Christoffel-Darboux-type Formulae for Orthonormal Rational Functions and Asymptotics

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Outline

1. Orthonormal rational functions
2. Christoffel-Darboux formulae
3. Asymptotics
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1 Orthonormal rational functions

2 Christoffel-Darboux formulae

3 Asymptotics
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1. Orthonormal rational functions
2. Christoffel-Darboux formulae
3. Asymptotics
Orthonormal rational functions

\[ f_n(x) = \frac{c_n x^n + c_{n-1} x^{n-1} + \ldots + c_0}{(1-x/\alpha_1)(1-x/\alpha_2) \ldots (1-x/\alpha_n)}, \; n = 1, 2, \ldots \]

Poles

- \(\alpha_1, \alpha_2, \alpha_3, \ldots\)
- Arbitrary complex or infinite, but different from \(\alpha_\emptyset = 0\)
- Fixed in advance

Function spaces

- \(\mathcal{L}_{-1} = \{0\}, \mathcal{L}_0 = \mathbb{C}\)
- \(n > 0: \mathcal{L}_n = \mathcal{L}\{\alpha_1, \ldots, \alpha_n\} = \text{space of rational functions with poles among } \{\alpha_1, \ldots, \alpha_n\}\)
- \(\mathcal{L} = \text{closure of } \bigcup_{n=0}^{\infty} \mathcal{L}_n\)
- \(\mathcal{L}_{-1} \subset \mathcal{L}_0 \subset \mathcal{L}_1 \subset \ldots \subset \mathcal{L}_n \subset \ldots \subset \mathcal{L}\)
Orthonormal rational functions

Rational basis

\[ b_k(x) = \prod_{j=1}^{k} \frac{x}{1 - x/\alpha_j}, \quad k > 0, \quad \alpha_j \in \mathbb{C}_0 \]

\[ \mathcal{L}_n = \text{span}\{1, b_1(x), \ldots, b_n(x)\} \]

Polynomials

\((\forall j : \alpha_j = \infty) \Rightarrow b_k(x) \equiv x^k\)

\[ \mathcal{L}_n \equiv \mathcal{P}_n = \text{span}\{1, x, \ldots, x^n\} \]
### Orthonormal rational functions

#### Rational basis

\[ b_k(x) = \prod_{j=1}^{k} \frac{x}{1 - x/\alpha_j}, \quad k > 0, \quad \alpha_j \in \overline{\mathbb{C}_0} \]

\[ \mathcal{L}_n = \text{span}\{1, b_1(x), \ldots, b_n(x)\} \]

#### Polynomials

\((\forall j : \alpha_j = \infty) \Rightarrow b_k(x) \equiv x^k\)

\[ \mathcal{L}_n \equiv \mathcal{P}_n = \text{span}\{1, x, \ldots, x^n\} \]
Orthonormal rational functions (ORFs)
Consider an inner product $\langle f, g \rangle = \mathcal{F}\{fg^c\}$, $f, g \in \mathcal{L}$, where $g^c(x) = g(\bar{x})$ and $\mathcal{F}$ is an hermitian positive-definite linear functional:

$$\forall f, g \in \mathcal{L} : \mathcal{F}\{fg^c\} = \mathcal{F}\{f^c g\} \quad \text{and} \quad \forall f \in \mathcal{L}\{0\} : \mathcal{F}\{ff^c\} > 0$$

$\rightarrow$ ORFs: $\varphi_k(x)$

$\varphi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$, $\varphi_k \perp \mathcal{L}_{k-1}$, and $\|\varphi_k\| = \sqrt{\langle \varphi_k, \varphi_k \rangle} = 1$.

Quasi-orthogonal rational functions (qORFs)

$Q_n \in \mathcal{L}_n$ is called a qORF if $Q_n \perp \mathcal{L}_{n-1}(\alpha_n)$ where

$$\mathcal{L}_k(\alpha) := \left\{ \frac{p_k}{\pi_k} \in \mathcal{L}_k : p_k(\alpha) = 0 \right\}.$$
Orthonormal rational functions

Consider an inner product \( \langle f, g \rangle = \mathcal{F}\{fg^c\} \), \( f, g \in \mathcal{L} \), where \( g^c(x) = \overline{g(x)} \) and \( \mathcal{F} \) is an hermitian positive-definite linear functional:

\[
\forall f, g \in \mathcal{L} : \mathcal{F}\{fg^c\} = \overline{\mathcal{F}\{f^cg\}} \quad \text{and} \quad \forall f \in \mathcal{L}\setminus\{0\} : \mathcal{F}\{ff^c\} > 0
\]

\( \rightarrow \) ORFs: \( \varphi_k(x) \)

\( \varphi_k \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}, \varphi_k \perp \mathcal{L}_{k-1}, \text{ and } \|\varphi_k\| = \sqrt{\langle \varphi_k, \varphi_k \rangle} = 1. \)

Quasi-orthogonal rational functions (qORFs)

\( Q_n \in \mathcal{L}_n \) is called a qORF if \( Q_n \perp \mathcal{L}_{n-1}(\overline{\alpha}_n) \) where

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\mathcal{L}_k(\alpha) := \left\{ \frac{p_k}{\pi_k} \in \mathcal{L}_k : p_k(\alpha) = 0 \right\}.
\]
Orthonormal rational functions

**Definitions**

Let \( f_k = \frac{p_k}{\pi_k} \in \mathcal{L}_k \setminus \mathcal{L}_{k-1} \) with \( p_k \in \mathcal{P}_k \). Then \( f_k \) is

- **exceptional** iff \( p_k(\alpha_{k-1}) = 0 \)
- **degenerate** iff \( p_k(\overline{\alpha}_{k-1}) = 0 \)
- **regular** iff not exceptional and not degenerate

By convention \( p_k(\infty) = 0 \Leftrightarrow p_k \in \mathcal{P}_{k-1} \).

**Auxiliary rational functions**

\[
\phi_n(x) = \left(1 - \frac{x}{\alpha_n}\right)\varphi_n(x) \\
\psi_n(x) = \left(1 - \frac{x}{\overline{\alpha}_n}\right)\varphi_n(x)
\]
Orthonormal rational functions

**Definitions**

Let $f_k = \frac{p_k}{\pi_k} \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$ with $p_k \in \mathcal{P}_k$. Then $f_k$ is

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**Auxiliary rational functions**

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\begin{align*}
\phi_n(x) &= (1 - x/\alpha_n)\varphi_n(x) \\
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\end{align*}
$$
Theorem (Three-term Recurrence)

Whenever \( \varphi_n \) is not exceptional, \( \varphi_{n-1} \) is regular, and \( \varphi_{n-2} \) is not degenerate, they satisfy a three-term recurrence relation:

\[
x \varphi_{n-1}(x) = \gamma_n \phi_n(x) + \rho_n \phi_{n-1}(x) + \tilde{\gamma}_{n-1} \psi_{n-2}(x),
\]

with initial conditions

\[
\varphi_{-1}(x) \equiv 0 \quad \text{and} \quad \varphi_0(x) \equiv \eta_0 \|1\|^{-1}, \quad |\eta_0| = 1,
\]

\[
\alpha_{-1} \in \overline{\mathbb{R}}_0 \quad \text{and} \quad \alpha_0 = \overline{C}_0,
\]

and \( \tilde{\gamma}_{n-1} \) depends on \( \gamma_{n-1}, \rho_n \) and \( \alpha_{n-1} \).
Reproducing kernels

Reproducing kernels

\[ k_n(x, y) = \sum_{k=0}^{n-1} \varphi_k(x) \overline{\varphi_k(y)}. \]

which has the property that

\[ \forall f \in \mathcal{L}_{n-1}, \forall y \in \overline{\mathbb{C}} : \langle f(\cdot), k_n(\cdot, y) \rangle = f(y). \]

Christoffel-Darboux formula (polynomial case)

\[ k_n(x, y) = \gamma_n \left( \frac{\varphi_n(x) \overline{\varphi_{n-1}(y)} - \varphi_n(y) \overline{\varphi_{n-1}(x)}}{x - \overline{y}} \right) \]
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Christoffel-Darboux formula (polynomial case)

\[ k_n(x, y) = \gamma_n \left( \frac{\varphi_n(x) \varphi_{n-1}(y) - \varphi_n(y) \varphi_{n-1}(x)}{x - \overline{y}} \right) \]
Theorem

\[ c_n k_n(x, y) = \frac{b_{n-1}^c(y)}{b_{n-1}(y)} \left( \frac{\phi_n(x) \psi_{n-1}(y) - \phi_n(y) \psi_{n-1}(x)}{x - y} \right) \]

\[ \bar{c}_n k_n(x, y) = \frac{b_{n-1}(x)}{b_{n-1}^c(x)} \left( \frac{\phi_n^c(x) \psi_{n-1}^c(y) - \phi_n^c(y) \psi_{n-1}^c(x)}{x - y} \right) \]

where

\[ c_n = \bar{\kappa}_{n-1}^{-1} \lim_{z \to \alpha_{n-1}} \frac{\varphi_n(z)}{b_n(z)} \]

and

\[ \kappa_{n-1} = \lim_{z \to \alpha_{n-1}} \frac{\varphi_{n-1}(z)}{b_{n-1}(z)}. \]
Outline of the proof:

- The second equality follows from $k_n(y, x) = k_n(x, y)$
- For the first equality we will use the connection with qORFs

Lemma

Let

$$\Phi_n(x, y) := \frac{x - y}{(1 - x/\alpha_n)(1 - y/\overline{\alpha}_n)} k_n(x, \bar{y}).$$

Then for fixed $y$, $\Phi_n(\cdot, y)$ is a qORF in $\mathcal{L}_n$. 
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Let

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\Phi_n(x, y) := \frac{x - y}{(1 - x/\alpha_n)(1 - y/\overline{\alpha_n})} k_n(x, \overline{y}).
$$

Then for fixed $y$, $\Phi_n(\cdot, y)$ is a qORF in $L_n$. 
Outline of the proof (cont.):

**Lemma**

Let \( Q_n \in \mathcal{L}_n \) be such that

\[
(1 - x/\alpha_n)Q_n(x) = a_n\phi_n(x) + b_n\psi_{n-1}(x),
\]

with \( a_n, b_n \in \mathbb{C} \). Then \( Q_n \) is a qORF.

Conversely, suppose \( \varphi_n \) is not degenerate and let \( Q_n \in \mathcal{L}_n \) be a qORF. Then there exist constants \( a_n, b_n \in \mathbb{C} \) such that (1) holds true.
Outline of the proof (cont.):
From the previous two lemmas it follows that, if $\varphi_n$ is not degenerate, then

$$\frac{x - y}{(1 - y/\alpha_n)} k_n(x, y) = a_n(y)\varphi_n(x) + b_n(y)\psi_{n-1}(x),$$

where

- $a_n, b_n \in \mathcal{L}_n^c = \mathcal{L}\{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$
- $a_n(y)\varphi_n(y) = -b_n(y)\psi_{n-1}(y)$
- $\varphi_n$ and $\psi_{n-1}$ have common zeros iff $\varphi_n$ is degenerate

hence, the Christoffel-Darboux formula follows.
Outline of the proof (cont.):
From the previous two lemmas it follows that, if $\varphi_n$ is not degenerate, then

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where
- $a_n, b_n \in \mathcal{L}^c_n = \mathcal{L}\{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\}$
- $a_n(y) \phi_n(y) = -b_n(y) \psi_{n-1}(y)$
- $\phi_n$ and $\psi_{n-1}$ have common zeros iff $\varphi_n$ is degenerate

hence, the Christoffel-Darboux formula follows.
If \( \varphi_n = \frac{p_n}{\pi_n} \) is degenerate, then

\[
\phi_n(x) \equiv \psi_n(x) - 1 \equiv \text{const}, \quad \text{hence} \quad \phi_n(x)\psi_{n-1}(\bar{y}) - \phi_n(\bar{y})\psi_{n-1}(x) \equiv 0
\]

\[
p_n(\bar{\alpha}_{n-1}) = 0, \quad \text{hence} \quad c_n = \kappa_{n-1}^{-1} \lim_{z \rightarrow \bar{\alpha}_{n-1}} \frac{\varphi_n(z)}{b_n(z)} = 0
\]

hence, the Christoffel-Darboux formula remains valid.
If $\varphi_n = \frac{p_n}{\pi_n}$ is degenerate, then

- $\frac{\phi_n(x)}{\psi_{n-1}(x)} \equiv \text{const}$, hence $\phi_n(x)\psi_{n-1}(\overline{y}) - \phi_n(\overline{y})\psi_{n-1}(x) \equiv 0$

- $p_n(\overline{\alpha}_{n-1}) = 0$, hence

$$c_n = \kappa_{n-1}^{-1} \lim_{z \to \overline{\alpha}_{n-1}} \frac{\varphi_n(z)}{b_n(z)} = 0$$

hence, the Christoffel-Darboux formula remains valid.
If $\varphi_n = \frac{p_n}{\pi_n}$ is degenerate, then

1. $\frac{\phi_n(x)}{\psi_{n-1}(x)} \equiv \text{const}$, hence $\phi_n(x)\psi_{n-1}(\overline{y}) - \phi_n(\overline{y})\psi_{n-1}(x) \equiv 0$
2. $p_n(\overline{\alpha}_{n-1}) = 0$, hence

$$c_n = \kappa_{n-1}^{-1} \lim_{z \to \overline{\alpha}_{n-1}} \frac{\varphi_n(z)}{b_n(z)} = 0$$

hence, the Christoffel-Darboux formula remains valid.
Asymptotics

**Additional assumptions**

- \( \mathcal{F}\{f\} = \int_{I} f(x) d\mu(x), \ I = [-1, 1] \)
- \( \int_{I} \frac{\log \mu'(x)}{\sqrt{1-x^2}} dx > -\infty \)
- \( \{\alpha_k\}_{k>0} \) is bounded away from \( I \)

**Joukowsky transform**

- \( x = \frac{1}{2}(z + z^{-1}) \) and \( y = \frac{1}{2}(u + u^{-1}) \), with \( |z| \leq 1 \) and \( |u| \leq 1 \)
- \( \alpha_k = \frac{1}{2}(\beta_k + \beta_k^{-1}) \), with \( |\beta_k| < 1 \)
- \( \hat{\mu}(E) = \mu(\{\cos \theta, \theta \in E \cap [0, \pi]\}) + \mu(\{\cos \theta, \theta \in E \cap [-\pi, 0]\}) \)
- \( \int_{I} f(x) d\mu(x) = \frac{1}{2} \int_{-\pi}^{\pi} \hat{f}(z) d\hat{\mu}(\theta), \ z = e^{i\theta} \)
Asymptotics

Additional assumptions

- $\mathcal{F}\{f\} = \int_I f(x) d\mu(x)$, $I = [-1, 1]$
- $\int_I \frac{\log \mu'(x)}{\sqrt{1-x^2}} \, dx > -\infty$
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Asymptotics

\[ c_n k_n(x, y) = \phi_n(x) \frac{b_{n-1}(y)}{b_{n-1}(y)} \psi_{n-1}(y) \left( \frac{1 - \frac{\phi_n(y)}{\psi_{n-1}(y)} \cdot \left[ \frac{\phi_n(x)}{\psi_{n-1}(x)} \right]^{-1}}{x - y} \right) \]

Lemma

\[ \lim_{n \to \infty} \left( \frac{1 - \frac{\phi_n(y)}{\psi_{n-1}(y)} \cdot \left[ \frac{\phi_n(x)}{\psi_{n-1}(x)} \right]^{-1}}{x - y} \right) \frac{(1 - \beta_n z)(\bar{u} - \bar{\beta}_{n-1})}{2z\bar{u}(1 - \beta_n \bar{\beta}_{n-1})} = \frac{1}{1 - z\bar{u}} \]

locally uniformly in \( \overline{C}_I \times \overline{C}_I \).
Asymptotics

\[ c_n k_n(x, y) = \phi_n(x) \frac{b_{n-1}(y)}{b_n(y)} \psi_{n-1}(y) \left( \frac{1 - \frac{\phi_n(y)}{\psi_{n-1}(y)} \cdot \left[ \frac{\phi_n(x)}{\psi_{n-1}(x)} \right]^{-1}}{x - y} \right) \]

**Lemma**

\[
\lim_{n \to \infty} \left( \frac{1 - \frac{\phi_n(y)}{\psi_{n-1}(y)} \cdot \left[ \frac{\phi_n(x)}{\psi_{n-1}(x)} \right]^{-1}}{x - y} \right) \frac{(1 - \beta_n z)(\bar{u} - \bar{\beta}_{n-1})}{2z\bar{u}(1 - \beta_n \bar{\beta}_{n-1})} = \frac{1}{1 - z\bar{u}}
\]

locally uniformly in \( \overline{C}_I \times \overline{C}_I \).
Consider the Blaschke products \( B_k(z) = \prod_{j=1}^{k} \frac{z-\beta_j}{1-\beta_j z}, \quad k > 0 \) and outer spectral factor \( \sigma(z) = \exp \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \mu'(\theta) d\theta \right\} \).

**Lemma**

\[ \exists \varepsilon_n, \rho_{n-1}, \ |\varepsilon_n| = |\rho_{n-1}| = 1, \text{ such that} \]

\[ \lim_{n \to \infty} \frac{\varepsilon_n \sqrt{2}z(1+\beta_n^2)}{\sqrt{1-|\beta_n|^2}(1-\beta_n z)} B_{n-1}(z) \phi_n(x) = \sigma^{-1}(z) \]

\[ \lim_{n \to \infty} \frac{\rho_{n-1} \sqrt{2}u(1+\overline{\beta}_{n-1}^2)}{\sqrt{1-|\beta_{n-1}|^2}(u-\overline{\beta}_{n-1})} B^c_{n-1}(\overline{u}) \frac{b_{n-1}(\overline{y})}{b_{n-1}(\overline{y})} \psi_n(y) = \sigma^{-1}(\overline{u}) \]

_locally uniformly in \( \overline{\mathbb{C}}_I \).\]
Asymptotics

Theorem

Suppose \( \int_I \frac{\log \mu'(x)}{\sqrt{1-x^2}} \, dx > -\infty \) and assume \( \{\alpha_k\}_{k>0} \) is bounded away from \( I \). Then \( \exists \lambda_n, |\lambda_n| = 1 \), such that locally uniformly in \( \overline{C}_I \times \overline{C}_I \)

\[
\lim_{n \to \infty} C_n B_{n-1}(z) B_{n-1}(u) k_n(x,y) = \left[ (1 - z\bar{u})\sigma(z)\sigma(u) \right]^{-1},
\]

where

\[
C_n = \frac{\lambda_n(1 + \beta_n^2)(1 + \bar{\beta}_{n-1}^2)c_n}{\sqrt{(1 - |\beta_n|^2)(1 - |\beta_{n-1}|^2)(1 - \beta_n\bar{\beta}_{n-1})}},
\]

\( x = \frac{1}{2}(z + z^{-1}), |z| < 1 \), \( y = \frac{1}{2}(u + u^{-1}), |u| < 1 \), and

\( \alpha_k = \frac{1}{2}(\beta_k + \beta_k^{-1}), |\beta_k| < 1 \).
Remark

Whenever at least one of $\alpha_{n-1}$ and $\alpha_n$ is real, then

$$|c_n| = |\gamma_n|^{-1}, \text{ and}$$

$$\lim_{n \to \infty} C_n = 2.$$ 

Whenever both $\alpha_{n-1}$ and $\alpha_n$ are not real, then certainly

$$|c_n| \neq |\gamma_n|^{-1}$$

but this does not necessarily imply that

$$\lim_{n \to \infty} C_n \neq 2.$$
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References


Thank you ...