CHAOS, QUANTIZATION AND THE CLASSICAL LIMIT ON THE TORUS[1]

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Abstract

The algebraic and the canonical approaches to the quantization of a class of classical symplectic dynamical systems on the two-torus are presented in a simple unified framework. This allows for ready comparison between the two very different approaches and is well adapted to the study of the semi-classical behaviour of the resulting models. Ergodic translations and skew translations, as well as the hyperbolic toral automorphisms and their Hamiltonian perturbations are treated. Ergodicity is proved for the algebraic quantum model of the translations and skew-translations and exponential mixing in the algebraic quantum model of the hyperbolic automorphisms. This latter result is used to show the non-commutativity of the classical and large time limits. Turning to the canonical model, recent results are reviewed on the behaviour in the classical limit of the eigenvalues and eigenvectors of the quantum propagators; the link with the ergodic or mixing properties of the underlying dynamics is explained. An example of the non-commutativity of the classical and large-time limits is proven here as well.

1. INTRODUCTION

Chaos in Hamiltonian dynamical systems is—to say the least—a much studied field. With the word “chaos” interpreted sufficiently vaguely, it encompasses a vast number of subjects such as (in)stability, periodic orbit theory, ergodicity, mixing, entropy, Lyapunov exponents, and hyperbolicity. Two types of questions present themselves rather naturally.

The first one is: “How should one define and study the notions of classical chaos in quantum mechanics”? One can for example legitimately ask what it means for a quantum dynamical system to be ergodic, mixing, hyperbolic or what the dynamical

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entropy of a quantum dynamical system is. These questions were originally addressed by the C^*-algebraic school in quantum statistical mechanics interested in understanding—among other things—the approach or return to equilibrium in systems with an infinite number of degrees of freedom, where such questions do indeed arise. A beautiful and recent review of this field is to be found in Ref.1, where special attention is paid to entropic questions.

Whereas this first question is—as stated—purely a problem of quantum dynamics, the second question is semi-classical in nature: "If a quantum system has a classical limit that is "chaotic", how does this reflect on the properties of the quantum dynamics"? In this context, one looks at systems with a finite (often small) number of degrees of freedom, having a quantum Hamiltonian with discrete spectrum. In the energy domain, for example, one studies the eigenvalues and eigenfunctions of this Hamiltonian in a small $\hbar$ dependent interval around a fixed energy value E_0. The question is to understand how the behaviour of these quantities is affected by the nature of the Hamiltonian dynamics of the classical Hamiltonian on the energy surface at energy E_0. In the time domain, one is interested in detecting differences between the quantal and classical evolution of observables, and in the (non-)commutativity of the $\hbar \to 0$ and $t \to \infty$ limits. Generally, one wishes to understand the signature of ergodic, mixing, or hyperbolic behaviour, the influence of periodic orbits, etc.. The physical motivation for this set of problems comes from nuclear, atomic and solid state physics. There is a vast literature on this subject (often referred to as quantum chaology), of varying degrees of mathematical rigour[2, 3, 4].

Of course, both questions are related. Somehow they refer both to the question of what "quantum chaos" is, if it exists at all. They nevertheless reflect different motivations and originate in completely different physical problems, which explains why they have been studied by two different schools of reseachers, studying different models. This situation seems to have generated some amount of Babylonic confusion (and even some bad temper) since in some cases, the same terms are used to denote entirely different notions: "quantum ergodicity" is a good example of this. Indeed, precisely because they have discrete spectrum, the Hamiltonians investigated by the second school can not generate a quantum dynamics that displays "chaotic" or even simply ergodic behaviour in the sense in which this is understood by the first school[1]. Nevertheless, they can "recuperate" ergodic (mixing, etc.) properties in the classical limit, and it is in this sense the term "quantum ergodic" is used by the second school[4, 5]. This perhaps somewhat unfortunate state of affairs should however not be construed to mean the concepts themselves are unclear.

There is one class of (toy) model classical systems, a quantum version of which has been introduced and studied by both schools. These are the hyperbolic automorphisms of the torus. I will describe both quantum models below and use them to illustrate the typical problems one addresses in each case. This will, I hope, allow the reader to appreciate both the differences and the similarities between the two lines of thought. I will profit from the occasion to discuss some semi-classical questions in the algebraic model, a problem that seems not to have been addressed until now. I will also prove, in both models, the non-commutativity of the large time and semi-classical limits (Theorem 5 and Theorem 7), which constitute the only new results in this contribution.

Let me now first describe the toral automorphisms. Let $\mathbb{T}^{(2)} = \mathbb{R}^2/\mathbb{Z}^2$ be the two-torus,with local coordinates $x = (q, p) \in [0, 1[\times[0, 1[$ and symplectic two-form $\omega = dq \wedge dp$. Let $A \in SL(2, \mathbb{Z})$; then A acts on $\mathbb{T}^{(2)}$ in the obvious way. Clearly A is a symplectic diffeomorphism of $\mathbb{T}^{(2)}$ referred to as an automorphism of the torus. The case

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

is often referred to as "Arnold's cat" . More generally, if $\mid TrA \mid > 2$, then A is an Anosov system. This means in particular that all orbits obtained by iterating A are exponentially unstable. This is therefore the prototypical example of a chaotic Hamiltonian system, which is in particular mixing and ergodic, as explained in some more detail below. Since it is essentially linear, it is also very simple. Hence it seems fair to hope that one should be able to quantize it and that its quantum version should shed some light on the two preceding questions.

Two completely different quantum analogs of this system have been proposed[6, 7] in the literature, one by each school.

The oldest "quantum Arnold cat" that roams the literature[6] is obtained by a "canonical quantization" that sticks closely to the "ordinary" quantum mechanics of systems with a finite number of degrees of freedom (in this case one). Here the first question is: "What is the quantum Hilbert space of states for a system having the torus as phase space?". The resulting space turns out to be finite dimensional, in agreement with the fact that the torus has finite volume. The observables are then identified with the self-adjoint operators on this Hilbert space. The classical limit corresponds to taking the dimension of the space to infinity. This quantization starts from the torus viewed as phase space of a classical system with one degree of freedom. It has an interpretation in terms of an optical system[6] and is completely analogous to the one of spin. In that case, the classical phase space is a sphere and the quantum Hilbert space is $\mathbb{C}^{2\ell+1}$ for spin $s = \hbar\ell$: the classical limit corresponds to $\ell \to \infty, \hbar \to 0, \hbar\ell = s$. The quantized automorphisms are realized as unitary operators on the finite dimensional Hilbert space of states and the interest is in the behaviour of its eigenelements in the classical limit. This model is described in section 3.

The algebraic "quantum Arnold cat" is the most recent of the two[7]. It puts the emphasis on the algebra of observables of the model, defined as an abstract von Neumann algebra, both in the classical and in the quantum model. The quantum algebra is a deformation of the classical one (providing the link with non-commutative geometry[8]) and it is in this sense only that it is viewed as a "quantization". Following the usual algebraic formulation of (non-)commutative dynamical systems, it considers "states" to be functionals on this algebra (interpreted as expectation values of observables). The dynamics is given by a discrete group of automorphisms of the algebra. The ergodic, mixing and entropic properties of this model have been studied in much detail[7, 9, 10, 11, 12] in the spirit of the algebraic school mentioned above. In those papers the model is presented as a quantum analog of the classical automorphisms, rather than as a quantization, and its classical limit does not seem to have been studied. I will present in section 2 a (slight variation on) the above quantum algebraic model. Rather than relying on C^* and von Neumann algebraic methods, I will use tools from ordinary quantum mechanics and Weyl quantization to define and study it. The resulting formulation allows for easy proofs of the various ergodic properties of the system (Theorems 3 and 4) and is well adapted to the study of the classical limit, a fact I will illustrate by showing the non-commutativity of the $\hbar \to 0$ and $t \to \infty$ limits (Theorem 5).

In standard quantum mechanics of a particle on the line, wave functions are functions (or tempered distributions) $\psi(x)$ of one variable $x \in \mathbb{R}$. Observables are self-adjoint operators on suitable domains of $L^2(\mathbb{R})$. Weyl (or canonical) quantization establishes a link between classical observables (i.e. functions on phase space) and quantum ones. For $f \in C^\infty(\mathbb{R}^2)$, write

$$f(q,p) = \int\int \tilde{f}(a,b) \exp -\frac{i}{\hbar}[ap - bq] \, \frac{dadb}{2\pi\hbar}, \tag{1.1}$$

and

$$Op^W f = \int\int \tilde{f}(a,b) U(a,b) \frac{dadb}{2\pi\hbar}, \tag{1.2}$$

where

$$U(a,b) = \exp -\frac{i}{\hbar}[aP - bQ] \tag{1.3}$$

are the phase space translation operators in the usual notations (See Ref.13 for further technical details).

If now a Hamiltonian $H(q,p)$ is given, yielding a classical dynamics, then the corresponding Schrödinger equation is

$$i\hbar\partial_t \psi_t = Op^W H \psi_t \tag{1.4}$$

which is solved by $\psi_t = U(t)\psi_0 = \exp -\frac{i}{\hbar} t Op^W H \ \psi_0$. Recalling that if H is a quadratic function ($H = p^2/2$ or $H = p^2/2 \pm x^2/2$, for example) then the corresponding classical flow is linear, one sees that to any $A \in SL(2,\mathbb{R})$ is in this manner associated a unitary propagator

$$(U(A)\psi)(x) = \Big(\frac{i}{2\pi\hbar a_{12}}\Big)^{\frac{1}{2}} \int e^{\frac{i}{\hbar} S_A(x,y)} \psi(y) dy \tag{1.5}$$

where S_A is the classical action associated to A.

With these elements of basic quantum mechanics recalled, it will be easy to understand both the algebraic and the canonical quantum "Arnold's cats". The next section is devoted to the definition and study of the algebraic model, the last one being reserved to the canonical one.

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2 THE ALGEBRAIC QUANTUM "ARNOLD'S CAT"

In the algebraic approach to dynamical systems the starting point are the observables of the theory and their evolution, as in the Heisenberg picture. The states of the system are defined as expectation values. In the present case, this approach can be understood as follows. First note that the classical observables for any system having the torus as phase space are real-valued functions $f(q,p)$ on the torus, which can usefully be thought of as periodic functions on the plane $\mathbb{R}^2$. The complex-valued C^k ($0 \le k \le \infty$) functions form a *-algebra under complex conjugation and multiplication. If $k = \infty$, it is also a Lie-algebra under the Poisson bracket. As in statistical mechanics, states of the system are described by probability measures that allow one to associate to each observable f its mean value (or expectation value), given by $\int_{\mathbb{T}^2} f \ d\mu$. For absolutely continuous measures with respect to Lebesgue measure, this becomes $\int_{\mathbb{T}^2} f(x)\rho(x) \ dx$ for some L^1-density ρ. The translations $x = (q,p) \in \mathbb{T}^2 \to (q+a, p+b) \in \mathbb{T}^2$ act by automorphisms on the function algebras via $f(x) \to \tau_{(a,b)} f(x) = f(q+a, p+b)$. Similarly, for any $H \in C^\infty(\mathbb{T}^2)$, the Hamiltonian flow Φ_t^H generates an automorphism $f \to f \circ \Phi_t^H$. Finally, the toral automorphisms A also define automorphisms $f \to \tau_A f = f \circ A$. Note that, by concentrating on the automorphisms of the function algebras, the emphasis is shifted away from the motion of individual points of the torus (i.e. away from individual trajectories). Lebesgue measure defines the only state invariant under all the above automorphisms. Nevertheless, for fixed Φ_t^H or A, there are many other invariant states, in particular the ones concentrated on one periodic orbit. They play an important role in the theory of classical dynamical systems and I will come back to them below.

The only algebras I will be needing for what follows are the above algebras of smooth functions, but in order to make the link with the algebraic literature, I will at each stage of the discussion briefly sketch how C^* and von Neumann algebras show up in this model. These remarks can be safely ignored by the uninterested as well as by the uninitiated reader.

The uniform closure of any of the above function algebras is the C^*-algebra $C^0(\mathbb{T}^2)$. The states on $C^0(\mathbb{T}^2)$ are in one to one correspondence with the probability measures μ on $\mathbb{T}^2$, associating to each observable f its mean value $\int_{\mathbb{T}^2} f\ d\mu$. For some purposes, the C^*-algebra of continuous functions is still too small. If one is convinced of the importance of Lebesgue measure by the above remarks, one might want to construct the GNS representation of $C^0(\mathbb{T}^2)$ associated to dx. This realizes $C^0(\mathbb{T}^2)$ as a norm closed algebra of multiplication operators on $L^2(\mathbb{T}^2, dx)$. Its weak closure is the von Neumann algebra $L^\infty(\mathbb{T}^2, dx)$. Normal states on $L^\infty(\mathbb{T}^2, dx)$ correspond to measures that are absolutely continuous with respect to Lebesgue measure. Note that measures supported on periodic orbits are not of this type. Actually, they do not define states on $L^\infty(\mathbb{T}^2, dx)$ at all. So where they are deemed important, it is likely that the the use of this von Neumann algebra should be avoided. At any rate, the von Neumann algebraic dynamical system associated to the classical dynamical system $(\mathbb{T}^2, A)$ is the triple $(L^\infty(\mathbb{T}^2, dx), dx, \tau_A)$. This is the algebraic version of the classical "Arnold's cat"[7, 1].

Let me briefly recall the basic dynamic properties of the classical hyperbolic toral automorphisms that will interest me here.

Theorem 1 *Let A be a hyperbolic toral automorphism and let μ be an absolutely continuous probability measure on $\mathbb{T}^2$ (with respect to dx). Then, for all $f \in L^\infty(\mathbb{T}^2, dx)$*

$$\lim_{k\to\infty} \int_{\mathbb{T}^2} f \circ A^k\ d\mu = \int_{\mathbb{T}^2} f\ dx. \tag{2.1}$$

This is a classical result, the proof of which I omit. It can at any rate also be obtained as a corollary of Theorem 2 below using a density argument. Theorem 1 is equivalent to saying that A is mixing. It can be stated in "physical terms" as follows. Thinking of dx as the "equilibrium state" of the system, any deviation from equilibrium, of the type $d\mu = \rho(x)\ dx$, with $\rho \in L^1(\mathbb{T}^2, dx)$, is washed away in the course of time. It shows, incidentally, that the only A-invariant measure, absolutely continuous with respect to dx, is dx itself. Note that (2.1) fails to hold in general for measures that are not absolutely continuous with respect to Lebesgue measure. In particular, it does of course not hold for the invariant measures supported by the periodic orbits of A. The return to equilibrium of Theorem 1 is actually exponential, in the following sense.

Theorem 2 *Let A be a hyperbolic toral automorphism* (Tr $A > 2$) *and $0 < \lambda_- < 1$ its smallest eigenvalue. Then there exists a constant C_A so that, for all $f, g \in C^1(\mathbb{T}^2)$ and for all $k \in \mathbb{N}$,*

$$\mid \int_{\mathbb{T}^2} f(A^k x) g(x)\ dx - \int_{\mathbb{T}^2} f(x)\ dx \int_{\mathbb{T}^2} g(x)\ dx \mid \leq C_A \lambda_-^k \parallel g \parallel_{2,1} \parallel f \parallel_{2,1}. \tag{2.2}$$

This is considered to be an important signature of chaos, since it can be interpreted as an exponential return to equilibrium at least for sufficiently smooth observables f and for measures μ with a C^1-density.

The above result is a well-known property of Anosov systems in general, but rather than relying on general results[14, 15] I prefer to give a simple proof of this particular case which has the added benefit of passing immediately to the quantum version, discussed below (Theorem 4).

Proof: The essence of the proof is contained in its last few lines, which you might wish to read first. I start however with some preliminaries. Let $Au_\pm = \lambda_\pm u_\pm, \lambda_+\lambda_- = 1, \lambda_+ > 1$. Write $u_\pm = (\cos\theta_\pm, \sin\theta_\pm)$ and $s_\pm = \tan\theta_\pm$. Note for later purposes that the slopes $s_\pm$ are quadratic irrationals.

It will obviously enough to show (2.2) for f, g of mean zero, so from now on I work in the subspace of $L^2(\mathbb{T}^2)$ orthogonal to the constant functions. Define there, on the obvious domain, the self-adjoint operators $D_\pm = \frac{1}{2\pi i}(u_\pm \cdot \nabla)$ with eigenfunctions $\chi_n = \exp i2\pi(n_2q - n_1p)$: $D_\pm\chi_n = (u_\pm \cdot n)\chi_n$. Also

$$D_\pm(f \circ A^k) = \lambda_\pm^k (D_\pm f) \circ A^k, \tag{2.3}$$

and

$$D_\pm^{-1} g = \sum_{n\in\mathbb{Z}^{2*}} g_n (u_\pm \cdot n)^{-1}\chi_n. \tag{2.4}$$

Note that here "$\cdot$" denotes the symplectic form: $x \cdot y = x_1y_2 - x_2y_1$, for $x, y \in \mathbb{R}^2$. The only technical ingredient of the proof is the following estimate:

$$\| D_\pm^{-1} g \|_2 \leq C_A \| g \|_{2,1} . \tag{2.5}$$

It is enough to write the proof of (2.5) for D_-. Note that this operator has dense pure point spectrum covering the entire axis, but that 0 is not an eigenvalue since s_- is irrational. As a result, D_-^{-1} is well-defined, but unbounded. The estimate (2.5) shows that any $g \in C^1$ of mean zero is in its domain. This follows from a simple calculation, using the observation that, since s_- is a quadratic irrational, basic properties of its continued fraction expansion imply that

$$\exists C > 0, \text{ so that } \quad \forall m, n > 0, \ \frac{1}{2Cm} <| n - ms_- | .$$

The proof of (2.2) is now immediate:

$$\begin{aligned} \int_{\mathbb{T}^2} (f \circ A^k)(x) g(x)\, dx &= < f \circ A^k, g > \\ &= < D_-(f \circ A^k), D_-^{-1} g > \\ &\leq \lambda_-^k \| D_-^{-1} g \|_2 \| D_- f \|_2, \end{aligned}$$

from which (2.2) follows. □

Recall finally that the translation $x \to x + \alpha$ is ergodic (but not mixing) iff the slope of α is irrational and $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \mathbb{Q}$. Also, the skew translation $x = (q, p) \to (q + \alpha, p + kq)$ is ergodic for $\alpha \notin \mathbb{Q}, k \in \mathbb{Z}^*$.

We are now ready to turn to the quantized version of the above model. I will follow ordinary quantum mechanics to obtain the quantum observables by Weyl quantization of the classical observables in the standard manner. For $f \in C^\infty(\mathbb{T}^2)$, with

$$f(q, p) = \sum_n f_n \exp 2\pi i(n_2q - n_1p), \tag{2.6}$$

(1.1)–(1.3) yield

$$Op^W f = \sum_{n\in\mathbb{Z}^2} f_n U(2\pi\hbar n_1, 2\pi\hbar n_2). \tag{2.7}$$

These operators form a *-algebra[16], that I will denote by $C^\infty_\hbar(\mathbb{T}^2)$ and which is the quantization of the classical *-algebra of periodic C^∞ functions. Note that (2.6)–(2.7) continues to make sense provided

$$\sum_{n\in\mathbb{Z}^2} |f_n| < \infty, \tag{2.8}$$

so that the sum in (2.6) is uniformly convergent (guaranteeing that $f \in C^0(\mathbb{T}^2)$) and that the one in (2.7) is norm-convergent. In the following, whenever I write $Op^W f$, (2.8) will be assumed to hold.

Whereas $C^\infty_\hbar(\mathbb{T}^2)$ provides an obvious quantum analog of $C^\infty(\mathbb{T}^2)$, I now make contact with the algebraic approach, by claiming a good quantum analog of the classical von Neumann algebra $L^\infty(\mathbb{T}^2, dx)$ is obtained by taking the weak closure of the above algebra $C^\infty_\hbar(\mathbb{T}^2)$, which I shall denote by $L^\infty_\hbar(\mathbb{T}^2)$. Similarly, one could define $C^0_\hbar(\mathbb{T}^2)$ as the C^*-algebra obtained by taking the uniform closure of $C^\infty_\hbar(\mathbb{T}^2)$. The elements of $L^\infty_\hbar(\mathbb{T}^2)$ are operators of the form (2.7), but with coefficients possibly not satisfying (2.8). Note furthermore that the algebra $L^\infty_\hbar(\mathbb{T}^2)$ is the commutant[12] of the algebra generated by $U(1,0)$ and $U(0,1)$, in accordance with the interpretation of these operators as phase space translation operators. It is indeed easy to check that

$$[U(a,b), U(k,\ell)] = 0, \quad \forall k,\ell \in \mathbb{Z} \iff \exists n \in \mathbb{Z}^2,\ a = 2\pi\hbar n_1, \quad b = 2\pi\hbar n_2,$$

from which one concludes that the ($\hbar$ dependent) algebra $L^\infty_\hbar(\mathbb{T}^2)$ of quantum observables is generated by the $U(2\pi\hbar n, 2\pi\hbar m)$.

In other words, and to summarize, the classical observables of a system on the torus are the subalgebra of all observables $f(q,p)$ for a system on the full plane having the periodicity property $f(q+k, p+\ell) = f(q,p)$, $\forall k,\ell \in \mathbb{Z}$. In the same way, the quantum observables are those operators F on $L^2(\mathbb{R})$ satisfying $U(k,\ell)FU(-k,-\ell) = F$.

Turning now to the states of the system, each density matrix ρ on $L^2(\mathbb{R})$ ($0 \le \rho \le 1$, $\mathrm{Tr}\ \rho = 1$) defines a (normal) state on $L^\infty_\hbar(\mathbb{T}^2)$ via

$$F \in L^\infty_\hbar(\mathbb{T}^2) \to \mathrm{Tr}\ F\rho \in \mathbb{C}.$$

I will argue below (Theorem 3) that these play the role played by the absolutely continuous measures in the classical model. This ends my discussion of the kinematical setup.

As for the dynamics, standard quantum mechanics, as embodied in equation (1.5), tells us how observables evolve: for $A \in SL(2,\mathbb{Z})$,

$$Op^W f \to \hat\tau_A(Op^W f) = U(A)^{-1}\ Op^W f\ U(A). \tag{2.9}$$

Similarly, translations act via

$$Op^W f \to \hat\tau_{(a,b)}(Op^W f) = U(a,b)^{-1}\ Op^W f\ U(a,b). \tag{2.10}$$

Combining (2.9)-(2.10), one can also quantize skew-translations. Recall[16] that

$$U(A)^{-1}\, Op^W f\, U(A) = Op^W(\tau_A f), \qquad U(a,b)^{-1}\, Op^W f\, U(a,b) = Op^W(\tau_{(a,b)} f), \tag{2.11}$$

which is a basic property of Weyl quantization related to the fact that when the dynamics is affine, "quantization and evolution commute" (as in a harmonic oscillator). As a result $\hat\tau_A$ and $\hat\tau_{(a,b)}$ define automorphisms of $C^\infty_\hbar(\mathbb{T}^2)$ and extend to automorphisms of $L^\infty_\hbar(\mathbb{T}^2)$. Note that, for $H \in C^\infty(\mathbb{T}^2)$, one similarly defines a one-parameter group of automorphisms by

$$Op^W f \to e^{\frac{i}{\hbar}tOp_W H}\ Op^W f\ e^{-\frac{i}{\hbar}tOp_W H}, \tag{2.12}$$

but now

$$e^{\frac{i}{\hbar}tOp_W H}\ Op^W f\ e^{-\frac{i}{\hbar}tOp_W H} \neq Op^W(f \circ \Phi^H_t), \tag{2.13}$$

a fact I shall come back to below.

To further support my claim that the algebras $C^\infty_\hbar(\mathbb{T}^2)$ and $L^\infty_\hbar(\mathbb{T}^2)$ are good quantum analogs of $C^\infty(\mathbb{T}^2)$ and $L^\infty(\mathbb{T}^2, dx)$ and that the above automorphisms are the analogs of their classical counterparts, I now prove the following theorem.

Theorem 3 *(i) Let ρ be a density matrix on $L^2(\mathbb{R})$. Then*

$$\mathrm{Tr}\ Op^W f\ \rho = \mathrm{Tr}\ U(a,b)^{-1} Op^W f\ U(a,b)\ \rho,\ \forall (a,b) \in \mathbb{R}^2,\ \forall f \in C^\infty(\mathbb{T}^2),$$

iff

$$\mathrm{Tr}\ Op^W f\ \rho = f_0 = \int_{\mathbb{T}^2} f(x)\ dx,\ \forall f \in C^\infty(\mathbb{T}^2).$$

(ii) There exists a density matrix ρ_0 so that, $\forall f \in C^\infty(\mathbb{T}^2)$,

$$\mathrm{Tr}\ Op^W f\ \rho_0 = \int_{\mathbb{T}^2} f(x)\ dx.$$

Moreover, ρ_0 can be chosen of the form $\rho_0 = Op^W \sigma_0$, with $\sigma_0 \in \mathcal{S}(\mathbb{R}^2)$.

(iii) In addition, for all $f, g \in C^\infty(\mathbb{T}^2)$,

$$\mathrm{Tr}\ Op^W f\ Op^W g\ \rho_0 = \mathrm{Tr}\ Op^W g\ Op^W f\ \rho_0 = \int_{\mathbb{T}^2} f(x) g(x)\ dx.$$

(iv) For any density matrix ρ, and $\forall f \in C^\infty(\mathbb{T}^2)$,

$$\lim_{K\to\infty} \mathrm{Tr}\ \Big[\frac{1}{K}\sum_{k=0}^{K-1} U(a,b)^{-k} Op^W f\ U(a,b)^k\Big]\rho = \mathrm{Tr}\ Op^W f\ \rho_0,$$

provided $x \to (q+a, p+b)$ is an ergodic translation.

(v) If ρ is a density matrix and A a hyperbolic toral automorphism, then

$$\lim_{k\to\infty} \mathrm{Tr}\ U(A)^{-k} Op^W f\ U(A)^k\ \rho = \mathrm{Tr}\ Op^W f\ \rho_0.$$

Statements (iii), (iv) and (v) remain true with $Op^W f, Op^W g$ replaced by any $F, G \in L^\infty_\hbar(\mathbb{T}^2)$.

Theorem 3 (v) was proven in Ref.7. Actually, the von Neumann algebra proposed in Ref.7 as the quantum analog (deformation) of the classical one differs from the one constructed here, but it is algebraically and topologically isomorphic to it[12]. In fact, it is just the von Neumann algebra obtained from the GNS representation of the faithful normal tracial state ρ_0. The interpretation of the above results is rather clear. The density matrix ρ_0 plays the role played by Lebesgue measure in the classical model. Note that, although ρ_0 is certainly not unique as a density matrix, (i) says that there is a unique translationally invariant (normal) state on $L^\infty_\hbar(\mathbb{T}^2)$. According to (iv), quantum translations are ergodic with respect to ρ_0; the same is true for the quantized ergodic skew-translations, as is easily checked. The quantized hyperbolic automorphisms are mixing, according to (v) (Compare to Theorem 1). In algebraic terms, $(L^\infty_\hbar(\mathbb{T}^2), \rho_0, \hat{\tau}_{(a,b)})$ is a quantum ergodic system and $(L^\infty_\hbar(\mathbb{T}^2), \rho_0, \hat{\tau}_A)$ is a quantum mixing system as defined in Ref.1. So it is clear that the ergodic properties of the classical model carry over unaltered and effortlessly to the algebraic quantum model: the reason for this is (2.11). While finishing these notes, I noticed that an equivalent of Theorem 3 is proven in Ref.17, but using anti-Wick quantization, working in the Bargmann representation.

Proof: Once (ii) is proven, (i), (iii) and (iv) follow immediately from (2.11), upon taking $f = \chi_n, g = \chi_{n'}$. To prove (ii), take, in the notations of (1.1)–(1.3), $\tilde{\sigma}_0 \in C_0^\infty(\mathbb{R}^2)$, $\tilde{\sigma}_0(0,0) = 1$, and such that supp $\tilde{\sigma}_0$ is contained in the ball of radius $\pi\hbar$. Since

$$\text{Tr } Op^W\sigma_0 \; U(a,b)^{-1} = \tilde{\sigma}_0(a,b)$$

it follows that, for all $f \in C^\infty(\mathbb{T}^2)$ (see (2.7))

$$\text{Tr } Op^W\sigma_0 \; Op^W f = f_0.$$

Note that $Op^W\sigma_0$ is a Hilbert-Schmidt operator. It remains to show it can be chosen positive: $Op^W\sigma_0 \geq 0$. Let $\eta_0(y) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp -\omega y^2/2\hbar$ and $W\eta_0$ its Wigner function. Lemma 2.85 in Ref.16 (or a direct computation) shows that, if $\tau \in \mathcal{S}(\mathbb{R}^2)$, $\tau \geq 0$, then $Op^W(\tau * W\eta_0) \geq 0$, where $*$ stands for convolution. Hence we need to solve $\sigma_0 = \tau * W\eta_0$ or $\tilde{\sigma}_0 = \tilde{\tau}\widetilde{W\eta_0}$ for $\tau \geq 0$. Taking $\tilde{\tau} = \bar{\tilde{k}} * \tilde{k}$, $\tilde{k} \in C_0^\infty(\mathbb{R}^2)$, with the support of $\tilde{k}$ sufficiently small around the origin, (ii) follows. To prove (v), note that it is enough to show that

$$\lim_{k\to\infty} < \psi, U(A)^{-k} Op^W \chi_n \; U(A)^k \psi >= 0,$$

for all $\psi \in L^2(\mathbb{R}), n \in \mathbb{Z}^{*2}$. Using (2.11), this follows readily from the Riemann-Lebesgue lemma together with the observation that, since $A \in SL(2,\mathbb{Z})$ is hyperbolic, both components of the vector $A^k n$ tend to infinity with k. □

I now state and prove, without further comment, the quantum equivalent of Theorem 2, a version of which is proven in Ref.9.

Theorem 4 *Let ρ be a trace-class operator on $L^2(\mathbb{R})$, such that $0 \leq \rho \leq 1$, $\text{Tr } \rho = 1$ and suppose ρ is of the form $\rho = Op^W\sigma_\rho$ with $\sigma_\rho, D_+\sigma_\rho \in L^1(\mathbb{R}^2)$. Suppose A is a hyperbolic toral automorphism, as in Theorem 2. Let $f \in C^0(\mathbb{T}^2)$ so that $\sum |n||f_n| < \infty$. Then there exists a constant C, depending on A, f and ρ, so that, for all $k \in \mathbb{N}$,*

$$| \text{ Tr } [U(A)^{-k} Op^W f \; U(A)^k \; \rho] - \int_{\mathbb{T}^2} f(x) \; dx \; |\leq C\lambda_-^k. \tag{2.14}$$

Proof: As before, we can suppose f is of mean zero and compute

$$\begin{aligned} \text{Tr } [Op^W(f \circ A^k)\rho] &= \int_{\mathbb{R}^2} (f \circ A^k)(x)\sigma_\rho(x) \; \frac{dx}{2\pi\hbar} \\ &= \int_{\mathbb{R}^2} D_+^{-1}(f \circ A^k)(x)(D_+\sigma_\rho)(x) \; \frac{dx}{2\pi\hbar} \end{aligned}$$

From (2.3), one infers

$$D_+^{-1}(f \circ A^k) = \lambda_-^k((D_+^{-1} f) \circ A^k)$$

so that

$$| \text{ Tr } [Op^W(f \circ A^k)\rho] \; |\leq \lambda_-^k \; \| \; D_+\sigma_\rho \; \|_1 \| \; D_+^{-1} f \; \|_\infty$$

whence the result follows with an estimate as in the proof of Theorem 2. □

This ends my description of the basic ergodic properties of the quantized translations, skew translations and toral automorphisms in their algebraic version. They mimic the classical properties almost perfectly and are easy to prove thanks essentially to (2.11) which, one way or another, reduces all quantum properties to classical ones. The situation would be less trivial and more interesting should we study the quantization of skew translations of the form $(q, p) \to (q+\alpha, p+h(q))$, which are easily quantized, or, better yet, Hamiltonian perturbed toral automorphisms, which are known to still be Anosov, and hence exponentially mixing. They can be constructed as follows[13]. Let $H \in C^\infty(\mathbb{T}^2)$, write Φ_t^H for the corresponding Hamiltonian flow, and consider

$$B_\epsilon = A \circ \Phi_\epsilon^H.$$

The structural stability of Anosov systems guarantees that this is still an Anosov symplectomorphism provided ϵ is taken small enough. If I now define the unitary operator

$$U(B_\epsilon) = U(A) \exp -i\frac{\epsilon}{\hbar} Op^W H \tag{2.15}$$

on $L^2(\mathbb{R})$, then, as in (2.9), it defines an automorphism of the algebras $C_\hbar^\infty(\mathbb{T}^2)$ and $L_\hbar^\infty(\mathbb{T}^2)$:

$$Op^W f \to \hat{\tau}_{B_\epsilon}(Op^W f) = U(B_\epsilon)^{-1} Op^W f\ U(B_\epsilon).$$

Note that in this case the equivalent of (2.11) does no longer hold in view of (2.13).

At any rate, it would perhaps be rather more fun to find ways in which the quantized systems differ radically from the classical ones. The obvious place to look for such phenomena is in manifestations of the uncertainty principle. As is shown in Theorem 5 below, the latter, together with the mixing property of the quantized hyperbolic automorphisms is indeed at the origin of the non-commutatitivity of the $\hbar \to 0$ and the $k \to \infty$ limits, which shows that at sufficiently long times, quantum mechanics "divorces" irremediably from classical mechanics.

Virtually all important notions in the study of classical dynamical systems involve the limit as time goes to infinity. On the other hand, the way in which quantum dynamical systems approach their classical limit is governed by the limit $\hbar \to 0$. Consequently, the commutativity or absence thereof of these two limiting procedures has been at the center of many problems in the study of "quantum chaos". One expects the two limits not to commute and this can be shown very easily in the present model, using Theorem 3. A more detailed analysis (in particular of the time scales involved) will be given elsewhere[18]. Recall first the definition of the standard Weyl-Heisenberg coherent states. For $z \in \mathbb{C}$, and $Imz > 0$, define the gaussian

$$\eta_{0,z}(y) = (\frac{Imz}{\pi\hbar})^{\frac{1}{4}} e^{\frac{i}{2\hbar}zy^2}, \quad y \in \mathbb{R}.$$

The coherent states are then defined, for all $x = (q, p) \in \mathbb{R}^2$, by

$$\eta_{x,z}(y) = (U(q, p)\eta_{0,z})(y)$$

in the standard manner. They are, as usual, interpreted as states, "optimally localized" at $x \in \mathbb{R}^2$. It will be useful on occasion to use the bra-ket notation of Dirac:

$$\eta_{x,z}(y) =< y \mid x, z > .$$

Note that two different coherent states $\mid x, z >$ and $\mid x', z >$ define different states (i.e. different functionals) on $L_\hbar^\infty(\mathbb{T}^2)$ iff $x' \neq x + n,\ \forall n \in \mathbb{Z}^2$! It is furthermore well known that

$$U(A) \mid x, z >=\mid Ax, A \cdot z >$$

where $A \cdot z$ denotes the homographic action of A on the upper half plane. Let us write

$$\rho(x_0) = \mid x_0, z >< x_0, z \mid$$

suppressing the z dependence in the notation. We then have

Theorem 5 *Let A be a hyperbolic toral automorphism, then, for all $x_0 \in \mathbb{T}^2$,*

$$\lim_{\hbar \to 0} \lim_{k \to \infty} \mathrm{Tr}\ \rho(x_0) Op^W(f \circ A^k) = \int_{\mathbb{T}^2} f(x)\ dx.$$

For almost all $x_0 \in \mathbb{T}^2$

$$\lim_{K \to \infty} \lim_{\hbar \to 0} \mathrm{Tr}\ \rho(x_0) \Big[\frac{1}{K} \sum_{k=0}^{K-1} U(A)^{-k} Op^W f\ U(A)^k\Big] = \int_{\mathbb{T}^2} f(x)\ dx.$$

However, if $x_0 \in \mathbb{Q} \times \mathbb{Q} \cap \mathbb{T}^2$, then

$$\lim_{K \to \infty} \lim_{\hbar \to 0} \mathrm{Tr}\ \rho(x_0) \Big[\frac{1}{K} \sum_{k=0}^{K-1} U(A)^{-k} Op^W f\ U(A)^k\Big] = \frac{1}{T} \sum_{k=0}^{T-1} f(A^k x_0),$$

where T is the period of the orbit of A through x_0.

Proof: The proof is immediate , since we know from standard semi-classical analysis that

$$\lim_{\hbar \to 0} \mathrm{Tr}\ \rho(x_0) Op^W(f \circ A^k) = f(A^k x_0)$$

so that the first and second statements follow from Theorem 3(v) and the ergodicity of A. The last statement follows since now x_0 is a periodic point of A. □

The interpretation of this result is clear. Since the coherent state is slightly spread out around its center x_0 (due to the uncertainty principle), the quantum evolution feels the neighbouring trajectories, most of which are ergodic. As a result, even if x_0 is a periodic point, after a sufficiently long time, for $\hbar$ fixed, the behaviour of the expected value is determined by the neighbouring trajectories, and hence we find the contribution $\int_{\mathbb{T}^2} f(x)\ dx$, coming from Theorem 3. If, on the other hand, we take $\hbar \to 0$ first, then only the trajectory through x_0 will play a role afterwards. The effect is perhaps the most striking for the case $x_0 = 0$. The wave packet then does not move, but the changing value of z reflects it being stretched in the unstable, and squeezed in the stable direction. This, together with the effective wrapping around the torus of the unstable manifold, due to the periodicity of the observable, produces the observed effect.

3 THE CANONICAL "QUANTUM ARNOLD'S CAT"

Let me now describe the canonical approach to quantization on the torus, initiated in Ref.6. I will follow the treatment of Ref.13, referring there for more details, and for further references to the literature on the subject. The mathematical machinery is precisely the same as in the previous section, but rather than looking first at observables with the required symmetry, one first constructs the quantum Hilbert space of states, by searching for states ψ having the symmetry of the torus, i.e. states periodic of period 1 both in the position and the momentum variable:

$$U(1,0)\psi = \exp -i\kappa_1 \psi \tag{3.1}$$

$$U(0,1)\psi = \exp i\kappa_2 \psi, \tag{3.2}$$

where one allows for the usual phase change. The space of solutions $\mathcal{H}_\hbar(\kappa)$ to (3.1)-(3.2) is a space of tempered distributions of which one proves readily that either it is zero-dimensional, or there exists a positive integer N so that

$$2\pi\hbar N = 1, \tag{3.3}$$

in which case it is N-dimensional. Equation (3.3), which is interpreted by saying that there is "one state per volume $2\pi\hbar$ in phase space", will henceforth always be assumed to hold. Note that no such Bohr-Sommerfeld condition shows up in the algebraic approach, although the observable algebras have special structure, depending on the arithmetic character of $\hbar$[7, 12] (see below). The solutions to (3.1)–(3.2) are easily computed. For $\kappa = 0$, for example, a basis of $\mathcal{H}_\hbar(\kappa)$ is

$$e_j(y) = \sqrt{\frac{1}{N}} \sum_n \delta(y - j\frac{1}{N} - n), \ y \in \mathbb{R}, \ 0 \le j < N. \tag{3.4}$$

Clearly, these vectors are to be interpreted as "position eigenvectors", localized at $\frac{j}{N}$. Condition (3.3) is equivalent to $\left[U(0,1), U(1,0)\right] = 0$, so that (3.1)–(3.2) is nothing but the problem of simultaneously diagonalizing $U(0,1)$ and $U(1,0)$. Since their spectra are continuous, this leads to a direct integral decomposition of $L^2(\mathbb{R})$:

$$L^2(\mathbb{R}) \cong \int_0^{2\pi}\int_0^{2\pi} \frac{d\kappa}{(2\pi)^2}\ \mathcal{H}_\hbar(\kappa).$$

This equips each "term" $\mathcal{H}_\hbar(\kappa)$ with a natural inner product. For each admissible $\hbar$ (i.e. $2\pi\hbar N = 1$), the spaces $\mathcal{H}_\hbar(\kappa)$, indexed by κ, are the quantum Hilbert spaces of states for systems having the torus as phase space. Turning now to the observables, since for each $f \in C^\infty(\mathbb{T}^2)$, $\left[Op^W f, U(k,\ell)\right] = 0$, it is clear that, for each κ, $Op^W f\ \mathcal{H}_\hbar(\kappa) \subset \mathcal{H}_\hbar(\kappa)$ and hence

$$Op^W f \cong \int_0^{2\pi}\int_0^{2\pi} \frac{d\kappa}{(2\pi)^2}\ Op^W_\kappa f,$$

where $Op^W_\kappa f$ denotes the restriction of $Op^W f$ to $\mathcal{H}_\hbar(\kappa)$ (This corresponds to the central decomposition of $L^\infty_\hbar(\mathbb{T}^2)$[12]). We conclude from (2.6)–(2.7) and (3.3) that

$$Op^W_\kappa f = \sum_{n\in\mathbb{Z}^2} f_n\ U_\kappa(\frac{n_1}{N}, \frac{n_1}{N}),$$

where I wrote $U_\kappa(\frac{n_1}{N}, \frac{n_1}{N})$ for the restriction of $U(\frac{n_1}{N}, \frac{n_1}{N})$ to $\mathcal{H}_\hbar(\kappa)$. This then gives the quantization of classical observables, associating to each one an operator acting on the quantum Hilbert space.

To summarize, the canonical quantization of the torus associates to it a finite dimensional quantum Hilbert space of states $\mathcal{H}_\hbar(\kappa)$ on which a discrete subgroup of phase space translation operators $U_\kappa(\frac{n}{N}, \frac{m}{N})$ acts irreducibly. For the quantization of observables an obvious adaptation of the usual Weyl quantization is used.

Note that now $U(a,b)\mathcal{H}_\hbar(\kappa) \subset \mathcal{H}_\hbar(\kappa)$ iff $(a,b) = (\frac{n_1}{N}, \frac{n_1}{N})$: the full translation group does no longer act on the quantum Hilbert space, in accordance with the intuition gained from (3.4). To quantize arbitrary translations (and skew translations) in spite of this, one has to work a little harder[19, 13].

To quantize the toral automorphisms on the other hand, the procedure is now straightforward. It turns out that for all N, there exist κ so that $U(A)\mathcal{H}_\hbar(\kappa) \subset \mathcal{H}_\hbar(\kappa)$; κ might depend on N, but I shall not indicate this dependence explicitly. For matrices of the form

$$A = \begin{pmatrix} 2g & 1 \\ 4g^2 - 1 & 2g \end{pmatrix}, \ g \in \mathbb{N}^*, \tag{3.5}$$

one can take $\kappa = 0$. I shall write $U_\kappa(A)$ for the resulting unitary operator on $\mathcal{H}_\hbar(\kappa)$. One still has

$$U(A)^{-1} Op^W_\kappa f \; U(A) = Op^W_\kappa (f \circ A),$$

as in (2.11). This completes the description of the quantized toral automorphisms in the canonical approach. The quantization of their Hamiltonian perturbations (see (2.15)) is now also straightforward. It is indeed obvious that, for the right choice of κ, $U(B_\epsilon)\mathcal{H}_\hbar(\kappa) \subset \mathcal{H}_\hbar(\kappa)$ so that we can define the restriction $U_\kappa(B_\epsilon)$ of $U(B_\epsilon)$ to $\mathcal{H}_\hbar(\kappa)$ to be the canonical quantization of B_ϵ.

The quantization of the toral automorphisms can also be obtained through geometric quantization[19], using real polarizations. Alternatively, one can use complex polarizations, or Toeplitz quantization. This latter approach has been worked out in Ref.20, but only for the case $\kappa = 0$, restricting the matrices A that can be quantized. In each case, the resulting quantum dynamical model is the same as the one obtained here (up to unitary equivalence of all relevant structures). The link with complex polarizations in geometric quantization is established easily using coherent states, as follows. The direct integral (3.4) leads to

$$\mid x, z > \cong \int_0^{2\pi} \int_0^{2\pi} \frac{d\kappa}{(2\pi)^2} \mid x, z, \kappa >$$

and one shows[13]

$$\int_0^1 \int_0^1 \frac{dx}{2\pi\hbar} \mid x, z, \kappa >< x, z, \kappa \mid = \mathrm{Id}_{\mathcal{H}_\hbar(\kappa)}$$

so that the map

$$\psi \in \mathcal{H}_\hbar(\kappa) \to < x, z, \kappa \mid \psi > \in L^2(\mathbb{T}^2, dx)$$

is an isometry. The image of $\mathcal{H}_\hbar(\kappa)$ under this map is a reproducing kernel subspace of $L^2(\mathbb{T}^2)$. It can be seen as a space of complex polarized sections of a prequantum bundle over the torus, determined by N and κ.

For fixed $\hbar$ (i.e. for fixed N), $\mathcal{H}_\hbar(\kappa)$ is a finite dimensional Hilbert space, on which the dynamics is generated by an N by N matrix, which can not display chaos in any sense of the word. This is a simple manifestation of the so-called "quantal suppression of classical chaos". Indeed, if ρ is a density matrix on $\mathcal{H}_\hbar(\kappa)$, any dynamical quantity of the type Tr $U_\kappa(B_\epsilon)^{-k} Op^W_\kappa f \; U_\kappa(B_\epsilon)^k \rho$, or of the type Tr $U_\kappa(B_\epsilon)^{-k} Op^W_\kappa f \; U_\kappa(B_\epsilon)^k Op^W_\kappa g$, is necessarily a quasi-periodic function and in particular no ergodic or mixing property as in Theorem 3(v) can ever hold for such a system. Actually, the limit as k goes to infinity simply does not exist for those quantities. As a result, if your main concern is to shed light on the first question raised in the introduction, the canonically quantized toral automorphisms are unlikely to be of any help to you, whereas their algebraic cousins might be. They do however form an interesting testing ground in connection with the second question, as I now explain.

In particular, writing

$$U_\kappa(B_\epsilon)\psi_j^{(N)} = \exp i\theta_j(N)\psi_j^{(N)},$$

for the eigenvalues and eigenvectors of $U_\kappa(B_\epsilon)$, one is interested in finding out how their behaviour in the semi-classical limit $N \to \infty$ is affected by the chaotic nature of the underlying dynamics.

Before addressing this question, let me point out one might expect this to be a relatively simple matter in the case $\epsilon = 0$ since the underlying classical system is linear and for such systems we expect to be able to find the answer to any question via explicit computations, as in the proofs of section 2. This expectation is not borne out for the case at hand. As an example, the eigenstates of $U_\kappa(A)$ have only been computed for A of the form (3.5), and even then it is quite an arduous task[21]. In addition, it turns out to be hard to study the asymptotic behaviour of these explicit expressions when they can be obtained. Nevertheless, some results such as trace formulas relating the trace of the quantum propagator $U_\kappa(A)$ to a sum over periodic orbits of the classical map can be obtained in closed form for these particular systems, because of their strong number theoretic properties, in particular when N is a prime number[22]. In the end, these systems are surprisingly complicated. Nevertheless, they are often criticized for being too special, and hence incapable of capturing the essence of the behaviour of quantum systems having a classical limit that is chaotic. One therefore wishes to study the properties of the quantized versions of larger classes of classically chaotic systems, for which closed form expressions can no longer be expected to exist. The easiest candidates in this perspective are perhaps the Hamiltonian perturbations described above. As already remarked, the failure of the equivalent of (2.11) in this case makes them more truely quantum mechanical.

I now turn to some of the semi-classical properties of $U_\kappa(B_\epsilon)$ and their link with the underlying dynamics.

Theorem 6 *Let $U(B_\epsilon)$ be as above and $f \in C^\infty(\mathbb{T}^2)$. Then*

$$V_N(f) = \frac{1}{N}\sum_{j=1}^{N} |< \psi_j^{(N)}, Op^W[f - \int_{\mathbb{T}^2} f(x)\ dx]\psi_j^{(N)} >|^2 \stackrel{N\to\infty}{\to} 0. \qquad (3.6)$$

For all N, there exists $E_N \subset \{1,\ldots,N\}$ with $\sharp E_N/N \to 1$ so that $\forall j_N \in E_N$, $\forall f \in C^\infty(\mathbb{T}^2)$

$$< \psi_{j_N}^{(N)},\ Op_\kappa^W f\ \psi_{j_N}^{(N)} > \stackrel{N\to\infty}{\to} \int_{\mathbb{T}^2} f(x)\ dx. \qquad (3.7)$$

Also,

$$\frac{|< x, z, \kappa \mid \psi_{j_N}^{(N)} >|^2}{2\pi\hbar} \to 1 \qquad (3.8)$$

in the sense of distributions.

Proof: Write $\bar f = \int_{\mathbb{T}^2} f(x)\ dx$ and $f_K = \frac{1}{K}\sum_{k=1}^K f\circ A^k$. Then, using the obvious inequality $|< \psi, A\psi >|^2 \le < \psi, A^*A\psi >$, together with the basic semi-classical estimates[13]

$$\begin{aligned} U(B_\epsilon)^{-k} Op^W f\ U(B_\epsilon)^k &= Op^W(f\circ B_\epsilon^k) + \mathcal{O}_k(\hbar), \qquad (3.9)\\ Op^W f\ Op^W g &= Op^W fg + \mathcal{O}(\hbar),\\ \frac{1}{N}\mathrm{Tr}\ Op^W g &= \int_{\mathbb{T}^2} g(x)\ dx + \mathcal{O}(\hbar). \end{aligned}$$

one easily sees that

$$\begin{aligned} V_N(f) &= \frac{1}{N}\sum_{j=1}^N |< \psi_j^{(N)}, Op^W[f-\bar f]_K \psi_j^{(N)} >|^2 + \mathcal{O}_K(\hbar)\\ &\le \frac{1}{N}\sum_{j=1}^N < \psi_j^{(N)}, Op^W\Big[\mid (f-\bar f)_K \mid^2\Big]\psi_j^{(N)} > + \mathcal{O}_K(\hbar)\\ &\le \int_{\mathbb{T}^2} \mid f_K - \bar f \mid^2\ dx + \mathcal{O}_K(\hbar). \end{aligned}$$

Hence, taking $N \to \infty$

$$\lim_{N\to\infty} V_N(f) \le \int_{\mathbb{T}^2} \mid f_K - \bar f\mid^2\ dx$$

whence the result follows upon invoking the ergodicity of the classical map. The rest of the theorem then follows from standard arguments that I will not repeat here. □

This result was proven in Ref.13, using a different argument. The idea of the above proof, and in particular the use of $V_N(f)$, goes back to Ref.5, where it is presented in a more abstract, C^*-algebraic framework.

Roughly speaking, the theorem says that, in the classical limit, most of the invariant states of the quantum system (i.e. its eigenstates) converge to Lebesgue measure. It is an open question if some eigenstates could converge to other invariant measures of the classical dynamics, in particular those concentrated on classical trajectories. The results of Theorem 6 also hold for quantized ergodic translations and skew-translations[13] and in fact for all quantized ergodic symplectic transformations for which an estimate of the type (3.9) holds. The theorem can be generalized in various ways. First of all, one might wonder what happens if one restricted the sum in (3.6) to those eigenvalues lying in a fixed interval I on the circle. In that case one shows[20, 23] that

$$\frac{\sharp\{j \mid \theta_j(N) \in I\}}{N} \stackrel{N\to\infty}{\to} \mid I \mid,$$

where $\mid I \mid$ denotes the Lebesgue measure of I. This is a kind of Weyl law, saying that the eigenvalues equidistribute over the circle. Furthermore[23],

$$V_N^I(f) = \frac{1}{\mid I \mid N} \sum_{j, \theta_j(N)\in I} \mid < \psi_j^{(N)}, Op^W[f - \int_{\mathbb{T}^2} f(x)\ dx]\psi_j^{(N)} > \mid^2 \to 0.$$

Since even perturbations of toral automorphisms are still expected to be atypical among chaotic systems, other models have been quantized and studied, most particularly the Baker map and the sawtooth maps[24, 25, 26]. These are discontinuous, uniformly hyperbolic systems, known to be exponentially mixing[15], and hence ergodic. In Ref.26 an equipartition result of the type of Theorem 6 is proven for those maps. Tracing through the proof of Theorem 6, it is clear that this amounts to showing an estimate of the type (3.9) for those maps. Due to their discontinuities, this is a non-trivial matter.

The attentive reader will have noticed that Theorem 6 only uses the ergodicity of the underlying classical system. Mixing has an effect on the off-diagonal matrix elements $< \psi_i^N, Op_\kappa^W f \psi_j^N >,\ i \neq j$ in the classical limit[27, 28, 5, 23]. Roughly speaking

$$< \psi_{i_N}^N, Op_\kappa^W f \psi_{j_N}^N > \to 0,\ i_N \neq j_N.$$

A result of this type is used in the proof of the following theorem, which is an analog of Theorem 5.

Theorem 7 *Let A be as in (3.5). Then there exists a sequence of integers $N_\ell \to \infty$ so that, $\forall \psi_{N_\ell} \in \mathcal{H}_\hbar(\kappa), \parallel \psi_{N_\ell} \parallel = 1$*

$$\lim_{N_\ell\to\infty} \lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} < \psi_{N_\ell}, U(A)^{-k} Op_\kappa^W f\ U(A)^k \psi_{N_\ell} > = \int_{\mathbb{T}^2} f(x)\ dx. \tag{3.10}$$

On the other hand, for almost all $x_0 \in \mathbb{T}^2$

$$\lim_{K\to\infty} \lim_{N\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} < x_0, z, \kappa \mid U(A)^{-k} Op_\kappa^W f\ U(A)^k \mid x_0, z, \kappa > = \int_{\mathbb{T}^2} f(x)\ dx, \tag{3.11}$$

whereas for all $x_0 \in \mathbb{T}^2 \cap \mathbb{Q} \times \mathbb{Q}$

$$\lim_{K\to\infty} \lim_{N\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} < x_0, z, \kappa \mid U(A)^{-k} Op_\kappa^W f\ U(A)^k \mid x_0, z, \kappa > = \frac{1}{T} \sum_{k=0}^{T-1} f(A^k x_0), \tag{3.12}$$

where T is the period of the orbit of A through x_0.

Remark: This result, in particular (3.12) corrects an erroneous statement in the Main Corollary of Ref.21, where the commutativity of both limits is claimed. In addition, equation (3.11) seems to contradict a claim made in Ref.29 about the supposed failure of the correspondence principle in the quantized Arnold cat: it is suggested there that only the behaviour of (3.12) could occur, (3.11) being excluded.
Proof: Decomposing $\psi_N \in \mathcal{H}_\hbar(\kappa)$ on the basis of eigenvectors of $U(A)$, we have

$$\psi_N = \sum_{r=1}^{N} c_r^{(N)} \psi_r^{(N)}.$$

It is easy to see that

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} < \psi_N, U(A)^{-k} Op_\kappa^W f \; U(A)^k \psi_N >= \sum_{\substack{r,s=1 \\ \theta_r(N)=\theta_s(N)}}^{N} \bar{c}_r^{(N)} c_s^{(N)} < \psi_r^{(N)}, Op_\kappa^W f \; \psi_s^{(N)} > .$$

It is proven in Ref.21 that there exists a sequence of integers $N_\ell \to \infty$ with the following properties: there exists a constant C so that

(i) $\sup_\ell \max\{\deg\, \theta_r(N_\ell) \mid r = 1, 2 \ldots, N\} < C$;

(ii) $|< \psi_r^{(N_\ell)}, Op_\kappa^W f \; \psi_s^{(N_\ell)} > -\delta_{rs} \int_{\mathbb{T}^2} f(x)\, dx \,|< \frac{C}{\sqrt{N_\ell}}$.

Hence

$$\begin{aligned} \Big| \lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} < \psi_N, U(A)^{-k} Op_\kappa^W f \; U(A)^k \psi_N > &- \int_{\mathbb{T}^2} f(x)\, dx \,\Big| \\ &\leq \frac{C}{\sqrt{N_\ell}} [1 + \sum_{\substack{r\neq s \\ \theta_r(N)=\theta_s(N)}} \bar{c}_r^{(N)} c_s^{(N)}]. \end{aligned}$$

The cross terms are controlled by remarking that, in an L-dimensional Euclidean space with orthonormal basis e_ℓ , $\ell = 1, \ldots, L$, one has, for any $\psi = \sum_{\ell=1}^{L} c_\ell e_\ell$

$$\sum_{\substack{\ell,\ell'=1 \\ \ell\neq\ell'}}^{L} |c_\ell||c_{\ell'}| \leq (L-1) \parallel \psi \parallel^2 .$$

Applying this argument to the sum over $r \neq s$ and recalling that (i) above shows each eigenspace is at most C dimensional, independently of N, the result follows. The two other statements of the theorem follow from[23]

$$\lim_{N\to\infty} < x, z, \kappa \mid Op_\kappa^W f \;\; \mid x, z, \kappa >= f(x).$$

□

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