gauss-Radau quadrature: First kind

- The nodes \( \{x_{nk}\}_{k=1}^{n} \) are rational Gaussian nodes: \( x_{nk} = x_{nk}^{(\hat{\mu}, \alpha)} \), \( k = 1, \ldots, n \) and \( \alpha \in \overline{\mathbb{R}}_I \)

- \( \{\varphi_0, \ldots, \varphi_{n-2}, k_n \varphi_{n-1}\} \) forms an orthonormal system for the inner product \( \langle \cdot, \cdot \rangle_{\hat{\mu}} \Rightarrow \)

\( \forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-2}^c \supset \mathcal{L}_{n-1} : \int_I f(x) d\mu(x) = \int_{\hat{\mathcal{S}}} f(x) d\hat{\mu}(x) \);

hence, the weights \( \{\lambda_{nk}\}_{k=1}^{n} \) are rational Gaussian weights:

\( \lambda_{nk} = \lambda_{nk}^{(\hat{\mu}, \alpha)} , k = 1, \ldots, n \) and \( \alpha \in \overline{\mathbb{R}}_I \)

- The nodes and weights can be computed by means of a modified GEP:

\( \hat{J}_n \hat{\nu}_{nk} = x_{nk} \hat{B}_n \hat{\nu}_{nk} \), where

\[
\hat{J}_n = \begin{pmatrix}
J_{n-1} & 0_{n-2} \\
0_{n-2}^T & \times
\end{pmatrix},
\hat{B}_n = \begin{pmatrix}
B_{n-1} & 0_{n-2} \\
0_{n-2}^T & \times
\end{pmatrix}
\]

and \( \hat{\nu}_{nk} \) is the corresponding normalized eigenvector.
Outline

1. ORFs
   - Notation
   - Orthonormal rational functions (ORFs)

2. RIQs
   - Rational interpolatory quadrature rules (RIQs)
   - Characterization

3. Gauss-type RIQs
   - Rational Gaussian quadrature
   - Rational Gauss-Radau quadrature
   - Rational Gauss-Lobatto quadrature

4. Conclusion and References
Rational Gauss-Lobatto quadrature: Second kind

Consider the qORF $Q_{n,2}(x)$ with fixed zero in $x_{n1}$ and $x_{n2}$:

$$Q_{n,2}(x) = \varphi_n(x) + a_{n1} \left( \frac{1 - x/\alpha_{n-1}}{1 - x/\alpha_n} \right) \varphi_{n-1}(x) + a_{n2} \left( \frac{1 - x/\alpha_{n-2}}{1 - x/\alpha_n} \right) \varphi_{n-2}(x)$$

Questions:

- Are the zeros $\{x_{nk}\}_{k=1}^n$ all real distinct and in $I$?
- Are the weights $\{\lambda_{nk}\}_{k=1}^n$ in the corresponding RIQ

$$\int_I f(x) d\mu(x) = \sum_{k=1}^n \lambda_{nk} f(x_{nk}), \ \forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{c_{n-2}}^c,$$ all in $\mathbb{R}^+_0$?

Answer: similar story as rational Gauss-Radau of the First kind, but a little bit more complicated.

[K. Deckers and A. Bultheel, Report TW 573, 2010]
Consider the qORF $Q_{n,2}(x)$ with fixed zero in $x_{n1}$ and $x_{n2}$:

$$Q_{n,2}(x) = \varphi_n(x) + a_{n1} \left( \frac{1 - x/\alpha_{n-1}}{1 - x/\alpha_n} \right) \varphi_{n-1}(x) +$$

$$a_{n2} \left( \frac{1 - x/\alpha_{n-2}}{1 - x/\alpha_n} \right) \varphi_{n-2}(x)$$

Questions:

- are the zeros \( \{x_{nk}\}_{k=1}^n \) all real distinct and in \( I \)?
- are the weights \( \{\lambda_{nk}\}_{k=1}^n \) in the corresponding RIQ \( \int_I f(x) d\mu(x) = \sum_{k=1}^n \lambda_{nk} f(x_{nk}), \forall f \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-2}^c \), all in \( \mathbb{R}_0^+ \)?

Answer: similar story as rational Gauss-Radau of the First kind, but a little bit more complicated.

[K. Deckers and A. Bultheel, Report TW 573, 2010]
Conclusion

- We characterized rational Gaussian, Gauss-Radau (two kinds) and Gauss-Lobatto (only Second kind) quadrature rules based on (quasi-)orthogonal rational functions.
- Still open for further investigation:
  - characterization of rational Gauss-Lobatto of the First kind,
  - other quasi-orthogonal rational functions leading to positive rational interpolatory quadrature rules
References


Thank you …