Recent results on quantum map eigenstates

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1 Introduction

One of the central problems in quantum chaos is to obtain a good understanding of the semi-classical behaviour of the eigenfunctions of quantum systems that have a chaotic Hamiltonian system as their classical limit. The pivotal result in this context, and the only general one to date, is the Schnirelman Theorem. Loosely speaking, it states that, if the underlying classical dynamics is ergodic on the appropriate energy surface, then 'most' eigenfunctions of the quantum system equidistribute on this energy surface. The challenge is to go beyond this theorem. I will report here on the progress that has been made in this direction in recent years for the special case of quantum maps on the torus.

Although the present text is meant to be reasonably self-contained, some familiarity with quantum chaos, for example at the level of [11], is assumed. I will also allow myself a rather loose discursive style, and present some speculations and open problems, referring the interested reader to the cited literature for the hard facts. I have limited the references almost exclusively to those concerning the quantum maps under study here. For a recent review of the analogous questions in the context of the geodesic flow and the Laplace-Beltrami operator on compact Riemannian manifolds, one may consult [26].

The very statement of the Schnirelman theorem immediately invites several questions. Given some class of models, one may first of all wonder whether the equidistribution property does indeed only hold for 'most' sequences of eigenfunctions or if, on the contrary, it holds for all. If you think the second alternative is true, you are a believer in what has been baptized "unique quantum ergodicity". It means you think the proof of the Schnirelman can be improved, and the restriction to 'most' eigenfunctions is due to a shortcoming in the proof. Alternatively, you may believe that the theorem can not in general be improved, and that exceptional eigensequences exist, that have a limit that does not equidistribute. In that case, you should try to provide examples. For billiards, for example, there does exist some numerical and theoretical evidence for "scars". This means for eigenfunction sequences that concentrate – at least partially – on periodic orbits of the dynamics, an observation that goes back to [19]. The precise sense in which this concentration takes place and if it is compatible or not with unique quantum ergodicity is not known, however.

I will explain in these pages what the answer is that has emerged in the case of a simple class of quantum maps, the so-called Continuous Automorphisms of the Torus, or CAT maps. I will also present some recent results on their perturbations.

2 Perturbed CAT maps: classical dynamics

Let me recall what CAT maps are. Consider the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and think of it as a toy classical phase space, which means that you equip the C^∞ functions on $\mathbb{T}^2$ with the usual Poisson bracket

$$\{f, g\}(x) = \partial_q f(x)\partial_p g(x) - \partial_p f(x)\partial_q g(x), \quad \text{where } x = (q, p) \in \mathbb{T}^2$$

or equivalently, that you think of it as a symplectic space with symplectic form $dq \wedge dp$. This implies in particular that you can associate to every smooth function g a Hamiltonian flow Φ_t^g, $(t \in \mathbb{R})$ which is defined by $\Phi_t^g(q, p) = (q(t), p(t))$, where $(q(t), p(t))$ is the solution of the Hamiltonian equations of motion $\dot{q}(t) = \partial_p g(x(t))$, $\dot{p}(t) = -\partial_q g(x(t))$ with initial condition $q(0) = q, p(0) = p$. Note that the Φ_t^g are symplectic maps (or canonical transformations), meaning they preserve the Poisson brackets:

$$\{f_1 \circ \Phi_t^g, f_2 \circ \Phi_t^g\} = \{f_1, f_2\} \circ \Phi_t^g, \quad \forall f_1, f_2 \in C^\infty(\mathbb{T}^2).$$

Now I will actually be interested in *discrete time* dynamical systems on $\mathbb{T}^2$. For that purpose, consider a matrix $A \in \mathrm{SL}(2, \mathbb{Z})$. That's a two by two matrix with integer entries and determinant one. Clearly A defines a linear map on $\mathbb{R}^2$ that passes naturally to the quotient $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ because it has integer entries. Because its determinant is one, it is easily checked that it is symplectic in the above sense:

$$\{f_1 \circ A^t, f_2 \circ A^t\} = \{f_1, f_2\} \circ A^t, \quad \forall f_1, f_2 \in C^\infty(\mathbb{T}^2),$$

where this time t is an integer rather than a real number. So A provides us with a dynamical system that is obtained by folding a linear map onto a torus and is therefore necessarily quite simple. When $|\mathrm{Tr}A| > 2$, such maps nevertheless have surprisingly rich properties. Indeed, in that case A has two real eigenvectors $v_\pm$ and two real eigenvalues $\exp \pm\gamma_0$, with the so-called Lyapounov exponent $\gamma_0 > 0$. This implies the dynamical system is exponentially unstable. Roughly speaking, if $x, x' \in \mathbb{T}^2$ are a small distance ϵ apart, then generically $A^t x$ and $A^t x'$ will be a distance $\epsilon \exp \gamma_0 t$ apart: the dynamics displays *sensitive dependence on initial conditions*, a well known source of chaotic behaviour. To put it graphically, if you know that initially the system point is in some set of linear size ϵ, all you can say at a time of order $\frac{1}{\gamma_0}|\ln\epsilon|$ later is that the system is in a set of size 1, which is not saying much more than that it is *somewhere* in the torus, and that is not saying very much at all. In other words, after a time of order $\frac{1}{\gamma_0}|\ln\epsilon|$ you lost all information on the whereabouts of the system. This is a very short time indeed, as you can see by imagining $\epsilon = \epsilon_0 10^{-k}$: it then grows only linearly in k. This instability has the consequence that the dynamics is exponentially mixing. This means that $\forall f_1, f_2 \in C^\infty(\mathbb{T}^2)$,

$$|\int_{\mathbb{T}^2}(f_1\circ A^t)(x)f_2(x)dx - \int_{\mathbb{T}^2} f_1(x)dx\int_{\mathbb{T}^2} f_2(x)dx \,|\leq C_{A,f_1} \parallel \nabla f_2 \parallel_1 \ \mathrm{e}^{-\gamma_0 t}.$$

Replacing f_1 and f_2 by characteristic functions of subsets B_1 and B_2 of the torus, this can be interpreted as follows: at large times, the probability that a trajectory starting in some set B_2 ends up in B_1 converges to the size of B_1, independently of where B_1 and B_2 are located on the torus. This is again a way of saying that all information on the whereabouts of the system is rapidly lost over time. Exponential mixing plays an important role in the following analysis. It implies furthermore that the dynamics is ergodic, meaning that for almost all initial conditions $x_0 \in \mathbb{T}^2$,

$$\frac{1}{T}\sum_{t=1}^{T} f(A^t x_0) \to \int_{\mathbb{T}^2} f(x)dx.$$

Note that this can not possibly be true for all initial conditions since there exists a dense set of periodic points for the dynamics: every point on the torus that has two rational coordinates is periodic. All these periodic orbits are unstable, of course. The simplest one is the fixed point $x = 0$.

These discrete dynamical systems are arguably the simplest fully chaotic Hamiltonian systems one can find: in particular, the ergodic properties I mentioned are readily proved by some simple Fourier analysis. This is in sharp contrast to what happens for other systems. Proving the Sinai or Bunimovich billiards are ergodic (let alone mixing) is very hard work and the geodesic flow on a negative curvature Riemannian manifold is also not a particularly simple object to study.

This is presumably why Hannay and Berry [18] came up with the idea of constructing a quantum mechanical equivalent to these simple systems, that I will describe in the next section.

Before turning to that task, let me point out that all the above properties of the dynamics are preserved if it is perturbed. Let g be a fixed smooth function on $\mathbb{T}^2$ and consider

$$\Phi_\epsilon = \Phi_\epsilon^g \circ A.$$

For small ϵ, this is still hyperbolic and exponentially mixing, with exponent γ_ϵ, $\lim_{\epsilon\to 0}\gamma_\epsilon = \gamma_0$ [3].

3 Quantum maps

What are the discrete quantum dynamical systems that correspond to the discrete classical dynamical systems described in the previous section? I will give a much simplified description and refer for details to [5] [11]. To avoid inessential technicalities, I will consider only the case where A is of the form ($m \in \mathbb{N}_*$)

$$A = \begin{pmatrix} 2m & 1 \\ 4m^2 - 1 & 2m \end{pmatrix}.$$

Introducing for each $a = (a_1, a_2) \in \mathbb{R}^2$ the phase space translation operator

$$U(a)\psi(y) = \mathrm{e}^{-\frac{i}{2\hbar} a_1 a_2} \mathrm{e}^{\frac{i}{\hbar} a_2 y} \psi(y - a_1) = \mathrm{e}^{-\frac{i}{\hbar}(a_1 P - a_2 Q)} \psi(y),$$

the quantum Hilbert space of the system is defined by

$$\mathcal{H}_\hbar = \{\psi \in \mathcal{S}'(\mathbb{R}) \mid U(1,0)\psi = \psi = U(0,1)\psi\}.$$

This space is N dimensional when $2\pi\hbar N = 1$ and zero-dimensional otherwise. Then any $\psi \in \mathcal{H}_\hbar$ can be written as follows:

$$\psi(y) = \sum_{\ell \in \mathbb{Z}} c_\ell \delta(y - \frac{\ell}{N}); \quad c_{\ell+N} = c_\ell.$$

If A were not of the above form, the definition of $\mathcal{H}_\hbar$ involves some extra phases that clutter up the page, which is the only reason why I avoid treating the general case here. That $\mathcal{H}_\hbar$ is finite dimensional is not surprising: since the classical phase space has finite area, and since the uncertainty principle tells us every quantum state 'takes up' an area $2\pi\hbar$, it is normal the Hilbert space is finite dimensional. This is reminiscent also of what happens when treating spin, where the classical phase space is a sphere, and the quantum Hilbert spaces are also finite dimensional.

The relation $2\pi\hbar N = 1$ will always be assumed to hold from now on. So when both $\hbar$ and N show up in the same formula, they are always related in that manner. Consequently, the semi-classical limit corresponds to taking N to infinity. In this spirit, I will occasionally find it useful to write $\mathcal{H}_N$ rather than $\mathcal{H}_\hbar$.

The standard Weyl quantization is readily adapted to the present situation. For $f \in C^\infty(\mathbb{T}^2), x = (q,p) \in \mathbb{T}^2$, write

$$f(x) = \sum_{n \in \mathbb{Z}^2} f_n \mathrm{e}^{-i2\pi(n_1 p - n_2 q)}$$

and define

$$\mathrm{Op}^{\mathrm{W}} f = \hat{f} = \sum_{n \in \mathbb{Z}^2} f_n \mathrm{e}^{-i2\pi(n_1 P - n_2 Q)} = \sum_{n \in \mathbb{Z}^2} f_n U(\frac{n}{N}) : \mathcal{H}_\hbar \to \mathcal{H}_\hbar.$$

The quantum dynamics is now defined as follows. For $A \in \mathrm{SL}(2,\mathbb{Z})$, $|\ \mathrm{Tr}A\ | > 2$, construct

$$M(A)\psi(y) = \left(\frac{i}{2\pi\hbar}\right)^{1/2} \int_{\mathbb{R}} \mathrm{e}^{\frac{i}{2\hbar}(2my^2 - yy' + 2my'^2)}\psi(y')dy'.$$

Then, for all $t \in \mathbb{Z}$,

$$M(A)\mathcal{H}_\hbar = \mathcal{H}_\hbar \quad \text{and} \quad M(A)^{-t}\ \mathrm{Op}^{\mathrm{W}} f\ M(A)^t - \mathrm{Op}^{\mathrm{W}}(f \circ A^t) = 0.$$

Now, for $\epsilon > 0$ define the unitary operator

$$U_\epsilon = \mathrm{e}^{-\frac{i}{\hbar}\epsilon \mathrm{Op}^{\mathrm{W}} g} M(A) : \mathcal{H}_\hbar \to \mathcal{H}_\hbar.$$

This is the quantum map we wish to study. It is naturally related to the discrete Hamiltonian dynamics on $\mathbb{T}^2$ obtained by iterating $\Phi_\epsilon = \Phi_\epsilon^g \circ A$. It acts on the N dimensional spaces $\mathcal{H}_\hbar$ and we are interested in the behaviour of its eigenfunctions and eigenvalues in the $N \to \infty$ limit:

$$U_\epsilon \psi_j^{(N)} = \mathrm{e}^{i\theta_j^{(N)}} \psi_j^{(N)}, \quad j = 1 \ldots N.$$

4 What is known?

Let me start by recalling a precise statement of the Schnirelman theorem in the present context:

Theorem 1. *[5] Let $\epsilon \geq 0$ and small. Then, for "almost all" sequences $\psi_{j_N}^{(N)} \in \mathcal{H}_\hbar$, so that $U_\epsilon \psi_{j_N}^{(N)} = \mathrm{e}^{i\theta_{j_N}^{(N)}} \psi_{j_N}^{(N)}$,*

$$\langle \psi_{j_N}^{(N)} \mid \mathrm{Op}^{\mathrm{W}} f \mid \psi_{j_N}^{(N)} \rangle \overset{N \to +\infty}{\to} \int_{\mathbb{T}^2} f(x)dx, \quad \forall f \in C^\infty(\mathbb{T}^2). \tag{1}$$

Note that here, for each N, a fixed basis of eigenfunctions has been chosen, and j_N is then an index in the set $\{1, \ldots N\}$. The "almost all" statement refers to the fact that one has to choose the j_N from a subset $G_N \subset \{1, \ldots, N\}$ such that $\#G_N/N \to 1$. As I already mentioned, this result is 'robust': it holds quite generally whenever the classical system is ergodic. It is proven in each different situation by adapting arguments that are used to prove it for other systems, such as Laplace-Beltrami operators on negatively curved manifolds. This is why it holds also for $\epsilon \neq 0$ and on higher dimensional tori. With (sometimes considerable) additional work, the proof can be even be adapted for maps that are not continuous, such as the Baker map [12] [14]. A version of the theorem for certain systems with a mixed phase space also exists [25].

As pointed out in the introduction, this theorem invites some immediate and obvious questions:

QUESTION 1 Do there exist exceptional sequences of eigenfunctions meaning ones that do not converge semi-classically to Lebesgue measure (this means (1) does not hold).
QUESTION 2 Can you characterize all possible semi-classical measures?

Here any measure ν on the torus for which there exists some sequence of eigenfunctions ψ_{N_ℓ} satisfying

$$\langle \psi_{N_\ell} | \mathrm{Op}^{\mathrm{W}} f | \psi_{N_\ell} \rangle \overset{N_\ell \to +\infty}{\to} \int_{\mathbb{T}^2} f(x) d\nu, \quad \forall f \in C^\infty(\mathbb{T}^2) \tag{2}$$

is called a semi-classical measure. Note that in this definition the sequence is not necessarily picked from a pre-assigned basis as is the case in the Schnirelman theorem.

There are of course still other questions that readily come to mind, related to the speed of convergence in the Schnirelman theorem, as well as to the fluctuations of the matrix elements that I shall not discuss here. Answers to the above two questions are not available in general systems. Only in some restricted classes of models have partial answers been obtained. For the (Hecke) eigenfunctions of the Laplace-Beltrami operator of a (class) of constant negative curvature surfaces the answer to this first question has been proven to be "NO!" [24]. On the other hand, for quantized hyperbolic toral automorphisms, the answer has been proven to be "YES!":

Theorem 2. *[15] Let $\epsilon = 0$. Let $0 \leq \alpha \leq \frac{1}{2}$, then there exists $N_k \to \infty$ and eigenfunctions $\psi_{N_k} \in \mathcal{H}_{N_k}$ so that*

$$\langle \psi_{N_k} | \mathrm{Op}^{\mathrm{W}} f | \psi_{N_k} \rangle \overset{N_k \to +\infty}{\to} \alpha f(0) + (1-\alpha) \int_{\mathbb{T}^2} f(x) dx, \quad \forall f \in C^\infty(\mathbb{T}^2). \tag{3}$$

This theorem holds for any choice of A with $|\mathrm{Tr} A| > 2$, not just for the particular ones discussed here. The sequence N_k depends on A. The construction (sketched below) yields such a sequence of eigenfunctions for any sequence of eigenangles $\theta^{(N_k)}_{j_{N_k}}$ of the quantum maps.

Note that this theorem is only stated for the quantized cat maps themselves, not for their perturbations. I strongly believe the result does not survive perturbation, a claim I will try to corroborate below.

Let me first sketch some of the ingredients that go into the proof of the result. I will indicate how the eigenfunctions with the above property are constructed in the case $\alpha = 1/2$. First of all, it is well known that the quantum maps $M(A)$ have a period: for each N, there exists an integer T_N so that $M(A)^{T_N} = \mathrm{e}^{i\varphi_N} \mathbb{1}_{\mathcal{H}_N}$. This implies that the eigenvalues $\exp i\theta^{(N)}_j$ have the property:

$$\theta^{(N)}_j \in \{ \frac{2\pi}{T_N} \ell + \frac{\varphi_N}{T_N} | \ell \in \mathbb{Z} \}.$$

Consequently, the N eigenangles $\theta_j^{(N)}$ can take on at most T_N distinct equally spaced values. Let me note in passing that it can be shown that the eigenvalues equidistribute around the circle in the classical limit [6]: this is the analog of Weyl's law in the present context. Now it is easy to see that

$$P_k = \frac{1}{T_N} \sum_{j=-T_N/2+1}^{T_N/2} \mathrm{e}^{-\frac{i}{T_N}(2\pi jk+\varphi_N j)} M(A)^j$$

is a projector. I am supposing T_N is even; if it is odd, you sum from $-(T_N-1)/2$ to $(T_N-1)/2$. *If this projector is non-zero*, it projects onto the eigenspace corresponding to the eigenvalue $\mathrm{e}^{\frac{i}{T_N}(2\pi k+\varphi_N)}$. In particular, if you can find some non-zero vector ϕ_N for which $P_k\phi_N \neq 0$, it certainly is an eigenvector. This suggests it is easy to construct at least some eigenvectors. Now, we are hunting for eigenvectors that are well localized around the origin in phase space. The idea is then to take for ϕ_N a coherent state centered on the origin. Recall that in the ordinary quantum mechanics of a system with one degree of freedom, those are of the form

$$\eta_z(y) = \left(\frac{\mathrm{Im}z}{\pi\hbar}\right)^{1/4} \exp i\frac{z}{2\hbar}y^2,$$

where z is a complex number in the upper half plane. By periodicizing this Gaussian in both the y variable and the dual (Fourier) variable, one obtains a family $|0,z\rangle$ of states in $\mathcal{H}_N$ that have two crucial properties for our purposes. First, they are localized at the origin in the sense that, as $N\to\infty$,

$$|\langle 0,z|\mathrm{Op}^{\mathrm{W}} f|0,z\rangle - f(0)| \leq C_f \frac{1+|z|^2}{\mathrm{Im}zN}. \tag{4}$$

Next, they behave nicely under the dynamics:

$$M(A)|0,z\rangle = \mathrm{e}^{i\sigma_N(A)}|0, A\cdot z\rangle. \tag{5}$$

Here $\sigma_N(A)$ is a phase that we will not play any role in what follows and

$$A\cdot z = \frac{2mz + (4m^2-1)}{z+2m}.$$

Let us now compute the matrix element of an observable $\mathrm{Op}^{\mathrm{W}} f$ in an evolved coherent state: $\langle 0, A^t\cdot z|\mathrm{Op}^{\mathrm{W}} f|0, A^t\cdot z\rangle$. Since $\mathrm{Im}(A^t\cdot z)$ behaves like $\exp -2\gamma_0|t|$ as $|t|\to\infty$ (the real part being uniformly bounded in time), the upper bound in (4) then guarantees that, for times t satisfying $|t|\leq \frac{1}{2}\tau_N$, where

$$\tau_N = \frac{1}{\gamma_0}\ln N,$$

this still converges to $f(0)$:

$$\langle 0, z|M(A)^{-t}\mathrm{Op}^{\mathrm{W}} f M(A)^{t}|0, z\rangle \stackrel{N\to+\infty}{\to} f(0). \tag{6}$$

But what happens for longer times? It was proven in [8] [9] [10] that, for times $|t|$ between $\frac{1}{2}\tau_N$ and $\frac{3}{2}\tau_N$, one has instead

$$\langle 0, z|M(A)^{-t}\mathrm{Op}^{\mathrm{W}} f M(A)^{t}|0, z\rangle \to \int_{\mathbb{T}^2} f(x)dx. \tag{7}$$

This is a consequence of the familiar 'spreading of the wavepacket', together with the exponential instability of the dynamics. One can indeed think of a coherent state as being concentrated in a region of linear size $\sqrt{\hbar}$ in both the position and momentum variables. Now in the semiclassical regime, evolving the coherent state with the exponentially unstable dynamics A stretches its support along the unstable direction v_+ to a size $\sqrt{\hbar}\exp\gamma_0 t$: this is no longer microscopic if t is of order $\frac{1}{2\gamma_0}\ln N$ or more so that the estimate (6) breaks down. Since the unstable manifold through 0, which is just the line $\mathbb{R}v_+$ through the origin, wraps itself ergodically around the torus, estimate (7) follows. To understand the reason for its breakdown beyond $\frac{3}{2}\tau_N$ we need to return to the quantum periods T_N.

The quantum periods T_N are very irregular functions of N, that grow roughly linearly in N [20], but that can, for some N, be very much smaller than that. It is shown in [8] that there exists a sequence N_k for which this period behaves like $2\tau_N$. But this means that, for those values of N

$$\langle 0, z|M(A)^{-(2\tau_N - s)}\mathrm{Op}^{\mathrm{W}} f M(A)^{2\tau_N - s}|0, z\rangle = \langle 0, z|M(A)^{s}\mathrm{Op}^{\mathrm{W}} f M(A)^{-s}|0, z\rangle$$

and consequently, for times s so that $|s| \leq \frac{1}{2}\tau_N$, and $N \to \infty$,

$$\langle 0, z|M(A)^{-(2\tau_N - s)}\mathrm{Op}^{\mathrm{W}} f M(A)^{2\tau_N - s}|0, z\rangle \to f(0).$$

This explains the breakdown of (7) for times beyond $\frac{3}{2}\tau_N$.

This breakdown can also be understood intuitively [8]. The exponential instability of the classical dynamics explained in Section 2 implies that *in the naïvest semi-classical picture of the quantum dynamics* the support of the Wigner function of the evolved coherent states is stretched out over a length $L_t = \sqrt{\hbar}\exp\gamma_0 t$ along the unstable direction. Now, this support is wound around the torus, and so successive windings are a distance $D_t = \hbar^{-1/2}\exp -\gamma_0 t$ apart. But on the torus, for any pair of conjugate variables Q, P, the uncertainty principle says $\Delta Q \Delta P \geq \hbar$, and, since we are on the torus $\Delta Q, \Delta P \leq 1$, so that both ΔQ and ΔP are *greater* than $\hbar$! This means, that on the torus, the Wigner function can not resolve details finer than on a scale $\hbar$. Hence one may expect that the above simple semi-classical picture of the evolution will break down when D_t is of order $\hbar$, and this is precisely when t is of order $\frac{3}{2}\tau_N$. At this time, interference effects start to take over. In the case of CAT maps, this leads eventually to the very "unclassical" recurrence at the quantum period, in which the original wave packet is reconstituted completely for $t = T_N$, and which may be as short as $2\tau_N$.

Let me now write

$$\psi_N = \sqrt{T_N} P_k |0, z\rangle.$$

These are our candidate eigenfunctions that will be proven to satisfy (3). The extra factor $\sqrt{T_N}$ is a normalization factor the origin of which will become clear in a second. It is enough to look at (4) and (7) to understand

(i) why this ought to work when $N = N_k$ and

(ii) what the remaining problems are.

First of all, let's be optimistic and suppose that in computing the matrix element $\langle \psi_N, \mathrm{Op}^{\mathrm{W}} f \psi_N \rangle$ we have to deal only with the diagonal matrix elements. Let us write

$$\langle \psi_{N_k} | \mathrm{Op}^{\mathrm{W}} f | \psi_{N_k} \rangle = \frac{1}{T_{N_k}} \sum_{t=-\frac{T_{N_k}}{2}+1}^{\frac{T_{N_k}}{2}} \langle 0, A^t \cdot z | \mathrm{Op}^{\mathrm{W}} f | 0, A^t \cdot z \rangle + \mathrm{OffDTerms}. \quad (8)$$

Now taking $N \to \infty$ and using (6) and (7), the sum of the diagonal terms above converges to the desired $\frac{1}{2} f(0) + \frac{1}{2} \int_{\mathbb{T}^2} f(x) dx$. This then proves Theorem 2 provided one can control the off diagonal terms, which requires more work.

One further idea that goes into this, and that is easily understood intuitively, is the following. For positive times t_+ larger than $\frac{1}{2}\tau_N$, the evolved coherent states stretches along the unstable manifold of $x = 0$ in the sense that its Wigner or Husimi function has a support that is concentrated along $\mathbb{R} v_+$ over a length $L_{t_+} = \sqrt{\hbar} \exp \gamma_o t_+$. Similarly, in the past, for negative times $-t_-$ so that $t_- \geq \frac{1}{2}\tau_N$, the evolved coherent state stretches along the stable manifold $\mathbb{R} v_-$ over a length L_{t_-}. Now these two intervals intersect, for each such time in only a finite (but growing!) set of homoclinic points. Since a finite number of points on a line should carry no weight (if all goes well), this suggests that the evolved states $M(A)^{t_+}|0, z\rangle$ and $M(A)^{-t_-}|0, z\rangle$ are almost orthogonal. This is indeed what is proven in [15], for times $t_- + t_+ < 2\tau_N$.

It should by now be clear why I claim the construction of strongly scarred eigenstates is particular to quantum CAT maps. Without the existence of quantum periods (which is already exceptional), and without the rather special arithmetic properties guaranteeing the existence of very short quantum periods, the construction would generate only quasimodes, not eigenvectors. The construction of strongly scarred eigenstates even breaks down on higher dimensional tori, since then the maps with short quantum periods do not exist [7]. All of this indicates that these strongly scarred eigenstates on isolated unstable periodic orbits are a particular feature of the unperturbed two-dimensional cat maps.

Having answered QUESTION 1 positively, we can now turn to QUESTION 2: what measures on $\mathbb{T}^2$ can be obtained as limits of the type $\langle \psi_N | \mathrm{Op}^{\mathrm{W}} f | \psi_N \rangle$, for *some* sequence of eigenfunctions ψ_N. It is easy to see that such measures have to be invariant under the dynamics generated by

A. Elaborating a little on the above ideas, one can show that, if μ is an A-invariant probability measure on $\mathbb{T}^2$ that is singular with respect to Lebesgue measure any probability measure on $\mathbb{T}^2$ of the form

$$\alpha\mu + (1-\alpha)\mathrm{dx} \tag{9}$$

is a limit measure provided $\alpha \leq 1/2$. This follows from a diagonalization trick and the well known fact that all invariant probability measures are weak limits of pure point measures [16]. This is optimal in the following sense:

Theorem 3. *[9][16] Let $\epsilon = 0$. Let μ be an A-invariant pure point measure. Let $\alpha > \frac{1}{2}$. Then there does not exist $N_k \to \infty$ and eigenfunctions $\psi_{N_k} \in \mathcal{H}_{N_k}$ so that*

$$\langle \psi_{N_k}, \mathrm{Op}^{\mathrm{W}} f \psi_{N_k} \rangle \stackrel{N_k \to +\infty}{\to} \alpha\mu(f) + (1-\alpha)\int_{\mathbb{T}^2} f(x)dx, \quad \forall f \in C^\infty(\mathbb{T}^2).$$

This result was obtained in [9] with a suboptimal bound $\alpha > 2/3$, which was then optimized in [16]. Still, this does not completely answer QUESTION 2. In particular, it is not known what happens to this result if μ is purely singular continuous or, more specifically, if a purely singular measure can be a semi-classical limit of the above type. This theorem does continue to hold on higher dimensional tori, but it is not clear if it is then optimal, for the reasons explained above.

The sequence N_k in the above construction is very sparse. It behaves roughly like $\exp(\gamma_0 k/2)$. It is legitimate to ask what happens if you stay away from this sequence: are exceptional sequences of eigenfunctions then still possible? First, one could consider sequences of N along which T_N behaves like $\sigma\tau_N$, with $\sigma > 2$. In that cases it is tempting to guess that any measure of the form (9) is now a semi-classical measure provided $\alpha \leq \sigma^{-1}$. The idea is that, in the eigenfunction construction sketched above, the fraction of terms that concentrates at the origin is σ^{-1}. Note however that our estimates do not imply such a result immediately, since we did not control the dynamics for times longer than $2\tau_N$. Still following the same line of thought, whenever the sequence of N is chosen so that $T_N/\ln N \to \infty$, one expects that the system should be uniquely quantum ergodic along that sequence. Here is a result that corroborates this conjecture:

Theorem 4. *[22] If $A \in SL(2,\mathbb{Z})$ is hyperbolic and $a_{11}a_{12} \equiv 0 \equiv a_{21}a_{22}$ mod 2, then there exists a density one sequence of integers $(\tilde{N}_\ell)_{\ell\in\mathbb{N}}$ along which*

$$\langle \psi_{\tilde{N}_\ell} | \mathrm{Op}^{\mathrm{W}} f | \psi_{\tilde{N}_\ell} \rangle \stackrel{\tilde{N}_\ell \to +\infty}{\to} \int_{\mathbb{T}^2} f(x)dx, \quad \forall f \in C^\infty(\mathbb{T}^2)$$

for any sequence of eigenfunctions $\psi_{\tilde{N}_\ell}$.

This is a generalization of the result of [13], where a result of this type is proven along a subsequence of primes. Clearly, the sequence $\tilde{N}_\ell$ showing up here is disjoint from the sequence N_k in Theorem 2. In fact $T_{\tilde{N}_\ell}$ is larger than $\sqrt{\tilde{N}_\ell}$. This result is of course a considerable strengthening of the Schnirelman theorem for the sequence of N concerned, because it says equidistribution holds for *all* sequences of eigenfunctions, not necessarily picked from a preassigned basis, and without any "density one" restriction, as opposed to what happens in the statement of the Schnirelman theorem.

To end this discussion, I would like to point out one further result that is fun to compare to Theorem 2:

Theorem 5. *[23] If $A \in SL(2, \mathbb{Z})$ is hyperbolic and $A \equiv \mathbb{I}_2 \mod 4$, then there exists for each N a basis $\{\psi_1, \psi_2, \ldots \psi_N\}$ of eigenfunctions of $M(A)$ so that*

$$\langle \psi_{j_N} | \mathrm{O}p^{\mathrm{W}} f | \psi_{j_N} \rangle \overset{N \to +\infty}{\to} \int_{\mathbb{T}^2} f(x) dx, \quad \forall f \in C^\infty(\mathbb{T}^2) \tag{10}$$

holds for any sequence ψ_{j_N}.

This again is an obvious strengthening of the Schnirelman theorem for the particular class of A considered. Note that the result holds for all N, without exceptions, so that a superficial reading of this result makes it look as if it is in contradiction with Theorem 2. The point is that the basis for which the result holds is explicitly described in the paper. Now, the eigenvalues of $M(A)$ may be degenerate so that it is possible that exceptional sequences of eigenfunctions not belonging to the above basis have a different semiclassical limit. This is precisely what happens with Theorem 2. Actually, since the quantum Hilbert spaces are N dimensional, the fact that the period is of order $\ln N_k$ for the special values of N implies that at least some of the eigenvalues must be highly degenerate in this case, with degeneracies of order $N_k / \ln N_k$. Although our construction of the exceptional eigenfunctions does not exploit this degeneracy explicitly, it clearly does so implicitly. Note finally that the sequence $\tilde{N}_\ell$ of Theorem 4 is such that the quantum periods $T_{\tilde{N}_\ell}$ are longer than $\sqrt{\tilde{N}_\ell}$. In fact, the authors of that paper show that, whenever the periods are at least this long along a sequence of integers, quantum unique ergodicity holds along that sequence. This implies that degeneracies as large as $\sqrt{\tilde{N}_\ell}$ do not suffice to produce strong scarring.

5 Perturbed cat maps

CAT maps are rather special: their linearity makes them easy to treat classically and endows their quantum counterparts with a number of arithmetic properties that can be exploited to prove a variety of results going beyond the Schnirelman theorem, as in [13] [22] [23] and to a lesser degree in [8][9][15] [16]. Since such results are not available in more complicated systems, the CAT maps provide a theoretical laboratory in which one can investigate with relative ease a certain number of questions in quantum chaos. But a real understanding of the issues of quantum chaos requires proofs that rely exclusively on semi-classical analysis and on the ergodic properties of the underlying classical dynamics of the systems considered. Indeed, only such proofs have a fighting chance to give insight into more complicated models, such as billiards or geodesic flows on negatively curved manifolds, and ultimately more realistic models with mixed phase space. Before attacking these systems, a good starting point is to look at the perturbed CAT maps described in Section 2. They are non-linear and preserve none of the simple arithmetic properties of the CAT maps themselves, once quantized. Their classical mechanics is sufficiently complicated that detailed information on their classical mixing rate has become available only rather recently [3]. On the quantum side, and contrary to what happens for quantized CAT maps, their eigenvalue statistics has been shown numerically to be generic [21].

Beyond the Schnirelman theorem, the only theorem in the previous section the proof of which has a chance to generalize to the case of perturbed maps is Theorem 3. The simplest version one could hope to prove would be:

> If $\psi_N \in \mathcal{H}_N$ is a sequence of eigenfunctions of U_ϵ, then there exist a function f so that
>
> $$\lim_{N\to\infty} \langle \psi_N, \mathrm{Op}^{\mathrm{W}} f \psi_N \rangle \neq f(0).$$

In other words, this statement says that for a perturbed cat map, the delta measure at the origin is not a semi-classical measure of the system.

A stragegy of proof of such a result is suggested by the main ingredient of the proof of Theorem 3 in [10]. The following result is proven there:

Theorem 6. *Let $\epsilon = 0$. Let $a_0 \in \mathbb{T}^2$. If $\varphi_N \in \mathcal{H}_\hbar$ is some sequence (not necessarily eigenfunctions!) with the property that, for all $f \in C^\infty(\mathbb{T}^2)$*

$$\langle \varphi_N | \mathrm{Op}^{\mathrm{W}} f | \varphi_N \rangle \overset{N\to+\infty}{\to} f(a_0)$$

then there exists a sequence of times $t_N \to \infty$ so that

$$\langle \varphi_N | U_0^{-t_N} \mathrm{Op}^{\mathrm{W}} f \ U_0^{t_N} | \varphi_N \rangle \overset{N\to+\infty}{\to} \int_{\mathbb{T}^2} f(x) dx.$$

If you can prove this result for $\epsilon \neq 0$, then that would imply immediately that a sequence of eigenfunctions can not concentrate on $x = 0$, since for eigenfunctions

$$\langle \psi_N | \mathrm{Op}^{\mathrm{W}} f | \psi_N \rangle = \langle \psi_N | M(A)^{-t} \mathrm{Op}^{\mathrm{W}} f M(A)^t | \psi_N \rangle.$$

The time scales t_N involved in the theorem are logarithmic in $\hbar$, more precisely, they are of the form $(1+\delta)\frac{1}{\gamma_0} \ln N$. So if you want to prove something like this for $\epsilon \neq 0$, you need to be able to control $U_\epsilon^{t_N}$ at such time scales.

Recall that for the Schnirelman theorem, such long time control is not needed since there, you take $\hbar \to 0$ first, and only after that exploit ergodicity by taking $t \to \infty$.

In [4], we made some progress on this question. To state the result, we need a definition:

Definition 1. *A sequence $\varphi_N \in \mathcal{H}_\hbar$ is said to localize at $a_0 \in \mathbb{T}^2$ if*

$$\langle \varphi_N | \mathrm{Op}^{\mathrm{W}} f | \varphi_N \rangle \overset{N \to +\infty}{\longrightarrow} f(a_0), \qquad \forall f \in C^\infty(\mathbb{T}^2).$$

This implies there exists $r_\hbar \to 0$ so that

$$\int_{|x-a| \geq r_\hbar} |\langle \varphi_N, x|z\rangle|^2 \frac{dx}{2\pi\hbar} \overset{N \to +\infty}{\longrightarrow} 0.$$

Here $|x, z\rangle$ is the coherent state localized at $x \in \mathbb{T}^2$.

Theorem 7. *Let U_ϵ be as before. There exists $\delta_0 > 0$ with the following property. Let ψ_N be a sequence of eigenfunctions of U_ϵ that localizes at some point $a \in \mathbb{T}^2$. Then $r_\hbar \geq \hbar^{\frac{1}{2} - \delta_0}$.*

One can paraphrase this result as follows:

Eigenfunctions do it slowly (if they do it at all).

In other words, a sequence of eigenfunctions can concentrate on a point only if it does so slowly, meaning that the region in which it concentrates is “large”, *i.e.* of linear size of order $\hbar^{\frac{1}{2} - \delta_0}$. The strategy of the proof is the following: we first prove an Egorov theorem for the system with control on the error term for time of order $\frac{2}{3\gamma_\epsilon} |\ln \hbar|$. We then use this result to get information on the propagation of coherent states, and finally use this to obtain the result. This result is of course far from optimal: this is largely due to the fact that we control the dynamics for insufficiently long times. Roughly speaking, and modulo some other technical problems, to show the desired result along those lines, one expects to need to control the dynamics up to times $(1+\delta)\frac{1}{\gamma_\epsilon} |\ln \hbar|$. This is consistent with the observation that such time scales are used in the proof of Theorem 6 for the unperturbed case. An improvement on the above result has been announced in [17]: Nonnenmacher has announced he can take $\delta_0(\epsilon)$ in such a way that it converges to $1/2$ as ϵ goes to zero. To do this, he controls the relevant dynamics for times up to $(1 - \delta(\epsilon))\frac{1}{\gamma_\epsilon} |\ln \hbar|$.

A different approach to characterizing semi-classical measures has very recently been proposed in [1] [2]. In [1], the author investigated the semi-classical measures on manifolds of negative curvature and shows, modulo an as yet unproven hypothesis on the eigenfunctions, that their support can not have small topological entropy. That result implies in particular that they can not concentrate on a (finite union of) periodic orbit(s). Recently, Anantharaman has been able to get rid of the extra unproven assumption [2]. If it is confirmed that this last argument does indeed work, then I expect it to go through also for the perturbed cat maps, with the same result: no semi-classical measures concentrated on a finite set of periodic orbits. In addition, since Anantharaman obtains these results with a control on the quantum evolution up to times that are only of the form $\delta\frac{1}{\gamma_\epsilon}|\ln\hbar|$, this is very good news for quantum chaos in general: it would mean very strong results on the eigenfunction behaviour can be obtained with rather minimal control on the long time quantum evolution, contrary to the general intuition confirmed by the work described in this note.

Although I have tried to explain that Theorem 2 is particular to quantized CAT maps and that, in my opinion, strong scars on isolated hyperbolic periodic orbits are not likely to show up in other systems, for those wanting to prove unique quantum ergodicity in exponentially unstable systems, there is still a lesson to be learned from Theorem 2: any general strategy based on semi-classical analysis and ergodic properties of the underlying dynamical system alone will work for CAT maps as well. But Theorem 2 shows CAT maps are not uniquely ergodic.

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