Positive rational interpolatory quadrature formulas on the unit circle and the interval

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Motivation and aim of the talk

- **Orthogonal Rational Functions** are a generalization of **Orthogonal Polynomials (poles at $\infty$)** and **Orthogonal Laurent Polynomials (poles at the origin and $\infty$).**
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- In the approximation of weighted integrals supported on the unit circle or the interval, if the integrand is a function with singularities (possible close to, but) outside the unit circle or the interval, then rational interpolatory quadrature formulas are often preferred than interpolatory rules.


- Aim: to present a connection between rational Gauss-type quadrature formulas and rational Szegő quadrature formulas.
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- **Aim:** to present a connection between rational Gauss-type quadrature formulas and rational Szegő quadrature formulas.
Notation

- $\mathbb{C}$: the complex plane. $\mathbb{R}$: the real line.
- $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: Riemann sphere.
- $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$: Extended real line.
- $T = \{z \in \mathbb{C} : |z| = 1\}$: unit circle.
- $D = \{z \in \mathbb{C} : |z| < 1\}$: open disk.
- $I = [-1, 1]$: the interval.
- $X_Y = \{t \in X : t \not\in Y\}$, for $X \subseteq \overline{\mathbb{C}}$ and $Y \subseteq \overline{\mathbb{C}}$.
- $\mathcal{P}_n$: the space of polynomials of degree less than or equal to $n$.
- Although $z$ and $x$ are both complex variables, we reserve the notation $z$ for the unit circle and $x$ for the interval.
For any complex function \( f(t) \), with \( t = z \) or \( t = x \) we define:

- **substr conjugate**: \( f_*(t) = \overline{f(1/t)} \).

- **super-c conjugate**: \( f^c(t) = \overline{f(t)} \).

Consequently, \( f^c_*(t) = f(1/t) \).

Note that if \( f(t) \) has a pole at \( t = p \), then

- \( f_*(t) \) has a pole at \( t = 1/p \),
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$f^c_*(t)$ has a pole at $t = 1/p$. 
Let a sequence of complex poles $\mathcal{A}_n = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \overline{\mathbb{C}}_I$ be fixed.

In what follows we will assume that $\alpha_n \in \overline{\mathbb{R}}_I$.

Define the factors

$$Z_k(x) = \frac{x}{1 - x/\alpha_k}, \quad k = 1, 2, \ldots, n,$$

and the basis functions

$$b_0(x) \equiv 1, \quad b_k(x) = b_{k-1}(x)Z_k(x), \quad k = 1, 2, \ldots, n.$$

These basis functions generate the nested spaces of rational functions with poles in $\mathcal{A}_n$:

$$\mathcal{L}_j = \text{span}\{b_0, \ldots, b_j\}, \quad 0 \leq j \leq n.$$
Spaces of rational functions

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Define

\[ \pi_0(x) \equiv 1, \quad \pi_j(x) = \prod_{k=1}^{j} (1 - x/\alpha_k), \quad 0 < j \leq n. \]

Then we may write equivalently  \[ \mathcal{L}_j = \{p_j/\pi_j : p_j \in \mathcal{P}_j\}. \]
Spaces of rational functions

- $\mathcal{L}_j^c = \{ f : f^c \in \mathcal{L}_j \}$.

- $\mathcal{L}_j$ and $\mathcal{L}_j^c$ are rational generalizations of $\mathcal{P}_j$: if all $\alpha_k = \infty$,
  
  \[ Z_k(x) = Z_k^c(x) = x \text{ and } b_k(x) = b_k^c(x) = x^k. \]

- Consider the integral
  
  \[ J_\mu(F) = \int_{-1}^{1} F(x)d\mu(x), \]
  
  $\mu$: a positive bounded Borel measure on $I$. 

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Positive rational interpolatory quadrature formulas.
An $n$th rational interpolatory rule is obtained by integrating an interpolating rational function of degree $n - 1$, and is of the form

$$J_n(F) = \sum_{k=1}^{n} \lambda_k F(x_k), \quad \{x_k\}_{k=1}^{n} \subset I, \quad \{\lambda_k\}_{k=1}^{n} \subset \mathbb{R},$$

$$x_j \neq x_k \text{ if } j \neq k,$$

so that $J_{\mu}(F) = J_n(F)$ for every $F \in \mathcal{R}_{p,q} = \mathcal{L}_p \cdot \mathcal{L}_q^c$, $p + q \leq 2n - 1$ and $0 \leq q \leq p \leq n$.

**Lemma**

The weights $\lambda_k$ in the quadrature formula $J_n(F)$ are real, if and only if,

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Orthogonal Rational Functions and Rational Gaussian quadrature formula

- \( \varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1} \): \( n \)th orthogonal rational function (ORF) w.r.t. the inner product

\[
\langle F, G \rangle_\mu = \int_{-1}^{1} F(x) G^c(x) d\mu(x).
\]

- Zeros \( x_k \) of \( \varphi_n(x) \): all distinct and in \((-1, 1)\). Hence, they can be chosen as nodes for \( J_n(F) \): \( n \)-point rational Gaussian quadrature formula, which has maximal domain of validity, i.e. the approximation is exact for every function \( F \in \mathcal{R}_{n,n-1} \) (taking \( p = n \) and \( q = n - 1 \)).

- From the previous Lemma, the weights are real. Moreover they are all positive.

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Positive rational interpolatory quadrature formulas.
Rational Gauss-type quadrature formula

- For any other choice of the nodes: the weights may be non-positive and the quadrature will only be exact in a smaller set of rational functions.

- For each node that is fixed in advance: the domain of validity will generally decrease by one.

Exactness at least for $p + q = n - 1$. 
Rational Gauss-type quadrature formula

Special cases:

- **If one node is fixed in advance:** the weights are all positive and the quadrature is exact for every $F \in \mathcal{R}_{n-1,n-1}$.

  *$n$-point rational Gauss-Radau quadrature formula.*

- Whenever **two nodes** in an $(n+1)$-point quadrature formula are fixed in advance, so that the weights are all positive and the quadrature is exact for every $F \in \mathcal{R}_{n,n-1}$, we obtain the $(n+1)$-point rational Gauss-Lobatto quadrature formula.

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- The existence depends on the choice of the prefixed nodes.

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- The existence depends on the choice of the prefixed nodes. Polynomial case characterized in A. Bultheel, R. C.-B., M. Van Barel. - On Gauss-type quadrature formulas with prescribed nodes anywhere on the real line, Calcolo. To appear (2009).
Another sequence of basic functions for the unit circle

Given a sequence of complex numbers

$$B_n = \{\beta_1, \beta_2, \ldots, \beta_n\} \subset \mathbb{D}, \text{ with } \beta_n \in (-1, 1),$$

we define the Blaschke factors for $B_n$ as

$$\zeta_k(z) = \eta_k \frac{z - \beta_k}{1 - \beta_k z}, \quad \eta_k = \begin{cases} -\frac{\beta_k}{|\beta_k|}, & \beta_k \neq 0 \\ 1, & \beta_k = 0 \end{cases} \quad , k = 1, 2, \ldots, n,$$

and the corresponding Blaschke products for $B_n$ as

$$B_0(z) \equiv 1, \quad B_k(z) = B_{k-1}(z)\zeta_k(z), \quad k = 1, 2, \ldots, n.$$
Another sequence of basic functions for the unit circle

- These Blaschke products generate the nested spaces of rational functions: \( \hat{L}_j = \text{span}\{B_0, \ldots, B_j\} \), \( 0 \leq j \leq n \).

- Define

\[
\hat{\pi}_0(z) \equiv 1, \quad \hat{\pi}_j(z) = \prod_{k=1}^{j} (1 - \bar{\beta}_k z), \quad 0 < j \leq n.
\]

Then, we may write equivalently

\[
B_j(z) = \nu_j \frac{\hat{\pi}_j^*(z)}{\hat{\pi}_j(z)}, \quad \nu_j = \prod_{k=1}^{j} \eta_k \in \mathbb{T},
\]

where \( \hat{\pi}_j^*(z) = z^j \hat{\pi}_j^*(z) \), and

\[
\hat{L}_j = \{p_j/\hat{\pi}_j : p_j \in \mathcal{P}_j\}.
\]
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Now,
\[ \hat{\mathcal{L}}_j^* = \{ f : f_\ast \in \hat{\mathcal{L}}_j \}, \quad \hat{\mathcal{L}}_j^c = \{ f : f^c \in \hat{\mathcal{L}}_j \} \text{ and} \]
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\( \hat{\mathcal{L}}_j \) and \( \hat{\mathcal{L}}_j^c \) are rational generalizations of \( \mathcal{P}_j \) too:
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Consider the integral
\[ I_\mu(F) = \int_{-\pi}^{\pi} F(z) d\mu(\theta), \ z = e^{i\theta}. \]

\( \mu \): a positive bounded Borel measure on \( \mathbb{T} \).
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Rational interpolatory quadrature formulas

Approximation of $I_{n}(F)$:

$$I_{n}(F) = \sum_{k=1}^{n} \lambda_{k} F(z_{k}), \quad \{z_{k} = e^{i\theta_{k}}\}_{k=1}^{n} \subset \mathbb{T}, \quad \{\lambda_{k}\}_{k=1}^{n} \subset \mathbb{R}, \quad z_{j} \neq z_{k} \text{ if } j \neq k,$$

so that $I_{\mu}(F) = I_{n}(F)$ for every $F \in \mathcal{R}_{p,q} = \mathcal{L}_{p} \cdot \mathcal{L}_{q,*}$, with $n - 1 \leq p + q \leq 2n - 2$ and $0 \leq q \leq p \leq n - 1$.

Lemma

The weights are real now iff $\mathcal{R}_{p,q} = \mathcal{R}_{(p,q)*}$, implying that $p = q$. 
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The weights are real now iff $\mathcal{R}_{p,q} = \mathcal{R}_{(p,q)^*}$, implying that $p = q$. 
Orthogonal Rational Functions and Rational Szegő quadrature formula

- \( \phi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1} \): \( n \)th ORF with respect to the inner product
  \[
  \langle F, G \rangle_{\tilde{\mu}} = \int_{-\pi}^{\pi} F(z)G_*(z)d\tilde{\mu}(\theta).
  \]

- Leading coefficient \( \kappa_n \), i.e. the coefficient of \( B_n(z) \) in the expansion of \( \phi_n(z) \) in the basis \( \{B_0, \ldots, B_n\} \), is then given by
  \( \kappa_n = \overline{\phi_n^*(\beta_n)} \), where \( \phi_n^*(z) = B_n(z)\phi_n^*(z) \).
  We will assume the ORF is monic, i.e. \( \phi_n^*(\beta_n) = 1 \).

- Para-orthogonal rational function (pORF):
  \[
  \hat{Q}_{n,\tau}(z) = \phi_n(z) + \tau \phi_n^*(z), \quad \tau \in \mathbb{T}.
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Zeros $z_k$ of $\tilde{Q}_{n,\tau}(z)$: all distinct and on $\mathbb{T}$. Hence, can be chosen as nodes for $I_n(F)$: 

$n$-point rational Szegő quadrature formula, which has maximal domain of validity $F \in \tilde{R}_{p,q} = \tilde{L}_{n-1} \cdot \tilde{L}_{(n-1)^*}$ ($p = q = n - 1$).

Again, it is well known that in this case the weights $\tilde{\lambda}_k$ are all positive.
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- Again, it is well known that in this case the weights $\tilde{\lambda}_k$ are all positive.
Rational Szegő-type quadrature formulas

- Unlike in the case of the interval: nodes and weights in an $n$-point rational Szegő quadrature formula are not unique.

- Consequently, an $n$-point rational Szegő-Radau quadrature formula (one fixed node) always has positive weights and maximal domain of validity too.

- An $n$-point rational Szegő-Lobatto quadrature formula (two fixed nodes) is at least exact for every $F \in \mathcal{R}_{n-2,n-2}$ and again always has positive weights.

Propaganda for Pablo’s talk in $\approx$ 45 minutes!
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Joukowski Transformation

- We denote \( x = \frac{1}{2}(z + z^{-1}) \) by \( x = J(z) \), mapping the open unit disc \( \mathbb{D} \) onto the cut Riemann sphere \( \overline{\mathbb{C}_I} \) and the unit circle \( \mathbb{T} \) onto the interval \( I \).

- When \( z = e^{i\theta} \), then \( x = J(z) = \cos \theta \).

- In what follows we will assume that \( x \) and \( z \) are related by this transformation.

- Inverse mapping: \( z = J^{\text{inv}}(x) \), chosen so that \( z \in \mathbb{D} \) if \( x \in \overline{\mathbb{C}_I} \).

- With the sequence \( \mathcal{A}_n = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \overline{\mathbb{C}_I} \) we associate a sequence \( \mathcal{B}_n = \{\beta_1, \beta_2, \ldots, \beta_n\} \subset \mathbb{D} \), so that \( \beta_k = J^{\text{inv}}(\alpha_k) \), and \( \hat{\mathcal{B}}_{2n} = \{\hat{\beta}_1, \ldots, \hat{\beta}_{2n}\} \subset \mathbb{D} \) with

\[
\hat{\beta}_{2k} = \overline{\hat{\beta}_{2k-1}} = \beta_k, \quad k = 1, \ldots, n.
\]
In what follows, the measures $\mu$ and $\tilde{\mu}$ are related by
$$\tilde{\mu}'(\theta) = \mu'((\cos \theta) |\sin \theta|).$$

By the Joukowksi Transform, a function $F(x)$ transforms into
a function $\tilde{F}(z) = (F \circ J)(z)$, so that $\tilde{F}(z) = \tilde{F}(z^{-1})$ and
$J_{\mu}(F) = \frac{1}{2} I_{\tilde{\mu}}(\tilde{F}).$

Moreover, every function $F \in \mathcal{L}_k \setminus \mathcal{L}_{k-1}$ transforms into a
function $\tilde{F} \in (\mathcal{L}_k^c \cdot \mathcal{L}_{k^*}) \setminus (\mathcal{L}_{k-1}^c \cdot \mathcal{L}_{(k-1)^*}).$
Lemma

Suppose the numbers \( \{\beta_1, \ldots, \beta_{n-1}\} \) are real or appear in complex conjugate pairs. Then,

1. the zeros of an \( n \)th pORF \( \hat{Q}_{n,\tau}(z) \) w.r.t. \( \hat{\mu} \) appear in complex conjugate pairs iff \( \tau = \pm 1 \).

2. the \( n \)th pORF \( \hat{Q}_{n,\tau_n}(z) \) w.r.t. \( \hat{\mu} \) has a zero in
   - \( z = 1 \) iff \( \tau_n = -\nu_n \),
   - \( z = -1 \) iff \( \tau_n = (-1)^{n+1}\nu_n \),

where \( \nu_n \in \{\pm 1\} \) is defined as before.

3. if \( I_n(F) = \sum_{k=1}^{n} \hat{\lambda}_k F(z_k) \) is an \( n \)-point rational Szegő quadrature formula for \( I_{\hat{\mu}}(F) \), based on the zeros of the pORF \( \hat{Q}_{n,\pm1}(z) \), then for \( k = 1, \ldots, n \), the weight \( \hat{\lambda}_k \) corresponding to the node \( z_k \) is equal to the weight \( \hat{\lambda}_j \) corresponding to the node \( z_j = \bar{z}_k \).
Generalization of Szegő’s relation to the rational case

- \( \varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1} \): an \( n \)th ORF with respect to the measure \( \mu \) on \( I \),
- \( \phi_n \in \hat{\mathcal{L}}_n \setminus \hat{\mathcal{L}}_{n-1} \): the \( n \)th monic ORF with respect to the measure \( \hat{\mu} \) on \( \mathbb{T} \).

- There exists a nonzero constant \( C_n \) so that

\[
\varphi_n(x) = C_n B_{n\ast}(z) \left\{ \hat{\phi}_{2n}^c(z) + \hat{\phi}_{2n}^*(z) \right\} = C_n B_{n\ast}(z) \hat{Q}_{2n,1}(z).
\]

In what follows, the hat refers to the sequence of numbers defined before.

- Generalization of the connection between Orthogonal Polynomials with respect to the measure \( \hat{\mu} \) on \( \mathbb{T} \) and Orthogonal Polynomials with respect to the measure \( \mu \) on the interval \( I \).
Theorem (Connection between Gauss q.f. and Szegö q.f.)

Let \( \hat{\mu} \) and \( \mu \) be positive Bounded Borel measures on \( \mathbb{T} \) and \( I \) respectively and related as before. Let \( \hat{I}_{2n}(F) = \sum_{k=1}^{2n} \hat{\lambda}_k F(z_k) \) be a 2n-point rational Szegö quadrature formula for \( I_{\hat{\mu}}(F) = \int_{-\pi}^{\pi} F(z) d\hat{\mu}(\theta) \), based on the zeros of the pORF \( \hat{Q}_{2n,1}(z) = \hat{\phi}_{2n}(z) + \hat{\phi}_{2n}^*(z) \).

Suppose \( z_k \neq z_j \) for every \( 1 \leq k < j \leq n \) and set \( z_k = e^{i\theta_k} \) for \( k = 1, \ldots, n \). Then, when taking \( x_k = \cos \theta_k \) and \( \lambda_k = \hat{\lambda}_k \) for \( k = 1, \ldots, n \), the formula \( J_n(F) = \sum_{k=1}^{n} \lambda_k F(x_k) \) coincides with the n-point rational Gaussian quadrature formula for \( J_\mu(F) = \int_{-1}^{1} F(x) d\mu(x) \), based on the zeros of an nth ORF \( \varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1} \).
In the opposite direction,

**Theorem (Connection between Gauss q.f. and Szegő q.f.)**

Let $J_n(F) = \sum_{k=1}^{n} \lambda_k F(x_k)$ be the $n$-point rational Gaussian quadrature formula for $J_{\mu}(F) = \int_{-1}^{1} F(x)d\mu(x)$, based on the zeros of the $n$th ORF $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$. Set $x_k = \cos \theta_k$ and define \( \{z_k\}_{k=1}^{2n} \) and \( \{\hat{\lambda}_k\}_{k=1}^{2n} \) by means of

\[
\begin{align*}
  z_k &= e^{i\theta_k}, & \hat{\lambda}_k &= \lambda_k \\
  z_{n+k} &= e^{-i\theta_k}, & \hat{\lambda}_{n+k} &= \lambda_k
\end{align*}
\]

Then $\hat{I}_{2n}(F) = \sum_{k=1}^{2n} \hat{\lambda}_k F(z_k) = \sum_{k=1}^{n} \hat{\lambda}_k [F(z_k) + F(\bar{z}_k)]$ coincides with a $2n$-point rational Szegő quadrature formula for $I_{\hat{\mu}}(F) = \int_{-\pi}^{\pi} F(z)d\hat{\mu}(\theta)$, when taking as nodes the zeros of the $pORF$ $\hat{Q}_{2n,1}(z) = \hat{\phi}_{2n}(z) + \hat{\phi}_{2n}^*(z)$. 
Consider the complex varying measure $\mu_{n-1}$ defined by

$$d\mu_{n-1}(x) = \frac{1}{2} \left( \beta_{n-1} - \frac{1}{\beta_{n-1}} \right) \sqrt{\alpha_n^2 - 1} \frac{(1 - x^2)}{(|\alpha_{n-1} - x| |\alpha_n - x|)} d\mu(x).$$

Let $\tilde{\varphi}_{n-1}(x)$ denote an $(n-1)$th ORF on $I$ w.r.t $\mu_{n-1}$, associated to the sequence $A_{n-1} \subset \overline{C}_I$.

Further, let $\phi_n \in \hat{\mathcal{L}}_n \setminus \hat{\mathcal{L}}_{n-1}$ denote the $n$th monic ORF with respect to the measure $\hat{\mu}$ on $\mathbb{T}$. 
There exists a nonzero constant $C_{n-1}$ so that

$$\tilde{\varphi}_{n-1}(x) = C_{n-1} \frac{B_{n*}(z)}{\zeta_n(z) - \zeta_{n*}(z)} \left\{ \hat{\phi}_c^c(z) - \hat{\phi}_2^*(z) \right\}$$

$$= \tilde{C}_{n-1} (1 - \beta_n z)^2 B_{(n-1)*}(z) \left\{ \frac{\hat{Q}_{2n-1}(z)}{z^2 - 1} \right\},$$

$$\tilde{C}_{n-1} = \frac{C_{n-1}}{1 - \beta_n^2}$$

Rational generalization of the connection between OPs with respect to the measure $\mu$ on the unit circle and OPs with respect to the measure $\tilde{\mu}$, with $d\tilde{\mu}(x) = (1 - x^2)d\mu(x)$, on the interval, which has been established by Szegő too.
Theorem (Connection between Gauss-Lobatto q.f. and Szegő q.f.)

Let \( \hat{I}_{2n}(F) = 2AF(-1) + 2BF(1) + \sum_{k=1}^{2n-2} \hat{\lambda}_k F(z_k) \) be a 2n-point rational Szegő quadrature formula for

\[
I_{\mu}(F) = \int_{-1}^{1} F(z) \, d\hat{\mu}(\theta),
\]

based on the zeros of the pORF \( \hat{Q}_{2n,-1}(z) = \hat{\phi}_{2n}(z) - \hat{\phi}_{2n}^*(z) \).

Suppose \( z_k \neq \bar{z}_j \) for every \( 1 \leq k < j \leq n - 1 \) and set \( z_k = e^{i\theta_k} \) for \( k = 1, \ldots, n - 1 \). Then, when taking \( x_k = \cos \theta_k \) and \( \lambda_k = \hat{\lambda}_k \) for \( k = 1, \ldots, n - 1 \), the formula

\[
J_{n+1}(F) = AF(-1) + BF(1) + \sum_{k=1}^{n-1} \lambda_k F(x_k)
\]

coincides with the \((n+1)\)-point rational Gauss-Lobatto quadrature formula for

\[
J_{\mu}(F) = \int_{-1}^{1} F(x) \, d\mu(x)
\]

with fixed nodes in \( 1 \) and \(-1 \) and based on the zeros of \( \tilde{\phi}_{n-1}(x) \).
In the opposite direction:

**Theorem (Connection between Gauss-Lobatto q.f. and Szegő q.f.)**

Let $J_{n+1}(F) = AF(-1) + BF(1) + \sum_{k=1}^{n-1} \lambda_k F(x_k)$ be the $(n + 1)$-point rational Gauss-Lobatto quadrature formula for $J_{\mu}(F) = \int_{-1}^{1} F(x) d\mu(x)$ with fixed nodes in 1 and $-1$. Set $x_k = \cos \theta_k$ and define $\{z_k\}_{k=1}^{2n-2}$ and $\{\hat{\lambda}_k\}_{k=1}^{2n-2}$ by means of

\[
\begin{align*}
    z_k &= e^{i\theta_k}, & \hat{\lambda}_k &= \lambda_k \\
    z_{n-1+k} &= e^{-i\theta_k}, & \hat{\lambda}_{n-1+k} &= \lambda_k
\end{align*}
\]

$k = 1, \ldots, n - 1$.

Then, \(\hat{I}_{2n}(F) = 2AF(-1) + 2BF(1) + \sum_{k=1}^{2n-2} \hat{\lambda}_k F(z_k) = 2AF(-1) + 2BF(1) + \sum_{k=1}^{2n-2} \hat{\lambda}_k [F(z_j) + F(\overline{z}_j)]\) coincides with a 2n-point rational Szegő quadrature formula for

\[
I_{\hat{\mu}}(F) = \int_{-\pi}^{\pi} F(z) d\hat{\mu}(\theta),
\]

when taking as nodes the zeros of the

\[pORF \hat{Q}_{2n-1}(z) = \hat{\phi}_{2n}(z) - \hat{\phi}^*_2(z).\]
Theorem

Let $\phi_n$ denote the monic ORF with respect to the measure $\mu$ on $\mathbb{T}$ and define $\varphi_{n-1} \in \mathcal{L}_{n-1}$ by

$$\varphi_{n-1}^{\pm}(x) = \frac{B_{(n-1)^*}}{1 \pm \eta_n \zeta_n(z)} \left\{ \hat{\phi}_{2n-1}(z) \pm \eta_n \hat{\phi}_{2n-1}^*(z) \right\}$$

$$= \frac{B_{(n-1)^*}}{1 \pm \eta_n \zeta_n(z)} \hat{Q}_{2n-1, \pm \eta_n}(z), \quad \eta_n = \hat{\nu}_{2n-1} \in \{\pm 1\}.$$

Then it holds that

$$Q_n^{\pm}(x) = \left( \frac{1 \pm x}{1 - x/\alpha_n} \right) \varphi_{n-1}^{\pm}(x)$$

is orthogonal to $\mathcal{L}_{n-1}(\alpha_n) = \left\{ \frac{(1-x/\alpha_n)p_{n-2}(x)}{\pi_{n-1}(x)} : p_{n-2} \in \mathcal{P}_{n-2} \right\}$ with respect to the measure $\mu$ on $I$. 

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Positive rational interpolatory quadrature formulas

- \( Q_n^\pm(x) \) orthogonal to \( L_{n-1}(\alpha_n) \) means that \( Q_n^\pm(x) \) is a quasi-orthogonal rational function of the form

\[
Q_n^\pm(x) = c_n \left\{ \varphi_n(x) + \rho^\pm \frac{Z_n(x)}{Z_{n-1}^c(x)} \varphi_{n-1}(x) \right\}, \quad c_n \in \mathbb{C}_0,
\]

where \( \varphi_k(x), \; k = n-1, n, \) denotes a \( k \)th ORF with respect to the measure \( \mu \) on \( I \), and

\[
\rho^\pm = -\frac{\varphi_n(\pm1)}{Z_n(\pm1)} \cdot \frac{Z_{n-1}^c(\pm1)}{\varphi_{n-1}(\pm1)}.
\]

- Previous Theorem is also a generalization of OPs with respect to the measure \( \tilde{\mu} \) and OPs with respect to the measure \( d\mu^\pm(x) = (1 \pm x)d\mu(x) \) on the interval.
Theorem (Connection between Gauss-Radau q.f. and Szegő q.f.)

Suppose $\xi \in \{\pm 1\}$. Let $\hat{I}_{2n}(F) = 2AF(\xi) + \sum_{k=1}^{2n-2} \hat{\lambda}_k F(z_k)$ be a $(2n - 1)$-point rational Szegő quadrature formula for $I_{\hat{\mu}}(F) = \int_{-\pi}^{\pi} F(z)d\hat{\mu}(\theta)$, based on the zeros of the pORF $\hat{Q}_{2n-1,-\xi\eta_n}(z) = \hat{\phi}_{2n-1}(z) - \xi\eta_n\hat{\phi}^*_{2n-1}(z)$.

Suppose $z_k \neq \overline{z}_j$ for every $1 \leq k < j \leq n - 1$ and set $z_k = e^{i\theta_k}$ for $k = 1, \ldots, n - 1$. Then, when taking $x_k = \cos \theta_k$ and $\lambda_k = \hat{\lambda}_k$ for $k = 1, \ldots, n$, the formula $J_n(F) = AF(\xi) + \sum_{k=1}^{n-1} \lambda_k F(x_k)$ coincides with the $n$-point rational Gauss-Radau quadrature formula for $J_{\mu}(F) = \int_{-1}^{1} F(x)d\mu(x)$ with fixed node in $\xi$. 
In the opposite direction:

**Theorem (Connection between Gauss-Radau q.f. and Szegő q.f.)**

Suppose $\xi \in \{\pm 1\}$. Let $J_n(F) = AF(\xi) + \sum_{k=1}^{n-1} \lambda_k F(x_k)$ be the $n$-point rational Gauss-Radau quadrature formula for $J_\mu(F) = \int_{-1}^{1} F(x) d\mu(x)$ with fixed node in $\xi$. Set $x_k = \cos \theta_k$ and define \( \{z_k\}_{k=1}^{2n-2} \) and \( \{\hat{\lambda}_k\}_{k=1}^{2n-2} \) by means of

\[
\begin{align*}
    z_k &= e^{i\theta_k}, & \hat{\lambda}_k &= \lambda_k \\
    z_{n-1+k} &= e^{-i\theta_k}, & \hat{\lambda}_{n-1+k} &= \lambda_k
\end{align*}
\]

Then

\[
\hat{I}_{2n-1}(F) = 2AF(\xi) + \sum_{k=1}^{2n-2} \hat{\lambda}_k F(z_k) = 2AF(\xi) + \sum_{k=1}^{n-1} \hat{\lambda}_k [F(z_k) + F(\bar{z}_k)]
\]

coincides with a $(2n-1)$-point rational Szegő quadrature formula for $I_{\hat{\mu}}(F) = \int_{-\pi}^{\pi} F(z) d\hat{\mu}(\theta)$, when taking as nodes the zeros of the pORF

\[
\hat{Q}_{2n-1,-\xi\eta_n}(z) = \hat{\phi}_{2n-1}(z) - \xi \eta_n \hat{\phi}_{2n-1}^*(z).
\]
A result for $\tau \neq \pm 1$

- For $\tau \neq \pm 1$ and under the condition that $\mathcal{L}_{n-1} = \mathcal{L}^c_{n-1}$, the $n$-point rational Szegő quadrature formulas transform into $n$-point rational interpolatory quadrature formulas on the interval with positive weights, which are only exact in $\mathcal{L}_{n-1}$ unless some particular choice of $\tau$ is made.

A result for $\tau \neq \pm 1$

**Theorem**

Suppose the poles $\{\alpha_1, \ldots, \alpha_{n-1}\}$ (and hence, the numbers $\{\beta_1, \ldots, \beta_{n-1}\}$) are real or appear in complex conjugate pairs, i.e. $\mathcal{L}_{n-1} = \mathcal{L}_{n-1}^c$ and $\mathcal{L}_{n-1} = \mathcal{L}_{n-1}^c$. Let $I_n(\hat{F}) = \sum_{k=1}^{n} \lambda_k \hat{F}(z_j)$ be an $n$th rational Szegő quadrature formula with respect to $\hat{\mu}$, where the nodes $\{z_k = e^{i\theta_k}\}_{k=1}^{n}$ are the zeros of the pORF $\hat{Q}_{n,\tau}(z) = \phi_n(z) + \tau \phi_n^*(z)$ with $\tau \neq \pm 1$. Set $x_k = \cos \theta_k$, $k = 1, \ldots, n$ and $\lambda_k = \hat{\lambda}_k/2 > 0$, and consider the $n$-point rational interpolatory quadrature formula based upon these nodes and weights $J_n^\tau(F) = \sum_{k=1}^{n} \lambda_k F(x_k)$ for $J_\mu(F) = \int_{-1}^{1} F(x) d\mu(x)$. Then this rational interpolatory quadrature formula is exact for every $F \in \mathcal{L}_{n-1}$. Furthermore, it is exact in $\mathcal{L}_n$ iff

$$\tau = \tau_{optimal} = -\phi_n(\beta_n) \pm i\sqrt{1 - \phi_n^2(\beta_n)}.$$
Numerical experiments.

Error bounds for the quadrature rules considered in the talk.

Thanks for your attention!
Numerical experiments.

Error bounds for the quadrature rules considered in the talk.

Thanks for your attention!