

# Where's that quantum?

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## **Abstract**

The nature and properties of the vacuum as well as the meaning and localization properties of one or many particle states have attracted a fair amount of attention and stirred up sometimes heated debate in relativistic quantum field theory over the years. I will review some of the literature on the subject and will then show that these issues arise just as well in non-relativistic theories of extended systems, such as free bose fields. I will argue they should as such not have given rise either to surprise or to controversy. They are in fact the result of the misinterpretation of the vacuum as “empty space” and of a too stringent interpretation of field quanta as point particles. I will in particular present a generalization of an apparently little known theorem of Knight on the non-localizability of field quanta, Licht's characterization of localized excitations of the vacuum, and explain how the physical consequences of the Reeh-Schlieder theorem on the cyclicity and separability of the vacuum for local observables are already perfectly familiar from non-relativistic systems of coupled oscillators.

## **1 Introduction**

Quantum field theory is the study of quantum systems with an infinite number of degrees of freedom. The simplest quantum field theories

are the free bose fields, which are essentially assemblies of an infinite number of coupled oscillators. Examples include the quantized electromagnetic field, lattice vibrations in solid state physics, and the Klein-Gordon field, some of which are relativistic while others are not. Particles show up in quantum field theory as “field quanta.” As we will see, these do not quite have all the properties of the usual particles of Newtonian physics or of Schrödinger quantum mechanics. I will in particular show (Section 3) that the quanta of free bose fields, relativistic or not, can not be perfectly localized in a bounded subset of space. This, in my opinion, shows conclusively that the difficulties encountered when attempting to define a position observable for the field quanta of relativistic fields, that continue to be the source of regular debate in the literature [Heg74] [Heg85] [Heg98] [BB98] [BY94] [FB99] [Hal01] [HC02] [Wal01b] [Wal01a], do not find their origin in any form of causality violation, as seems to be generally thought. Instead, they result from an understandable but ill-fated attempt to force too stringent a particle interpretation on the states of the quantum field containing a finite number of quanta.

States of free bose fields that are perfectly localized in bounded sets exist, but necessarily contain an infinite number of quanta. They can be classified quite easily, following the ideas of Licht [Lic63]. This is done in Sections 5 and 6. It turns out that those states do not form a vector subspace of the quantum Hilbert space. This has on occasion been presented as surprising or paradoxical within the context of relativistic quantum field theory, but it isn't: these properties of localized states are familiar already from finite systems of coupled harmonic oscillators.

The previous issues are intimately related to certain properties of the vacuum of free bose fields, that also have stirred up animated debate in the literature. I will in particular explain the physical implications of the Reeh-Schlieder theorem for free bose fields and their link with the localization issue. The Reeh-Schlieder theorem has been proven in the context of relativistic field theory, but holds equally well for (finite or infinite) non-relativistic systems of coupled oscillators. By studying it in that context, one easily convinces oneself that its implications for the quantum theory of measurement, for example, have nothing particularly surprising or paradoxical, and do not lead to causality violation, but are the result of the usual “weirdness” of quantum mechanics, since they are intimately linked to the observation that the vacuum is an entangled state. This thesis is developed

in Section 7.

The paper is organized as follows. After defining the class of free bose fields under consideration (Section 2), I will briefly discuss the question of the “localizability” of states with a finite number of quanta in Section 3. I will in particular recall from [Biè06] a generalization to free bose fields of a little known result of Knight about the Klein-Gordon field [Kni61], which shows they can not be perfectly localized excitations of the vacuum. I will then discuss the various, sometimes contradictory viewpoints on the question of particle localization prevalent in quantum field theory textbooks in Section 4 and compare them to the one presented here. I will argue in detail that the latter could help to clarify the situation considerably.

Section 5 contains the precise definition of “localized excitation of the vacuum” that I am using and some technical material needed for the presentation of Licht’s theorem on the characterization of local states in Section 6. Section 7 explains the physical implications of the Reeh-Schlieder theorem for free bose fields and their link with the localization issue.

Some of the proofs missing here, as well as a detailed critique of the Newton-Wigner position operator from the present viewpoint on localization can be found in [Biè06] which, together with the present work, is itself a short version of Chapter 6 of [Biè]. I refer to these references for more details.

## 2 Free bose fields

The simple physical systems under study here obey an equation of the form

$$\ddot{q} + \Omega^2 q = 0, \tag{1}$$

where  $\Omega$  is a self-adjoint, positive operator on a dense domain  $\mathcal{D}(\Omega)$  in a real Hilbert space  $\mathcal{K}$  and having a trivial kernel. I will always suppose  $\mathcal{K}$  is of the form  $\mathcal{K} = L^2_{\mathbb{r}}(K, d\mu)$ , where  $K$  is a topological space and  $\mu$  a Borel measure on  $K$ . Here the subscript “r” indicates that we are dealing with the real Hilbert space of real-valued functions. In fact, all examples of interest I know of are of this type. Those include:

- (i) Finite dimensional systems of coupled oscillators, where  $\mathcal{K} = \mathbb{R}^n$  and  $\Omega$  is a positive definite matrix;
- (ii) Lattices or chains of coupled oscillators, where  $\mathcal{K} = \ell^2(\mathbb{Z}^d, \mathbb{R})$  and  $\Omega^2$  is usually a bounded finite difference operator with a possibly

unbounded inverse;

(iii) The wave and Klein-Gordon equations, where  $\mathcal{K} = L^2(K, \mathbb{R})$ ,  $K \subset \mathbb{R}^d$  and  $\Omega^2 = -\Delta + m^2$  with suitable boundary conditions. More precisely, the Klein-Gordon equation is the equation

$$\partial_t^2 q(t, x) = -(-\Delta + m^2)q(t, x).$$

Equation (1) can be seen as a Hamiltonian system with phase space

$$\mathcal{H} = \mathcal{K}_{1/2} \oplus \mathcal{K}_{-1/2}, \quad \text{where} \quad \mathcal{K}_{\pm 1/2} = [\mathcal{D}(\Omega^{\pm 1/2})].$$

Here the notation  $[ \ ]$  means that we completed  $\mathcal{D}$  in the topology induced by  $\| \Omega^{\pm 1/2} q \|$  where  $\| \cdot \|$  is the Hilbert space norm of  $\mathcal{K}$ . On  $\mathcal{H}$ , the Hamiltonian ( $X = (q, p) \in \mathcal{H}$ )

$$H(X) = \frac{1}{2} p \cdot p + \frac{1}{2} q \cdot \Omega^2 q, \quad (2)$$

defines a Hamiltonian flow with respect to the symplectic structure

$$s(X, X') = q \cdot p' - q' \cdot p.$$

The Hamiltonian equations of motion  $\dot{q} = p$ ,  $\dot{p} = -\Omega^2 q$  are equivalent to (1). Note that I use  $\cdot$  for the inner product on  $\mathcal{K}$ .

The quantum mechanical description of these systems can be summarized as follows. Given a harmonic system determined by  $\mathcal{K}$  and  $\Omega$ , one chooses as the quantum Hilbert space of such a system the symmetric Fock space  $\mathcal{F}^+(\mathcal{K}^{\mathbb{C}})$ , and as quantum Hamiltonian the second quantization of  $\Omega$ :  $H = d\Gamma(\Omega)$ . Note that this is a positive operator and that the Fock vacuum is its ground state, with eigenvalue 0. In terms of the standard creation and annihilation operators on this Fock space, the quantized fields and their conjugates are then *defined* by ( $\eta \in \mathcal{K}_{-1/2}^{\mathbb{C}}$ ):

$$\eta \cdot Q := \frac{1}{\sqrt{2}} (a(\Omega^{-1/2} \bar{\eta}) + a^\dagger(\Omega^{-1/2} \eta)), \quad (3)$$

and, similarly ( $\eta \in \mathcal{K}_{1/2}^{\mathbb{C}}$ ),

$$\eta \cdot P := \frac{i}{\sqrt{2}} (a^\dagger(\Omega^{1/2} \eta) - a(\Omega^{1/2} \bar{\eta})). \quad (4)$$

For later purposes, I define, for each  $\xi \in \mathcal{K}^{\mathbb{C}}$ , the Weyl operator

$$W_{\text{F}}(\xi) = \exp(a^\dagger(\xi) - a(\xi)). \quad (5)$$

I will also need, for each  $X = (q, p) \in \mathcal{H}$

$$z_\Omega(X) = \frac{1}{\sqrt{2}}(\Omega^{1/2}q + i\Omega^{-1/2}p) \in \mathcal{K}^\mathbb{C}. \quad (6)$$

It follows easily that

$$W_F(z_\Omega(X)) = \exp -i(q \cdot P - p \cdot Q).$$

Loosely speaking, the observables of the theory are “all functions of  $Q$  and  $P$ .” For mathematical precision, one often uses various algebras (called CCR-algebras) generated by the Weyl operators  $W_F(z_\Omega(X))$ ,  $X \in \mathcal{H}$ , as we will see in some more detail below.

Things are particularly simple when the set  $K$  is discrete, as in examples (i) and (ii) above. One can then define the displacements  $Q_j$  and momenta  $P_j$  of the individual oscillators, with  $j = 1, \dots, n$  in the first case and  $j \in \mathbb{Z}^d$  in the second case. These examples are a helpful guide to the intuition, as we will see below.

Apart from the vacuum  $|0\rangle$ , which is the ground state of the system, excited states of the form

$$a^\dagger(\eta_1) \dots a^\dagger(\eta_k)|0\rangle,$$

play a crucial role and are referred to as states with  $k$  “quanta.” The quanta of the Klein-Gordon field, for example are thought of as spinless particles of mass  $m$ . In the case of an oscillator chain or lattice, they are referred to as “phonons.” To the extent that these quanta are thought of as “particles,” the question of their whereabouts is a perfectly natural one. It is to its discussion I turn next.

### 3 So, where’s that quantum?

Let me start with an informal discussion of the issue under consideration. Among the interesting observables of the oscillator systems we are studying are certainly the “local” ones. I will give a precise definition in Section 5, but thinking for example of a finite or infinite oscillator chain, “the displacement  $q_7$  or the momentum  $p_7$  of the seventh oscillator” is certainly a “local” observable. In the same way, if dealing with a wave equation, “the value  $q(x)$  of the field at  $x$ ” is a local observable. Generally, “local observables” are functions of the fields and conjugate fields in a bounded region of space. In the case

of the oscillator chain or lattice, space is  $K = \mathbb{Z}^d$  ( $d \geq 1$ ), and for a finite chain, space is simply the index set  $K = \{1, \dots, n\}$ .

I find this last example personally most instructive. It forces one into an unusual point of view on a system of  $n$  coupled oscillators that is well suited for making the transition to the infinite dimensional case. Think therefore of a system of  $n$  oscillators characterized by a positive  $n$  by  $n$  matrix  $\Omega^2$ . A local observable of such a system is a function of the positions and momenta of a fixed finite set  $B$  of oscillators. In this case,  $\mathcal{K} = \mathbb{R}^n$ , which I view as  $L^2_r(K)$ , where  $K$  is simply the set of  $n$  elements. Indeed,  $q \in \mathbb{R}^n$  can be seen as a function  $q : j \in \{1, \dots, n\} \mapsto q(j) \in \mathbb{R}$ , obviously square integrable for the counting measure.

Consider now a subset  $B$  of  $K$ , say  $B = \{3, 6, 9\}$ . A local observable over  $B$  is then a finite linear combination of operators on  $L^2(\mathbb{R}^n)$  of the form ( $a_j, b_j \in \mathbb{R}, j \in B$ ):

$$\exp -i \left( \sum_{j \in B} (a_j P_j - b_j Q_j) \right).$$

Note that those form an algebra. More generally it is an operator of the form

$$\int (\prod_{j \in B} da_j db_j) f(a_j, b_j) e^{-i(\sum_{j \in B} (a_j P_j - b_j Q_j))},$$

for some function  $f$  in a reasonable class. In other words, it is a function of the position and momentum operators of the oscillators inside the set  $B$ .

Better yet, if you write, in the Schrödinger representation,

$$L^2(\mathbb{R}^n) \cong L^2(\mathbb{R}^{\#B}, \prod_{j \in B} dx_j) \otimes L^2(\mathbb{R}^{n-\#B}, \prod_{j \notin B} dx_j),$$

then it is clear that the weak closure of the above algebra of local observables is

$$\mathcal{B}(L^2(\mathbb{R}^{\#B}, \prod_{j \in B} dx_j)) \otimes \mathbb{1}.$$

So, indeed, a local observable is clearly one that acts only on the degrees of freedom indexed by elements of  $B$ . The definition of a local observable over a finite subset  $B$  of any oscillator lattice is perfectly analogous, where this time the  $Q_j, P_j$  are defined on Fock space, as

explained in the previous section. This definition is natural and poses no problems.

Now what is a local state of such a system? More precisely, I want to define what a “strictly local excitation of the vacuum” is. The equivalent classical notion is readily described. The vacuum, being the ground state of the system, is the quantum mechanical equivalent of the global equilibrium  $X = 0$ , which belongs of course to the phase space  $\mathcal{H}$ , and a local perturbation of this equilibrium is an initial condition  $X = (q, p)$  with the support of  $q$  and of  $p$  contained in a subset  $B$  of  $K$ . An example of a local perturbation of an oscillator lattice is a state  $X \in \mathcal{H}$  where only  $q_0$  and  $p_0$  differ from 0. In the classical theory, local perturbations of the equilibrium are therefore states that differ from the equilibrium state only inside a bounded subset  $B$  of  $K$ . It is this last formulation that is readily adapted to the quantum context, through the use of the notion of “local observable” introduced previously. Turning again to the above example, let me write  $|0, \Omega\rangle$  for the ground state of the oscillator system in the Schrödinger representation; then a strictly local excitation of the vacuum over  $B$  is a state  $\psi \in L^2(\mathbb{R}^n, dx)$  so that, for all  $a_j, b_j \in \mathbb{R}$ ,

$$\langle \psi | \exp -i \sum_{j \notin B} (a_j P_j - b_j Q_j) | \psi \rangle = \langle 0, \Omega | \exp -i \sum_{j \notin B} (a_j P_j - b_j Q_j) | 0, \Omega \rangle.$$

In other words, outside  $B$ , the states  $\psi$  and  $|0, \Omega\rangle$  coincide. Any measurement performed on a degree of freedom outside  $B$  gives the same result, whether the system is in the vacuum state or in the state  $\psi$ . The mean kinetic or potential energy of any degree of freedom outside  $B$  is identical in both cases as well. All this certainly expresses the intuitive notion of “localized excitation of the vacuum”. Note that it is based on the idea of viewing the full system as composed of two subsystems: the degrees of freedom inside  $B$  and the degrees of freedom outside  $B$ . The analogous definition of strictly local excitation of the vacuum for the general class of bose fields considered in the previous section is easily guessed and given in Section 5.

The following question arises naturally. Let’s consider a free bose field and suppose we consider some state containing one quantum, meaning a state of the form  $a^\dagger(\xi)|0\rangle$ . Can such a state be perfectly localized in a bounded set  $B$ ? Since we like to think of these quanta as particles, one could a priori expect the answer to be positive, but the answer is simply: “NO, not in any model of interest.” This is the content of the generalization of Knight’s theorem proven in [Biè06]

(see Theorem 7.1 (iv)). States containing only one quantum (or even a finite number of them), cannot be strictly localized excitations of the vacuum. This is a little surprising at first, but perfectly natural. In fact, it is true even in finite chains of oscillators, as I will now show.

A special case of the result is indeed easily proven by hand, and clearly brings out the essential ingredient of the general phenomenon. Consider a finite oscillator chain, and suppose simply  $\Omega^2$  does not have any of the canonical basis vectors  $e_i$  of  $\mathbb{R}^n$  as an eigenmode. This means that each degree of freedom is coupled to at least one other one and is certainly true for the translationally invariant finite chain, to give a concrete example. I will show by a direct computation that in this situation there does not exist a one quantum state  $a^\dagger(\xi)|0\rangle$  ( $\xi \in \mathbb{C}^n, \bar{\xi} \cdot \xi = 1$ ), that is a perturbation of the vacuum strictly localized on one of the degrees of freedom, say the first one  $i = 1$ . In other words, there is no such state having the property that, for all  $Y = (a, b) \in \mathbb{R}^{2n}, a_1 = 0 = b_1$ , one has

$$\langle 0 | a(\xi) \exp -i \sum_{j=2}^n (a_j P_j - b_j Q_j) a^\dagger(\xi) | 0 \rangle = \langle 0 | \exp -i \sum_{j=2}^n (a_j P_j - b_j Q_j) | 0 \rangle. \quad (7)$$

Now, a simple computation with the Weyl operators shows that this last condition is equivalent to

$$\bar{\xi} \cdot z_\Omega(Y) = 0,$$

for all such  $Y$ . Here  $z_\Omega(Y) = \frac{1}{\sqrt{2}}(\Omega^{1/2}a + i\Omega^{-1/2}b)$ , and simple linear algebra then implies that this last condition can be satisfied for some choice of  $\xi$  if and only if  $e_1$  is an eigenvector of  $\Omega$ . But this implies it is an eigenvector of  $\Omega^2$ , which is a situation I excluded. Hence  $a^\dagger(\xi)|0\rangle$  is not a strictly localized excitation of the vacuum at site  $i = 1$  for any choice of  $\xi$ .

Of course, if the matrix  $\Omega^2$  is diagonal, this means that the degrees of freedom at the different sites are not coupled, and then the result breaks down. But in all models of interest, the degrees of freedom at different points in space are of course coupled.

The main ingredient for the proof of the generalization of Knight's theorem to free bose fields given in [Biè06] is the non-locality of  $\Omega$ . A precise definition will follow below, but the idea is that the operator  $\Omega$ , which is the square root of a finite difference or of a second order differential operator in all models of interest, does not preserve supports. The upshot is that states of free bose fields with a finite number of

particles, and a fortiori, one-particle states, are never strictly localized in a bounded set  $B$ . This gives a precise sense in which the elementary excitations of the vacuum in a bosonic field theory (relativistic or not) differ from the ordinary point particles of non-relativistic mechanics: their Hilbert space of states contains no states in which they are perfectly localized.

Having decided that one-quantum states cannot be strictly localized excitations of the vacuum on bounded sets, the question arises if such strictly localized states exist. Sticking to the simple example of the chain, any excitation of the vacuum strictly localized on the single site  $i = 1$  can be proven to be of the type  $\exp iF(Q_1, P_1)|0\rangle$ , where  $F(Q_1, P_1)$  is a self-adjoint operator, function of  $Q_1$  and  $P_1$  alone. For a precise statement, see Theorem 7.1. Coherent states  $\exp -i(a_1P_1 - b_1Q_1)|0\rangle$  are of this type. So there are plenty such states. Note however that the linear superposition of two such states is not usually again such a state: the strictly localized excitations of the vacuum on the site  $i = 1$  do not constitute a vector subspace of the space of all states. This is in sharp contrast to what happens when, in the non-relativistic quantum mechanics of a system of  $n$  particles moving in  $\mathbb{R}$ , we ask the question: “What are the states  $\psi(x_1, \dots, x_n)$  for which all particles are in some interval  $I \subset \mathbb{R}$ ?” These are all wave functions supported on  $I \times \dots \times I$ , and they clearly form a vector space. In that case, to the question “Are all particles inside  $I$ ?” corresponds therefore a projection operator  $P_I$  with the property that the answer is “yes” with probability  $\langle \psi | P_I | \psi \rangle$ . But in oscillator lattices there is no projection operator corresponding to the question “Is the state a strictly local excitation of the vacuum inside  $B$ ?”. This situation reproduces itself in relativistic quantum field theory, and does not any more constitute a conceptual problem there as in the finite oscillator chain. I will discuss this point in more detail in Section 7.

In view of the above, it is clear that no position operator for the quanta of, for example, the Klein-Gordon field can exist. Those quanta simply do not have all attributes of the point particles of our classical mechanics or non-relativistic quantum mechanics courses. But, since the same conclusion holds for the quanta of a lattice vibration field, this has nothing to do with causality or relativity, as seems to be generally believed. It nevertheless seems that this simple lesson of quantum field theory has met and still continues to meet with a lot of resistance, as we will see in Section 4.

So, to sum it all up, one could put it this way. To the question

Why is there no sharp position observable for particles?

the answer is

It is the non-locality of  $\Omega$ , stupid!

More details on this aspect of the story have been published in [Biè06], where it is in particular argued more forcefully that the Newton-Wigner position operator is not a good tool for describing sharp localization properties of the quanta of quantum fields.

## 4 Various viewpoints on localization

Now, what is currently the standard view in the physics community on the question of localization of particles or quanta in field theory? Let me first point out that this is not necessarily easy to find out from reading the textbooks on quantum field theory or relativistic quantum physics. Indeed, the least one can say is that the whole localization issue does not feature prominently in these books. One seems to be able to detect three general attitudes. First of all, some books make no mention of it at all: [MG84] [BS83] [Cha90] [Hua98] [Der01]. In [BS83], for example, when discussing the attributes of particles in an introductory chapter, the authors identify those as rest mass, spin, charge and lifetime, but do not mention any notion of position or localizability.

A second group of authors give an intuitive discussion, based on the uncertainty principle, together with the non-existence of superluminal speeds, to explain single massive particles should not be localizable within a region smaller than their Compton wavelength: [BD65] [Sak67] [BLP71] are examples. The idea is that, since  $\Delta X \Delta P \geq \hbar$ , whenever  $\Delta X$  is of the order of the Compton wavelength  $\hbar/mc$ ,  $\Delta P \geq mc$ . Since the gap between positive and negative eigenstates is  $2mc^2$ , this is indicating that to obtain such sharp localization, negative energy eigenstates are “needed.” Let me point out that this reasoning does not by any means exclude the possibility of having one-particle states strictly localized in regions bigger than the Compton wavelength. The argument is then further used to support the idea that in a fully consistent relativistic quantum theory, one is unavoidably led to a many body theory with particle/anti-particle creation, and to field theory.

The essential idea underlying this type of discussion is that, even though the solutions to the relevant wave equation (Klein-Gordon or

Dirac, mostly) do not, as such, have a satisfactory probabilistic interpretation, the coupling to other (classical or quantum) fields is done via the solution and so the particle is present essentially where this wave function is not zero: the particle is where its energy density is. In this spirit, Björken and Drell, for example write, at the end of the section where they address some of the problems associated with the single particle interpretation of the Klein-Gordon and Dirac equations (including a brief discussion of the Klein paradox at a potential step): “We shall tackle and resolve these questions in Chapter 5. Before doing this let us look in the vast, if limited, domain of physical problems where the application forces are weak and smoothly varying on a scale whose energy unit is  $mc^2$  and whose distance unit is  $\frac{\hbar}{mc}$ . Hence we may expect to find fertile fields for application of the Dirac equation for positive energy solutions.”

This attitude means that whenever a conceptual problem arises, it is blamed on the single-particle approach taken. The trouble is that, once these authors treat the full field theory, they do not come back to the localization issue at all, be it for particle states or for general states of the field.

A third group of authors give a slightly more detailed discussion, including of the Newton-Wigner position operator, but fail to indicate the problems associated with the latter, leaving the impression that strict particle localization is quite possible after all: [Sch61] [Gre90] [Ste93] [Str98]. Serman, for example, when discussing one-quantum states of the field, writes: “To merit the term ‘particle’, however, such excitations [of the quantum field] should be localisable.” He then discusses the Newton-Wigner operator as the solution to this last problem, without pointing out the difficulties associated with it. Something similar happens in Schweber who writes: “By a single particle state we mean an entity of mass  $m$  and spin 0 which has the property that the events caused by it are localized in space.” Interestingly, neither of them makes any mention of the contradicting intuitive argument which completely rules out perfect single particle localization. This is in contrast to Greiner, who does give this argument, but does not point the apparent contradiction with the notion of a position operator, the (generalized) eigenstates of which are supposed to correspond to a perfectly localized particle. Of course, this last point is obscured, as in many cases, by the observation that the Klein-Gordon wave function corresponding to such a state is exponentially decreasing with a localization length which is the Compton

wavelength.

Let me note in passing that it is generally admitted that a photon is not localisable at all, not even approximately. This is sometimes argued by pointing out it admits no Newton-Wigner position operator, or intuitively, based on the uncertainty principle argument which blames this on its masslessness: its Compton wavelength is infinite. Note however that massless spinless particles *are* localisable in the Newton-Wigner sense, so that this argument is not convincing. In fact, it was proven in [BB98] that one-photon states can be localized with sub-exponential tails in the sense that the field energy density of the state decreases sub-exponentially away from a localization center.

To summarize, it seems there is no simple, generally agreed upon and clearly argued textbook viewpoint on the question of localization in quantum field theory, even for free fields. The general idea seems to be that the above intuitive limitations on particle localization give a sufficient understanding since, at any rate, the whole issue is not very important. As Bacry puts it, not without irony, in [Bac88]: “The position operator is only for students and . . . for people interested in the sex of the angles, this kind of people you find among mathematical physicists, even among the brightest ones such as Schrödinger and Wigner.”

In my view, it is the absence of a clear definition of “localized state” – such as the one provided by Knight – that has left the field open for competing speculations on how to circumvent the various problems with the idea of sharp localization of particles and in particular with the Newton-Wigner operator. Some authors seem to believe strongly in the need for a position observable for particles, claiming the superluminal speeds it entails do not constitute a problem after all, essentially because the resulting causality violation is too small to be presently observable [Rui81] [FB99]. Others have provided alternative constructions with non-commuting components [Bac88] [FB99].

To me, this is similar to clinging at all cost to ether theory in the face of the “strange” properties of time implied by Einstein’s special relativity. Between giving up causality or giving up position operators for field quanta, I have made my choice. This, together with the definition of Knight and the accompanying theorem, which could easily be explained in simple terms in physics books, as Section 3 shows, would go a long way in clarifying the situation. On top of that, it would restore the “democracy between particles” [Bac88]. In fact, contrary to what is usually claimed and although I have not worked out this

in detail here, photons are no worse or better than electrons when it comes to localization, a point of view that I am not the first one to defend. Peierls, for example, in [Pei73], compares photon and electron properties with respect to localization and although he starts off with the statement “On the other hand, one of the essentially particle-like properties of the electron is that its position is an observable, there is no such thing as the position of the photon,” he concludes the discussion as follows, after a more careful analysis of the relativistic regime: “If we work at relativistic energies, the electron shows the same disease. So in this region, the electron is as bad a particle as the photon.”

At any rate, if you find my point of view difficult to accept and are reluctant to do so, you are in good company. Here is what Wigner himself says about it in [Wig83], almost forty years after his paper with Newton: “One either has to accept this [referring to non-causality] or deny the possibility of measuring position precisely or even giving significance to this concept: a very difficult choice.” In spite of this, in the conclusion of this same article, he writes, apparently joining my camp: “Finally, we had to recognize, every attempt to provide a precise definition of a position coordinate stands in direct contradiction to relativity.”

Having advocated Knight’s definition of “strictly local excitations of the vacuum”, I turn in Section 6 to their further study. First, we need some slightly more technical material.

## 5 All things local

Let’s recall we consider harmonic systems over a real Hilbert space  $\mathcal{K}$  of the form  $\mathcal{K} = L_r^2(K, d\mu)$ , where  $K$  is a topological space and  $\mu$  a Borel measure on  $K$ . I need to give a precise meaning to “local observables in  $B \subset K$ ”, for every Borel subset of  $K$ . To do that, I introduce the notion of “local structure”, which is a little abstract, but the examples given below should give you a good feel for it.

**Definition 5.1.** *A local structure for the oscillator system determined by  $\Omega$  and  $\mathcal{K} = L_r^2(K, d\mu)$  is a subspace  $\mathcal{S}$  of  $\mathcal{K}$  with the following properties:*

1.  $\mathcal{S} \subset \mathcal{K}_{1/2} \cap \mathcal{K}_{-1/2}$ ;
2. Let  $B$  be a Borel subset of  $K$ , then  $\mathcal{S}_B := \mathcal{S} \cap L_r^2(B, d\mu)$  is dense in  $L_r^2(B, d\mu)$ .

In addition, we need

$$\mathcal{H}(B, \Omega) \stackrel{\text{def}}{=} \mathcal{S}_B \times \mathcal{S}_B.$$

Note that, thanks to the density condition in the definition, this is a symplectic subspace of  $\mathcal{H}$ . This is a pretty strange definition, and I will turn to the promised examples in a second, but let me first show how to use this definition to define what is meant by “local observables”.

**Definition 5.2.** *Let  $\mathcal{K} = L^2_r(K, d\mu), \Omega, \mathcal{S}$  be as above and let  $B$  be a Borel subset of  $K$ . The algebra of local observables over  $B$  is the algebra*

$$\text{CCR}_0(\mathcal{H}(B, \Omega)) = \text{span} \{W_F(z_\Omega(Y)) \mid Y \in \mathcal{S}_B \times \mathcal{S}_B\}.$$

Here “CCR” stands for Canonical Commutation Relations. The algebras  $\text{CCR}_0(\mathcal{H}(B, \Omega))$  form a net of local algebras in the usual way [Emc72] [Hor90] [Haa96]. Note that  $\Omega$  plays a role in the definition of  $\mathcal{S}$  through the appearance of the spaces  $\mathcal{K}_{\pm 1/2}$ . The first condition on  $\mathcal{S}$  guarantees that  $\mathcal{S} \times \mathcal{S} \subset \mathcal{H}$  so that, in particular, for all  $Y \in \mathcal{S} \times \mathcal{S}$ ,  $s(Y, \cdot)$  is well defined as a function on  $\mathcal{H}$  which is important for the definition of the local observables to make sense.

For the wave or Klein-Gordon equation, one can choose  $\mathcal{S}$  to be either the space of Schwartz functions or  $C_0^\infty(\mathbb{R}^d)$ . Similarly, on a lattice, one can use the space of sequences of rapid decrease or of finite support. In the simple example of a finite system of oscillators,  $\mathcal{S} = \mathbb{R}^n$  will do.

Finally, I need the following definition:

**Definition 5.3.**  *$\Omega$  is said to be strongly non-local on  $B$  if there does not exist a non-vanishing  $h \in \mathcal{K}_{1/2}$  with the property that both  $h$  and  $\Omega h$  vanish outside  $B$ .*

Here I used the further definition:

**Definition 5.4.** *Let  $h \in \mathcal{K}_{\pm 1/2}$  and  $B \subset K$ . Then  $h$  is said to vanish in  $B$  if for all  $\eta \in \mathcal{S}_B$ ,  $\eta \cdot h = 0$ . Similarly, it is said to vanish outside  $B$ , if for all  $\eta \in \mathcal{S}_{B^c}$ ,  $\eta \cdot h = 0$ .*

Intuitively, a strongly non-local operator is one that does not leave the support of any function  $h$  invariant. In the examples cited, this is always the case (see [Biè] for details).

Finally, we can give the general definition of “strictly local state,” which goes back to Knight [Kni61] for relativistic fields, and generalizes (7).

**Definition 5.5.** *If  $B$  is a Borel subset of  $K$ , a strictly local excitation of the vacuum with support in  $B$  is a normalized vector  $\psi \in \mathcal{F}^+(\mathcal{K}^{\mathbb{C}})$ , different from the vacuum itself such that*

$$\langle \psi | W_{\mathbb{F}}(z_{\Omega}(Y)) | \psi \rangle = \langle 0 | W_{\mathbb{F}}(z_{\Omega}(Y)) | 0 \rangle \quad (8)$$

for all  $Y = (q, p) \in \mathcal{H}(B^c, \Omega)$ .

So it is a state which is indistinguishable from the vacuum outside  $B$ .

## 6 Characterizing the strictly local excitations

Having established in Section 3 that in no models of interest finite particle states can be strictly localized excitations of the vacuum, it is natural to wonder which states do have this property. I mentioned that coherent states are in this class, as Knight already pointed out. Knight also conjectured that all states that are strictly localized excitations of the vacuum over some open set  $B$ , are obtained by applying a unitary element of the local algebra to the vacuum. This was subsequently proven for relativistic fields by Licht in [Lic63]. Here is a version of this result adapted to our situation [Biè].

**Theorem 6.1.** *Suppose we are given a harmonic system determined by  $\Omega$  and  $\mathcal{K} = L^2_{\mathbb{F}}(K, d\mu)$ , and with a local structure  $\mathcal{S}$ . Let  $B \subset K$  and suppose  $\Omega$  is strongly non-local over  $B$ . Let  $\psi \in \mathcal{F}^+(\mathcal{K}^{\mathbb{C}})$ . Then the following are equivalent:*

- (i)  $\psi \in \mathcal{F}^+(\mathcal{K}^{\mathbb{C}})$  is a strictly local excitation of the vacuum inside  $B$ ;
- (ii) There exists a partial isometry  $U$ , belonging to the commutant of  $\text{CCR}_0(\mathcal{H}(B^c, \Omega))$  so that

$$\psi = U|0\rangle.$$

As we have seen,  $\Omega$  tends to be strongly non-local over bounded sets in all examples of interest, so the result gives a complete characterization of the localized excitations of the vacuum over bounded sets in those cases.

Since the condition in the definition of localized excitation is quadratic in the state, there is no reason to expect the set of localized

excitations inside  $B$  to be closed under superposition of states. Of course it is closed under the taking of convex combinations (mixtures). Licht gives a simple criterium in the cited 1963 paper allowing to decide whether the linear combination of two strictly local excitations is still a strictly local excitation. Both the statement and the proof are again easily adapted to our situation [Biè].

**Theorem 6.2.** *Suppose we are given a harmonic system determined by  $\Omega$  and  $\mathcal{K} = L^2_r(K, d\mu)$ , and with a local structure  $\mathcal{S}$ . Let  $B \subset K$  and suppose  $\Omega$  is strongly non-local over  $B$ . Let  $\psi_1, \psi_2 \in \mathcal{F}^+(\mathcal{K}^{\mathbb{C}})$  be strictly local excitations of the vacuum inside  $B$ , so that  $\psi_i = U_i|0\rangle$ , with  $U_1, U_2 \in (\text{CCR}_0(\mathcal{H}(B^c, \Omega)))'$ . Then*

$$\psi = (\alpha\psi_1 + \beta\psi_2) / (\|\alpha\psi_1 + \beta\psi_2\|)$$

*is a strictly local excitation of the vacuum inside  $B$  for all choices of  $(\alpha, \beta) \neq (0, 0)$  iff  $U_2^*U_1$  is a multiple of the identity operator.*

The conclusion is then clear. Whenever  $\Omega$  is strongly non-local over  $B$ , the superposition of strictly localized states does typically *not* yield a strictly localized state over  $B$ . In fact, taking  $U_1 = W_F(z_\Omega(Y_1))$ , and  $U_2 = W_F(z_\Omega(Y_2))$ , with  $Y_1, Y_2 \in \mathcal{H}(B, \Omega)$ , it is clear that  $U_2^*U_1$  is a multiple of the identity only if  $Y_1 = Y_2$  so that the localized states do certainly not form a vector space in that situation. This is in sharp contrast to what we are used to in the non-relativistic quantum mechanics of systems of a finite number  $N$  of particles. In that case, a wave function  $\psi(x_1, \dots, x_N)$  describes a state of the system with all the particles in a subset  $B$  of  $\mathbb{R}^3$  iff it vanishes as soon as one of the variables is outside of  $B$ . The corresponding states make up the subspace  $L^2(B^N)$  of  $L^2(\mathbb{R}^N)$ . In that case, to the question “Are all the particles in the set  $B$ ?” corresponds a projection operator  $P_B$  with the property that the answer is “yes” with probability  $\langle\psi|P_B|\psi\rangle$ . To the question “Is the state a strictly local excitation of the vacuum in  $B$ ?” cannot correspond such a projection operator! I will belabour this point in Section 7.

## 7 Surprises?

Here is a list of three mathematical truths that have originally been proven in the context of relativistic quantum field theory, and that seem to have generated a fair amount of surprise and/or debate:

1. The vacuum is a cyclic vector for the local algebras over open regions.
2. The vacuum is a separating vector for the local algebras over open regions.
3. The set of strictly local states (in the sense of Knight) over an open region is not closed under superposition of states.

In that context, the open regions referred to are open regions of Minkowski space time. And by relativistic, I mean of course invariant under the Poincaré group. Recall that a vector  $\phi$  in a Hilbert space  $\mathcal{V}$  is cyclic for an algebra  $\mathcal{A}$  of bounded operators on  $\mathcal{V}$  if  $\overline{\text{span} \mathcal{A} \phi} = \mathcal{V}$ . And it is separating if  $A \in \mathcal{A}$ ,  $A\phi = 0$ , implies  $A = 0$ .

It turns out that, as soon as  $\Omega$  is a non-local operator, suitably adapted analogous statements hold for the harmonic systems that are the subject of this paper. We already saw this for the third statement in the previous section. The results are summed up in the following theorem (proven in [Biè]).

**Theorem 7.1.** *Let  $\mathcal{K} = L^2(K, d\mu)$ ,  $\Omega^2 \geq 0$  and  $\mathcal{S}$  be as before. Suppose that, for some  $B \subset K$ ,  $\Omega$  is strongly non-local over  $B$ . Then*

- (i) *The vacuum is a cyclic vector for  $\text{CCR}_0(\mathcal{H}(B^c, \Omega))$ ;*
- (ii) *The vacuum is a separating vector for  $\text{CCR}_w(\mathcal{H}(B, \Omega))$ ;*
- (iii) *The set of strictly local states over  $B$  do NOT form a vector space.*
- (iv) *There do not exist finite particle states that are strictly local states over  $B$ .*

Since, as we pointed out, the hypotheses of the theorem hold in large classes of examples, and for various sets  $B$ , so do the conclusions. Note that they therefore do not have a particular link with relativistic invariance. Note also that for lattices, the vacuum cannot be cyclic for a bounded set, since then  $\mathcal{H}(B, \Omega)$  is finite dimensional so that  $\overline{\text{span}_{\mathbb{C}} z_{\Omega}(\mathcal{H}(B, \Omega))}$  is a strict subspace of  $\mathcal{K}^{\mathbb{C}}$ . However, it is easily shown in translationally invariant models, for example, that the vacuum is cyclic for the the CCR-algebra over the complement of any bounded set  $B$  (See [Biè] for details).

My goal in this section is to explain in each case why the above statements have generated surprise in the context of relativistic field theory, then to argue that none of these properties should have surprised anyone precisely since they hold for simple systems of  $n$  coupled oscillators, and for free bose fields in rather great generality as

the previous theorem shows. In particular, they therefore hold for the Klein-Gordon equation on Minkowski spacetime, which happens to be relativistic (meaning here Poincaré invariant). Why should it then be contrary to anyone's physical intuition if they continue to hold for interacting relativistic fields?

**Statement 1.** This was proven for relativistic quantum fields by Reeh and Schlieder in [RS61] and has been a well-known feature of *relativistic* quantum field theory ever since. As already mentioned, in that context, the open regions referred to are open regions of Minkowski space time. For a textbook formulation in the context of axiomatic, respectively algebraic relativistic quantum field theory you may consult [SW64], respectively [Haa96] [Hor90].

Rather than giving the full proof of Theorem 7.1 (i), I will once again restrict myself to a system of  $n$  oscillators and consider the subspace of the state space  $L^2(\mathbb{R}^n)$  containing all vectors of the form

$$\sum_{j=1}^L c_j \exp -i(a_j P_1 - b_j Q_1) |0, \Omega\rangle,$$

for all choices of  $c_1 \dots c_L$ ,  $L \in \mathbb{N}$ . Note that, since only the operators  $Q_1$  and  $P_1$  occur in the exponents, it follows from the considerations of the previous sections that such vector is a linear combination of excitations of the vacuum that are strictly localized at site  $i = 1$ . Nevertheless, one can easily prove that that those vectors form a dense subset of the full Hilbert space, which means that the vacuum is a cyclic vector for the local algebra over that site. A very pedestrian proof goes as follows: thinking for simplicity of the case  $n = 2$ , it is easy to see, taking limits, that the vectors  $Q_1^k |0, \Omega\rangle$  and  $P_1^\ell |0, \Omega\rangle$  belong to the closure of the span of the above vectors. Now, using that  $\Omega$  is not diagonal, one easily concludes that therefore all wave functions of the form

$$p(x_1, x_2) \exp -\frac{1}{2} x \cdot \Omega x,$$

with  $p(x_1, x_2)$  any polynomial belong to the space. Taking Hermite polynomials, for example, one obtains a basis for the full state space  $L^2(\mathbb{R}^2)$ . Note that, again,  $\Omega$  has to be non-diagonal for this to work. So indeed, the vacuum is a cyclic vector for the algebra of local observables over  $B$  (here  $B = \{1\}$ ). It holds in much more generality, provided  $\Omega$  is non-local: this is the content of Theorem 7.1. There are

various poetic and misleading ways to express this result, for example by saying: “Local operations on the vacuum can produce instantaneous and arbitrary changes to the state vector arbitrarily far away.” Lest one enjoys confusing oneself, it is a good idea to stay clear of such loose talk, as I will further argue below.

Let me now corroborate my claim that the cyclicity of the vacuum came as a surprise when it was proven for relativistic fields. Segal writes in [Seg63] it is “particularly striking” and Segal and Goodman say it is “quite surprising” and a “bizarre phenomenon” in [SG65]. Streater and Wightman call it “a surprise” in [SW64]. Even rather recently, Haag refers to it as “startling” in [Haa96] (p. 102), although he reduces this qualification to “(superficially) paradoxical” later on in his book (p. 254). Redhead in [Red95] similarly calls it “surprising, even paradoxical”.

The surprise finds its origin in an apparent contradiction between the above mathematical statement and some basic physical intuition on the behaviour of quantum mechanical systems. Segal for example writes in [Seg63] that Reeh-Schlieder is particularly striking because it apparently means that “the entire state vector space of the field could be obtained from measurements in an arbitrarily small region of space-time.” This, he argues, is “quite at variance with the spirit of relativistic causality.” Similar arguments can be found for example in [Red95] or in [Haa96] and in [FB99], the authors write that “it is hard to square with naïve, or even educated, intuitions about localization.”

This supposed contradiction is, as we shall see, directly related to the misconception of the vacuum as “empty space,” which already was part of the problem with the debate surrounding the Newton-Wigner position operator. To understand this, let me explain what the contradiction translates to in our present context of harmonic systems: it is the too naïve and, as we shall see, erroneous expectation that the mathematical operation of applying to the vacuum vector a local observable  $A$  belonging to the local algebra over some subset  $B$  of  $K$  yields a state  $A|0\rangle/\langle 0|A^*A|0\rangle^{1/2}$  which is a strictly localized excitation of the vacuum in  $B$  in the sense of Knight’s Definition 5.5 (for brevity called “local states” in what follows) [HS65]. This, if it were true, would of course be in blatant contradiction with the cyclicity of the vacuum. Indeed, if the vacuum is cyclic, any state of the system, including one that differs from the vacuum very far away from  $B$  can be approximated by one of the above form. Fortunately, we know from Licht’s result that applying a local observable to the vac-

uum yields a strictly local excitation only if the local observable is a partial isometry. Of course, this only shifts the paradox, because one may choose to find Licht's result paradoxical. Indeed, when  $A$  is a projector, one can, according to the standard interpretational rules of quantum mechanics, and in particular the "collapse of the wave function" prescription, prepare (in principle!) an ensemble of systems, all in the state  $A|0\rangle/\langle 0|A^*A|0\rangle^{1/2}$ . Now, if the projector  $A$  is a local observable, these measurements can correctly be thought of as being executed within  $B$ . Now, *if you think of the vacuum as empty space*, it is inconceivable on physical grounds that such a measurement could instantaneously change something outside  $B$ . But this is then in contradiction with the mathematical result of Licht which asserts that if  $A$  is a projector, the state  $A|0\rangle/\langle 0|A^*A|0\rangle^{1/2}$  is *not* local and therefore *does* differ from the vacuum outside  $B$ . The way out is obvious: the vacuum is not empty space but the ground state of an extended system. To see what is happening, the example of the chain of  $n$  coupled oscillators is again instructive. Concentrate for example on the seventh oscillator of the chain and consider the question: "Does the displacement of the seventh oscillator fall within the interval  $[a, b]$ ?" To this corresponds the projector  $\chi_{[a,b]}(Q_7)$ . The outcome of the corresponding preparation procedure will be an ensemble of systems, all in the state

$$\chi_{[a,b]}(Q_7)|0\rangle/\langle 0|\chi_{[a,b]}(Q_7)|0\rangle$$

with the non-vanishing probability  $\langle 0|\chi_{[a,b]}(Q_7)|0\rangle$ . But it is obvious that this state differs from the vacuum on the neighbouring site 8 and even on very far away sites! So even though the above projector corresponds to a local physical operation it does nevertheless not lead to a local excitation of the vacuum because even a local physical operation (here a measurement of the displacement of a single oscillator on one site) on the vacuum will instantaneously change the state of the system everywhere else. This is obviously the result here of the fact that the vacuum exhibits correlations between (commuting!) observables at different sites along the ring, a perfectly natural and expected phenomenon. After all, the oscillators on different sites are connected by springs. The ultimate reason for this phenomenon is therefore that the ground state of a typical oscillator system characterized by an  $n$  by  $n$  matrix  $\Omega^2$  is an "entangled" state in  $L^2(\mathbb{R}^n) \cong L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \cdots \otimes L^2(\mathbb{R})$ , unless of course  $\Omega^2$  is diagonal so that the oscillators are uncoupled to begin with. This entanglement can be seen in the fact that the ground state is a Gaus-

sian with correlation matrix  $\Omega$ , which is not diagonal. The change far away that the vacuum undergoes in the above measurement process is therefore nothing new, but a version of the usual “weirdness” of quantum mechanics, at the origin also of the EPR paradox.

In conclusion, I would therefore like to claim that in oscillator systems such as the ones under study here, one should expect that physical changes (such as measurements) operated on the vacuum vector inside some set  $B \subset K$  (for example as the result of a measurement) will alter the state of the system outside  $B$  instantaneously. This is already true for finite dimensional systems as explained above and remains true for infinite dimensional ones such as oscillator lattices or the Klein-Gordon equation. That the latter has the additional feature of being Poincaré invariant does not in any way alter this conclusion nor does it lead to additional paradoxes. It does in particular not cause any causality problems in the sense that the phenomenon cannot be used to send signals, as one can easily see.

In short, the cyclicity of the vacuum does not lead to any contradictions with basic physical intuition, provided the latter is correctly used. It is perhaps interesting to note that Licht’s 1963 result which is of great help in understanding the situation, is not cited in any of the other works on the subject I mentioned (although his paper sometimes is to be found in the bibliography . . .).

**Statement 2.** What about the separability of the vacuum? The separability also gives rise to an apparent contradiction that is equally easily dispelled with. Indeed, the “surprise” can be formulated as follows. If a non-trivial projector  $P$  belongs to the local algebra  $CCR_w(\mathcal{H}(B, \Omega))$ , then  $P|0\rangle \neq 0$  since the vacuum is a separating vector. Now this means that if the system is in its vacuum state and one measures locally some property of the system, such as, for example, whether the displacement of the oscillator on site 69 has a value between 7 and 8, then the answer is “yes” with non-zero probability. This consequence of separability (possibly first mentioned in [HK70]) can be paraphrased suggestively as follows:

“When the system is in the vacuum state, anything that can happen will happen,”

or, alternatively, as in [SW85], “every local detector has a non-zero vacuum rate.” This result is certainly paradoxical if you think of the vacuum as being empty space. Indeed, how can any measurement, local or not, give a non-trivial result in empty space?

To see why there is no reason to be surprised, let us look again at my favourite example, a system of  $n$  coupled oscillators. Its ground state is a Gaussian with correlation matrix  $\Omega$ , so the probability that the displacement of the oscillator on site 69 has a value between 7 and 8 is obviously non-zero! That a similar property survives in the quantized Klein-Gordon field does not strike me as particularly odd, since the mathematical structure of both models is exactly identical, as should be clear from the previous sections.

**Statement 3.** In the context of relativistic quantum field theory, this was proven by Licht in his cited 1963 paper. He uses as a basic ingredient the result of Reeh and Schlieder. In our context here, it is the content of Theorem 6.2, which is a consequence of the strong non-locality of  $\Omega$ .

Now, Theorem 6.2 has an interesting consequence for quantum measurement theory. Indeed, given a set  $B$  so that  $\Omega$  is strongly non-local over  $B$ , there cannot exist a projection operator  $P_B$  on  $\mathcal{F}^+(\mathcal{K}^{\mathbb{C}})$  with the property that  $P_B\psi = \psi$  if and only if  $\psi \in \mathcal{F}^+(\mathcal{K}^{\mathbb{C}})$  is a strictly local excitation of the vacuum over  $B$ . Indeed, if such an operator existed, the strictly localized excitations would be stable under superposition of states, of course. So there is no projector associated to the “yes-no” question: “Is the system strictly localized in  $B$ ?” This is different from what we are used to in the non-relativistic quantum mechanics of a finite number of particles. There questions such as “Are all particles in  $B$ ?” have a projector associated to them. But of course, we are dealing here with extended systems, such as oscillator lattices, and asking questions about excitations of the vacuum, not about the whereabouts of the individual oscillators, for example! It is only when you forget that, and try to interpret all statements about the fields in terms of particles that you run into trouble with your intuition.

As I pointed out before, this third statement above follows from the second, the second from the first and the main ingredient of the proof of the first by Reeh and Schlieder is relativistic invariance and the spectral property. This seems to have lead to the impression that these three properties, and especially the last one, are typical of relativistic fields, and absent in non-relativistic ones. For example, Redhead says in [Red95] that “to understand why the relativistic vacuum behaves in such a remarkable way, let us begin by contrasting the situation with nonrelativistic quantum field theory”. It is furthermore said in [HS65] that “in a relativistic field theory it is not possible to define

a class of states strictly localized in a finite region of space within a given time interval if we want to keep all the properties which one would like to associate with localization.” The authors include in those properties the fact that it has to be a linear manifold. Quite recently still, a similar argument is developed in some detail in [BY94]. They explain that “there are marked differences between non-relativistic and relativistic theories, which manifest themselves in the following alternative structure of the set of vectors”  $\psi$  representing states that are strictly localized excitations of the vacuum. They then go on to explain that, in the non-relativistic case, the set of localized states are closed under superposition, whereas in relativistic quantum field theory, they are not.

But these statements are potentially misleading since precisely the same phenomena produce themselves in the eminently non-relativistic systems of coupled oscillators that I have been describing as is clear from the results of the previous sections. These phenomena are a consequence of the strong non-locality of  $\Omega$ , and have nothing to do with relativistic invariance. The problem is that one has the tendency to compare relativistic field theories with the second quantization of the Schrödinger field, which is of course a non-relativistic field theory. In that context, the above three statements do not hold and in particular, the set of local states is a linear subspace of the Hilbert space. But if you compare, as you should, relativistic field theories to the equally non-relativistic harmonic systems, such as lattices of coupled oscillators, you remark that many of the features of the relativistic fields are perfectly familiar from the non-relativistic regime. They should therefore not come as a surprise, and not generate any paradoxes. It should be noted that already in [SG65] the source of the Reeh-Schlieder properties for free relativistic fields is identified to be the non-locality of  $(-\Delta + m^2)^{1/2}$ . This is further exploited in [Mas68] [Mas73] and [Ver94] where the anti-locality of potential perturbations of  $-\Delta$ , respectively of the Laplace-Beltrami operator on Riemannian manifolds is proven, which then yields a proof of the Reeh-Schlieder property in these situations.

As a further remark along those lines, I would like to point out that the fact that the strictly local states are not stable under superposition of states is sometimes related to the type of the local algebras of observables [Lic63] [BY94]. In relativistic quantum field theory, they are known to be of type III [Dri75], a result that generated a fair amount of excitement when it was discovered, since type III fac-

tors were thought to be esoteric objects [Seg63]. It should be noted, however, that the local algebras of observables in oscillator lattices are type I factors, and that the strictly local excitations nevertheless are not stable under superposition. In addition, just as in relativistic quantum field theory, pure states look locally like mixtures: this is a consequence of entanglement, not of relativity.

A further paradox related to the third statement is the following. Suppose you have a strictly local excitation of the vacuum  $\psi$  over some set  $B$ . Now ask yourself the question if a local measurement inside  $B$  can prepare this state. In other words, is the projector onto  $\psi$  a local observable? In relativistic theories, the answer is in the negative [Red95]. Indeed, since the algebras are of type III, they contain no finite dimensional projectors. But even in oscillator lattices, the answer is negative, since the local algebras do not contain finite dimensional projectors either, and this despite the fact that they are of type I. Of course, this is not in the least little bit surprising: intuitively also, to fix the state of an extended system such as an oscillator lattice, you expect to need to make measurements on every site of the lattice. In particular, if the state is a strictly local excitation of the vacuum on the fifth site, you should check it coincides with the vacuum on all other sites. So you can neither measure nor prepare such a state by working only on a few lattice sites. Again, the situation is very different from the one of, for example, a one-electron system, where local states can be prepared locally.

Of course, by now, I hope I have brainwashed you into agreeing that it is really a bad idea to think of the vacuum as empty space. But should you not be convinced, again, you are in excellent company. This is how Schwinger talks about the vacuum in [Sch73]: “With [quantum field theory] the vacuum becomes once again a physically reasonable state with no particles in evidence. The picture of an infinite sea of negative energy electrons is now but regarded as an historical curiosity, and forgotten. Unfortunately, this episode, and discussions of vacuum fluctuations, seem to have left people with the impression that the vacuum, the physical state of nothingness (under controlled physical circumstances), is actually the scene of wild action.” And a bit further down in the same article, he insists again: “I recall that for us the vacuum is the state in which no particles exist. It carries no physical properties: it is structureless and uniform. I emphasize this by saying that the vacuum is not only the state of minimal energy, it is the state of *zero* energy, zero momentum,

zero charge, zero whatever. Physical properties, structure, come into existence only when we disturb the vacuum, when we excite it.”

## 8 Conclusions

As long as one studies only a finite number of oscillators, the imaginative description of the quantum states of harmonic systems in terms of quanta is rather cute but not terribly useful or important. It is however a crucial element of relativistic and non-relativistic quantum field theories, which have an infinite number of degrees of freedom. In that case, the quanta are traditionally interpreted as particles. Photons, for example, are the quanta of the electromagnetic field and phonons are those of the vibration field. Electrons are similarly quanta of the Dirac field.

Now if you want to interpret the quanta as particles, you automatically are lead to the question that features as the title of this manuscript. One thinks of a particle as a localized object, and so it seems perfectly natural to wish to have a position operator for it, or at least some way to answer questions such as : “What is the probability of finding the particle in such and such a region of space?” In fact, as I have argued via the generalization of Knight’s theorem, since the particles of quantum field theory are quanta, the situation is similar to the one we discovered already with finite systems of oscillators: the quanta cannot be perfectly localized and therefore there is no way to associate a position operator to them, and in that sense the question above does not really make any sense at all. It should be noted that whereas Knight’s definition is regularly referred to in discussions of localization issues, his theorem, which is very helpful in understanding the issues at hand, seems to never be mentioned.

That quanta cannot be perfectly localized does not constitute a problem. A good notion of localized states exist: it is the one provided by considering localized excitations of the vacuum and goes back to Knight. Those states differ from the vacuum only inside a set  $B$  and there are plenty of them. They do however not form a vector subspace of the quantum Hilbert space, and no projection operator is associated with the localized excitations over a fixed set  $B$ . As I have explained, this feature of relativistic quantum field theories is also familiar from non-relativistic oscillator chains, and as such not related to relativistic invariance. If the right analogies between relativistic

and non-relativistic theories are used, it is not counter-intuitive or in any way surprising.

Let me mention that a complete discussion of the localization properties of field states should also analyse notions of approximate localization, allowing for exponential or algebraic decay of the expectation values outside the set  $B$ . These notions are implicit in the physics literature, where states with exponential tails, for example, are often thought of as localized. They have been developed by several authors [Amr69] [HS65] [BB98] [Haa96] [Wal01b]. A complete discussion should finally also address those questions for other fields, such as complex bose fields and Fermi fields. These issues will be adressed elsewhere [Tis].

In conclusion, the morale of the story is this: when testing your understanding of a notion in quantum field theory, try to see what it gives for a finite system of oscillators. Examples of situations where this algorithm seems to meet with some success are the various puzzles associated with particle localization and the Reeh-Schlieder theorem and its consequences, as I have argued here. In particular, if the notion under study, when adapted to finite or infinite oscillator chains, looks funny there, it is likely to lead you astray in the context of quantum field theory as well: an example is the Newton-Wigner position operator. Of course, I am not the first one to point these analogies out. In [Pei73], one can read: “The radiation field differs from atomic systems principally by ...having an infinite number of degrees of freedom. This may cause some difficulties in visualizing the physical problem, but is not, in itself, a difficulty of the formalism.”

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