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Recent Results on p-Stable Convex Compact Sets with Applications

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This paper is dedicated to Miklós Csörgő on the occasion of his 70th birthday

Abstract. This is a survey paper on stable compact convex sets in Banach spaces. Particular case of stable random zonotopes is given main attention. One unexpected application to statistics is discussed.

1 Introduction

In the paper we present a small survey of recent results on stable convex compact sets in Banach spaces. We are interested in limit theorems for sums of random sets (summation of sets everywhere in this paper is in the sense of Minkowski), and at once it is necessary to say that at present there exists the general theory of summation of random sets, developed mainly in 1975–1985. During this period there was a big international group of probabilists working on Probability in Banach spaces, and the application of methods and results from this theory (it is necessary to have in mind that a set of all compact convex sets in a Banach space is not a Banach space itself) allowed to create a general theory of infinitely divisible laws and limit theorems for sums of independent random convex compact (cc) sets (see, for example [11], [12] and references therein).

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After the main theory of summation of random sets was created, there was some loss of interest in random sets, but during last 5-6 years this interest arose again. Here it is necessary to mention series of papers by G. F. Lawler, O. Schramm and W. Werner (see, for example, [16], [17],[18]), concerning self-avoiding walks on lattices and planar Brownian motion, which were inspired by problems and conjectures in theoretical physics. Another topic where random sets appear is the so-called convexification of stochastic processes, see the survey paper by Yu. Davydov and R. Zitikis in this volume [8].

One more motivation to return to random sets was to understand more deeply the structure of stable laws in the space of compact sets even on plane (not speaking about general Banach spaces). What is meant by the word "structure" now we shall explain in more details, but at first we shall introduce some notation.

Let $\mathbf{B}$ denote a separable Banach space with a norm $||\cdot||$. Its dual space then is denoted by $\mathbf{B}^*$ and duality by $f(x),\ f\in\mathbf{B}^*,\ x\in\mathbf{B}$. For subsets $A, C\subset\mathbf{B}$ and a real number α Minkowski addition and homothetics, respectively, are defined by

$$A+C=\{a+c: a\in A,\ c\in C\},\quad \alpha A=\{\alpha a: a\in A\}.$$

Let $\mathcal{K}(\mathbf{B})$ be the collection of nonempty compact subsets of $\mathbf{B}$. It becomes a complete separable metric space when endowed with the Hausdorff distance δ

$$\begin{aligned}\delta(A,C) &= \max\{\sup_{a\in A}\inf_{c\in C}||a-c||,\ \sup_{c\in C}\inf_{a\in A}||a-c||\} \\ &= \inf\{\varepsilon>0: A\subset C^\varepsilon,\ C\subset A^\varepsilon\},\end{aligned}$$

where $D^\varepsilon=\{x\in\mathbf{B}:\delta(x,D)<\varepsilon\}$ and $\delta(x,D)=\inf\{||x-y||: y\in D\}$ for $D\subset\mathbf{B}$. We also denote $||A||=\delta(\{0\},A)=\sup\{||a||: a\in A\}$ for $A\in\mathcal{K}(\mathbf{B})$. Two important subsets of $\mathcal{K}(\mathbf{B})$ are the set $co\mathcal{K}(\mathbf{B})$ of cc subsets of $\mathbf{B}$ and the set $co\mathcal{K}_0(\mathbf{B})$, consisting of cc sets containing 0.

In a metric space S we can define atomic distribution, which prescribes all mass to at most countable set of points, and diffuse measure, which assigns zero mass for each point $s\in S$. If additionally there is a linear structure in S and it is infinite dimensional, then we shall say that a measure μ is trivially diffuse, if it is supported by one dimensional linear manifold generated by some fixed element of S. For example, if $a(t)$ is a fixed continuous function on $[0,1]$ and X is a random variable with a density, then random functions $a(t)+X$ and $Xa(t)$ give us examples of trivially diffuse measures on $C([0,1])$, while the distribution of Brownian motion is a non-trivially diffuse measure on the same space. The space $\mathcal{K}(\mathbf{B})$ is not a linear space, but we can define trivially diffuse measure in this space as follows. We say that a measure μ on $\mathcal{K}(\mathbf{B})$ is trivially diffuse if it is concentrated on the sets of the form $tA,\ t\geq 0$ or $A+x,\ x\in\mathbf{B}$, where A is a fixed cc set in Banach space $\mathbf{B}$. In many spaces Gaussian measures are the most simple examples of non-trivially diffuse measures, but it is known (see, for example [11], [23]) that in $\mathcal{K}(\mathbf{B})$ Gaussian random cc sets are of the form $A+X$, where A is a fixed cc set in $\mathbf{B}$ and X is a Gaussian vector in $\mathbf{B}$. The same is true for all stable random convex compact sets with stability index $1\leq\alpha<2$, while in case $0<\alpha<1$ we can get non-trivially diffuse measures. It is an interesting question what is the support for such non-trivially diffuse measures, that is, how look like typical sets on which these measures are concentrated. As in the case of Brownian motion in $C([0,1])$ we can expect some irregularity. Special case of stable random cc sets are the so-called countable stable random zonotopes, which were introduced in [7] and later studied in [6]. It was realized that even on plane the structure of the boundary of these random zonotopes is rather complicated. Although at present we have no final answer about the boundary of such random cc sets, in section 3 we will introduce these sets and formulate several results from [4], [21], which confirms conjecture of the authors of [6] on Hausdorff dimension of the set of extremal points of the boundary of stable countable random zonotopes.

Rather unexpectedly study of the boundary of random stable zonotopes gave rise to the new idea how to estimate parameters of multivariate stable laws and later on was used in estimating tail index of univariate distributions. We will present this idea and results from papers [6], [5], [20].

The paper is organized as follows. In section 2 we give three types of description of stable cc sets in Banach space and introduce the notion of Lévy motion with values in $\mathcal{K}(\mathbf{B})$. In section 3 we introduce countable stable random zonotopes and present results, connected with the structure of the boundary of such zonotopes on plane. In section 4 we formulate some recent results from [20], on estimation of tail index and parameters of stable laws.

2 Representation of α-stable convex compact sets, $0 < \alpha < 1$

A random compact set K in a Banach space $\mathbf{B}$ is a Borel measurable function from a probability space $(\Omega, \mathcal{F}, P)$ into $\mathcal{K}(\mathbf{B})$. If $K \in co\mathcal{K}(\mathbf{B})$ a.s. then K is called *a random compact convex set* (random cc set).

We shall deal with stable cc sets and use the following definition (see [11])

Definition 2.1 A random compact convex set K is called *α-stable*, $0 < \alpha \leq 2$ if for any K_1, K_2 independent and distributed as K and for all $a, b \geq 0$ there exist sets $C, D \in co\mathcal{K}(\mathbf{B})$ such that

$$aK_1 + bK_2 + C \stackrel{D}{=} (a^\alpha + b^\alpha)^{1/\alpha} K + D.$$

K is *strictly α-stable* if C and D can be chosen to be $\{0\}$. If $p = 2$, then K is called Gaussian.

In what follows $\stackrel{D}{=}$ means equality of distributions.

We consider only the case $0 < \alpha < 1$, since it was proved in [11] that if K is α-stable random convex compact set and $1 \leq \alpha \leq 2$, then $K = \{\xi\} + D$ a.s., where ξ is α-stable random element in $\mathbf{B}$ and D is a fixed nonrandom cc set.

Let $\mathcal{K}_1 = \{A \in co\mathcal{K}(\mathbf{B}) : ||A|| = 1\}$ and let σ be a finite Borel measure on $\mathcal{K}_1$. Without loss of generality we may assume that σ is a probability distribution, that is $\sigma(\mathcal{K}_1) = 1$. Let θ be a real positive α-stable random variable, $0 < \alpha < 1$.

By M_α we denote *a positive independently scattered random measure* on Borel sets of $\mathcal{K}_1$ such that $M_\alpha(A) \stackrel{D}{=} (\sigma(A))^{1/\alpha}\theta$ for each Borel set A. Then M_α is called a positive α-stable random measure with the control measure σ.

The first representation of an α-stable cc set involves stochastic integral with respect random measure M_α and was proved in [11].

Theorem 2.2 *Let $0 < \alpha < 1$. A random set $K(\omega)$ is α–stable compact convex set in $\mathbf{B}$ if and only if*

$$K = A + \int_{\mathcal{K}_1} x M_\alpha(dx) \quad a.s.,$$

where $A \in co\mathcal{K}(\mathbf{B})$ and M_α is a positive α–stable random measure on $\mathcal{K}_1$ with a control measure σ.

The second representation uses Poisson random measure and integral. Such representation for the countable zonotopes in $\mathbf{R}^d$ was obtained in [7], general case was considered in [6]. In order to formulate this result we need some more notation.

Let $(S, \mathcal{S}, n)$ be a measure space, and let $\mathcal{S}_0 = \{A \in \mathcal{S} : n(A) < \infty\}$. *A Poisson random measure* N on $(S, \mathcal{S}, n)$ is an independently scattered σ-finite random measure such that for each set $A \in \mathcal{S}_0$ the random variable $N(A)$ has a Poisson distribution with mean $n(A)$. Then n is called the control measure of N.

Theorem 2.3 *Let* Π *be a random Poisson measure on* $\mathcal{K}_1 \times \mathbf{R}_+$ *with control measure* $\sigma \times \gamma$*, where* γ *is the measure on* $\mathbf{R}_+$ *with the density* $x^{-1/(\alpha+1)}$, $0 < \alpha < 1$. *Then a random set* $K(\omega)$ *is* α*–stable compact convex set in* $\mathbf{B}$ *if and only if*

$$K \stackrel{D}{=} A + \int_{\mathcal{K}_1 \times \mathbf{R}_+} xt\Pi(dx, dt),$$

where $A \in co\mathcal{K}(\mathbf{B})$.

σ will be called spectral measure of K.

Now we formulate the third representation of α-stable random cc sets. This series representation usually is called LePage or Lévy–LePage representation, but the name of Khinchine probably must be added, since he was the first to derive such representation , (see [14]). We recommend J. Rosinski paper [22] for different series representations of Lévy processes and especially for his historical remarks on this topic.

To formulate the series representation we need some more notation. Let (η_i) be independent identically distributed (iid) random variables with exponential distribution , that is, $P\{\eta_i > t\} = e^{-t},\ t \geq 0$. Set $\Gamma_j = \sum_{i=1}^{j} \eta_i,\ \ j \geq 1$. The sequence (Γ_j) defines the successive times of jumps of a standard Poisson process.

Let ζ be a positive real valued random variable such that $||\zeta||_\alpha^\alpha = E\zeta^\alpha < \infty$ and let (ζ_j) be independent copies of ζ. Throughout

$$c_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}.$$

Theorem 2.4 *Assume that* Y_α *is a strictly* α*-stable random convex compact set in* $\mathbf{B}$ *with* $0 < \alpha < 1$ *and corresponding spectral probability measure* σ*. Let* (ε_j) *be a sequence of independent random elements on* $\mathcal{K}_1$ *having distribution* σ*, and assume that the sequences* $(\Gamma_k), (\zeta_k)$ *and* (ε_k) *are independent. Then the series*

$$c_\alpha ||\zeta||_\alpha^{-1} \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \zeta_k \varepsilon_k, \tag{2.1}$$

converges almost surely in $\mathcal{K}(\mathbf{B})$ *and this series is distributed as* Y_α.

Each of these three representations is useful in particular problems connected with stable random elements. In the context of random cc sets the series representation is useful when one tries to define Lévy motion with values in $co\mathcal{K}(\mathbf{B})$. Here it is appropriate to mention that $\mathcal{K}(\mathbf{B})$ is not a linear space and the difference of cc sets A and B can be defined not for all A and B. But if there exists a set C for which $A + C = B$, then it is easy to see that it is unique. Therefore if one considers a process $X(t),\ t \in [0, 1]$ with values in $co\mathcal{K}(\mathbf{B})$, then the problem to define a process with independent increments is not a trivial one. In [6] we proposed the following two definitions.

Definition 2.5 Let $s < t$, $s, t \in [0, 1]$. The increment $X(t) - X(s)$ of a random cc set process, if it exists, is such a random cc set, that

$$X(s) + (X(t) - X(s)) = X(t) \quad a.s.$$

It is possible to formulate (see [11]) a criterion, expressed in terms of a support process (we shall introduce this notion later) and which ensures the possibility to define increments for a process with values in $\mathcal{K}(\mathbf{B})$. Now we define Lévy motion in $co\mathcal{K}(\mathbf{B})$.

Definition 2.6 Let $0 < \alpha < 1$. A cc set process $\{Y(t),\ t \in [0, 1]\}$ is called *α-stable cc set Lévy motion* if

i) $Y(0) = \{0\}$;
ii) the increments of the process $(Y(t), t \in [0, 1])$ are well defined and are independent: random compact convex sets $Y(t_2) - Y(t_1), \ldots, Y(t_n) - Y(t_{n-1})$ are independent for any $0 \leq t_1 < \cdots < t_n \leq 1$;
iii) if $0 \leq s < t \leq 1$ then $Y(t) - Y(s)$ has α-stable distribution and $Y(t) - Y(s) \stackrel{D}{=} (t-s)^{1/\alpha} Y(1)$.

It is possible to prove (see [6]) that the process

$$Y_\alpha(t) = c_\alpha \sum_{k=1}^{\infty} 1_{[0,t]}(U_k) \Gamma_k^{-1/\alpha} \varepsilon_k, \quad t \in [0, 1] \tag{2.2}$$

is α-stable cc set Lévy motion. Here $\{U_k\}$ are independent uniformly distributed random variables on $[0, 1]$, $\{\varepsilon_k\}$ are iid on $\mathcal{K}_1(\mathbf{B})$ with a distribution σ and $0 < \alpha < 1$. All sequences $(\Gamma_k), (U_k)$, and (ε_k) are assumed to be independent.

Having defined Lévy motion in $\mathcal{K}(\mathbf{B})$, the next step is to prove the invariance principle and this was done in [6].

An interesting (and most probably, difficult) problem is the rate of convergence in the series representation even on $\mathcal{K}(\mathbf{R}^2)$, not speaking about general case of $\mathcal{K}(\mathbf{B})$. Namely, we would like to know how close to stable random cc set are the partial sums of the series (2.1), that is, we would like to compare distributions of $\sum_{k=1}^{n} \Gamma_k^{-1\alpha/} \varepsilon_k$ and $\sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \varepsilon_k$ over some class of sets (here the terminology is a little bit ambiguous - we speak about distribution of a random cc set as a function on Borel σ-algebra of $\mathcal{K}(\mathbf{B})$). At present the optimal rates of convergence in series representation are obtained only in the case of one-dimensional or finite-dimensional random variables ε_k's (see [2], [3]).

The rates of convergence in series representation of stable laws are interesting for the following interpretation (it was not formulated explicitly in both papers [2], [3]). For the simplicity of formulation let us consider one-dimensional case. It is well known, that classical limit theorems and rates of convergence are quite different in case of the Gaussian limit law ($\alpha = 2$) and stable limit laws ($0 < \alpha < 2$). But if we consider the rates of convergence in series representation as the rates in special summation scheme, then these results resemble classical summation results in Gaussian case. In order to demonstrate this similarity better, we present the results in parallel two columns, on the left one presenting results in series convergence ($0 < \alpha < 2$), on the right one - classical results in CLT with Gaussian limit law ($\alpha = 2$). Let us denote

$$\Lambda_r = \Lambda_r(X) = \sup_{t \geq 0} t^r P\{|X| > t\}.$$

On the left column Δ_n stands for $\sup|P(S_n \in A) - G_\alpha(A)|$ and sup is taken over all Borel subsets A, while on the right column Δ_n stands for the usual $\sup_x |P(Z_n \le x) - \Phi(x)|$, where Φ stands for the standard normal law (in this column without loss of generality we assume that $EX_1 = 0$ and $EX_1^2 = 1$).

$S_n = \sum_{k=1}^{n} \Gamma_k^{-1/\alpha} X_k$	$Z_n = n^{-1/2} \sum_{k=1}^{n} X_k$
limit law G_α, if $E\|X_1\|^\alpha < \infty$	limit law Φ, if $EX_1^2 < \infty$
if $\Lambda_{\alpha+\delta} < \infty$	if $\Lambda_{2+\delta} < \infty$
then $\Delta_n \le Cn^{-\delta/\alpha}$	then $\Delta_n \le Cn^{-\delta/2}$
if $E\|X_1\|^{\alpha+\delta} < \infty$	if $E\|X_1\|^{2+\delta} < \infty$
then $\Delta_n = o(n^{-\delta/\alpha})$	then $\Delta_n = o(n^{-\delta/2})$
?	if $E\|X_1\|^k < \infty,\ k \ge 3$, then
?	there are asymptotic expansions

Question marks on the left column mean that at present we do not know how to construct asymptotic expansions in the series representation. There are some additional restrictions on δ, we refer to above cited papers [2], [3] for precise formulations, here our aim was to show similarity between convergence to Gaussian and stable laws: in both cases main parameters, determining convergence and convergence rates, are the moments of summands.

3 Structure of the boundary of countable stable zonotopes

In this section we consider the so-called countable stable zonotopes in Banach spaces $\mathbf{B}$, which were introduced in [7] in the case of $\mathbf{B} = \mathbf{R}^k$ and general case of Banach space $\mathbf{B}$ was considered in [6]. At first we shall formulate general theorem giving the relation between limit theorems in $\mathbf{B}$ and limit theorems for zonotopes in $\mathcal{K}(\mathbf{B})$. By a zonotope we call a set $\sum_{k=1}^{n}[0, x_k]$, where $x_k \in \mathbf{B},\ k = 1, \dots, n$. As usual, $[0, x] = \{y = ax,\ 0 \le a \le 1\}$, and summation is in the Minkowski sense. A countable zonotope is $\sum_{k=1}^{\infty}[0, x_k]$, if the series converges in $\mathcal{K}(\mathbf{B})$. If we take random end points of segments, then we get a random cc set

$$Z_n = \sum_{k=1}^{n} [0, \xi_k].$$

Suppose that $\xi_i,\ i \ge 1$ are iid random elements in $\mathbf{B}$ and let us denote

$$S_n = \sum_{k=1}^{n} \xi_k.$$

It is natural to expect that limit relations for Z_n and S_n imply each other. For $0 < \alpha < 1$ let us consider limit relations:

$$b_n^{-1} Z_n \xrightarrow{D} Y_\alpha, \tag{3.1}$$

$$b_n^{-1} S_n \xrightarrow{D} \eta_\alpha, \tag{3.2}$$

where Y_α is a α-stable compact convex set in $\mathbf{B}$ and η_α is a α-stable random element in $\mathbf{B}$.

Theorem 3.1 *Limit relations (3.1) and (3.2) are equivalent. Moreover, Y_α is a countable zonotope, i.e.*

$$Y_\alpha \stackrel{D}{=} \sum_{k=1}^{\infty}[0, \tilde{\varepsilon_k}],$$

with $\tilde{\varepsilon_k}$ being random vectors in $\mathbf{B}$.

Proof The implication (3.2) $\Rightarrow$ (3.1) was proved in [6], the implication (3.1) $\Rightarrow$ (3.2) seems to be a new fact, therefore we sketch its proof. Now we need a notion of the support process of a compact random set. Let $\mathbf{B}_1^* = \{f \in \mathbf{B}^* : ||f|| \le 1\}$.

Definition 3.2 The *support function* of a subset $A \subset \mathbf{B}$ is the function s_A defined on $\mathbf{B}_1^*$ by the equality

$$s_A(f) := \sup_{x \in A} f(x).$$

If A is compact, then $s_A(f) < \infty$ for all $f \in \mathbf{B}_1^*$, moreover (see [12])

$$\delta(A, B) = ||s_A - s_B||_\infty$$

for all $A, B \in co\mathbf{B}$, in particular $||A|| = ||s_A||_\infty$. Here $||s_A||_\infty := \sup_{f \in \mathbf{B}_1^*} |s_A(f)|$. Let us introduce the metric d^* on $\mathbf{B}_1^*$

$$d^*(f, g) = \sum 2^{-i}|f(x_i) - g(x_i)|, \quad f, g \in \mathbf{B}_1^*$$

where $x_1, x_2, \ldots$ is a fixed dense set in the unit sphere $\mathbf{S}_1 := \mathbf{S}_1(\mathbf{B}) = \{x \in \mathbf{B} : ||x|| = 1\}$. Then it is known that $T := (\mathbf{B}_1^*, d^*)$ is a metric compact. Let $C(T)$ stand for the usual separable Banach space of continuous functions on metric compact T with supremum norm. If K is a random cc set , then its support process s_K (we on purpose use "process" instead of "function" , since this is a function with values in a function space) is a random element with values in $C(T)$.

Relation (3.1) means that random cc set $[0, \xi_1]$ belongs to the domain of the attraction of some stable random cc set and we shall write this fact in short $[0, \xi_1] \in DA_\alpha(\mathcal{K}(\mathbf{B}))$. Similarly, the relation (3.2) will be denoted as $\xi_1 \in DA_\alpha(\mathbf{B})$.

Taking into account that every Banach space $\mathbf{B}$ is of type α-stable for $\alpha < 1$ we have the following result on the $DA_\alpha(\mathbf{B})$ (see, Theorem 7.11 and Corollary 6.20 in [1], where a reader can find also the definition of α-stable type space).

Theorem 3.3 *Let $\mathbf{B}$ be a separable Banach space, $\alpha < 1$, and σ - a finite measure on unit sphere $\mathbf{S}_1$. Let X be a $\mathbf{B}$-valued random element. The following conditions:*

i) $t^\alpha P(||X|| > t)$ is a slowly varying function,
ii) for every σ-continuity set $A \subset \mathbf{S}_1$,

$$\lim_{t \to \infty} \frac{P\{X/||X|| \in A, \; ||X|| > t\}}{P\{||X|| > t\}} = \frac{\sigma(A)}{\sigma(\mathbf{S}_1)},$$

are necessary and sufficient for $X \in DA_\alpha(\mathbf{B})$ with the α-stable limit law with spectral measure σ.

Now, having all these preparations done, the proof of the implication (3.1) $\Rightarrow$ (3.2) is easy. Using isometry map from $co\mathcal{K}(\mathbf{B})$ to $C(T)$, given by $A \to s_A$ (see [11]) and the fact that $[0,\xi_1] \in DA_\alpha(\mathcal{K}(\mathbf{B}))$ we get $s_{[0,\xi_1]} \in DA_\alpha(C(T))$. Since $C(T)$ is the separable Banach space, the condition $i)$ for $s_{[0,\xi_1]}$ implies that $t^\alpha P(\|X\| > t)$ is a slowly varying function, that is, we have $i)$ for ξ_1. Here we used the equalities $\|A\| = \|s_A\|_\infty$ and $\|[0,\xi_1]\| = \|\xi_1\|$.

The second condition $ii)$ for ξ_1 can be obtained from the relation (see [11])

$$\lim_{n\to\infty} \alpha t^\alpha P\Big\{\frac{[0,\xi_1]}{||[0,\xi_1]||} \in A, ||\xi_1|| \geq b_n\Big\} = \sigma(A),$$

where σ is a spectral measure of Y_α, that is, a measure on the unit ball $\mathcal{K}_1$. From this relation in a standard way we can derive $ii)$ for ξ_1 with $\tilde\sigma(C)$, where C is a Borel set on the unit sphere of $\mathbf{B}$ and $\tilde\sigma(C) = \sigma(A_C)$ with $A_C = \bigcup_{x\in C}[0,x]$. Here we use the evident relation

$$\Big\{\frac{[0,\xi_1]}{||[0,\xi_1]||} \in A_C\Big\} = \Big\{\frac{\xi_1}{||\xi_1||} \in C\Big\}.$$

Having both conditions of Theorem 3.3 for ξ_1, from this theorem we get $\xi_1 \in DA_\alpha(\mathbf{B})$, that is, relation (3.2). ☐

On one hand, the distribution of random countable zonotopes Y_α presents the most simple example of non-trivially diffuse measures on $co\mathcal{K}(\mathbf{B})$, on the other hand, the sets, on which these measures are concentrated, have not so simple structure. Even on the plane still there are unanswered questions about the support of these measures, therefore from now on we restrict ourselves to the case $\mathbf{B} = \mathbf{R}^2$.

Taking into account series representation, countable stable random zonotope can be written

$$Y_\alpha \stackrel{D}{=} \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha}[0,\varepsilon_k], \tag{3.3}$$

where , as usual, $\Gamma_k = \eta_1 + \cdots + \eta_k$ and $\varepsilon_k,\ k \geq 1$ is a sequence of iid random vectors on unit circle on the plane. Let σ stand for the distribution of ε_1. Let us assume that σ is a continuous distribution. One can think of the random set (3.3) as a polygon with countably many sides, the length of k-th side is $\Gamma_k^{-1/\alpha}$ and the direction of this side is given by ε_k (to be more precise, for each k there are two parallel sides of the same length). The structure of the boundary of such a set is rather complicated, since between any two sides of such zonotope there is again infinitely many sides. So if one takes the set of extremal points of the boundary, it resembles Cantor set (from the boundary of the set from each side we remove interior points of this side, leaving only end points). The question about Hausdorff dimension of this random set of extremal points on the plane remains as open problem. Although it was not stated in [6], all three co-authors of this paper conjecture that, with probability one, the Hausdorff dimension of this set is α.

As a first step in solving this problem in [4] there was considered a non-random Cantor set on real line, constructed by means of a sequence $\lambda_k = k^{-p},\ k \geq 1$ with $p = 1/\alpha > 1$. Such sequence was chosen for the reason that the lengths of sides of a stable countable random zonotope on plane are $\Gamma_k^{-1/\alpha}, k \geq 1$ and $E\Gamma_k^{-1/\alpha}$ is asymptotically equivalent to $k^{-1/\alpha}$ (see formula (5.5) in [19]). Therefore the sum $2\sum_{k=1}^{\infty} \lambda_k$ well approximates the mean value of the length of the boundary of countable random stable zonotope. In order to formulate the main results from [4] we shall repeat the construction of a Cantor set obtained by a sequence. Let $\lambda = \{\lambda_k\}_{k\in\mathbf{N}}$ be a sequence of non-negative numbers, such that $\sum_k \lambda_k = K < +\infty$. We can associate to any sequence λ a Cantor set C_λ in the following way. We start with a closed interval $I_0 = [0, K]$ of length K. In the first step we remove from I_0 an open interval of length λ_1, getting two intervals of step 1, namely I_0^1 and I_1^1, and a gap of length λ_1. The position of this gap will be determined if we define the length of I_0^1, and this will be done when we explain our further steps. From interval I_0^1 we remove λ_2, from I_1^1 we remove λ_3, and so on. The length of I_0^1 is the sum of the lengths of all intervals which will be removed from this interval by the construction, and it is easy to see that

$$|I_0^1| = \sum_{n=1}^{\infty} \sum_{j=0}^{2^{n-1}-1} \lambda_{2^n+j}, \qquad |I_1^1| = \sum_{n=1}^{\infty} \sum_{j=2^{n-1}}^{2^n-1} \lambda_{2^n+j}.$$

In what follows, we use the notation $\mathrm{diam}(I) = |I|$ for any interval.

In step k of this construction we have 2^k intervals, I_0^k, I_1^k, $\ldots$, $I_{2^k-1}^k$, and for each ℓ, $0 \le \ell \le 2^k - 1$, we remove from I_ℓ^k an open interval of length $\lambda_{2^k+\ell}$, forming two intervals $I_{2\ell}^{k+1}$ and $I_{2\ell+1}^{k+1}$ and a gap of the length $\lambda_{2^k+\ell}$, so the following relation holds

$$|I_\ell^k| = |I_{2\ell}^{k+1}| + \lambda_{2^k+\ell} + |I_{2\ell+1}^{k+1}|.$$

It is not difficult to verify that for all $k = 0, 1, \ldots$ and $\ell = 0, 1, \ldots, 2^k - 1$, we have

$$|I_\ell^k| = \sum_{n=k}^{\infty} \sum_{j=\ell 2^{n-k}}^{(\ell+1)2^{n-k}-1} \lambda_{2^n+j}.$$

Let us denote

$$C_\lambda^k = \bigcup_{i=0}^{2^k-1} I_i^k,$$

then

$$C_\lambda = \bigcap_{k=0}^{\infty} C_\lambda^k$$

is the constructed set, and we shall call it Cantor set associated to the sequence λ.

As it was noted in [4], this construction is quite general, since in fact any Cantor set (of zero Lebesgue measure) can be obtained in this way for an appropriate choice of the sequence. Also it is clear from the construction of the Cantor set associated to a given sequence, that the specific order in which the gaps appear in the sequence determines the resulting Cantor set. Let $\sigma : \mathbf{N} \to \mathbf{N}$ be a bijective map, we say that the sequence $\{\lambda_{\sigma(k)}\}_{k\in\mathbf{N}}$ is a *rearrangement* of λ and denote it by $\sigma(\lambda)$. In general, a rearrangement of the original sequence yields a different Cantor set. Finally, before the formulation of the results, we recall the definitions of Hausdorff measure and dimension (see, for example, [10]).

Definition 3.4 Let $A \subset \mathbf{R}$ be a Borel-measurable set and $\alpha > 0$. For $\delta > 0$ let

$$\mathcal{H}_\delta^\alpha(A) = \inf\Big\{ \sum (\mathrm{diam}(E_i))^\alpha \ : \ E_i \text{ open}, \ \cup E_i \supset A, \ \mathrm{diam}(E_i) \le \delta \Big\}.$$

Then, the *α-dimensional Hausdorff measure of A*, $\mathcal{H}^\alpha(A)$, is defined as

$$\mathcal{H}^\alpha(A) = \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(A),$$

and the Hausdorff dimension of A is,

$$\dim_H(A) = \sup\{\alpha : \mathcal{H}^\alpha(A) > 0\}.$$

If for some choice $\alpha = s$, $0 < \mathcal{H}^s(A) < \infty$, then A is called an s-set. As in [4] we will use only the Hausdorff dimension therefore from now on we omit the subscript H.

Now the main results from [4] can be stated as follows

Theorem 3.5 *Let $\lambda = \{\lambda_k\}_{k\in\mathbf{N}}$ be defined by $\lambda_k = (1/k)^p$, $p > 1$. Then C_λ is a $1/p$-set, precisely,*

$$\frac{1}{8}\left(\frac{2^p}{2^p - 2}\right)^{1/p} \le \mathcal{H}^{\frac{1}{p}}(C_\lambda) \le 2\left(\frac{1}{p-1}\right)^{1/p}$$

and

$$dim\ C_\lambda = \frac{1}{p}.$$

If $\sigma(\lambda)$ is any rearrangement of the sequence $\lambda = k^{-p}$, $k \in \mathbf{N}$, with $p > 1$, then

$$0 \le dim\, C_{\sigma(\lambda)} \le \frac{1}{p}.$$

Furthermore, for each $0 < s \le 1/p$, there exists a rearrangement $\sigma_s(\lambda)$ such that $C_{\sigma_s(\lambda)}$ is an s-set.

The next step is to allow for the lengths of intervals to be random, having in mind that the length of sides of a stable zonotope is Γ_k^{-p}, and this was done in a recent paper [21]. We need one more notation. If $\lambda_k = \mu_k^{-p}$ where $\mu = \{\mu_k\}$, $\mu_k > 0$, and $p > 0$ are such that λ is summable, then C_λ will be denoted by $C_{\mu,p}$.

Now suppose that we have a sequence of real non-negative random variables $X_n, n \ge 1$, defined on some probability space $(\Omega, \mathcal{F}, P)$. If series $\sum_{k=1}^\infty X_k^{-p}$ converges a.s. for some $p > 0$ then $C_{X,p}$ is well defined random Cantor set associated to the sequence X_k^{-p}. The main result from [21] can be formulated as follows.

Theorem 3.6 *Let a sequence $X_n, n \ge 1$ of random variables introduced above satisfy the following condition*

$$\sum_{k=1}^\infty P\{|X_k - k| > h(k)\} < \infty \tag{3.4}$$

with some function $h(x)$, which is monotone, $0 < h(x) < x$, $h(x)x^{-1} \downarrow 0$, as $x \to \infty$ and

$$\sum_{j=1}^\infty h(2^j)2^{-j} < \infty.$$

Then for any $p > 1$

$$P\{dim_H(C_{X,p}) = p^{-1}\} = 1$$

and

$$P\{0 < \mathcal{H}^{1/p}(C_{X,p}) < \infty\} = 1,$$

It is appropriate to mention that no assumption about independence of random variables X_k is made, the only assumption is (3.4) which means that as $k \to \infty$ random variables X_k must become more and more concentrated around k. From this general result one can derive results in the case where X_k is a sum of iid random variables, in particular $X_k = \Gamma_k$. Let $\Gamma = \{\Gamma_k\}$, with $\Gamma_k = \sum_{j=1}^k \eta_j$, where $\eta_i, i \geq 1$ are i.i.d. random variables with standard exponential distribution, i.e., $P\{\eta_1 > x\} = \exp(-x),\ x > 0$. It is not difficult to see that $\{\Gamma_k\}$ satisfies conditions of Theorem 3.6 and we have

Corollary 3.7 *For any $p > 1$*

$$P\{dim_H(C_{\Gamma,p}) = p^{-1},\ \ 0 < \mathcal{H}^{1/p}(C_{\Gamma,p}) < \infty\} = 1.$$

All above formulated results concern sets on the line, so they only indirectly support our conjecture about the Hausdorff dimension of the set of extremal points of a countable stable random zonotope on the plane. And at present there is only a small hope to investigate the boundary of such zonotopes in higher dimensions.

4 A new estimator of the tail index

In this section we will formulate some results from [20] (see also [6], [5]) where some new estimator of tail index was proposed. The idea of this new estimator came when considering random stable zonotope. If we take random stable zonotope on plane, it's the largest side has the length $\Gamma_1^{-1/\alpha}$ and the second largest is $\Gamma_2^{-1/\alpha}$. Simple calculations show that

$$E\Big(\frac{\Gamma_2^{-1/\alpha}}{\Gamma_1^{-1/\alpha}}\Big) = E\Big(\frac{\eta_1}{\eta_1+\eta_2}\Big)^{1/\alpha} = \frac{\alpha}{\alpha+1}.$$

Since the quantity $\alpha/(\alpha+1)$ is a simple function of α, so estimating this quantity we will get an estimate for tail index α, too.

In [6] and [5] this idea was applied to estimate parameters - index of stability and spectral measure - of multivariate stable laws, later it was noticed that it works in general problem of estimating the tail index for heavy-tailed distributions. We shall formulate main result from [20].

Let us assume that we have a sample $X_1, \ldots, X_N$ from distribution F, which satisfies the second order asymptotic relation (as $x \to \infty$)

$$1 - F(x) = C_1 x^{-\alpha} + C_2 x^{-\beta} + o(x^{-\beta}), \tag{4.1}$$

with some parameters $0 < \alpha < \beta \leq \infty$. The case $\beta = \infty$ corresponds to Pareto distribution, $\beta = 2\alpha$ – to stable distribution with exponent $0 < \alpha < 2$. Here may be it is appropriate to mention, that in many papers, devoted to tail index estimation, the second order asymptotic relation is used in different form with different parameters γ, ρ (see, for example, [13], [9]) but both forms are equivalent and there is a simple relation between (α, β) and (γ, ρ).

We divide the sample into n groups $V_1, \ldots, V_n$, each group containing m random variables, that is, we assume that $N = n \cdot m$. (In practice we choose m and then $n = [N/m]$, where $[a]$ stands for the integer part of a number $a > 0$.) Let

$$M_{ni}^{(1)} = \max\{X_j : X_j \in V_i\}$$

and let $M_{ni}^{(2)}$ denote the second largest element in the same group V_i. Now let us denote

$$\kappa_{ni} = \frac{M_{ni}^{(2)}}{M_{ni}^{(1)}}, \quad S_n = \sum_{i=0}^{n} \kappa_{ni}, \quad Z_n = n^{-1} S_n.$$

To simplify the formulations, let us denote

$$p = \frac{\alpha}{1+\alpha}, \quad \hat{p} = n^{-1} S_n.$$

The following result was proved in [20].

Theorem 4.1 *Let us suppose that* F *satisfies (2) with* $\alpha < \beta \leq \infty$. *If we choose*

$$n = \varepsilon_N N^{2\zeta/(1+2\zeta)}, \; m = \varepsilon_N^{-1} N^{1/(1+2\zeta)},$$

where $\varepsilon_N \to 0$, *as* $N \to \infty$ *and* $\zeta = (\beta - \alpha)/\alpha$, *then*

$$\sqrt{n}(\hat{p} - p) \xrightarrow{\mathrm{D}}_{N\to\infty} N(0, \sigma^2),$$

where $\sigma^2 = \lim_{n\to\infty} \sigma_n^2 = \alpha((\alpha+1)^2(\alpha+2))^{-1}$.

Since this topic of tail estimation is rather far from convex compact random sets, we shall not go into details, sending the interested reader to the papers [6], [5], and especially to [20], where some simulation results are provided. Our aim was to demonstrate, that sometimes rather abstract mathematical results give rise to practical applications.

5 Concluding remarks

1. There are some interesting open problems, the first one, of course, is the Hausdorff dimension of the set of extremal points of the boundary of a countable stable zonotope on the plain, more precisely, the lower bound for this dimension, since the upper bound (easy part) we essentially had when we were writing the paper [6]. Most probably, the problem in higher dimensions is more difficult, but we conjecture that the Hausdorff dimension of α-stable random zonotope is α in all finite-dimensional spaces, provided that the spectral measure is non-atomic . Surprisingly, it seems that the structure of the boundary of countable stable zonotope in infinite–dimensional case is more simple. For the simplicity of explanation (the same picture must be in all Banach spaces) let us assume that $\mathbf{B} = l_2$ and a distribution of ε_1 is non-trivially diffuse and infinite dimensional in the sense that $P(\varepsilon_1 \in L) = 0$ for any finite-dimensional subspace $L \subset l_2$. Then the countable stable zonotope

$$Y_\alpha = \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} [0, \varepsilon_k],$$

is infinite-dimensional parallelepiped, since if we take the direction ε_1 and denote by L_1 the subspace, generated by ε_1, then with probability 1 ε_2 will be in $l_2 \backslash L_1$. Similarly, with probability 1 ε_3 will be in $l_2 \backslash L_2$, where L_2 is a subspace generated by ε_1 and ε_2, and so on. Thus, there will be no such effect as on plane, when all sequence of sides of a zonotope are on the same plane. But even in such comparatively simple situation it is not difficult to see that the set of extremal points of the boundary of such infinite-dimensional parallelepiped is of continuum cardinality, and the question about Hausdorff dimension of such set is non-trivial.

2. Singularity of distributions of countable stable zonotopes was investigated in [6] in the finite-dimensional case, the question remains open in infinite-dimensional space $\mathbf{B}$.

3. It should be interesting to study more deeply relations between random zonotopes considered in the paper, the random zonoids, used by Koshevoy and Mosler to generalize Lorenz curves to the multidimensional setting (see [15]) and convexification.

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