Poisson approximations for the Ising model

D. Coupier

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MAP5, CNRS UMR 8145, Université René Descartes, Paris

E-Mail address : coupier@math-info.univ-paris5.fr

Mail address : MAP5, CNRS UMR 8145,
Université Paris 5, 45 rue des Saints-Pères
75270 PARIS Cedex 06, FRANCE.

Telephone : 33 1 44 55 35 29
Fax : 33 1 44 55 35 35

Abstract

A $d$-dimensional Ising model on a lattice torus is considered. As the size $n$ of the lattice tends to infinity, a Poisson approximation is given for the distribution of the number of copies in the lattice of any given local configuration, provided the surface potential $a = a(n)$ tends to $-\infty$ and the pair potential $b$ remains fixed. Using the Stein-Chen method, a bound is given for the total variation error in the ferromagnetic case ($b$ positive).

Key words : Poisson approximation, Ising model, ferromagnetic interaction, Stein-Chen method.

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1 Introduction

The following situation, called “the law of small numbers”, is very classical in probability theory. Suppose \( \{ I_\lambda \}_{\lambda \in \Lambda} \) is a finite family of indicator random variables, with the properties that the probabilities \( \mathbb{P}(I_\lambda = 1) \) are small and that there is not too much dependence between the \( I_\lambda \)'s. Then, it is reasonable to expect the distribution of \( \sum_{\lambda \in \Lambda} I_\lambda \) to be approximately Poisson. In the theory of random graphs, inaugurated by Erdös and Rényi [6], such results are frequent (see [3] or [12] for a general reference). The \( I_\lambda \) can, for instance, indicate the places in the random graph where a given subgraph appear. Some analogous results hold for random colorings of a lattice graph in dimension 2, corresponding to the context of random images [4]. In both cases, the models are built on a large number of independent random variables: the edges of a random graph or the pixels of a random image. In this article, we shall study Poisson approximations for sums of indicators defined from a large number of dependent random variables, namely the spins of an Ising model.

Let us consider a lattice graph in dimension \( d \geq 1 \), with periodic boundary conditions (lattice torus). The vertex set is \( V_n = \{0, \ldots, n-1\}^d \). The integer \( n \) will be called the size of the lattice. The edge set, denoted by \( E_n \), will be specified by defining the set of neighbors \( \mathcal{V}(x) \) of a given vertex \( x \):

\[
\mathcal{V}(x) = \{ y \neq x \in V_n, \| y - x \|_p \leq \rho \},
\]

where the subtractions is taken componentwise modulo \( n \), \( \| \cdot \|_p \) stands for the \( L_p \) norm in \( \mathbb{R}^d \) \((1 \leq p \leq \infty)\), and \( \rho \) is a fixed parameter. For instance, the square lattice is obtained for \( p = \rho = 1 \). Replacing the \( L_1 \) norm by the \( L_\infty \) norm adds the diagonals. From now on, all operations on vertices will be understood modulo \( n \).

A configuration is a mapping from the vertex set \( V_n \) to the state space \( W = \{-1, +1\} \). Their set is denoted by \( \mathcal{X}_n = W^{V_n} \) and called the configuration set. Here, we shall deal with one of the simplest and most widely studied parametric families of random field distributions: the Ising model (see e.g. [10, 11]).

**Definition 1.1** Let \( G_n = (V_n, E_n) \) be an undirected graph structure with finite vertex set \( V_n \) and edge set \( E_n \). Let \( a \) and \( b \) be two reals. The Ising model with parameters \( a \) and \( b \) is the probability measure \( \mu_{a,b} \) on \( \mathcal{X}_n = \{-1, +1\}^{V_n} \) defined by:

\[
\forall \sigma \in \mathcal{X}_n, \quad \mu_{a,b}(\sigma) = \frac{1}{Z_{a,b}} \exp \left( a \sum_{x \in V_n} \sigma(x) + b \sum_{(x,y) \in E_n} \sigma(x)\sigma(y) \right),
\]

where the normalizing constant \( Z_{a,b} \) is such that \( \sum_{\sigma \in \mathcal{X}_n} \mu_{a,b}(\sigma) = 1 \).

Following the definition of [11] p. 2, the measure \( \mu_{a,b} \) defined above is a Gibbs measure associated to potentials \( a \) and \( b \). Expectations relative to \( \mu_{a,b} \) will be denoted by \( \mathbb{E}_{a,b} \).

In the classical presentation of statistical physics, the elements of \( \mathcal{X}_n \) are spin configurations; each vertex of \( V_n \) is an atom whose spin is either positive or negative. Here, we shall
simply talk about positive or negative vertices instead of positive or negative spins. The parameters $a$ and $b$ are respectively the surface potential and the pair potential. The model remaining unchanged by swapping positive and negative vertices and replacing $a$ by $-a$, we choose to study only negative values of the surface potential $a$. Various “laws of small numbers” have been already proved for the Ising model. Fernández et al. [7] have established the asymptotic Poisson distribution of contours in the nearest-neighbor Ising model at low temperature (i.e. $b$ large enough) and zero surface potential. Barbour and Greenwood [1] have applied the Stein-Chen method to a class of Markov random fields; the bounds that they obtained for the Ising model are not quite explicit. Using the same method, Ganesh et al. [9] have studied the Ising model for positive values of $b$. Provided the surface potential $a$ tends to infinity, they proved that the distribution of the number of negative vertices is approximately Poisson.

Our goal is to generalize the convergence in distribution given by [9] to any value of the pair potential $b$ and to objects more elaborated than a single vertex.

We are interested in the occurrences in the graph $G_n$ of a fixed local configuration $\eta$ (see Section 2 for a precise definition and Figure 1 for an example). Such a configuration is called “local” in the sense that the vertex set on which it is defined is fixed and does not depend on $n$.

As the size $n$ of the lattice tends to infinity, the potential $a = a(n)$ is allowed to depend on $n$ whereas the potential $b$ remains fixed. The case where $a(n)$ tends to $-\infty$ corresponds to rare positive vertices among a majority of negative ones. In this context, the local configuration $\eta$ may occur or not in the graph, depending on its number of positive vertices $k(\eta)$. See Proposition 4.2 of [5] for a precise description of this phenomenon. As a consequence, throughout this paper, the surface potential $a(n)$ will satisfy the identity

$$2k(\eta)a(n) = \log \left( \frac{e}{n^d} \right),$$

where $c$ is a positive constant. The Poisson approximation for the number of occurrences of $\eta$ will depend on $k(\eta)$ through (3), but also on the geometry of $\eta$ through its perimeter, denoted by $\gamma(\eta)$. Our main result (Theorem 3.1) states that the number of occurrences $X_n(\eta)$ of the local configuration $\eta$ in the lattice converges in distribution to the Poisson distribution with parameter $ce^{-2b\gamma(\eta)}$. The proof is based on the moment method (see [3] p. 25 or Lemma 3.2 below), and requires estimates based on the local energy of $\eta$ (Definition 2.1). The result of Ganesh et al. [9] is obtained as a particular case when the pair potential $b$ is positive and $\eta$ is a single positive vertex: $k(\eta) = 1$ and $\gamma(\eta) = 4$.

The Stein-Chen method (see [2] for a very complete reference) makes it possible to obtain good estimates on the accuracy of the Poisson approximation. If the pair potential $b$ is positive, the Gibbs measure $\mu_{a,b}$ defined in (2) satisfies the FKG inequality [8]. In this case, thanks to the Stein-Chen method, bounds on the first two moments of $X_n(\eta)$ lead to Theorem 4.1: the total variation distance between the probability distribution of the random variable $X_n(\eta)$ and the Poisson distribution with parameter $ce^{-2b\gamma(\eta)}$ is of order $n^{-d/k(\eta)}$.

The paper is organized as follows. The notion of local configuration $\eta$ and its number
$X_n(\eta)$ of occurrences in the lattice are defined in Section 2. The integers $k(\eta)$ and $\gamma(\eta)$ are also introduced. They naturally occur in the expression of the local energy of $\eta$ (Definition 2.1). Under the hypothesis (3), the conditional probability of $\eta$ to occur in the graph is bounded in Lemma 2.2. Then, the limit of the expectation of $X_n(\eta)$ is deduced (formula (9)). Section 3 is devoted to the proof of Theorem 3.1. A sum $\overline{X}_n(\eta)$ of increasing (in the sense of the FKG inequality) indicators is introduced in Section 4. The Stein-Chen method is applied to this random variable in Lemma 4.4. Finally, precise estimates on the first two moments of $\overline{X}_n(\eta)$ (Lemmas 4.2 and 4.5) conclude the proof of Theorem 4.1.

2 Conditionally probable of a local configuration

Let us start with some notations and definitions. Given $\sigma \in \mathcal{X}_n = W^{V_n}$ and $V \subset V_n$, we denote by $\sigma_V$ the natural projection of $\sigma$ over $W^V$. If $U$ and $V$ are two disjoint subsets of $V_n$ then $\sigma_U \sigma'_V$ is the configuration on $U \cup V$ which is equal to $\sigma$ on $U$ and $\sigma'$ on $V$. Let us denote by $\delta V$ the neighborhood of $V$ (corresponding to (1)):

$$\delta V = \{ y \in V_n \setminus V, \exists x \in V, \{x, y\} \in E_n \},$$

and by $\overline{V}$ the union of the two disjoint sets $V$ and $\delta V$. Moreover, $|V|$ denotes the cardinality of $V$ and $\mathcal{F}(V)$ the $\sigma$-algebra generated by the configurations of $W^V$. In what follows, $\sigma, \sigma'$ will denote elements of $\mathcal{X}_n$ and $\eta, \eta'$ local configurations.

As usual, the graph distance $\text{dist}$ is defined as the minimal length of a path between two vertices. We shall denote by $B(x, r)$ the ball of center $x$ and radius $r$:

$$B(x, r) = \{ y \in V_n; \text{dist}(x, y) \leq r \}.$$

In the case of balls, $\overline{B(x, r)} = B(x, r + 1)$. In order to avoid unpleasant situations, like self-overlapping balls, we will always assume that $n > 2\rho r$. If $n$ and $n'$ are both larger than $2\rho r$, the balls $B(x, r)$ in $G_n$ and $G_{n'}$ are isomorphic. Two properties of the balls $B(x, r)$ will be crucial in what follows. The first one is that two balls with the same radius are translated of each other:

$$B(x + y, r) = y + B(x, r).$$

The second one is that for $n > 2\rho r$, the cardinality of $B(x, r)$ depends only on $r$ and not on $x$ nor $n$; it will be denoted by $\beta(r)$; so does the number of edges $\{y, z\} \in E_n$ with $y, z \in B(x, r)$ which will be denoted by $\alpha(r)$.

Let $r$ be a positive real, and consider a fixed ball with radius $r$, say $B(0, r)$. We denote by $D_r = W^{B(0, r)}$ the set of configurations on that ball. They will be called local configurations of radius $r$. A local configuration $\eta \in D_r$ is determined by its subset $V_+(\eta) \subset B(0, r)$ of positive vertices:

$$V_+(\eta) = \{ x \in B(0, r), \eta(x) = +1 \}.$$

The cardinality of this set will be denoted by $k(\eta)$ and its complementary set in $B(0, r)$, i.e. the set of negative vertices of $\eta$, by $V_-(\eta)$. Of course, there exists only a finite number of local configurations of radius $r$ (precisely $2^{\beta(r)}$).
Figure 1: A local configuration $\eta$ with $k(\eta) = |V_+(\eta)| = 10$ positive vertices, in dimension $d = 2$ and on a ball of radius $r = 3$ (with $\rho = 1$ and relative to $\| \cdot \|_{\infty}$).

Let $\eta \in D_r$. For each vertex $x \in V_n$, denote by $\eta_x$ the translation of $\eta$ onto the ball $B(x, r)$ (up to periodic boundary conditions):

$$\forall y \in V_n, \text{ dist}(0, y) \leq r \implies \eta_x(x + y) = \eta(y).$$

Let us denote by $I^n_x$ the indicator function defined on $\mathcal{X}_n$ as follows: $I^n_x(\sigma)$ is 1 if the restriction of the configuration $\sigma \in \mathcal{X}_n$ to the ball $B(x, r)$ is $\eta_x$ and 0 otherwise. Finally, let us define the random variable $X_n(\eta)$ which counts the number of copies of the local configuration $\eta$ in $G_n$:

$$X_n(\eta) = \sum_{x \in V_n} I^n_x.$$

Due to periodicity, this sum bears over $n^d$ indicator functions $I^n_x$, which have the same distribution.

In order to control the random variable $X_n(\eta)$, we describe its “local behavior” by introducing the local energy of $\eta$. Let us start with the following definition.

**Definition 2.1** Let $x \in V_n$ and $\sigma \in W^{B(x, r+1)}$. The local energy $H^{B(x, r)}(\sigma)$ of the configuration $\sigma$ on the ball $B(x, r)$ is defined by:

$$H^{B(x, r)}(\sigma) = a(n) \sum_{y \in B(x, r)} \sigma(y) + b \sum_{\{y, z\} \in E_n} \sigma(y)\sigma(z).$$

Let us fix a vertex $x$ and denote merely by $B$ the ball $B(x, r)$. For any local configuration $\eta \in D_r$ and for any $\sigma \in W^{B}$, the local energy $H^B(\eta_x\sigma)$ on $B$ of the configuration which is equal to $\eta_x$ on $B$ and $\sigma$ on $\partial B$ can be expressed as:

$$H^B(\eta_x\sigma) = a(n)(2k(\eta) - \beta(r)) + b \left( \sum_{\{y, z\} \in E_n} \eta_x(y)\eta_x(z) + \sum_{\{y, z\} \in E_n} \eta_x(y)\sigma(z) \right).$$
Actually, this notion of local energy allows us to explicitly write the conditional probability \( \mu_{a,b}(I_x^n = 1|\sigma), \sigma \in \mathcal{W}^\delta \):

\[
\mu_{a,b}(I_x^n = 1|\sigma) = \frac{e^{H^B(\eta,\sigma)}}{\sum_{\eta' \in \mathcal{D}_x} e^{H^B(\eta',\sigma)}}.
\]

(4)

As we shall see in Lemma 2.2, bounding the above conditional probability is central in our study.

An easy way to connect the number of copies of the local configuration \( \eta \) to its local energy consists in writing, for any given vertex \( x \):

\[
\mathbb{E}_{a,b}[X_n(\eta)] = \mathbb{E}_{a,b}[n^d \mu_{a,b}(I_x^n = 1|\mathcal{F}(\delta B))] .
\]

Here, \( \mu_{a,b}(I_x^n = 1|\mathcal{F}(\delta B)) \) represents a \( \mathcal{F}(\delta B) \)-measurable random variable and, for \( \sigma \in \mathcal{W}^\delta \), \( \mu_{a,b}(I_x^n = 1|\mathcal{F}(\delta B))(\sigma) = \mu_{a,b}(I_x^n = 1|\sigma) \) a conditional probability. Remark that the set \( \delta B \) has bounded cardinality (not depending on \( n \)). Then, from a convergence result on the random variable \( n^d \mu_{a,b}(I_x^n = 1|\mathcal{F}(\delta B)) \) it will be easy to obtain a similar result for its expectation, i.e. for \( \mathbb{E}_{a,b}[X_n(\eta)] \).

We will now give the reason for the hypothesis (3), that links the surface potential \( a(n) \) to the number of positive vertices of the local configuration \( \eta \). From now on, assume that \( \eta \) has at least one positive vertex; \( k(\eta) \geq 1 \). The event \( X_n(\eta) > 0 \) corresponds to the appearance of \( \eta \) in the graph \( G_n \). In [5] Proposition 4.2, it has been proved that:

\[
\begin{align*}
\text{if } & \lim_{n \to \infty} e^{2a(n)k(\eta)} n^d = 0 \text{ then } \lim_{n \to \infty} \mu_{a,b}(X_n(\eta) > 0) = 0 \text{ and } \lim_{n \to \infty} \mathbb{E}_{a,b}[X_n(\eta)] = 0 ; \\
\text{if } & \lim_{n \to \infty} e^{2a(n)k(\eta)} n^d = +\infty \text{ then } \lim_{n \to \infty} \mu_{a,b}(X_n(\eta) > 0) = 1 \text{ and } \lim_{n \to \infty} \mathbb{E}_{a,b}[X_n(\eta)] = +\infty .
\end{align*}
\]

(5)

(6)

Using the vocabulary of the random graph theory, these two statements essentially mean that the quantity \( n^{-d/k(\eta)} \) is the \textit{threshold function} (for \( e^{2a(n)} \)) of the property \( X_n(\eta) > 0 \). It does not depend on the radius \( r \) of the ball on which the local configuration \( \eta \) is defined: \( r \) is just a phantom parameter which serves only to ensure that \( \eta \) is a local configuration. Actually, the function \( n^{-d/k(\eta)} \) only depends on the number of positive vertices of \( \eta \). Roughly speaking, if \( e^{2a(n)} \) is small compared to \( n^{-d/k(\eta)} \), then asymptotically, there is no copy of \( \eta \) in \( G_n \). If \( e^{2a(n)} \) is large compared to \( n^{-d/k(\eta)} \), then at least one copy of \( \eta \) can be found in the graph, with probability tending to 1.

For the rest of this article, (3) is satisfied, i.e.

\[
2k(\eta)a(n) = \log \left( \frac{c}{n^2} \right) ,
\]

for some positive constant \( c \). We shall impose a boundary condition on the local configuration \( \eta \) that will simplify the proofs, without actually restricting generality. If \( e^{2a(n)} \) is of order \( n^{-d/k(\eta)} \) (hypothesis (3)), then (5) implies that asymptotically there will be no local
configurations with (strictly) more than $k(\eta)$ positive vertices in the lattice. So, if $\eta_x$ occurs on the ball $B(x,r)$ then vertices of $B(x,R) \setminus B(x,r)$, $R \geq r$, have negative vertices with probability tending to 1. The subset $V_-(\eta)$ of negative vertices of $\eta$ has no influence: more than the local configuration $\eta$, it is actually its subset of positive vertices $V_+(\eta)$ that we are really interested in. Consequently, replacing $r$ with $r+1$ and without loss of generality, we can assume that vertices of the reference ball $B(0,r)$ which are at distance $r$ from the center 0, all belong to $V_-(\eta)$ (as in Figure 1).

The geometry (in the sense of the graph structure) of the set $V_+(\eta)$ of positive vertices of $\eta$ will play a role. Precisely, let us define the perimeter $\gamma(\eta)$ of the local configuration $\eta$ by the formula:

$$\gamma(\eta) = |\{(x,y) \in V_+(\eta) \times V_-(\eta) \ , \ \{x,y\} \in E_n\}| .$$

The perimeter $\gamma(\eta)$ represents the number of pairs of neighboring vertices $x$ and $y$ of $B(0,r)$ having opposite spins under $\eta$. It is easy to see that the perimeter of a local configuration is always an even integer. For instance, that of Figure 1 is equal to 30.

The following lemma will play an essential role in the proofs of Theorem 3.1 and 4.1: it gives a uniform bound for the random variable $n^d\mu_{a,b}(I_x^n = 1|\mathcal{F}(\delta B(x,r)))$.

**Lemma 2.2** There exists a constant $M = M(b,r) > 0$ such that $\forall n, \forall x \in V_n, \forall \sigma \in X_n$,

$$ce^{-2\gamma(\eta)} \left(1 - Me^{2a(n)}\right) \leq n^d\mu_{a,b}(I_x^n = 1|\sigma_{\delta B(x,r)}) \leq ce^{-2\gamma(\eta)} ,$$

and

$$ce^{-2\gamma(\eta)} \left(1 - Me^{2a(n)}\right) \leq E_{a,b}[X_n(\eta)] \leq ce^{-2\gamma(\eta)} .$$

Since the quantity $e^{2a(n)}$ tends to 0 as $n$ tends to infinity, the inequalities (7) and (8) yield the two following limits. For any vertex $x$ and any configuration $\sigma$,

$$\lim_{n \to +\infty} n^d\mu_{a,b}(I_x^n = 1|\sigma_{\delta B(x,r)}) = \lim_{n \to +\infty} E_{a,b}[X_n(\eta)] = ce^{-2\gamma(\eta)} .$$

**Proof:** Let $x$ be a vertex of $V_n$ and denote by $B$ the ball $B(x,r)$. Since the expectations of the variables $X_n(\eta)$ and $n^d\mu_{a,b}(I_x^n = 1|\mathcal{F}(\delta B))$ are equal, (8) is an immediate consequence of (7). So, let us prove this relation.

Let us start with inserting the perimeter $\gamma(\eta)$ in the expression of the local energy of $\eta$. Assume that $\eta_x$ occurs on $B$. Then, there are $\gamma(\eta)$ edges $\{y,z\} \in E_n$ with $y, z \in B$ satisfying $\eta_x(y)\eta_x(z) = -1$ and $\alpha(r) - \gamma(\eta)$ ones satisfying $\eta_x(y)\eta_x(z) = 1$. Hence, for all $\sigma \in X_n$, the local energy $H^B(\eta_x,\sigma_{\delta B})$ can be expressed as:

$$H^B(\eta_x,\sigma_{\delta B}) = a(n)(2k(\eta) - \beta(r)) + b \left(\alpha(r) - 2\gamma(\eta) + \sum_{\{y,z\} \in E_n \atop y \in B, z \in \delta B} (-1)\sigma_{\delta B}(z)\right) .$$

The factor $(-1)$ in the last sum comes from the fact that, by hypothesis, vertices at distance $r$ from $x$ are all negative. Let $\eta' \in D_r$ be a local configuration of radius $r$ with $k(\eta')$ positive
vertices. Then, the difference $H^B(\eta'_x, \sigma_{\delta B}) - H^B(\eta_x, \sigma_{\delta B})$ between the local energies of $\eta'_x$ and $\eta_x$ is equal to:

$$H^B(\eta'_x, \sigma_{\delta B}) - H^B(\eta_x, \sigma_{\delta B}) = 2a(n)(k(\eta') - k(\eta)) + b(2\gamma(\eta) + Q(\eta'_x)),$$

where

$$Q(\eta'_x) = \sum_{\{y,z\} \in E_n} \eta'_x(y)\eta'_x(z) - \alpha(r) + \sum_{\{y,z\} \in E_n} (\eta'_x(y) + 1)\sigma_{\delta B}(z).$$

The real $Q(\eta'_x)$ does not depend on $n$ and satisfies $|Q(\eta'_x)| \leq 2\alpha(r + 1)$. Among the local configurations of radius $r$, one of them has only negative vertices. Denote by $\eta^0$ this unique element of $\mathcal{D}_r$. It plays an important role here since $Q(\eta^0) = 0$. So, using (3), the quantity $H^B(\eta'_x, \sigma_{\delta B}) - H^B(\eta_x, \sigma_{\delta B})$ becomes:

$$H^B(\eta'_x, \sigma_{\delta B}) - H^B(\eta_x, \sigma_{\delta B}) = -2a(n)k(\eta) + 2b\gamma(\eta) = \log \frac{n^d}{C} + 2b\gamma(\eta),$$

for any configuration $\sigma \in \mathcal{X}_n$. Then, using the explicit formula for the conditional probability $\mu_{a,b}(I^n_x = 1|\sigma_{\delta B})$ (relation (4)), we get:

$$\mu_{a,b}(I^n_x = 1|\sigma_{\delta B}) \leq e^{H^B(\eta_x, \sigma_{\delta B}) - H^B(\eta'_x, \sigma_{\delta B})} \leq \frac{C^{2b\gamma(\eta)}}{n^d},$$

i.e. the upper bound of (7). The lower bound is obtained as follows. For any configuration $\sigma \in \mathcal{X}_n$:

$$n^d \mu_{a,b}(I^n_x = 1|\sigma_{\delta B}) = \frac{n^d}{\sum_{\eta' \in \mathcal{D}_r} e^{H^B(\eta'_x, \sigma_{\delta B}) - H^B(\eta_x, \sigma_{\delta B})}} = \frac{C^{2b\gamma(\eta)}}{1 + \sum_{\eta', k(\eta') > 0} e^{2a(n)k(\eta') + bQ(\eta'_x)}} \geq \frac{C^{2b\gamma(\eta)}}{1 + e^{2a(n)\mathcal{D}_r} e^{2b\alpha(r + 1)}}.$$

Let $M$ denote the quantity $|\mathcal{D}_r| e^{2b\alpha(r + 1)}$, it only depends on the pair potential $b$ and the radius $r$. Finally, the inequality

$$\forall u > -1, \quad \frac{1}{1 + u} \geq 1 - u$$

implies the lower bound of (7). \qed
3 Poisson approximation

This section is devoted to the proof of the following theorem which gives the limit distribution for the random variable $X_n(\eta)$. Previous notations and hypotheses still hold: in particular, the relation (3) between the pair potential $a(n)$ and the number $k(\eta)$ of positive vertices in $\eta$.

**Theorem 3.1** As $n$ tends to infinity, the distribution of $X_n(\eta)$ converges weakly to the Poisson distribution with parameter $ce^{-2b \gamma(\eta)}$.

Before proving this result, it is worth pointing out here the role of the pair potential $b$. First, remark that local configurations of radius $r$ having the same number of positive vertices can have different perimeters. Theorem 3.1 assures that the probability (for $\mu_{a,b}$) of the local configuration $\eta$ of occurring in the graph is asymptotically equal to

$$1 - e^{-ce^{-2b \gamma(\eta)}}.$$

So, if $b > 0$ (resp. $b < 0$), this asymptotic probability is a decreasing (resp. increasing) function of the perimeter $\gamma(\eta)$. In other words, if $b > 0$ (resp. $b < 0$), among the local configurations having the same number of positive vertices, those having the highest asymptotic probability of occurring in the infinite graph are those having the smallest (resp. largest) perimeter.

If the pair potential $b$ is null then the perimeter $\gamma(\eta)$ of the local configuration $\eta$ has no influence. All local configurations having the same number of positive vertices have the same asymptotic probability $1 - e^{-c}$ of occurring in the graph. In the 2-dimensional case, this is Theorem 2.4 of [4].

In order to prove Theorem 3.1, we use the moment method based on the following lemma ([3] p. 25).

**Lemma 3.2** Let $(Y_n)_{n \in \mathbb{N}^*}$ be a sequence of integer valued, nonnegative random variables and $\lambda$ be a strictly positive real. For all $n, l \in \mathbb{N}^*$ define $M_l(Y_n)$, the $l$-th moment of $Y_n$, by

$$M_l(Y_n) = \sum_{k \geq l} \mathbb{P}(Y_n = k) \frac{k!}{(k - l)!}.$$

If, for all $l \in \mathbb{N}^*$, $\lim_{n \to \infty} M_l(Y_n) = \lambda^l$ then the distribution of $Y_n$ converges weakly as $n$ tends to infinity to the Poisson distribution with parameter $\lambda$.

**Proof (of Theorem 3.1):** Thanks to Lemma 3.2, we just need to prove the convergence of $M_l(X_n(\eta))$ to $(ce^{-2b \gamma(\eta)})^l$ for every $l \in \mathbb{N}^*$. The case $l = 1$ has already been treated at the end of the previous section, where it was proved that $M_1(X_n(\eta)) = \mathbb{E}_{a,b}[X_n(\eta)]$ tends to $ce^{-2b \gamma(\eta)}$. From now on, fix an integer $l \geq 2$. In our case, the variable $X_n(\eta)$ counts the number of copies in the graph $G_n$ of the local configuration $\eta$. Then, the quantity
$M_l(X_n(\eta))$ can be interpreted as the expected number of ordered $l$-tuples of copies of $\eta$. If $B$ represents a set of balls of radius $r$ whose centers belong to $V_n$, then two elements $B(y, r)$ and $B(z, r)$ of $B$ will be said to be connected if there exists an integer $m$ and balls $B_1, \ldots, B_m \in B$ such that $B_1 = B(y, r)$, $B_m = B(z, r)$ and for $j = 1, \ldots, m - 1$, $\overline{B}_j \cap \overline{B}_{j+1} \neq \emptyset$. The connectivity is an equivalence relation on the set $B$.

For $s = 1, \ldots, l$, denote by $C_l(s)$ the set of $l$-tuples of vertices $(x_1, \ldots, x_l)$ such that the set of balls $\{B(x_1, r), \ldots, B(x_l, r)\}$ is composed of $s$ equivalence classes for the connectivity relation. Then, the $l$-th moment of $X_n(\eta)$ becomes:

$$M_l(X_n(\eta)) = \sum_{s=1}^{l} E_{a,b} \left[ \sum_{(x_1, \ldots, x_s) \in C_l(s)} I_{x_1}^n \times \cdots \times I_{x_s}^n \right].$$

The term corresponding to $s = l$ in the above sum will be denoted by $M'_l(X_n(\eta))$ and the remaining sum by $M''_l(X_n(\eta))$. We are going to prove the two following limits

$$\lim_{n \to \infty} M'_l(X_n(\eta)) = (ce^{-2\eta(\eta)})^l, \quad (10)$$
$$\lim_{n \to \infty} M''_l(X_n(\eta)) = 0, \quad (11)$$

from which Theorem 3.1 follows.

Let us first estimate the cardinality of $C_l(l)$. We want to choose $l$ vertices $x_1, \ldots, x_l$ of $V_n$ such that the balls of radius $r + 1$ centered on those vertices are two by two disjoint. For the first vertex $x_1$, there are $n^d$ possibilities. Let $2 \leq j \leq l$ and suppose vertices $x_1, \ldots, x_{j-1}$ have been chosen. For the $j$-th choice, the set of all vertices $x$ such that $\text{dist}(x, x_k) \leq 2(r + 1)$ for some $1 \leq k \leq j - 1$, must be avoided. The cardinality of this set is bounded above by $(j - 1) \times \beta(2(r + 1))$ whatever $x_1, \ldots, x_{j-1}$. This bound does not depend on $n$. So, asymptotically the number of choices for the $j$-th element is $n^d$, and consequently the cardinality of $C_l(l)$ is equivalent to $n^{dl}$. Then, the two quantities $M'_l(X_n(\eta))$ and $n^{dl} \mu_{a,b}(I_{x_1}^n \times \cdots \times I_{x_l}^n = 1)$ have the same limit as $n$ tends to infinity, for any given $l$-tuple $(x_1, \ldots, x_l)$ belonging to $C_l(l)$.

Let $(x_1, \ldots, x_i) \in C_l(l)$. By definition of the connectivity relation, remark that, for $1 \leq i, j \leq l$ and $i \neq j$, no vertex of the ball $B(x_i, r)$ can be a neighbor of a vertex of the ball $B(x_j, r)$. The Gibbs measure $\mu_{a,b}$ yields a Markov random field with respect to neighborhoods defined in (1) (see for example [11], Lemma 3 p. 7). As a consequence,

$$n^{dl} \mu_{a,b} \left( \prod_{i=1}^{l} I_{x_i}^n = 1 \right) = E_{a,b} \left[ n^{dl} \mu_{a,b} \left( \prod_{i=1}^{l} I_{x_i}^n = 1 \right| \mathcal{F}(\cup_{j=1}^{l} \delta B(x_j, r)) \right]$$

$$= E_{a,b} \left[ n^{dl} \prod_{i=1}^{l} \mu_{a,b} \left( I_{x_i}^n = 1 \right| \mathcal{F}(\cup_{j=1}^{l} \delta B(x_j, r)) \right]$$

$$= E_{a,b} \left[ n^{dl} \prod_{i=1}^{l} \mu_{a,b} \left( I_{x_i}^n = 1 \right| \mathcal{F}(\delta B(x_i, r)) \right].$$
Due to Lemma 2.2, the quantity \( n^d \mu_{a,b}(I_{x_i}^n = 1|\sigma_{\delta B(x_i, r)}) \) tends to \( ce^{-2b\gamma(n)} \), for all indices \( i \) and configurations \( \sigma \). Hence,

\[
\forall \sigma \in X_n, \lim_{n \to \infty} n^d \prod_{i=1}^{l} \mu_{a,b}(I_{x_i}^n = 1|\sigma_{\delta B(x_i, r)}) = (ce^{-2b\gamma(n)})^l.
\]

Finally, since the set of configurations on \( \cup_{j=1}^{l} \delta B(x_j, r) \) is finite (its cardinality only depends on \( l \) and \( r \), (10) follows:

\[
\lim_{n \to \infty} M_1'(X_n(\eta)) = \lim_{n \to \infty} \mathbb{E}_{a,b} \left[ n^d \prod_{i=1}^{l} \mu_{a,b}(I_{x_i}^n = 1|\mathcal{F}(\delta B(x_i, r))) \right] = (ce^{-2b\gamma(n)})^l.
\]

There remains to prove that \( M_1''(X_n(\eta)) \) tends to 0 as \( n \) tends to infinity. The intuition is that if the local configuration \( \eta \) occurs on two balls \( B(x, r) \) and \( B(x', r) \) with \( \text{dist}(x, x') \leq 2(r + 1) \), then locally (strictly) more than \( k(\eta) \) positive vertices are present in a ball of radius \( 2(r + 1) \). This has vanishing probability by (5).

Let us prove that every term of the sum defining \( M_1''(X_n(\eta)) \) tends to 0: fix an integer \( 1 \leq s \leq l - 1 \). If vertices \( \{x_1, \ldots, x_m\} \) are such that the balls \( B(x_1, r), \ldots, B(x_m, r) \) are all equivalent for the connectivity relation, then there are only \( O(n^d) \) different ways to choose them in \( V_n \). Consequently, there exists a constant \( C > 0 \) which does not depend on \( n \), such that the cardinality of \( \mathcal{C}_l(s) \) is bounded by \( Cn^{d s} \). Hence

\[
\mathbb{E}_{a,b} \left[ \sum_{(x_1, \ldots, x_l) \in \mathcal{C}_l(s)} I_{x_1}^n \times \ldots \times I_{x_l}^n \right] \leq Cn^{d s} \mu_{a,b} \left( \prod_{i=1}^{l} I_{x_i}^n = 1 \right),
\]

where \( (x_1, \ldots, x_l) \) is a fixed element of \( \mathcal{C}_l(s) \). The set of balls with radius \( r \), centered at these vertices, splits into \( s \) equivalence classes, say \( EC(1), \ldots, EC(s) \). Let us denote by \( C_j \) the union of balls belonging to the equivalence class \( EC(j) \). Once again, we use the Markovian character of the Gibbs measure \( \mu_{a,b} \):

\[
n^d \mu_{a,b} \left( \prod_{i=1}^{l} I_{x_i}^n = 1 \right) = \mathbb{E}_{a,b} \left[ n^d \mu_{a,b} \left( \prod_{i=1}^{l} I_{x_i}^n = 1\right) \mathcal{F}(\cup_{j=1}^{s} \delta C_j) ) \right] = \mathbb{E}_{a,b} \left[ n^d \prod_{j=1}^{s} \mu_{a,b} \left( \prod_{i, B(x_i, r) \in EC(j)} I_{x_i}^n = 1\right) \mathcal{F}(\delta C_j) \right].
\]

As a consequence of \( s \leq l - 1 \), there exists at least one connected component, say \( EC(1) \), having at least two elements. For every \( j = 2, \ldots, s \), denote by \( x(j) \) one of centers of balls
belonging to $EC(j)$. Then, we can write:

$$
\mu_{a,b} \left( \prod_{i, B(x_i, r) \in EC(j)} I_{x_i}^n = 1 \big| \mathcal{F}(\delta C_j) \right) \leq \mu_{a,b} \left( I_{x(j)}^n = 1 \big| \mathcal{F}(\delta C_j) \right)
$$

$$
\leq \mathbb{E}_{a,b} \left[ \mathbb{E}_{a,b} \left[ I_{x(j)}^n \big| \mathcal{F}(\delta C_j \cup \delta B(x(j), r)) \right] \big| \mathcal{F}(\delta C_j) \right]
$$

$$
\leq \mathbb{E}_{a,b} \left[ \mathbb{E}_{a,b} \left[ I_{x(j)}^n \big| \mathcal{F}(\delta B(x(j), r)) \right] \big| \mathcal{F}(\delta C_j) \right]
$$

$$
\leq \frac{ce^{-2b\gamma(\eta)}}{n^d}
$$

by Lemma 2.2. This last inequality allows us to write, with $C' = C(ce^{-2b\gamma(\eta)})^{s-1}$:

$$
\mathbb{E}_{a,b} \left[ \sum_{(x_1, \ldots, x_l) \in C_i(s)} I_{x_1}^n \times \cdots \times I_{x_l}^n \right] \leq C' n^d \mathbb{E}_{a,b} \left[ \mu_{a,b} \left( \prod_{i, B(x_i, r) \in EC(j)} I_{x_i}^n = 1 \big| \mathcal{F}(\delta C_1) \right) \right]
$$

$$
\leq C' n^d \mu_{a,b} \left( \prod_{i, B(x_i, r) \in EC(j)} I_{x_i}^n = 1 \right)
$$

$$
\leq C' n^d \mu_{a,b}(I_{x_1}^n = I_{x_2}^n = 1)
$$

(12)

where $x_1$ and $x_2$ are two different vertices, centers of balls belonging to $EC(1)$ and satisfying $B(x_1, r+1) \cap B(x_2, r+1) \neq \emptyset$. So, in order to have (11), it is enough to prove that (12) tends to 0. Let $R$ be an integer such that $B(x_1, R)$ contains both balls $B(x_1, r)$ and $B(x_2, r)$. Denote by $\mathcal{D}_R^{>k(\eta)}$ the (finite) set of local configurations of radius $R$ having at least $k(\eta) + 1$ positive vertices. Then, the event $I_{x_1}^n = I_{x_2}^n = 1$ implies that one of the elements of $\mathcal{D}_R^{>k(\eta)}$ occurs in $B(x_1, R)$. It follows that:

$$
n^d \mu_{a,b}(I_{x_1}^n = I_{x_2}^n = 1) \leq n^d \sum_{\Delta \in \mathcal{D}_R^{>k(\eta)}} \mathbb{E}_{a,b}[I_{x_1}^\Delta = 1]
$$

$$
\leq \sum_{\Delta \in \mathcal{D}_R^{>k(\eta)}} \mathbb{E}_{a,b}[X_n(\Delta)]
$$

where $X_n(\Delta)$ counts the number of copies of $\Delta$ in the graph $G_n$. Now, any given $\Delta \in \mathcal{D}_R^{>k(\eta)}$ has $k(\Delta) > k(\eta)$ positive vertices. Then $e^{2a(n)n^{d/k(\Delta)}}$ tends to 0 as $n$ tends to infinity and so does $\mathbb{E}_{a,b}[X_n(\Delta)]$ by relation (5). Hence, we conclude that (12) tends to 0 and Theorem 3.1 follows.  \[\square\]
4 The ferromagnetic case

Let us denote by $\mathcal{L}(X)$ the probability distribution of a random variable $X$ and by $\mathcal{P}(\lambda)$ the Poisson distribution with parameter $\lambda$. If $\mu$ and $\nu$ are two probability distributions, the total variation distance between $\mu$ and $\nu$ is

$$d_{TV}(\mu, \nu) = \sup_A |\mu(A) - \nu(A)| ,$$

where the supremum is taken over all measurable sets. In this section, our goal is to bound the total variation distance between $\mathcal{L}(X_n(\eta))$ and its Poisson approximation $\mathcal{P}(ce^{-2b\gamma(\eta)})$.

**Theorem 4.1** Assume that the surface potential $a(n)$ satisfies (3) and that the pair potential $b$ is positive. Then,

$$d_{TV}(\mathcal{L}(X_n(\eta)), \mathcal{P}(ce^{-2b\gamma(\eta)})) = \mathcal{O}(n^{-d/k(\eta)}) .$$

We believe that $n^{-d/k(\eta)}$ is the real speed at which $d_{TV}(\mathcal{L}(X_n(\eta)), \mathcal{P}(ce^{-2b\gamma(\eta)}))$ tends to zero. However, we could only prove an upper bound. In the case where the local configuration $\eta$ represents a single positive vertex (hence $k(\eta) = 1$ and $\gamma(\eta) = 4$), Ganesh et al. [9] proved that

$$\frac{\log d_{TV}(\mathcal{L}(X_n(\eta)), \mathcal{P}(ce^{-8b}))}{\log n^{-d}} \to 1 ,$$

as $n$ tend to infinity.

The rest of this section is devoted to the proof of Theorem 4.1. Let us start with some notations and definitions. There is a natural partial ordering on the configuration set $\mathcal{X}_n = \{-1, +1\}^{V_n}$ defined by $\sigma \leq \sigma'$ if $\sigma(x) \leq \sigma'(x)$ for all vertices $x \in V_n$. A function $f : \mathcal{X}_n \rightarrow \mathbb{R}$ is increasing if $f(\sigma) \leq f(\sigma')$ whenever $\sigma \leq \sigma'$. For a positive value of the pair potential $b$, the Gibbs measure $\mu_{a,b}$ defined by (2) satisfies the FKG inequality, i.e.

$$\mathbb{E}_{a,b}[fg] \geq \mathbb{E}_{a,b}[f] \mathbb{E}_{a,b}[g] ,$$

for all increasing functions $f$ and $g$ on $\mathcal{X}_n$ (see for instance Section 3 of [8]).

From the local configuration $\eta$, let us define the subset $\mathcal{D}_r(\eta)$ of $\mathcal{D}_r$ by:

$$\mathcal{D}_r(\eta) = \{\eta' \in \mathcal{D}_r, V_+ (\eta') \supset V_+ (\eta)\} .$$

Each local configuration of $\mathcal{D}_r^*(\eta) = \mathcal{D}_r(\eta) \setminus \{\eta\}$ has at least $k(\eta) + 1$ positive vertices. Moreover, by definition of $\mathcal{D}_r(\eta)$ and for all $x \in V_n$, the indicator $\overline{T_x^\eta}$ defined by

$$\overline{T_x^\eta} = \sum_{\eta' \in \mathcal{D}_r(\eta)} T_x^{\eta'}$$

is an increasing function. Let us introduce the corresponding random variable $\overline{X}_n(\eta)$:

$$\overline{X}_n(\eta) = \sum_{x \in V_n} \overline{T_x^\eta} = \sum_{\eta' \in \mathcal{D}_r(\eta)} X_n(\eta') .$$
whose expectation $E_{a,b}[\mathcal{X}_n(\eta)]$ will be simply denoted by $\lambda_n$.

The proof of Theorem 4.1 is organized as follows. The total variation distance between $\mathcal{L}(X_n(\eta))$ and $\mathcal{P}(ce^{-2b\gamma(\eta)})$ is bounded by:

$$d_{TV}(\mathcal{L}(X_n(\eta)), \mathcal{L}(\overline{X}_n(\eta))) + d_{TV}(\mathcal{L}(\overline{X}_n(\eta)), \mathcal{P}(\lambda_n)) + d_{TV}(\mathcal{P}(\lambda_n), \mathcal{P}(ce^{-2b\gamma(\eta)})) .$$

Let us respectively denote by $T_1$, $T_2$ and $T_3$ the three terms of the above sum. We are going to prove that each of them is of order $O(n^{-d/k(\eta)})$. Firstly, the difference $\overline{X}_n(\eta) - X_n(\eta)$ needs to be controlled. The inequality (14) of Lemma 4.2 allows us to bound $T_1$. The term $T_3$ is dealt with using Lemma 4.3. Applied to the family of indicators $\{\overline{I}_x, x \in V_n\}$, the Stein-Chen method gives an upper bound for $T_2$ (Lemma 4.4). Finally, Lemma 4.5 implies that this upper bound is a $O(n^{-d/k(\eta)})$.

Let us start with an estimate of $\lambda_n$. As the expectation of $X_n(\eta)$, that of $\overline{X}_n(\eta)$ tends to $ce^{-2b\gamma(\eta)}$:

**Lemma 4.2** There exist two positive constants $M' = M'(b, r)$ and $M'' = M''(b, r)$ such that $\forall x \in V_n$ and $\forall \sigma \in \mathcal{X}_n$:

$$\sum_{\eta' \in \mathcal{D}_r^*(\eta)} n^d \mu_{a,b}(I_x' = 1 | \sigma_{\delta B(x, r)}) \leq M' e^{-2b\gamma(\eta)} e^{2a(n)} ,$$

and

$$ce^{-2b\gamma(\eta)} (1 - M'' e^{2a(n)}) \leq \lambda_n \leq ce^{-2b\gamma(\eta)} (1 + M'' e^{2a(n)}) .$$

**Proof:** Fix a vertex $x$ and denote merely by $B$ the ball $B(x, r)$. Let $\sigma \in \mathcal{X}_n$ and $\eta'$ be a local configuration of $\mathcal{D}_r^*(\eta)$. The proof of (14) is very close to that of the upper bound in (7). If $\eta^0$ represents the element of $\mathcal{D}_r$ having only negative vertices, we have already seen that:

$$H^B(\eta'_x \sigma_{\delta B}) - H^B(\eta^0_x \sigma_{\delta B}) = 2a(n) k(\eta') + bQ(\eta'_x) \leq 2a(n)(k(\eta) + 1) + 2b\alpha(r + 1) ,$$

by definition of the set $\mathcal{D}_r^*(\eta)$. Then, from (4) and (3),

$$n^d \mu_{a,b}(I_x' = 1 | \sigma_{\delta B}) \leq n^d e^{H^B(\eta'_x \sigma_{\delta B}) - H^B(\eta^0_x \sigma_{\delta B})} \leq ce^{2a(n)} e^{2b\alpha(r + 1)} .$$

With $M' = |\mathcal{D}_r^*(\eta)| e^{2b\gamma(\eta)} e^{2b\alpha(r+1)}$, the inequality (14) is proved. Let us remark that (16) does not depend on the geometry of $\eta'$. So, it is valid for any local configuration having at least $k(\eta) + 1$ positive vertices and not only for those belonging to the set $\mathcal{D}_r^*(\eta)$.

The random variable $\overline{X}_n(\eta)$ can be written as

$$\overline{X}_n(\eta) = X_n(\eta) + \sum_{\eta' \in \mathcal{D}_r^*(\eta)} X_n(\eta') .$$
Hence, (14) implies that
\[ \mathbb{E}_{a,b}[X_n(\eta)] \leq \lambda_n \leq \mathbb{E}_{a,b}[X_n(\eta)] + M' e^{-2\gamma(\eta)} e^{2a(n)}. \]

Thanks to relation (8), the expectation of the random variable \( X_n(\eta) \) is now bounded:
\[ ce^{-2\gamma(\eta)} (1 - M e^{2a(n)}) \leq \lambda_n \leq ce^{-2\gamma(\eta)} (1 + M' e^{2a(n)}). \]

Finally, relation (15) follows by letting \( M'' = \max\{M, M'\}. \quad \square \)

The total variation distance between two probability distributions can be written as
\[ d_{TV}(\mu, \nu) = \inf \{P(X \neq Y), \mathcal{L}(X) = \mu \text{ and } \mathcal{L}(Y) = \nu\}. \]

Using this characterization and the inequality (14), the term \( T_1 \) is bounded:
\[ T_1 = d_{TV}(\mathcal{L}(X_n(\eta)), \mathcal{L}(\mathbb{X}_n(\eta))) \leq \mu_{a,b}(X_n(\eta) \neq \mathbb{X}_n(\eta)) \]
\[ \leq \mu_{a,b} \left( \sum_{\eta' \in \mathcal{D}_n(\eta)} X_n(\eta') > 0 \right) \]
\[ \leq \sum_{\eta' \in \mathcal{D}_n(\eta)} \mathbb{E}_{a,b}[X_n(\eta')] \]
\[ \leq M' e^{-2\gamma(\eta)} e^{2a(n)}. \]

So, \( T_1 \) is of order \( \mathcal{O}(n^{-d/k(\eta)}) \) by relation (3).

Using (15), we shall now bound \( T_3 \):

**Lemma 4.3** One has:
\[ T_3 = d_{TV}(\mathcal{P}(\lambda_n), \mathcal{P}(ce^{-2\gamma(\eta)})) = \mathcal{O}(n^{-d/k(\eta)}). \]

**Proof:** The total variation distance between two probability distributions on the set of integers can be expressed as:
\[ d_{TV}(\mu, \nu) = \frac{1}{2} \sum_{m \geq 1} |\mu(m) - \nu(m)|. \]

Let \( m \geq 1 \) and denote simply by \( \lambda \) the quantity \( ce^{-2\gamma(\eta)} \). Thanks to relation (15), the difference \( \lambda^m e^{-\lambda n} - \lambda^m e^{-\lambda} \) is easily controlled. Thus, \( d_{TV}(\mathcal{P}(\lambda_n), \mathcal{P}(\lambda)) \) is bounded by \( \frac{1}{2} \sum_{m \geq 1} \max\{\alpha_m, \beta_m\} \) where:
\[ \alpha_m = \frac{\lambda^m e^{-\lambda} m!}{m!} \left( e^{\lambda M'' e^{2a(n)}} (1 + M'' e^{2a(n)})^m - 1 \right) \]

15
and
\[
\beta_m = \frac{\lambda^m e^{-\lambda}}{m!} \left(1 - e^{-\lambda M'' e^{2a(n)}} (1 - M'' e^{2a(n)})^m\right).
\]

Using the convexity of the function
\[
f_m : \] \longrightarrow \mathbb{R},\ x \longmapsto (1 + x)^m e^{\lambda x},
\]
one easily checks that \(\alpha_m \geq \beta_m\), for all \(m \in \mathbb{N}^\ast\). As a consequence,

\[
d_{TV}(\mathcal{P}(\lambda_n), \mathcal{P}(\lambda)) \leq \frac{1}{2} \sum_{m \geq 1} \alpha_m \leq \frac{1}{2} \left(e^{2\lambda M'' e^{2a(n)}} - 1\right)
\]

which is of order \(O(n^{-d/k(n)})\) by relation (3). \(\Box\)

There remains to bound the term \(T_2 = d_{TV}(\mathcal{L}(\overline{X}_n(\eta)), \mathcal{P}(\lambda_n))\). The random variable \(\overline{X}_n(\eta)\) has been constructed as a sum of (dependent) increasing random indicators \(\overline{I}_x, x \in V_n\). The Stein-Chen method (see [2], Corollary 2.C.4, p. 26) applies to \(\overline{X}_n(\eta)\) and gives an upper bound for the total variation distance between \(\mathcal{L}(\overline{X}_n(\eta))\) and the Poisson distribution with parameter \(\lambda_n\). This is the only place where the hypothesis \(b > 0\) is actually used in the proof.

**Lemma 4.4** The following inequality holds:

\[
d_{TV}(\mathcal{L}(\overline{X}_n(\eta)), \mathcal{P}(\lambda_n)) \leq \frac{1}{\lambda_n} \left(Var_{a,b}[\overline{X}_n(\eta)] - \lambda_n + 2 \sum_{x \in V_n} E_{a,b}[\overline{I}_x]^{2}\right).
\]

The Poisson distribution has the property that the variance is equal to the mean. Although this is not a characterization, the above bound indicates that, as \(n \to +\infty\), \(\sum_{x \in V_n} E_{a,b}[\overline{I}_x]^{2}\) is small and the distance to the Poisson approximation is essentially the difference between the variance and the expectation of \(\overline{X}_n(\eta)\). Using good estimates on the first two moments of the random variable \(\overline{X}_n(\eta)\), this difference will be bounded. The case of the first moment of \(\overline{X}_n(\eta)\), i.e. its expectation \(\lambda_n\), has been treated in Lemma 4.2. The following result concerns its second moment:

**Lemma 4.5** The second moment \(M_2(\overline{X}_n(\eta)) = E_{a,b}[\overline{X}_n(\eta) (\overline{X}_n(\eta) - 1)]\) satisfies the inequality:

\[
M_2(\overline{X}_n(\eta)) \leq \left(ce^{-2\gamma(\eta)}\right)^2 + O(n^{-d/k(n)})\).
\]

Writing the variance of the variable \(\overline{X}_n(\eta)\) as \(M_2(\overline{X}_n(\eta)) + \lambda_n - \lambda_n^2\) and the sum \(\sum_{x \in V_n} E_{a,b}[\overline{I}_x]^{2}\) as the ratio \(\lambda_n^2/n^d\), we deduce from Lemma 4.4 that:

\[
T_2 \leq \frac{1}{\lambda_n} \left(M_2(\overline{X}_n(\eta)) + \lambda_n^2 \left(\frac{2}{n^d} - 1\right)\right).
\]

16
The inequalities given by (15) and Lemma 4.5 allow us to control the expectation \( \lambda_n \) and the second moment \( M_2(\overline{X}_n(\eta)) \) of the random variable \( \overline{X}_n(\eta) \). This implies:

\[
T_2 \leq \mathcal{O}(e^{2\alpha(n)}) + \frac{2(ce^{-2\rho(\eta)})^2}{n^d},
\]

which is a \( \mathcal{O}(e^{2\alpha(n)}) = \mathcal{O}(n^{-d/k(\eta)}) \) since \( k(\eta) \geq 1 \).

Let us finish the proof of Theorem 4.1 by proving Lemma 4.5.

**Proof (of Lemma 4.5):** As in the proof of Theorem 3.1, the second moment of the random variable \( X_n(\eta'), \eta' \in \mathcal{D}_r(\eta) \), can be expressed as:

\[
M_2(X_n(\eta')) = \mathbb{E}_{a,b} [X_n(\eta')(X_n(\eta') - 1)] = 2 \sum_{s=1} \mathbb{E}_{a,b} \left[ \sum_{(x_1,x_2) \in \mathcal{C}_2(s)} I_{x_1}^{\eta'} \times I_{x_2}^{\eta'} \right],
\]

where \( \mathcal{C}_2(s) \) denotes the set of couples of vertices \( (x_1,x_2) \) such that the set of balls \( \{B(x_1,r), B(x_2,r)\} \) is composed of \( s \) equivalence classes for the connectivity relation. Once again, the terms of the sum above corresponding to \( s = 2 \) and \( s = 1 \) will be respectively denoted by \( M_2'(X_n(\eta')) \) and \( M_2''(X_n(\eta')) \). Then, the second moment \( M_2(\overline{X}_n(\eta)) \) becomes:

\[
M_2'(X_n(\eta)) + \sum_{\eta' \in \mathcal{D}_r(\eta)} M_2'(X_n(\eta')) + M_2''(X_n(\eta)) + \sum_{\eta' \in \mathcal{D}_r(\eta)} M_2''(X_n(\eta')).
\]

We are going to prove that the first term of the above sum is bounded by \( (ce^{-2\rho(\eta)})^2 \) and the 3 others are \( \mathcal{O}(n^{-d/k(\eta)}) \).

The cardinality of the set \( \mathcal{C}_2(2) \) is bounded by \( n^{2d} \). Let \( (x_1,x_2) \) be one of its elements. This set was introduced in order to write:

\[
M_2'(X_n(\eta)) \leq n^{2d} \mathbb{E}_{a,b} [I_{x_1}^{\eta} \times I_{x_2}^{\eta}]
\]

\[
\leq \mathbb{E}_{a,b} \left[ n^{2d} \mu_{a,b} (I_{x_1}^{\eta} = I_{x_2}^{\eta} = 1|\mathcal{F}(\delta B(x_1,r) \cup \delta B(x_2,r))) \right]
\]

\[
\leq \mathbb{E}_{a,b} \left[ \prod_{i=1}^{2} n^d \mu_{a,b} (I_{x_i}^{\eta} = 1|\mathcal{F}(\delta B(x_i,r))) \right].
\]

Using (7), the latter quantity is bounded by \( (ce^{-2\rho(\eta)})^2 \), so does \( M_2'(X_n(\eta)) \). In the same way, relation (16) implies:

\[
\sum_{\eta' \in \mathcal{D}_r(\eta)} M_2'(X_n(\eta')) \leq \sum_{\eta' \in \mathcal{D}_r(\eta)} n^{2d} \mathbb{E}_{a,b} [I_{x_1}^{\eta'} \times I_{x_2}^{\eta'}]
\]

\[
\leq \mathbb{E}_{a,b} \left[ \sum_{\eta' \in \mathcal{D}_r(\eta)} \prod_{i=1}^{2} n^d \mu_{a,b} (I_{x_i}^{\eta'} = 1|\mathcal{F}(\delta B(x_i,r))) \right]
\]

\[
\leq |\mathcal{D}_r(\eta)| \left( ce^{2\alpha(n)} e^{2\rho(\eta+1)} \right)^2.
\]

17
which is of course a $O(e^{2\alpha(n)}) = O(n^{-d/k(\eta)})$. There are $\beta(2r+1)n^d$ different ways to choose an element of $C_2(1)$. Let $(x_1, x_2)$ be one of them. Then,

$$\sum_{\eta' \in D_\gamma^1(\eta)} M'_2(X_n(\eta')) \leq \beta(2r+1) \sum_{\eta' \in D_\gamma^1(\eta)} E_{a,b} \left[n^d I_{x_1}^\eta \right]$$

$$\leq \beta(2r+1) E_{a,b} \left[ \sum_{\eta' \in D_\gamma^1(\eta)} n^d \mu_{a,b} \left(I_{x_1}^\eta = 1 \mid F(\delta B(x_1, r)) \right) \right]$$

$$\leq \beta(2r+1) M'_{cc} e^{-2\alpha(n)} e^{2\alpha(n)} = O(n^{-d/k(\eta)}).$$

The last inequality is deduced from (14). Finally, there remains to prove that $M''_2(X_n(\eta))$ is a $O(n^{-d/k(\eta)})$. We are going to proceed as in the proof of Theorem 3.1. Let $R$ be an integer such that $B(x_1, R)$ contains both balls $B(x_1, r)$ and $B(x_2, r)$. Recall that $D_{R}^{>k(\eta)}$ represents the set of local configurations of radius $R$ having at least $k(\eta) + 1$ positive vertices. The event $I_{x_1}^n = I_{x_2}^n = 1$ implies that one element of $D_{R}^{>k(\eta)}$ occurs in $B(x_1, R)$. As a consequence:

$$M''_2(X_n(\eta)) \leq \beta(2r+1) E_{a,b} \left[n^d I_{x_1}^n \times I_{x_2}^n \right]$$

$$\leq \beta(2r+1) E_{a,b} \left[ \sum_{\Delta \in D_{R}^{>k(\eta)}} n^d \mu_{a,b} \left(I_{x_1}^\Delta = 1 \mid F(\delta B(x_1, R)) \right) \right]. \ (17)$$

Moreover, it has been precised that inequality (16) was valid for any local configuration having at least $k(\eta) + 1$ positive vertices. So, applied to one element $\Delta$ of $D_{R}^{>k(\eta)}$, (16) produces:

$$n^d \mu_{a,b}(I_{x}^\Delta = 1 \mid \sigma_{\delta B(x_1, R)}) \leq ce^{2\alpha(n)} e^{2\alpha(R+1)},$$

for all $\sigma \in X_n$. So, it is easy to bound (17) and to conclude that $M''_2(X_n(\eta))$ is a $O(n^{-d/k(\eta)})$.

\[\square\]

**References**


18


[8] C.M. Fortuin, P.W. Kasteleyn, and J. Ginibre. Correlation inequalities on some par-


