Convergence analysis of a DDFV scheme for a system describing miscible fluid flows in porous media ☆

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Abstract

In this paper, we prove the convergence of a discrete duality finite volume scheme for a system of partial differential equations describing miscible displacement in porous media. This system is made of two coupled equations: an anisotropic diffusion equation on the pressure and a convection-diffusion-dispersion equation on the concentration. We first establish some a priori estimates satisfied by the sequences of approximate solutions. Then, it yields the compactness of these sequences. Passing to the limit in the numerical scheme, we finally obtain that the limit of the sequence of approximate solutions is a weak solution to the problem under study.

Keywords:
finite volume method, convergence analysis, porous medium, miscible fluid flows

1. Introduction

The Peaceman model has been introduced by Bear in [7] and Douglas in [19]. It describes the single-phase displacement of one fluid by another in a porous medium; the fluids are assumed incompressible and the gravity is neglected. This model is constituted of an anisotropic diffusion equation on the pressure of the mixture and a convection-diffusion-dispersion on the concentration of the invading fluid. We refer to the work [30] by Feng for the theoretical analysis of this system of partial differential equations.

Many different schemes have already been proposed for the Peaceman model, since the beginning of the 1980's: finite element schemes for both equations [20, 21, 35], finite element schemes for the pressure combined with method of characteristics for the concentration [39, 26, 25], or combined with Eulerian Lagrangian Localized Adjoint Method for the concentration [40, 41]. The first finite volume scheme scheme proposed for the Peaceman model is a Mixed Finite Volume scheme [9]. In this paper, Chainais-Hillairet and Droniou establish the convergence of the MFV scheme for the Peaceman model. In [6], Bartels, Jensen and Müller provide the convergence analysis of a combined Mixed Finite Element method for the pressure and a Discontinuous Galerkin method for the concentration.

The discrete duality finite volume (DDFV) schemes are devoted to the numerical approximation of anisotropic diffusion operators. They are based on two fundamental ideas: integration of the equations on a primal and a dual meshes, as suggested by Hermeline [32, 33], reconstruction of discrete gradients on a diamond mesh, as in the work by Coudière, Vila and Villedieu [15]. Developing these two ideas, Domelevo and Omnes in [18] introduced the DDFV schemes for the Laplace equation and established the fundamental duality property between discrete gradient and discrete divergence. Since ten years, the DDFV strategy has then been applied for several linear and nonlinear problems: convection-diffusion problems in [4, 12]; the nonlinear diffusion equations for Leray-Lions operators in [5, 8]; Stokes equations in [36, 37, 17]. We can also mention [13] where the DDFV method is adapted to solve numerically a bi-domain problem arising in electrocardiology. We have proposed in a recent work [10] some discrete

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duality finite volume schemes for the Peaceman model. In [10], we have focused on the a priori estimates satisfied by the schemes and on the study of the numerical efficiency. The numerical experiments showed the good convergence behaviour of the schemes and also good qualitative results. In the present paper, we will now consider the convergence analysis (when time and space steps go to 0) of a DDFV scheme for the Peaceman model.

1.1. Presentation of the problem

Let assume that Ω is a connected polygonal domain of $\mathbb{R}^2$ and let $T > 0$. We denote by $\partial\Omega$ the boundary of Ω. The unknowns of the Peaceman model are the pressure in a fluid mixture, $\bar{p}$, its Darcy velocity $\bar{\mathbf{U}}$ and the concentration of some invading fluid $\bar{c}$. As proposed by Chainais-Hillairet and Droniou in [9], we consider a synthesized form of the Peaceman model. It writes:

$$\operatorname{div}\left(\bar{\mathbf{U}}\right) = q^+ - q^- \qquad \text{in }]0,T[\times\Omega, \tag{1a}$$

$$\bar{\mathbf{U}} = -\mathbb{A}(\cdot,\bar{c})\nabla\bar{p} \qquad \text{in }]0,T[\times\Omega, \tag{1b}$$

$$\bar{\mathbf{U}}\cdot\mathbf{n} = 0 \qquad \text{on }]0,T[\times\partial\Omega, \tag{1c}$$

$$\int_\Omega \bar{p}(\cdot,x)\,dx = 0 \qquad \text{on }]0,T[, \tag{1d}$$

$$\Phi\partial_t\bar{c} - \operatorname{div}(\mathbb{D}(\cdot,\bar{\mathbf{U}})\nabla\bar{c}) + \operatorname{div}(\bar{c}\bar{\mathbf{U}}) + q^-\bar{c} = q^+\widehat{c} \qquad \text{in }]0,T[\times\Omega, \tag{2a}$$

$$\mathbb{D}(\cdot,\bar{\mathbf{U}})\nabla\bar{c}\cdot\mathbf{n} = 0 \qquad \text{on }]0,T[\times\partial\Omega, \tag{2b}$$

$$\bar{c}(0,\cdot) = c_0 \qquad \text{on } \Omega. \tag{2c}$$

In this system, q^+ and q^- denote the injection and production terms, $\hat{c}$ the injected concentration, Φ the porosity of the porous medium. The tensor $\mathbb{A}$ contains the effect of the permeability of the porous medium and the viscosity of the fluid mixture. The tensor $\mathbb{D}$ is the diffusion-dispersion tensor; it includes molecular diffusion and mechanical dispersion. The dependency of $\mathbb{A}$ with respect to permeability and viscosity and the dependency of $\mathbb{D}$ with respect to diffusion and dispersion will be detailed in Section 6, with the presentation of numerical experimentations. For the theoretical analysis of the scheme, we need the follow assumptions on the data:

$$\begin{gathered}(q^+,q^-)\in L^\infty(0,T;L^2(\Omega)) \text{ are nonnegative functions such that}\\ \int_\Omega q^+(\cdot,x)\,dx = \int_\Omega q^-(\cdot,x)\,dx \text{ a.e. on }]0,T[,\end{gathered} \tag{3}$$

$$\begin{gathered}\mathbb{A}:\Omega\times\mathbb{R}\to\mathcal{M}_2(\mathbb{R}) \text{ is a Caratheodory matrix-valued function satisfying:}\\ \exists\alpha_A>0 \text{ such that } \mathbb{A}(x,s)\boldsymbol{\xi}\cdot\boldsymbol{\xi}\ge\alpha_A|\boldsymbol{\xi}|^2 \text{ for a.e. } x\in\Omega, \text{ all } s\in\mathbb{R} \text{ and all } \boldsymbol{\xi}\in\mathbb{R}^2,\\ \exists\Lambda_A>0 \text{ such that } |\mathbb{A}(x,s)|\le\Lambda_A \text{ for a.e. } x\in\Omega \text{ and all } s\in\mathbb{R},\end{gathered} \tag{4}$$

$$\begin{gathered}\mathbb{D}:\Omega\times\mathbb{R}^2\to\mathcal{M}_2(\mathbb{R}) \text{ is a Caratheodory matrix-valued function satisfying:}\\ \exists\alpha_D>0 \text{ s.t. } \mathbb{D}(x,\mathbf{W})\xi\cdot\xi\ge\alpha_D(1+|\mathbf{W}|)|\xi|^2 \text{ for a.e. } x\in\Omega, \text{ all } \mathbf{W}\in\mathbb{R}^2 \text{ and all } \xi\in\mathbb{R}^2,\\ \exists\Lambda_D>0 \text{ such that } |\mathbb{D}(x,\mathbf{W})|\le\Lambda_D(1+|\mathbf{W}|) \text{ for a.e. } x\in\Omega \text{ and all } \mathbf{W}\in\mathbb{R}^2,\end{gathered} \tag{5}$$

$$\Phi\in L^\infty(\Omega) \text{ and there exists } \Phi_*>0 \text{ such that } \Phi_*\le\Phi\le\Phi_*^{-1} \text{ a.e. in } \Omega, \tag{6}$$

$$\widehat{c}\in L^\infty(]0,T[\times\Omega) \text{ satisfies: } 0\le\widehat{c}\le 1 \text{ a.e. in }]0,T[\times\Omega, \tag{7}$$

$$c_0\in L^\infty(\Omega) \text{ satisfies: } 0\le c_0\le 1 \text{ a.e. in } \Omega. \tag{8}$$

The following definition (similar to the one in [30]) of weak solution to (1)—(2) makes sense.

Definition 1.1. *Under assumptions* (3)—(8)*, a weak solution to* (1)—(2) *is a triple* $(\bar{p},\bar{\mathbf{U}},\bar{c})$ *satisfying*

$$\bar{p}\in L^\infty(0,T;H^1(\Omega))\,,\quad \bar{\mathbf{U}}\in L^\infty(0,T;L^2(\Omega))^2\,,\quad \bar{c}\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))\,,$$

$$\int_\Omega \bar{p}(t,\cdot) = 0\, for\ a.e.\ t \in]0,T[\,, \quad \bar{\mathbf{U}} = -\mathbb{A}(\cdot,\bar{c})\nabla\bar{p}\ \ a.e.\ on\ ]0,T[\times\Omega\,,$$

$$\forall\varphi \in C^\infty([0,T]\times\bar{\Omega})\,, \quad -\int_0^T\int_\Omega \bar{\mathbf{U}}\cdot\nabla\varphi = \int_0^T\int_\Omega (q^+ - q^-)\varphi\,, \tag{9}$$

$$\forall\varphi \in C_c^\infty([0,T[\times\bar{\Omega})\,, \quad -\int_0^T\int_\Omega \Phi\bar{c}\partial_t\varphi + \int_0^T\int_\Omega \mathbb{D}(\cdot,\bar{\mathbf{U}})\nabla\bar{c}\cdot\nabla\varphi - \int_0^T\int_\Omega \bar{c}\bar{\mathbf{U}}\cdot\nabla\varphi + \int_0^T\int_\Omega q^-\bar{c}\varphi$$
$$- \int_\Omega \Phi c_0\varphi(0,\cdot) = \int_0^T\int_\Omega q^+\hat{c}\varphi. \tag{10}$$

1.2. Aim of the paper and outline

Different development of new finite volume schemes for diffusion equations have been done since twenty years. Their aim is to reconstruct some discrete gradient which has no serious restriction on meshes and strong enough convergence for handling the nonlinear coupling of the equations. Let us cite for instance the Multi Points Flux Approximation schemes by Aavatsmark, Barkve, Boe and Mannseth [1, 2], the Discrete Duality Finite Volume (DDFV) schemes by Domelevo and Omnes [18, 5], the Mixed Finite Volume schemes by Droniou and Eymard [24, 22], the Scheme Using Stabilization and Hybrid Interfaces by Eymard, Gallouët and Herbin [28, 29]. We refer to [23] where Droniou presents a review on finite volume methods for diffusion equations, with a focus on coercivity and minimum-maximum principles.

In [10], we have proposed a DDFV scheme for the Peaceman system (1)-(2). The DDFV scheme requires unknowns on both vertices and "centers" of control volumes. These two sets of unknowns allow to define a two-dimensional discrete gradient (piecewise constant on new geometric elements called diamonds) and a discrete divergence operator. These two operators satisfy a duality property in a discrete sense, which gives its name to the method.

In order to prove the convergence of the scheme, we need to add a penalization operator in the discretization of the convection-diffusion-dispersion equation. Such a penalization operator has already been introduced by Andreianov, Bendahmane and Karlsen in the numerical approximation of a degenerate hyperbolic-parabolic equation [4]. It ensures that both reconstructions of the concentration, either on the primal mesh or on the dual mesh, converge to the same limit. It is crucial when passing to the limit in the concentration equation. However, the numerical experiments will show that the penalization operator is not necessary in practice.

In Section 2, we present the different meshes and the associated notations. After having introduced the different discrete operators, we present the DDFV scheme in Section 2.5. The main result of the paper (convergence of the DDFV scheme) is stated in Theorem 2.6.

In order to prove this Theorem, we apply a similar strategy as [30] on our numerical approximation instead of the regularization of the Peaceman model. We establish in Section 3 some *a priori* estimates satisfied by the numerical solution to the scheme. These first estimates lead to the well-posedness of the scheme and provide first results of compactness in space. In order to get the compactness in space and time of the sequences of approximate solution, we will apply a discrete counterpart of Aubin-Simon Theorem, proved by Gallouët and Latché in [31]. Therefore we prove an estimate on the discrete time derivative in Section 3 and some properties satisfied by the discrete functional spaces in Section 4. Then, the proof of Theorem 2.6 is concluded by passing to the limit into the scheme in Section 5. In Section 6, we provide some numerical experiments. The efficiency of the DDFV scheme has already been shown in [10]. In this last Section, we focus on the effect of the penalization term introduced in the scheme. We show that the penalization operator, necessary for the proof of convergence, can be set to 0 in practice. We give also some results concerning the overshooting and the undershooting effects observed on different meshes.

2. Presentation of the numerical scheme and of the main results

2.1. Meshes and notations

In order to define a DDFV scheme, as for instance in [18, 5], we need to introduce three different meshes – the primal mesh, the dual mesh and the diamond mesh – and some associated notations.

The mesh construction starts from the partition $\mathfrak{M}$, the partition of the computational domain Ω, with disjoint open polygonal control volumes $\mathcal{K} \subset \Omega$ such that $\cup \bar{\mathcal{K}} = \bar{\Omega}$. This partition $\mathfrak{M}$ is called the interior primal mesh. We denote by $\partial\mathfrak{M}$ the set of boundary edges, which are considered as degenerate control volumes. Then, the primal mesh is composed of $\mathfrak{M} \cup \partial\mathfrak{M}$, denoted by $\overline{\mathfrak{M}}$. To construct the two others meshes, we need to associate at each primal cell $\mathcal{K} \in \overline{\mathfrak{M}}$, a point $x_{\mathcal{K}} \in \mathcal{K}$, called the center of the primal cell. Notice that for $\mathcal{K}$ a degenerate control volume, the point $x_{\mathcal{K}}$ is necessarily the midpoint of $\mathcal{K}$. This family of centers is denoted by $X = \{x_{\mathcal{K}}, \mathcal{K} \in \overline{\mathfrak{M}}\}$ and these will determine the two others meshes.

Let X^* denote the set of the vertices of the primal control volumes in $\overline{\mathfrak{M}}$. Distinguishing the interior vertices from the vertices lying on the boundary, we split X^* into $X^* = X^*_{int} \cup X^*_{ext}$. To any point $x_{\mathcal{K}^*} \in X^*_{int}$, we associate the polygon $\mathcal{K}^*$, whose vertices are $\{x_{\mathcal{K}} \in X / x_{\mathcal{K}^*} \in \bar{\mathcal{K}}, \mathcal{K} \in \mathfrak{M}\}$. The set of these polygons defines the interior dual mesh denoted by $\mathfrak{M}^*$. To any point $x_{\mathcal{K}^*} \in X^*_{ext}$, we then associate the polygon $\mathcal{K}^*$, whose vertices are $\{x_{\mathcal{K}^*}\} \cup \{x_{\mathcal{K}} \in X / x_{\mathcal{K}^*} \in \bar{\mathcal{K}}, \mathcal{K} \in \overline{\mathfrak{M}}\}$. The set of these polygons is denoted by $\partial\mathfrak{M}^*$ called the boundary dual mesh and the dual mesh is $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$, denoted by $\overline{\mathfrak{M}^*}$.

In order to define the diamond mesh, we first introduce the notion of edges. For all neighboring primal cells $\mathcal{K}$ and $\mathcal{L}$, we assume that $\partial\mathcal{K} \cap \partial\mathcal{L}$ is a segment, corresponding to an edge of the mesh $\mathfrak{M}$, denoted by $\sigma = \mathcal{K}|\mathcal{L}$. Let $\mathcal{E}$ be the set of such edges. We similarly define the set $\mathcal{E}^*$ of the edges of the dual mesh $\overline{\mathfrak{M}^*}$: $\mathcal{E}^* = \left\{\sigma^*, \sigma^* = \mathcal{K}^*|\mathcal{L}^* \text{ with } \mathcal{K}^*, \mathcal{L}^* \in \overline{\mathfrak{M}^*}\right\}$. Let us note that, if $\mathcal{K} \in \overline{\mathfrak{M}}$, all its edges belong to $\mathcal{E}$ and if $\mathcal{K}^* \in \mathfrak{M}^*$, all its edges belong to $\mathcal{E}^*$. But, if $\mathcal{K}^* \in \partial\mathfrak{M}^*$, then it has edges inside the domain and also on its boundary: the interior edges belong to $\mathcal{E}^*$ while the boundary edges belong to $\mathcal{E}$.

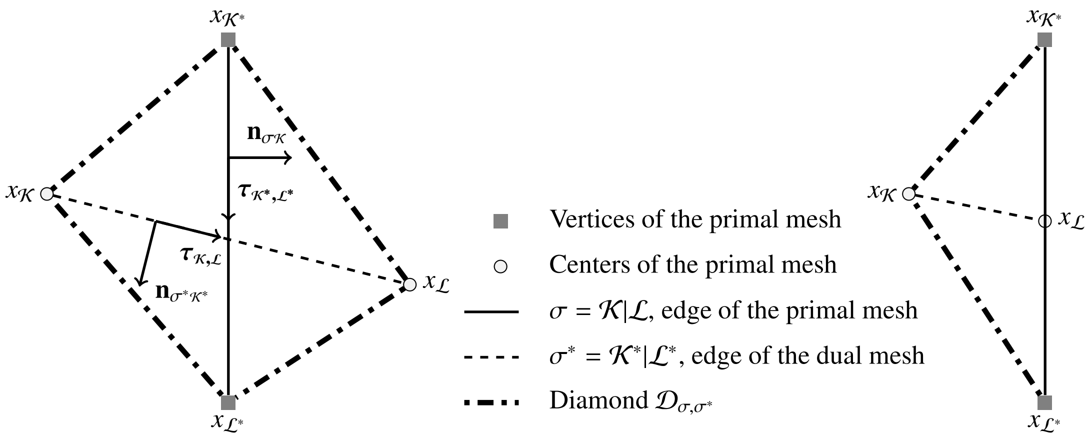


Figure 2.1: Definition of the diamonds $\mathcal{D}_{\sigma,\sigma^*}$

For each couple $(\sigma, \sigma^*) \in \mathcal{E} \times \mathcal{E}^*$ such that $\sigma = \mathcal{K}|\mathcal{L} = (x_{\mathcal{K}^*}, x_{\mathcal{L}^*})$ and $\sigma^* = \mathcal{K}^*|\mathcal{L}^* = (x_{\mathcal{K}}, x_{\mathcal{L}})$, we define the quadrilateral diamond cell $\mathcal{D}_{\sigma,\sigma^*}$ whose diagonals are σ and σ^*. If $\sigma \in \mathcal{E} \cap \partial\Omega$, we note that the diamond degenerates into a triangle. The set of the diamond cells defines the diamond mesh $\mathfrak{D}$. It verifies $\bar{\Omega} = \bigcup_{\mathcal{D} \in \mathfrak{D}} \mathcal{D}$. We have as many diamond cells as primal edges. We can rewrite $\mathfrak{D} = \mathfrak{D}^{ext} \cup \mathfrak{D}^{int}$ where $\mathfrak{D}^{ext}$ is the set of all the boundary diamonds (associated to the boundary edges) and $\mathfrak{D}^{int}$ the set of all the interior diamonds.

Finally, the DDFV mesh is made of the $\mathcal{T} = (\overline{\mathfrak{M}}, \overline{\mathfrak{M}^*})$ and $\mathfrak{D}$. Let us now introduce some notations associated to the meshes $\mathcal{T}$ and $\mathfrak{D}$. For each primal or dual cell V ($V \in \overline{\mathfrak{M}}$ or $V \in \overline{\mathfrak{M}^*}$), we define m_V the measure of V, $\mathcal{E}_V$ the set of the edges of V (it coincides with the edge $\sigma = V$ if $V \in \partial\mathfrak{M}$), $\mathfrak{D}_V$ the set of diamonds $\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}$ such that $\mathrm{m}(\mathcal{D}_{\sigma,\sigma^*} \cap V) > 0$, and d_V the diameter of V.

For a diamond $\mathcal{D}_{\sigma,\sigma^*}$, whose vertices are $(x_{\mathcal{K}}, x_{\mathcal{K}^*}, x_{\mathcal{L}}, x_{\mathcal{L}^*})$, we define, as shown on Figure 2.1: $x_{\mathcal{D}}$ the center of the diamond cell $\mathcal{D}$: $\{x_{\mathcal{D}}\} = \sigma \cap \sigma^*$, m_σ the length of the primal edge σ, m_{σ^*} the length of the dual edge σ^*, $\mathrm{m}_{\mathcal{D}}$ the measure of $\mathcal{D}$, $d_{\mathcal{D}}$ its diameter, $\theta_{\mathcal{D}}$ the angle between $(x_{\mathcal{K}}, x_{\mathcal{L}})$ and $(x_{\mathcal{K}^*}, x_{\mathcal{L}^*})$. We will also use two direct basis $(\boldsymbol{\tau}_{\mathcal{K}^*,\mathcal{L}^*}, \mathbf{n}_{\sigma\mathcal{K}})$ and $(\mathbf{n}_{\sigma^*\mathcal{K}^*}, \boldsymbol{\tau}_{\mathcal{K},\mathcal{L}})$, where $\mathbf{n}_{\sigma\mathcal{K}}$ is the unit normal to σ, outward $\mathcal{K}$, $\mathbf{n}_{\sigma^*\mathcal{K}^*}$ is the unit normal to σ^*, outward $\mathcal{K}^*$, $\boldsymbol{\tau}_{\mathcal{K}^*,\mathcal{L}^*}$ is the unit tangent vector to σ, oriented from $\mathcal{K}^*$ to $\mathcal{L}^*$, $\boldsymbol{\tau}_{\mathcal{K},\mathcal{L}}$ is the unit tangent vector to σ^*, oriented from

$\mathcal{K}$ to $\mathcal{L}$.

We introduce now the size of the mesh, $\mathrm{size}(\mathcal{T}) = \max_{\mathcal{D}\in\mathfrak{D}} d_{\mathcal{D}}$. We assume that the diamonds cannot be flat: there exists a unique $\theta_{\mathcal{T}} \in]0, \frac{\pi}{2}]$ such that $\sin(\theta_{\mathcal{T}}) := \min_{\mathcal{D}\in\mathfrak{D}}(|\sin(\theta_{\mathcal{D}})|)$. We also need some regularity of the mesh, as in [5]. We assume that there exists $\zeta > 0$ such that

$$\sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}}} \mathrm{m}_{\sigma}\mathrm{m}_{\sigma^*} \leq \frac{\mathrm{m}_{\mathcal{K}}}{\zeta}, \forall \mathcal{K} \in \mathfrak{M}, \qquad \text{and} \qquad \sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}} \mathrm{m}_{\sigma}\mathrm{m}_{\sigma^*} \leq \frac{\mathrm{m}_{\mathcal{K}^*}}{\zeta}, \forall \mathcal{K}^* \in \overline{\mathfrak{M}^*}, \tag{11a}$$

$$\mathrm{m}_{\mathcal{D}} \leq \frac{\mathrm{m}_{\mathcal{K}\cap\mathcal{K}^*}}{\zeta}, \forall \mathcal{D} \in \mathfrak{D}, \mathcal{K} \in \mathfrak{M}, \mathcal{K}^* \in \overline{\mathfrak{M}^*} \text{ such that } \mathrm{m}(\mathcal{D}\cap\mathcal{K}) \neq 0 \text{ and } \mathrm{m}(\mathcal{D}\cap\mathcal{K}^*) \neq 0. \tag{11b}$$

2.2. *Set of discrete unknowns*

We need several types of degrees of freedom to represent scalar and vector fields in the discrete setting. Let us introduce :

- $\mathbb{R}^{\mathcal{T}}$ the linear space of scalar fields constant on the cells of $\overline{\mathfrak{M}}$ and $\overline{\mathfrak{M}^*}$:

$$\mathbb{R}^{\mathcal{T}} = \left\{ u_{\mathcal{T}} = \left((u_{\mathcal{K}})_{\mathcal{K}\in\overline{\mathfrak{M}}}, (u_{\mathcal{K}^*})_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} \right), \text{ with } u_{\mathcal{K}} \in \mathbb{R},\ \forall \mathcal{K} \in \overline{\mathfrak{M}}, \text{ and } u_{\mathcal{K}^*} \in \mathbb{R},\ \forall \mathcal{K}^* \in \overline{\mathfrak{M}^*} \right\}.$$

- $\left(\mathbb{R}^2\right)^{\mathfrak{D}}$ the linear space of vector fields constant on the cells of $\mathfrak{D}$:

$$\left(\mathbb{R}^2\right)^{\mathfrak{D}} = \left\{ \boldsymbol{\xi}_{\mathfrak{D}} = (\boldsymbol{\xi}_{\mathcal{D}})_{\mathcal{D}\in\mathfrak{D}}, \text{ with } \boldsymbol{\xi}_{\mathcal{D}} \in \mathbb{R}^2,\ \forall \mathcal{D} \in \mathfrak{D} \right\}.$$

Similarly, we may define $\mathbb{R}^{\mathfrak{D}}$, $\mathbb{R}^{\mathfrak{D}^{ext}}$, $\mathbb{R}^{\partial\mathfrak{M}}$ the spaces of scalar fields constant respectively on $\mathfrak{D}$, $\mathfrak{D}^{ext}$ and $\partial\mathfrak{M}$ and $(\mathbb{R}^2)^{\mathfrak{D}^{ext}}$ the space of vector fields constant on $\mathfrak{D}^{ext}$. It permits to introduce two trace operators, defined respectively on $\mathbb{R}^{\mathcal{T}}$ and $\left(\mathbb{R}^2\right)^{\mathfrak{D}}$. The first one is $\gamma^{\mathcal{T}} : u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}} \mapsto \gamma^{\mathcal{T}}(u_{\mathcal{T}}) = \left(\gamma_{\mathcal{L}}(u_{\mathcal{T}})\right)_{\mathcal{L}\in\partial\mathfrak{M}} \in \mathbb{R}^{\partial\mathfrak{M}}$, defined by :

$$\gamma_{\mathcal{L}}(u_{\mathcal{T}}) = \frac{u_{\mathcal{K}^*} + 2u_{\mathcal{L}} + u_{\mathcal{L}^*}}{4}, \qquad \forall\ \mathcal{L} = [x_{\mathcal{K}^*}, x_{\mathcal{L}^*}] \in \partial\mathfrak{M}. \tag{12}$$

The second one is $\gamma^{\mathfrak{D}} : \boldsymbol{\varphi}^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}} \mapsto (\boldsymbol{\varphi}_{\mathcal{D}})_{\mathcal{D}\in\mathfrak{D}_{ext}} \in (\mathbb{R}^2)^{\mathfrak{D}_{ext}}$.

We define the scalar products $[\![\cdot,\cdot]\!]_{\mathcal{T}}$ on $\mathbb{R}^{\mathcal{T}}$ and $(\cdot,\cdot)_{\mathfrak{D}}$ on $\left(\mathbb{R}^2\right)^{\mathfrak{D}}$ by

$$\begin{aligned}
[\![v_{\mathcal{T}}, u_{\mathcal{T}}]\!]_{\mathcal{T}} &= \frac{1}{2}\left(\sum_{\mathcal{K}\in\mathfrak{M}} \mathrm{m}_{\mathcal{K}} u_{\mathcal{K}} v_{\mathcal{K}} + \sum_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} \mathrm{m}_{\mathcal{K}^*} u_{\mathcal{K}^*} v_{\mathcal{K}^*} \right), \quad \forall u_{\mathcal{T}}, v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}},\\
(\boldsymbol{\xi}_{\mathfrak{D}}, \boldsymbol{\varphi}_{\mathfrak{D}})_{\mathfrak{D}} &= \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}}\, \boldsymbol{\xi}_{\mathcal{D}} \cdot \boldsymbol{\varphi}_{\mathcal{D}}, \quad \forall \boldsymbol{\xi}_{\mathfrak{D}}, \boldsymbol{\varphi}_{\mathfrak{D}} \in \left(\mathbb{R}^2\right)^{\mathfrak{D}}.
\end{aligned}$$

The corresponding norms are denoted by $\|\cdot\|_{2,\mathcal{T}}$ and $\|\cdot\|_{2,\mathfrak{D}}$. More generally, we set for all $u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, $\boldsymbol{\xi}_{\mathfrak{D}} \in \left(\mathbb{R}^2\right)^{\mathfrak{D}}$ and $1 \leq p < +\infty$:

$$\begin{aligned}
\|u_{\mathcal{T}}\|_{p,\mathcal{T}} &= \left(\frac{1}{2} \sum_{\mathcal{K}\in\mathfrak{M}} \mathrm{m}_{\mathcal{K}} |u_{\mathcal{K}}|^p + \frac{1}{2} \sum_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} \mathrm{m}_{\mathcal{K}^*} |u_{\mathcal{K}^*}|^p \right)^{1/p}, & \left\|\boldsymbol{\xi}_{\mathfrak{D}}\right\|_{p,\mathfrak{D}} &= \left(\sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}} |\boldsymbol{\xi}_{\mathcal{D}}|^p \right)^{1/p},\\
\|u_{\mathcal{T}}\|_{\infty,\mathcal{T}} &= \max\left(\max_{\mathcal{K}\in\mathfrak{M}} |u_{\mathcal{K}}|, \max_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} |u_{\mathcal{K}^*}| \right), & \left\|\boldsymbol{\xi}_{\mathfrak{D}}\right\|_{\infty,\mathfrak{D}} &= \max_{\mathcal{D}\in\mathfrak{D}} \left|\boldsymbol{\xi}_{\mathcal{D}}\right|.
\end{aligned} \tag{13}$$

We also define the bilinear form $\langle\cdot,\cdot\rangle_{\partial\Omega}$ on $\mathbb{R}^{\mathfrak{D}_{ext}} \times \mathbb{R}^{\partial\mathfrak{M}}$ by

$$\langle \phi_{\mathfrak{D}}, v_{\partial\mathfrak{M}} \rangle_{\partial\Omega} = \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} \mathrm{m}_{\sigma} \phi_{\mathcal{D}} v_{\sigma}, \quad \forall\ \phi_{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}_{ext}}, \forall v_{\partial\mathfrak{M}} = (v_{\sigma})_{\sigma\in\partial\mathfrak{M}} \in \mathbb{R}^{\partial\mathfrak{M}}.$$

To a given vector $u_{\mathcal{T}} = \left((u_{\mathcal{K}})_{\mathcal{K}\in\overline{\mathfrak{M}}}, (u_{\mathcal{K}^*})_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}}\right) \in \mathbb{R}^{\mathcal{T}}$ defined on a DDFV mesh $\mathcal{T}$ of size h, we associate the approximate solution:

$$u_h = \frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} u_{\mathcal{K}}\mathbf{1}_{\mathcal{K}} + \frac{1}{2}\sum_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} u_{\mathcal{K}^*}\mathbf{1}_{\mathcal{K}^*}. \tag{14}$$

With this definition, we use simultaneously the values on the primal mesh and the values on the dual mesh. Indeed, we have $u_h = \frac{1}{2}(u_{h,\mathfrak{M}} + u_{h,\overline{\mathfrak{M}^*}})$, where $u_{h,\mathfrak{M}}$ and $u_{h,\overline{\mathfrak{M}^*}}$ are two different reconstructions based either on the primal values or the dual values:

$$u_{h,\mathfrak{M}} = \sum_{\mathcal{K}\in\mathfrak{M}} u_{\mathcal{K}}\mathbf{1}_{\mathcal{K}} \quad \text{and} \quad u_{h,\overline{\mathfrak{M}^*}} = \sum_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} u_{\mathcal{K}^*}\mathbf{1}_{\mathcal{K}^*}.$$

The space of the approximate solutions is denoted by $H_{\mathcal{T}}$:

$$H_{\mathcal{T}} = \left\{ u_h \in L^1(\Omega) \;/\; \exists u_{\mathcal{T}} = \left((u_{\mathcal{K}})_{\mathcal{K}\in\overline{\mathfrak{M}}}, (u_{\mathcal{K}^*})_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}}\right) \in \mathbb{R}^{\mathcal{T}} \text{ such that } u_h = \frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} u_{\mathcal{K}}\mathbf{1}_{\mathcal{K}} + \frac{1}{2}\sum_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} u_{\mathcal{K}^*}\mathbf{1}_{\mathcal{K}^*} \right\}. \tag{15}$$

In the sequel, we will also need some reconstruction of the approximate solutions on the diamond cells. Therefore, we associate to a given $u_h \in H_{\mathcal{T}}$ the piecewise constant function on diamond cells $u_{h,\mathfrak{D}}$, defined by:

$$u_{h,\mathfrak{D}}(x) = \sum_{\mathcal{D}\in\mathfrak{D}} u_{\mathcal{D}}\mathbf{1}_{\mathcal{D}} \quad \text{with} \quad u_{\mathcal{D}} = \frac{1}{\mathrm{m}_{\mathcal{D}}}\int_{\mathcal{D}} u_h(y)\mathrm{d}y \quad \forall \mathcal{D}\in\mathfrak{D}. \tag{16}$$

2.3. Discrete operators and duality formula

In this section, we recall the definition of the discrete operators: discrete gradient, discrete divergence operator and discrete convection operator. The discrete gradient has been introduced in [16] and developed in [18]. The discrete divergence has been introduced in [18].

Definition 2.1. *The discrete gradient is a mapping from* $\mathbb{R}^{\mathcal{T}}$ *to* $\left(\mathbb{R}^2\right)^{\mathfrak{D}}$ *defined for all* $u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ *by* $\nabla^{\mathfrak{D}} u_{\mathcal{T}} = \left(\nabla^{\mathcal{D}} u_{\mathcal{T}}\right)_{\mathcal{D}\in\mathfrak{D}}$, *where for* $\mathcal{D}\in\mathfrak{D}$ *:*

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{\sin(\theta_{\mathcal{D}})}\left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{\mathrm{m}_{\sigma^*}}\mathbf{n}_{\sigma\mathcal{K}} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{\mathrm{m}_{\sigma}}\mathbf{n}_{\sigma^*\mathcal{K}^*}\right).$$

Definition 2.2. *The discrete divergence operator* $\mathrm{div}^{\mathcal{T}}$ *is a mapping from* $\left(\mathbb{R}^2\right)^{\mathfrak{D}}$ *to* $\mathbb{R}^{\mathcal{T}}$ *defined for all* $\boldsymbol{\xi}_{\mathfrak{D}} \in \left(\mathbb{R}^2\right)^{\mathfrak{D}}$ *by*

$$\mathrm{div}^{\mathcal{T}}\boldsymbol{\xi}_{\mathfrak{D}} = \left(\mathrm{div}^{\mathfrak{M}}\boldsymbol{\xi}_{\mathfrak{D}}, \mathrm{div}^{\partial\mathfrak{M}}\boldsymbol{\xi}_{\mathfrak{D}}, \mathrm{div}^{\mathfrak{M}^*}\boldsymbol{\xi}_{\mathfrak{D}}, \mathrm{div}^{\partial\mathfrak{M}^*}\boldsymbol{\xi}_{\mathfrak{D}}\right),$$

with $\mathrm{div}^{\mathfrak{M}}\boldsymbol{\xi}_{\mathfrak{D}} = (\mathrm{div}_{\mathcal{K}}\boldsymbol{\xi}_{\mathfrak{D}})_{\mathcal{K}\in\mathfrak{M}}$, $\mathrm{div}^{\partial\mathfrak{M}}\boldsymbol{\xi}_{\mathfrak{D}} = 0$, $\mathrm{div}^{\mathfrak{M}^*}\boldsymbol{\xi}_{\mathfrak{D}} = (\mathrm{div}_{\mathcal{K}^*}\boldsymbol{\xi}_{\mathfrak{D}})_{\mathcal{K}^*\in\mathfrak{M}^*}$ *and* $\mathrm{div}^{\partial\mathfrak{M}^*}\boldsymbol{\xi}_{\mathfrak{D}} = (\mathrm{div}_{\mathcal{K}^*}\boldsymbol{\xi}_{\mathfrak{D}})_{\mathcal{K}^*\in\partial\mathfrak{M}^*}$ *such that:*

$$\forall\, \mathcal{K}\in\mathfrak{M}, \mathrm{div}_{\mathcal{K}}\boldsymbol{\xi}_{\mathfrak{D}} = \frac{1}{\mathrm{m}_{\mathcal{K}}}\sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}}\\ \mathcal{D}=\mathcal{D}_{\sigma,\sigma^*}}} \mathrm{m}_{\sigma}\,\boldsymbol{\xi}_{\mathcal{D}}\cdot\mathbf{n}_{\sigma\mathcal{K}},$$

and analogous definitions for $\mathrm{div}_{\mathcal{K}^*}\boldsymbol{\xi}_{\mathfrak{D}}$ *for* $\mathcal{K}^*\in\overline{\mathfrak{M}^*}$ *(see [10]).*

Discrete Duality Finite Volume methods are based on the discrete duality formula recalled in Theorem 2.3 and proved for instance in [18]. This is the discrete counterpart of the Green formula.

Theorem 2.3. *For all* $(\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}}) \in \left(\mathbb{R}^2\right)^{\mathfrak{D}} \times \mathbb{R}^{\mathcal{T}}$, *we have*

$$[\![\mathrm{div}^{\mathcal{T}}\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}}]\!]_{\mathcal{T}} = -(\boldsymbol{\xi}_{\mathfrak{D}}, \nabla^{\mathfrak{D}} v_{\mathcal{T}})_{\mathfrak{D}} + \langle\gamma^{\mathfrak{D}}(\boldsymbol{\xi}_{\mathfrak{D}})\cdot\mathbf{n}, \gamma^{\mathcal{T}}(v_{\mathcal{T}})\rangle_{\partial\Omega},$$

where $\mathbf{n}$ *is the exterior unit normal to* Ω.

The discrete convection operator has been introduced in [10]. It is similar with previous definitions given by Andreianov, Bendahmane and Karlsen in [4] and by Coudière and Manzini in [12].

Definition 2.4. *The discrete convergence operator* $\mathrm{divc}^{\mathcal{T}}$ *is a mapping from* $\left(\mathbb{R}^2\right)^{\mathfrak{D}} \times \mathbb{R}^{\mathcal{T}}$ *to* $\mathbb{R}^{\mathcal{T}}$ *defined for all* $\boldsymbol{\xi}_{\mathfrak{D}} \in \left(\mathbb{R}^2\right)^{\mathfrak{D}}$ *and* $v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ *by*

$$\mathrm{divc}^{\mathcal{T}}(\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}}) = \left(\mathrm{divc}^{\mathfrak{M}}(\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}}), \mathrm{divc}^{\partial\mathfrak{M}}(\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}}), \mathrm{divc}^{\mathfrak{M}^*}(\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}}), \mathrm{divc}^{\partial\mathfrak{M}^*}(\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}})\right),$$

with $\mathrm{divc}^{\mathfrak{M}}(\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}}) = (\mathrm{divc}_{\mathcal{K}}(\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}}))_{\mathcal{K}\in\mathfrak{M}}$, $\mathrm{divc}^{\partial\mathfrak{M}}(\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}}) = 0$, $\mathrm{divc}^{\mathfrak{M}^*}(\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}}) = (\mathrm{divc}_{\mathcal{K}^*}(\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}}))_{\mathcal{K}^*\in\mathfrak{M}^*}$ *and* $\mathrm{divc}^{\partial\mathfrak{M}^*}(\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}}) = (\mathrm{divc}_{\mathcal{K}^*}(\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}}))_{\mathcal{K}^*\in\partial\mathfrak{M}^*}$ *such that:*

$$\forall\, \mathcal{K} \in \mathfrak{M},\ \mathrm{divc}_{\mathcal{K}}(\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}}) = \frac{1}{\mathrm{m}_{\mathcal{K}}} \sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}}\\ \mathcal{D}=\mathcal{D}_{\sigma,\sigma^*}}} \mathrm{m}_{\sigma} \left((\boldsymbol{\xi}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}})^+ v_{\mathcal{K}} - (\boldsymbol{\xi}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}})^- v_{\mathcal{L}}\right),$$

where $x^+ = \max(x, 0)$ *and* $x^- = -\min(x, 0)$ *for all* $x \in \mathbb{R}$*, and analogous definitions for* $\mathrm{divc}_{\mathcal{K}^*}(\boldsymbol{\xi}_{\mathfrak{D}}, v_{\mathcal{T}})$ *for* $\mathcal{K}^* \in \overline{\mathfrak{M}^*}$ *(see [10]).*

2.4. A penalization operator

Let us introduce now a penalization operator as in [4]. This operator has not been introduced in our previous work [10]. However, we will see that it is essential when passing to the limit in the scheme, especially in the convection term in (2a). Indeed, the penalization operator will ensure that the reconstructions of the concentration on the primal mesh and on the dual mesh converge to the same limit.

Definition 2.5. *Let* $\beta \in]0, 2[$*. The penalization operator* $\mathcal{P}^{\mathcal{T}} : \mathbb{R}^{\mathcal{T}} \to \mathbb{R}^{\mathcal{T}}$ *is defined for all* $u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$*, by:*

$$\mathcal{P}^{\mathcal{T}} u_{\mathcal{T}} = \left(\mathcal{P}^{\mathfrak{M}} u_{\mathcal{T}}, \mathcal{P}^{\partial\mathfrak{M}} u_{\mathcal{T}}, \mathcal{P}^{\mathfrak{M}^*} u_{\mathcal{T}}, \mathcal{P}^{\partial\mathfrak{M}^*} u_{\mathcal{T}}\right),$$

with $\mathcal{P}^{\mathfrak{M}} u_{\mathcal{T}} = (\mathcal{P}_{\mathcal{K}} u_{\mathcal{T}})_{\mathcal{K}\in\mathfrak{M}}$, $\mathcal{P}^{\partial\mathfrak{M}} u_{\mathcal{T}} = 0$, $\mathcal{P}^{\mathfrak{M}^*} u_{\mathcal{T}} = (\mathcal{P}_{\mathcal{K}^*} u_{\mathcal{T}})_{\mathcal{K}^*\in\mathfrak{M}^*}$ *and* $\mathcal{P}^{\partial\mathfrak{M}^*} u_{\mathcal{T}} = (\mathcal{P}_{\mathcal{K}^*} u_{\mathcal{T}})_{\mathcal{K}^*\in\partial\mathfrak{M}^*}$ *such that*

$$\forall\, \mathcal{K} \in \mathfrak{M},\ \mathcal{P}_{\mathcal{K}} u_{\mathcal{T}} = \frac{1}{\mathrm{m}_{\mathcal{K}}} \frac{1}{\mathrm{size}(\mathcal{T})^{\beta}} \sum_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} \mathrm{m}_{\mathcal{K}\cap\mathcal{K}^*} (u_{\mathcal{K}} - u_{\mathcal{K}^*}),$$

$$\forall\, \mathcal{K}^* \in \overline{\mathfrak{M}^*},\ \mathcal{P}_{\mathcal{K}^*} u_{\mathcal{T}} = \frac{1}{\mathrm{m}_{\mathcal{K}^*}} \frac{1}{\mathrm{size}(\mathcal{T})^{\beta}} \sum_{\mathcal{K}\in\mathfrak{M}} \mathrm{m}_{\mathcal{K}\cap\mathcal{K}^*} (u_{\mathcal{K}^*} - u_{\mathcal{K}}).$$

The penalization operator clearly satisfies the following property:

$$[\![\mathcal{P}^{\mathcal{T}} u_{\mathcal{T}}, u_{\mathcal{T}}]\!]_{\mathcal{T}} = \frac{1}{2} \frac{1}{\mathrm{size}(\mathcal{T})^{\beta}} \sum_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} \sum_{\mathcal{K}\in\mathfrak{M}} \mathrm{m}_{\mathcal{K}\cap\mathcal{K}^*} (u_{\mathcal{K}} - u_{\mathcal{K}^*})^2 = \frac{1}{2} \frac{1}{\mathrm{size}(\mathcal{T})^{\beta}} \|u_{h,\mathfrak{M}} - u_{h,\overline{\mathfrak{M}^*}}\|^2_{L^2(\Omega)}. \tag{17}$$

2.5. The numerical scheme

Let $(\mathcal{T}, \mathfrak{D})$ be a DDFV mesh of Ω (as presented in Section 2.1) and $\delta t > 0$ be a time step. We set $N_T = T/\delta t$ (we always choose time steps such that N_T is an integer) and we define $t_n = n\delta t$ for $n \in \{0, \dots, N_T\}$.

First, we discretize all the data of the problem. Therefore, we introduce $\mathbb{P}_{\mathcal{K}}$ (respectively $\mathbb{P}_{\mathcal{K}^*}$) the L^2 projection over an interior primal (respectively dual) cell. We then define $c^0_{\mathcal{T}} = \left((\mathbb{P}_{\mathcal{K}} c_0)_{\mathcal{K}\in\mathfrak{M}}, 0, (\mathbb{P}_{\mathcal{K}^*} c_0)_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}}\right) \in \mathbb{R}^{\mathcal{T}}$. and $\Phi_{\mathcal{T}} = \left((\mathbb{P}_{\mathcal{K}} \Phi)_{\mathcal{K}\in\mathfrak{M}}, 0, (\mathbb{P}_{\mathcal{K}^*} \Phi)_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}}\right) \in \mathbb{R}^{\mathcal{T}}$. In a similar way, for all $n \geq 1$, we define $(q^{+,n}_{\mathcal{T}}, q^{-,n}_{\mathcal{T}}, \hat{c}^n_{\mathcal{T}}) \in (\mathbb{R}^{\mathcal{T}})^3$ by taking the mean values of q^+, q^- and $\hat{c}$ on the primal and dual cells crossed with the time interval (t_{n-1}, t_n). For $w = q^+, q^-, \hat{c}$, it writes:

$$w^n_{\mathcal{T}} = \frac{1}{\delta t} \int_{t_{n-1}}^{t_n} \left((\mathbb{P}_{\mathcal{K}} w(.,t))_{\mathcal{K}\in\mathfrak{M}}, 0, (\mathbb{P}_{\mathcal{K}^*} w(.,t))_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}}\right) dt.$$

At each time step n, the numerical solution will be given by $(p^n_{\mathcal{T}}, \mathbf{U}^n_{\mathfrak{D}}, c^n_{\mathcal{T}}) \in \mathbb{R}^{\mathcal{T}} \times \left(\mathbb{R}^2\right)^{\mathfrak{D}} \times \mathbb{R}^{\mathcal{T}}$ and the computation of the pressure and the velocity $(p^n_{\mathcal{T}}, \mathbf{U}^n_{\mathfrak{D}})$ will be decoupled from the computation of the concentration $(c^n_{\mathcal{T}})$. Due to the coupling in the Darcy law (1b), we need to reconstruct some approximate values on the diamond cells $c^{n-1}_{\mathfrak{D}} = (c^{n-1}_{\mathcal{D}})_{\mathcal{D}\in\mathfrak{D}}$ from $c^{n-1}_{\mathcal{T}}$ following (16). We may also introduce the approximate tensors

$$\mathbb{A}_{\mathcal{D}}(s) = \frac{1}{\mathrm{m}_{\mathcal{D}}} \int_{\mathcal{D}} \mathbb{A}(x,s)dx \;\forall s \in \mathbb{R}, \quad \mathbb{D}_{\mathcal{D}}(\mathbf{W}) = \frac{1}{\mathrm{m}_{\mathcal{D}}} \int_{\mathcal{D}} \mathbb{D}(x,\mathbf{W})dx, \;\forall \mathbf{W} \in \mathbb{R}^2.$$

It permits to define $\mathbb{A}_{\mathfrak{D}}(c^{n-1}_{\mathfrak{D}}) = \left(\mathbb{A}_{\mathcal{D}}(c^{n-1}_{\mathcal{D}})\right)_{\mathcal{D}\in\mathfrak{D}}$ and $\mathbb{D}_{\mathfrak{D}}\left(\mathbf{U}^n_{\mathfrak{D}}\right) = \left(\mathbb{D}_{\mathcal{D}}(\mathbf{U}^n_{\mathcal{D}})\right)_{\mathcal{D}\in\mathfrak{D}}$.

Then, the scheme for (1) writes:

$$\mathrm{div}^{\mathcal{T}}\left(\mathbf{U}^n_{\mathfrak{D}}\right) = q^{+,n}_{\mathcal{T}} - q^{-,n}_{\mathcal{T}}, \;\forall 1 \le n \le N_T, \tag{18a}$$

$$\mathbf{U}^n_{\mathfrak{D}} = -\mathbb{A}_{\mathfrak{D}}(c^{n-1}_{\mathfrak{D}})\nabla^{\mathfrak{D}} p^n_{\mathcal{T}}, \;\forall 1 \le n \le N_T, \tag{18b}$$

$$\mathbf{U}^n_{\mathcal{D}} \cdot \mathbf{n} = 0, \quad \forall \mathcal{D} \in \mathfrak{D}_{ext}, \;\forall 1 \le n \le N_T, \tag{18c}$$

$$\sum_{\mathcal{K}\in\mathfrak{M}} \mathrm{m}_{\mathcal{K}} p^n_{\mathcal{K}} = \sum_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} \mathrm{m}_{\mathcal{K}^*} p^n_{\mathcal{K}^*} = 0, \;\forall 1 \le n \le N_T, \tag{18d}$$

and the scheme for (2) writes:

$$\Phi_{\mathcal{T}} \frac{c^n_{\mathcal{T}} - c^{n-1}_{\mathcal{T}}}{\delta t} - \mathrm{div}^{\mathcal{T}}\left(\mathbb{D}_{\mathfrak{D}}\left(\mathbf{U}^n_{\mathfrak{D}}\right)\nabla^{\mathfrak{D}} c^n_{\mathcal{T}}\right) + \mathrm{divc}^{\mathcal{T}}\left(\mathbf{U}^n_{\mathfrak{D}}, c^n_{\mathcal{T}}\right) + q^{-,n}_{\mathcal{T}} c^n_{\mathcal{T}} + \lambda\mathcal{P}^{\mathcal{T}}(c^n_{\mathcal{T}}) = q^{+,n}_{\mathcal{T}} \hat{c}^n_{\mathcal{T}}, \;\forall 1 \le n \le N_T, \tag{19a}$$

$$\mathbb{D}_{\mathcal{D}}\left(\mathbf{U}^n_{\mathcal{D}}\right)\nabla^{\mathcal{D}} c^n_{\mathcal{T}} \cdot \mathbf{n} = 0, \forall \mathcal{D} \in \mathfrak{D}_{ext}, \;\forall 1 \le n \le N_T. \tag{19b}$$

Note that λ is a positive constant. The scheme (18)–(19) comes down to a resolution of two linear systems: starting from $c^{n-1}_{\mathcal{T}}$, $(p^n_{\mathcal{T}}, \mathbf{U}^n_{\mathfrak{D}})$ is obtained by solving the linear system (18a)–(18d) and then $c^n_{\mathcal{T}}$ is computed by solving the linear system (19a)-(19b). Existence and uniqueness of a solution to each linear system has been proved in [10] in the case where $\lambda = 0$. This result is based on the *a priori* estimates satisfied by the discrete pressure and the discrete concentration. It remains true in the case where $\lambda > 0$ because the same *a priori* estimates on the pressure and the concentration still hold (see Lemma 3.1 and Lemma 3.2 in Section 3).

2.6. Definition of the functional spaces for approximate solutions

As we are interested in the numerical analysis of the scheme (and particularly in its convergence analysis), we need to define some functional spaces for the approximate solutions.

We have already defined in (15) the space of approximate solutions $H_{\mathcal{T}}$. For a function $u_h \in H_{\mathcal{T}}$, we define its approximate gradient $\nabla^h u_h$ by

$$\nabla^h u_h = \sum_{\mathcal{D}\in\mathfrak{D}} \nabla^{\mathcal{D}} u_{\mathcal{T}} \mathbf{1}_{\mathcal{D}}.$$

This approximate gradient is a piecewise constant function on each diamond. The space of such functions is denoted by $H_{\mathfrak{D}}$:

$$H_{\mathfrak{D}} = \left\{ \mathbf{U}_h \in (L^1(\Omega))^2 \;/\; \exists \mathbf{U}_{\mathfrak{D}} \in \left(\mathbb{R}^2\right)^{\mathfrak{D}} \text{ such that } \mathbf{U}_h = \sum_{\mathcal{D}\in\mathfrak{D}} \mathbf{U}_{\mathcal{D}} \mathbf{1}_{\mathcal{D}} \right\}.$$

Then, we define the space-time approximation spaces $H_{\mathcal{T},\delta t}$ and $H_{\mathfrak{D},\delta t}$ based respectively on $H_{\mathcal{T}}$ and $H_{\mathfrak{D}}$:

$$\begin{aligned}
H_{\mathcal{T},\delta t} &= \left\{ u_{h,\delta t} \in L^1([0,T]\times\Omega) \text{ such that } u_{h,\delta t}(t,x) = u^n_h(x) \;\forall t \in [t_{n-1}, t_n), \text{ with } u^n_h \in H_{\mathcal{T}}, \;\forall 1 \le n \le N_T \right\}, \\
H_{\mathfrak{D},\delta t} &= \left\{ \mathbf{U}_{h,\delta t} \in (L^1([0,T]\times\Omega))^2 \text{ such that } \mathbf{U}_{h,\delta t}(t,x) = \mathbf{U}^n_h(x) \;\forall t \in [t_{n-1}, t_n), \text{ with } \mathbf{U}^n_h \in H_{\mathfrak{D}}, \;\forall 1 \le n \le N_T \right\}.
\end{aligned}$$

We still keep the notation ∇^h to define the approximate gradient of $u_{h,\delta t} \in H_{\mathcal{T},\delta t}$:

$$\nabla^h u_{h,\delta t}(x,t) = \nabla^h u^n_h(x) \;\forall t \in [t_{n-1}, t_n).$$

Therefore, for all $u_{h,\delta t} \in H_{\mathcal{T},\delta t}$, we have $\nabla^h u_{h,\delta t} \in H_{\mathfrak{D},\delta t}$. Furthermore, we introduce the following reconstructions

$$u_{h,\delta t,\mathfrak{M}}(t,x) = u^n_{h,\mathfrak{M}}(x) = \sum_{\mathcal{K}\in\mathfrak{M}} u^n_{\mathcal{K}} \mathbf{1}_{\mathcal{K}}(x), \qquad \forall t \in [t_{n-1}, t_n), \tag{20a}$$

$$u_{h,\delta t,\overline{\mathfrak{M}^*}}(t,x) = u^n_{h,\overline{\mathfrak{M}^*}}(x) = \sum_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} u^n_{\mathcal{K}^*} \mathbf{1}_{\mathcal{K}^*}(x), \qquad \forall t \in [t_{n-1}, t_n), \tag{20b}$$

$$u_{h,\delta t,\mathfrak{D}}(t,x) = u^n_{h,\mathfrak{D}}(x) = \sum_{\mathcal{D}\in\mathfrak{D}} u^n_{\mathcal{D}}\, \mathbf{1}_{\mathcal{D}}(x), \qquad \forall t \in [t_{n-1}, t_n). \tag{20c}$$

We may now define some norms on $H_{\mathcal{T}}$, $H_{\mathcal{T},\delta t}$. First, we define some discrete $W^{1,p}$-norms ($1 \le p \le +\infty$) and a discrete $W^{1,-1}$-norm on $H_{\mathcal{T}}$. For all $u_h \in H_{\mathcal{T}}$, we set

$$\begin{aligned}
\|u_h\|_{1,p,\mathcal{T}} &= \left(\|u_{\mathcal{T}}\|^p_{p,\mathcal{T}} + \left\|\nabla^{\mathfrak{D}} u_{\mathcal{T}}\right\|^p_{p,\mathfrak{D}}\right)^{1/p}, \quad \forall 1 \le p < +\infty,\\
\|u_h\|_{1,\infty,\mathcal{T}} &= \|u_{\mathcal{T}}\|_{\infty,\mathcal{T}} + \left\|\nabla^{\mathfrak{D}} u_{\mathcal{T}}\right\|_{\infty,\mathfrak{D}},\\
\|u_h\|_{1,\infty\star,\mathcal{T}} &= \|u_h\|_{1,\infty,\mathcal{T}} + [\![\mathcal{P}^{\mathcal{T}} u_{\mathcal{T}}, u_{\mathcal{T}}]\!]^{\frac{1}{2}}_{\mathcal{T}},\\
\|u_h\|_{1,-1,\mathcal{T}} &= \max\left\{ [\![v_{\mathcal{T}}, u_{\mathcal{T}}]\!]_{\mathcal{T}}, \forall v_h \in H_{\mathcal{T}} \text{ verifying } \|v_h\|_{1,\infty\star,\mathcal{T}} \le 1\right\},
\end{aligned}$$

where the norms $\|\cdot\|_{p,\mathcal{T}}$ and $\|\cdot\|_{p,\mathfrak{D}}$ have been defined by (13) and the penalization operator $\mathcal{P}^{\mathcal{T}}$ is given in Definition 2.5. Then, we define some discrete $L^1(0,T;W^{1,p}(\Omega))$ ($1 \le p < +\infty$), $L^\infty(0,T;W^{1,\infty}(\Omega))$ and $L^\infty(0,T;L^p(\Omega))$-norms on $H_{\mathcal{T},\delta t}$. For all $u_{h,\delta t} \in H_{\mathcal{T},\delta t}$, we set:

$$\begin{aligned}
\left\|u_{h,\delta t}\right\|_{1;1,p,\mathcal{T}} &= \sum_{n=1}^{N_T} \delta t \left\|u^n_h\right\|_{1,p,\mathcal{T}}, \quad \forall 1 \le p < +\infty,\\
\left\|u_{h,\delta t}\right\|_{\infty;1,\infty,\mathcal{T}} &= \max_{n\in\{1,\cdots,N_T\}} \left\|u^n_h\right\|_{1,\infty,\mathcal{T}},\\
\left\|u_{h,\delta t}\right\|_{\infty;0,p,\mathcal{T}} &= \max_{n\in\{1,\cdots,N_T\}} \left(\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} \mathrm{m}_{\mathcal{K}} |u^n_{\mathcal{K}}|^p + \frac{1}{2}\sum_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} \mathrm{m}_{\mathcal{K}^*} |u^n_{\mathcal{K}^*}|^p\right)^{1/p}, \quad \forall 1 \le p < +\infty.
\end{aligned}$$

Let us also remark that, for all $\mathbf{U}_{h,\delta t} \in H_{\mathfrak{D},\delta t}$ and for $1 \le p < +\infty$, we have

$$\begin{aligned}
\|\mathbf{U}_{h,\delta t}\|_{(L^\infty(0,T;L^p(\Omega)))^2} &= \max_{n\in\{1,\cdots,N_T\}} \left(\sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}} |\mathbf{U}^n_h|^p\right)^{1/p},\\
\|\mathbf{U}_{h,\delta t}\|_{(L^p((0,T)\times\Omega))^2} &= \left(\sum_{n=1}^{N_T} \delta t \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}} |\mathbf{U}^n_h|^p\right)^{1/p}.
\end{aligned}$$

2.7. Main result

We may now state the main result of the paper.

Theorem 2.6. *Let Ω be an open bounded connected polygonal domain of $\mathbb{R}^2$ and $T > 0$. Assume* (3)–(8) *hold, $\lambda > 0$ and $\beta \in]0,2[$. Let $(\mathcal{T}_m)_{m\ge1}$ be a sequence of DDFV meshes such that $h_m = size(\mathcal{T}_m) \underset{m\to\infty}{\longrightarrow} 0$ while the regularity parameters ζ_m and θ_m verifying:*

$$\exists \theta > 0, \zeta > 0 \text{ such that}, \forall m, \quad \theta_m \ge \theta \text{ and } \zeta_m \le \zeta. \tag{21}$$

Let $(\delta t_m)_{m\ge1}$ be a sequence of time steps such that $T/\delta t_m$ is an integer and $\delta t_m \underset{m\to\infty}{\longrightarrow} 0$. Then, the scheme (18)–(19) *defines a sequence of approximate solutions $(p_m = p_{h_m,\delta t_m}, \mathbf{U}_m = \mathbf{U}_{h_m,\delta t_m}, c_m = c_{h_m,\delta t_m}) \in H_{\mathcal{T}_m,\delta t_m} \times H_{\mathfrak{D}_m,\delta t_m} \times H_{\mathcal{T}_m,\delta t_m}$,* *there exists $\bar{p} \in L^\infty(0,T;H^1(\Omega))$,$\bar{\mathbf{U}} \in L^\infty(0,T;L^2(\Omega))^2$ and $\bar{c} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, and, up to a subsequence, we have the following convergence results when $m \to \infty$:*

$$\begin{aligned}
&p_m \to \bar{p} && \textit{weakly-}* \textit{ in } L^\infty(0,T;L^2(\Omega)) \textit{ and strongly in } L^p(0,T;L^q(\Omega)), \forall p<\infty, q<2;\\
&\nabla^{h_m} p_m \to \nabla\bar{p} && \textit{weakly-}* \textit{ in } (L^\infty(0,T;L^2(\Omega)))^2 \textit{ and strongly in } (L^2((0,T)\times\Omega))^2;\\
&\mathbf{U}_m \to \bar{\mathbf{U}} && \textit{weakly-}* \textit{ in } (L^\infty(0,T;L^2(\Omega)))^2 \textit{ and strongly in } (L^2((0,T)\times\Omega))^2;\\
&c_m \to \bar{c} && \textit{weakly-}* \textit{ in } L^\infty(0,T;L^2(\Omega)) \textit{ and strongly in } L^p(0,T;L^q(\Omega)), \forall p<\infty, q<2;\\
&\nabla^{h_m} c_m \to \nabla\bar{c} && \textit{weakly in } (L^2((0,T)\times\Omega))^2.
\end{aligned}$$

Moreover, $(\bar{p}, \bar{\mathbf{U}}, \bar{c})$ is a weak solution to (1)-(2).

In order to prove this result, we split the proof in different steps. Firstly, we establish some *a priori* estimates satisfied by the scheme (Section 3). Then, thanks to these estimates and to some properties of the spaces of approximate solution (Section 4), it allows us to apply a discrete counterpart of Aubin-Simon theorem, proved by Gallouët-Latché [31] and to show the compactness of the sequences of approximate concentrations and of approximate pressures. Then we can pass to the limit in the scheme for the pressure and in the scheme for the concentration.

For the sake of simplicity, we will restrict the proof of Theorem 2.6 to the case where the porosity Φ is constant on the whole domain ($\Phi = \Phi^*$). Indeed, in this case, the proof of the compactness of the sequence of approximate concentration is simpler and based on the paper by Gallouët-Latché [31].

3. Estimates on the approximate solution

In this Section, we first prove *a priori* estimates satisfied by a solution to the scheme. Lemma 3.1 gives *a priori* estimates on the pressure, the gradient of the pressure and the Darcy's velocity at the discrete level, while Lemma 3.2 gives *a priori* estimates on the approximate concentration and its approximate gradient. These *a priori* estimates imply that the two linear systems (18)-(19) have a unique solution, as in [10]. They also provide some compactness in space of the sequences of approximate solution. Then, Lemma 3.3 shows that the reconstructions of the concentration on the primal and dual meshes will necessarily converge to the same limit (when convergence occurs). In Lemma 3.4, we give an estimate on the discrete time derivatives of the approximate concentration, which will be essential to apply the compactness result by Gallouët-Latché [31].

Lemma 3.1. *Let assume the hypotheses of Theorem 2.6. If their exists a solution, denoted by $(p_{h,\delta t}, \mathbf{U}_{h,\delta t}, c_{h,\delta t}) \in H_{\mathcal{T},\delta t} \times H_{\mathfrak{D},\delta t} \times H_{\mathcal{T},\delta t}$, to the scheme* (18)–(19)*, then this solution satisfies*

$$\left\|p_{h,\delta t}\right\|_{\infty;0,2,\mathcal{T}} + \left\|\nabla^h p_{h,\delta t}\right\|_{(L^\infty(0,T;L^2(\Omega)))^2} + \left\|\mathbf{U}_{h,\delta t}\right\|_{(L^\infty(0,T;L^2(\Omega)))^2} \le C\|q^+ - q^-\|_{L^\infty(0,T;L^2(\Omega))}. \tag{22}$$

where $C > 0$ depends only on Ω, ζ, θ, α_A and Λ_A.

Proof. Inequality (22) is a direct consequence of Lemma 3.2 in [10]. □

Lemma 3.2. *Let assume the hypotheses of Theorem 2.6. If their exists a solution, denoted by $(p_{h,\delta t}, \mathbf{U}_{h,\delta t}, c_{h,\delta t}) \in H_{\mathcal{T},\delta t} \times H_{\mathfrak{D},\delta t} \times H_{\mathcal{T},\delta t}$, to the scheme* (18)–(19)*, then this solution satisfies*

$$\left\|c_{h,\delta t}\right\|^2_{\infty;0,2,\mathcal{T}} + \left\|\nabla^h c_{h,\delta t}\right\|^2_{(L^2((0,T)\times\Omega))^2} + \left\||\mathbf{U}_{h,\delta t}|^{\frac{1}{2}} \nabla^h c_{h,\delta t}\right\|^2_{(L^2((0,T)\times\Omega))^2} \le C\left(\|c_0\|^2_{L^2(\Omega)} + \|q^+\|^2_{L^\infty(0,T,L^2(\Omega))}\right). \tag{23}$$

$$\lambda \sum_{n=1}^{N_T} \delta t [\![\mathcal{P}^{\mathcal{T}}(c^n_{\mathcal{T}}), c^n_{\mathcal{T}}]\!]_{\mathcal{T}} \le C\left(\|c_0\|^2_{L^2(\Omega)} + \|q^+\|^2_{L^\infty(0,T,L^2(\Omega))}\right). \tag{24}$$

where $C > 0$ depends only on Ω, T, ζ, θ, Φ_* and α_D.

Proof. The proof is very close to the proof of Lemma 3.3 in [10]. We multiply the scheme (19a) by $c_{\mathcal{T}}^n$. It yields

$$\left[\!\!\left[\Phi_{\mathcal{T}}\frac{c_{\mathcal{T}}^n - c_{\mathcal{T}}^{n-1}}{\delta t}, c_{\mathcal{T}}^n\right]\!\!\right]_{\mathcal{T}} - \left[\!\!\left[\operatorname{div}^{\mathcal{T}}\left(\mathbb{D}_{\mathfrak{D}}\left(\mathbf{U}_{\mathfrak{D}}^n\right)\nabla^{\mathfrak{D}}c_{\mathcal{T}}^n\right), c_{\mathcal{T}}^n\right]\!\!\right]_{\mathcal{T}} + \left[\!\!\left[\operatorname{divc}^{\mathcal{T}}\left(\mathbf{U}_{\mathfrak{D}}^n, c_{\mathcal{T}}^n\right), c_{\mathcal{T}}^n\right]\!\!\right]_{\mathcal{T}} + [\![q_{\mathcal{T}}^{-,n}c_{\mathcal{T}}^n, c_{\mathcal{T}}^n]\!]_{\mathcal{T}}$$
$$+ \lambda[\![\mathcal{P}^{\mathcal{T}}(c_{\mathcal{T}}^n), c_{\mathcal{T}}^n]\!]_{\mathcal{T}} = \left[\!\!\left[q_{\mathcal{T}}^{+,n}\hat{c}_{\mathcal{T}}^n, c_{\mathcal{T}}^n\right]\!\!\right]_{\mathcal{T}}.$$

Following the same computations as in [10], we get

$$\frac{1}{2\delta t}\left([\![\Phi_{\mathcal{T}}, (c_{\mathcal{T}}^n)^2]\!]_{\mathcal{T}} - [\![\Phi_{\mathcal{T}}, (c_{\mathcal{T}}^{n-1})^2]\!]_{\mathcal{T}}\right) + \alpha_D\left(\left\|\nabla^{\mathfrak{D}}c_{\mathcal{T}}^n\right\|_{2,\mathfrak{D}}^2 + \left\||\mathbf{U}_{\mathfrak{D}}^n|^{\frac{1}{2}}\nabla^{\mathfrak{D}}c_{\mathcal{T}}^n\right\|_{2,\mathfrak{D}}^2\right) + \lambda[\![\mathcal{P}^{\mathcal{T}}(c_{\mathcal{T}}^n), c_{\mathcal{T}}^n]\!]_{\mathcal{T}} \le \left\|q_{\mathcal{T}}^{+,n}\right\|_{2,\mathcal{T}}\left\|c_{\mathcal{T}}^n\right\|_{2,\mathcal{T}}.$$

Multiplying by $2\delta t$ and summing over $n = 1, \ldots, N$ with $1 \le N \le N_T$, we get

$$\Phi_*\left\|c_{\mathcal{T}}^N\right\|_{2,\mathcal{T}}^2 + 2\alpha_D\sum_{n=1}^{N}\delta t\left(\left\|\nabla^{\mathfrak{D}}c_{\mathcal{T}}^n\right\|_{2,\mathfrak{D}}^2 + \left\||\mathbf{U}_{\mathfrak{D}}^n|^{\frac{1}{2}}\nabla^{\mathfrak{D}}c_{\mathcal{T}}^n\right\|_{2,\mathfrak{D}}^2\right) + 2\lambda\sum_{n=1}^{N}\delta t[\![\mathcal{P}^{\mathcal{T}}(c_{\mathcal{T}}^n), c_{\mathcal{T}}^n]\!]_{\mathcal{T}}$$
$$\le \Phi_*^{-1}\|c_0\|_{L^2(\Omega)}^2 + \frac{2T^2}{\Phi_*}\|q^+\|_{L^\infty(0,T,L^2(\Omega))}^2 + \frac{\Phi_*}{2}\sup_{1\le n\le N_T}\|c_{\mathcal{T}}^n\|_{2,\mathcal{T}}^2. \quad (25)$$

Thanks to (17), the contribution of the penalization is positive and therefore we conclude the proof of (23) by taking the supremum over $1 \le N \le N_T$. Then, restarting from (25), we obtain (24).

□

Thanks to Lemma 3.1 and 3.2, we have the existence and uniqueness of a solution $(p_{h,\delta t}, \mathbf{U}_{h,\delta t}, c_{h,\delta t}) \in H_{\mathcal{T},\delta t} \times H_{\mathfrak{D},\delta t} \times H_{\mathcal{T},\delta t}$ to the linear scheme (18)–(19). We refer to the proof of Theorem 3.4 in [10].

Lemma 3.3. *Under the hypotheses of Theorem 2.6, there exists $C > 0$ depending only on Ω, T, ζ, θ, Φ_* and α_D such that the solution $(p_{h,\delta t}, \mathbf{U}_{h,\delta t}, c_{h,\delta t}) \in H_{\mathcal{T},\delta t} \times H_{\mathfrak{D},\delta t} \times H_{\mathcal{T},\delta t}$ to the scheme* (18)–(19) *verifies*

$$\|c_{h,\delta t,\mathfrak{M}} - c_{h,\delta t,\overline{\mathfrak{M}^*}}\|_{L^2(0,T;L^2(\Omega))}^2 \le \frac{C}{\lambda}h^\beta\left(\|c_0\|_{L^2(\Omega)}^2 + \|q^+\|_{L^\infty(0,T,L^2(\Omega))}^2\right). \quad (26)$$

Moreover,

$$\sum_{n=1}^{N_T}\delta t\sum_{\mathcal{D}\in\mathfrak{D}}\mathrm{m}_{\mathcal{D}}|c_{\mathcal{D}}^n - c_{\mathcal{K}}^n|^2 \to 0, \qquad \sum_{n=1}^{N_T}\delta t\sum_{\mathcal{D}\in\mathfrak{D}}\mathrm{m}_{\mathcal{D}}|c_{\mathcal{D}}^n - c_{\mathcal{K}^*}^n|^2 \to 0, \quad \textit{when } h, \delta t \to 0. \quad (27)$$

Proof. The property (17) of the penalization operator yields

$$\sum_{n=1}^{N_T}\delta t[\![\mathcal{P}^{\mathcal{T}}(c_{\mathcal{T}}^n), c_{\mathcal{T}}^n]\!]_{\mathcal{T}} = \frac{1}{2h^\beta}\sum_{n=1}^{N_T}\delta t\sum_{\mathcal{K}\in\mathfrak{M}}\sum_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}}\mathrm{m}_{\mathcal{K}\cap\mathcal{K}^*}(c_{\mathcal{K}}^n - c_{\mathcal{K}^*}^n)^2 = \frac{1}{2h^\beta}\|c_{h,\delta t,\mathfrak{M}} - c_{h,\delta t,\overline{\mathfrak{M}^*}}\|_{L^2(0,T;L^2(\Omega))}^2.$$

Then, we deduce (26) from Lemma 3.2. In order to prove (27), let us rewrite $c_{\mathcal{D}}^n$:

$$c_{\mathcal{D}}^n = \frac{1}{\mathrm{m}_{\mathcal{D}}}\int_{\mathcal{D}}c_h^n(x)\mathrm{d}x = \frac{\mathrm{m}_{\mathcal{D}\cap\mathcal{K}}}{2\mathrm{m}_{\mathcal{D}}}c_{\mathcal{K}}^n + \frac{\mathrm{m}_{\mathcal{D}\cap\mathcal{K}^*}}{2\mathrm{m}_{\mathcal{D}}}c_{\mathcal{K}^*}^n + \frac{\mathrm{m}_{\mathcal{D}\cap\mathcal{L}}}{2\mathrm{m}_{\mathcal{D}}}c_{\mathcal{L}}^n + \frac{\mathrm{m}_{\mathcal{D}\cap\mathcal{L}^*}}{2\mathrm{m}_{\mathcal{D}}}c_{\mathcal{L}^*}^n$$

Therefore, we have

$$c_{\mathcal{D}}^n - c_{\mathcal{K}}^n = \frac{\mathrm{m}_{\mathcal{D}\cap\mathcal{L}}}{2\mathrm{m}_{\mathcal{D}}}(c_{\mathcal{L}}^n - c_{\mathcal{K}}^n) + \frac{\mathrm{m}_{\mathcal{D}\cap\mathcal{L}^*}}{2\mathrm{m}_{\mathcal{D}}}(c_{\mathcal{L}^*}^n - c_{\mathcal{K}^*}^n) + \frac{1}{2}(c_{\mathcal{K}^*}^n - c_{\mathcal{K}}^n).$$

Using the fact that $c_{\mathcal{L}}^n - c_{\mathcal{K}}^n = \mathrm{m}_{\sigma^*}(\nabla^{\mathcal{D}}c_{\mathcal{T}}^n)\cdot\boldsymbol{\tau}_{\mathcal{K},\mathcal{L}}$ and $c_{\mathcal{L}^*}^n - c_{\mathcal{K}^*}^n = \mathrm{m}_\sigma(\nabla^{\mathcal{D}}c_{\mathcal{T}}^n)\cdot\boldsymbol{\tau}_{\mathcal{K}^*,\mathcal{L}^*}$, we obtain:

$$\sum_{\mathcal{D}\in\mathfrak{D}}\mathrm{m}_{\mathcal{D}}|c_{\mathcal{D}}^n - c_{\mathcal{K}}^n|^2 \le \frac{3}{2}h^2\sum_{\mathcal{D}\in\mathfrak{D}}\mathrm{m}_{\mathcal{D}}|\nabla^{\mathcal{D}}c_{\mathcal{T}}^n|^2 + \frac{3}{4}\sum_{\mathcal{D}\in\mathfrak{D}}\mathrm{m}_{\mathcal{D}}|c_{\mathcal{K}^*}^n - c_{\mathcal{K}}^n|^2.$$

Thanks to the regularity of the mesh (11b), we get:

$$\sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}}|c^n_{\mathcal{K}^*} - c^n_{\mathcal{K}}|^2 \leq \frac{1}{\zeta}\sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{K}\cap\mathcal{K}^*}|c^n_{\mathcal{K}^*} - c^n_{\mathcal{K}}|^2 \leq \frac{1}{\zeta}\|c^n_{h,\mathfrak{M}} - c^n_{h,\overline{\mathfrak{M}^*}}\|^2_{L^2(\Omega)}.$$

We deduce that

$$\sum_{n=1}^{N_T}\delta t \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}}|c^n_{\mathcal{D}} - c^n_{\mathcal{K}}|^2 \leq \frac{3}{2}h^2\|\nabla^h c_{h,\delta t}\|^2_{(L^2((0,T)\times\Omega))^2} + \frac{3}{4\zeta}\|c_{h,\delta t,\mathfrak{M}} - c_{h,\delta t,\overline{\mathfrak{M}^*}}\|^2_{L^2(0,T;L^2(\Omega))}.$$

It yields the first part of (27), thanks to (26) and (23). The second part of (27) is obtained similarly.

□

The *a priori* estimates given in Lemma 3.1 and Lemma 3.2 will lead to compactness in space of the sequences of approximate solutions. But, as the problem is evolutive in time, we also need compactness in time for the sequence of approximate concentration. Therefore, we need an *a priori* estimate on the discrete time derivatives of the approximate concentration.

For a given function $u_{h,\delta t} \in H_{\mathcal{T},\delta t}$, we recall that we have $u_{h,\delta t}(\cdot,t) = u^n_h(\cdot) \in H_{\mathcal{T}}$ for all $t \in [t_{n-1}, t_n)$. Let us define the discrete time derivative $\partial_{t,\mathcal{T}} u_{h,\delta t} \in H_{\mathcal{T},\delta t}$ by

$$\partial_{t,\mathcal{T}} u_{h,\delta t}(\cdot,t) = \frac{u^n_h(\cdot) - u^{n-1}_h(\cdot)}{\delta t}, \quad \forall t \in [t_{n-1}, t_n).$$

Then, we note $\partial_{t,\mathcal{T}} u^n_{h,\delta t} = \dfrac{u^n_h - u^{n-1}_h}{\delta t} \in H_{\mathcal{T}}$, associated to the vector of values

$$\partial_{t,\mathcal{T}} u^n_{\mathcal{T}} = \left(\left(\frac{u^n_{\mathcal{K}} - u^{n-1}_{\mathcal{K}}}{\delta t}\right)_{\mathcal{K}\in\overline{\mathfrak{M}}}, \left(\frac{u^n_{\mathcal{K}^*} - u^{n-1}_{\mathcal{K}^*}}{\delta t}\right)_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}}\right).$$

Lemma 3.4. *Under the hypotheses of Theorem 2.6, there exists $C > 0$ depending only on T, Ω, ζ, θ, q^+, q^-, c_0, α_A, Λ_A, Λ_D, Φ_* and α_D such that the approximate solution $(p_{h,\delta t}, \mathbf{U}_{h,\delta t}, c_{h,\delta t}) \in H_{\mathcal{T},\delta t}\times H_{\mathfrak{D},\delta t}\times H_{\mathcal{T},\delta t}$ to the scheme (18)–(19) satisfies:*

$$\sum_{n=1}^{N_T}\delta t \left\|\Phi_{\mathcal{T}}\partial_{t,\mathcal{T}} c^n_{h,\delta t}\right\|_{1,-1,\mathcal{T}} \leq C. \tag{28}$$

Proof. Let $w_h \in H_{\mathcal{T}}$ and $n \in \{1,\cdots,N_T\}$. Multiplying the scheme (19a) by $w_{\mathcal{T}}$, we get :

$$\left[\!\!\left[\Phi_{\mathcal{T}}\frac{c^n_{\mathcal{T}} - c^{n-1}_{\mathcal{T}}}{\delta t}, w_{\mathcal{T}}\right]\!\!\right]_{\mathcal{T}} = \left[\!\!\left[\mathrm{div}^{\mathcal{T}}\left(\mathbb{D}_{\mathfrak{D}}\left(\mathbf{U}^n_{\mathfrak{D}}\right)\nabla^{\mathfrak{D}}c^n_{\mathcal{T}}\right), w_{\mathcal{T}}\right]\!\!\right]_{\mathcal{T}} - \left[\!\!\left[\mathrm{divc}^{\mathcal{T}}\left(\mathbf{U}^n_{\mathfrak{D}}, c^n_{\mathcal{T}}\right), w_{\mathcal{T}}\right]\!\!\right]_{\mathcal{T}} - \left[\!\!\left[q^{-,n}_{\mathcal{T}}c^n_{\mathcal{T}}, w_{\mathcal{T}}\right]\!\!\right]_{\mathcal{T}}$$
$$- \lambda[\![\mathcal{P}^{\mathcal{T}}(c^n_{\mathcal{T}}), w_{\mathcal{T}}]\!]_{\mathcal{T}} + \left[\!\!\left[q^{+,n}_{\mathcal{T}}\hat{c}^n_{\mathcal{T}}, w_{\mathcal{T}}\right]\!\!\right]_{\mathcal{T}}.$$

We will now bound separately each term, denoted by T_i for $1 \leq i \leq 5$, of the right-hand-side of this equality.

Using the discrete duality formula (Theorem 2.3) and the boundary conditions, we first obtain that

$$T_1 = \left[\!\!\left[\mathrm{div}^{\mathcal{T}}\left(\mathbb{D}_{\mathfrak{D}}\left(\mathbf{U}^n_{\mathfrak{D}}\right)\nabla^{\mathfrak{D}}c^n_{\mathcal{T}}\right), w_{\mathcal{T}}\right]\!\!\right]_{\mathcal{T}} = -\left(\mathbb{D}_{\mathfrak{D}}\left(\mathbf{U}^n_{\mathfrak{D}}\right)\nabla^{\mathfrak{D}}c^n_{\mathcal{T}}, \nabla^{\mathfrak{D}}w_{\mathcal{T}}\right)_{\mathfrak{D}} = -\sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}}\mathbb{D}_{\mathcal{D}}\left(\mathbf{U}^n_{\mathcal{D}}\right)\nabla^{\mathcal{D}}c^n_{\mathcal{T}}\cdot\nabla^{\mathcal{D}}w_{\mathcal{T}}.$$

Then, the hypothesis (5) on $\mathbb{D}$ implies :

$$|T_1| \leq \Lambda_D\,\|w_h\|_{1,\infty,\mathcal{T}}\sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}}\left(1 + \left|\mathbf{U}^n_{\mathcal{D}}\right|\right)\left|\nabla^{\mathcal{D}}c^n_{\mathcal{T}}\right|. \tag{29}$$

The second term $T_2 = -\left[\!\left[\mathrm{divc}^{\mathcal{T}}\left(\mathbf{U}^n_{\mathfrak{D}}, c^n_{\mathcal{T}}\right), w_{\mathcal{T}}\right]\!\right]_{\mathcal{T}}$ can be split into the sum of a primal term $T_{2,p}$ and a dual term $T_{2,d}$. Let us consider the primal term

$$T_{2,p} = -\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} \mathrm{m}_{\mathcal{K}} \mathrm{divc}_{\mathcal{K}}(\mathbf{U}^n_{\mathfrak{D}}, c^n_{\mathcal{T}}) w_{\mathcal{K}} = -\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} \sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}}\\ \mathcal{D}=\mathcal{D}_{\sigma,\sigma^*}}} \mathrm{m}_\sigma \left(\left(\mathbf{U}^n_{\mathcal{D}}\cdot\mathbf{n}_{\sigma\mathcal{K}}\right)^+ c^n_{\mathcal{K}} - \left(\mathbf{U}^n_{\mathcal{D}}\cdot\mathbf{n}_{\sigma\mathcal{K}}\right)^- c^n_{\mathcal{L}}\right) w_{\mathcal{K}}.$$

Rewriting $T_{2,p}$ as a sum on all the primal edges of the mesh and using the relations $x = x^+ - x^-$, we get:

$$T_{2,p} = -\frac{1}{2}\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} \mathrm{m}_\sigma \left(\mathbf{U}^n_{\mathcal{D}}\cdot\mathbf{n}_{\sigma\mathcal{K}}\right) c^n_{\mathcal{K}}(w_{\mathcal{K}} - w_{\mathcal{L}}) - \frac{1}{2}\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} \mathrm{m}_\sigma (\mathbf{U}^n_{\mathcal{D}}\cdot\mathbf{n}_{\sigma\mathcal{K}})^-(c^n_{\mathcal{K}} - c^n_{\mathcal{L}})(w_{\mathcal{K}} - w_{\mathcal{L}}). \tag{30}$$

But, by definition, we have $(w_{\mathcal{K}} - w_{\mathcal{L}}) = \mathrm{m}_{\sigma^*}\nabla^{\mathcal{D}} w_{\mathcal{T}}\cdot\boldsymbol{\tau}_{\mathcal{K},\mathcal{L}}$ and therefore $|w_{\mathcal{K}} - w_{\mathcal{L}}| \le \|w_h\|_{1,\infty,\mathcal{T}}\,\mathrm{m}_{\sigma^*}$. It yields:

$$\left|\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} \mathrm{m}_\sigma \left(\mathbf{U}^n_{\mathcal{D}}\cdot\mathbf{n}_{\sigma\mathcal{K}}\right) c^n_{\mathcal{K}}(w_{\mathcal{K}} - w_{\mathcal{L}})\right| \le \|w_h\|_{1,\infty,\mathcal{T}} \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} \mathrm{m}_{\sigma^*}\mathrm{m}_\sigma |\mathbf{U}^n_{\mathcal{D}}|\,|c^n_{\mathcal{K}}|.$$

For the second term in $T_{2,p}$, we use the bound $|w_{\mathcal{K}} - w_{\mathcal{L}}| \le 2\,\|w_h\|_{1,\infty,\mathcal{T}}$ to get:

$$\left|\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} \mathrm{m}_\sigma (\mathbf{U}^n_{\mathcal{D}}\cdot\mathbf{n}_{\sigma\mathcal{K}})^-(c^n_{\mathcal{K}} - c^n_{\mathcal{L}})(w_{\mathcal{K}} - w_{\mathcal{L}})\right| \le 2\,\|w_h\|_{1,\infty,\mathcal{T}} \left(\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} \mathrm{m}_\sigma\mathrm{m}_{\sigma^*} |\mathbf{U}^n_{\mathcal{D}}|\,|\nabla^{\mathcal{D}} c^n_{\mathcal{T}}|\right).$$

As we may treat similarly the dual term $T_{2,d} = \dfrac{1}{2}\displaystyle\sum_{n=1}^{N_T}\delta t \sum_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} \mathrm{m}_{\mathcal{K}^*}\mathrm{divc}_{\mathcal{K}^*}(\mathbf{U}^n_{\mathfrak{D}}, c^n_{\mathcal{T}}) w_{\mathcal{K}^*}$, we deduce that

$$|T_2| \le \|w_h\|_{1,\infty,\mathcal{T}} \left(\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} \mathrm{m}_{\sigma^*}\mathrm{m}_\sigma |\mathbf{U}^n_{\mathcal{D}}|\,|c^n_{\mathcal{K}}| + 2\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} \mathrm{m}_\sigma\mathrm{m}_{\sigma^*} |\mathbf{U}^n_{\mathcal{D}}|\,|\nabla^{\mathcal{D}} c^n_{\mathcal{T}}|\right). \tag{31}$$

Let us now consider

$$T_3 = -\lambda[\![\mathcal{P}^{\mathcal{T}}(c^n_{\mathcal{T}}), w_{\mathcal{T}}]\!]_{\mathcal{T}} = -\frac{\lambda}{2}\sum_{\mathcal{K}\in\mathfrak{M}}\sum_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} \frac{1}{h^\beta}\mathrm{m}_{\mathcal{K}\cap\mathcal{K}^*}(c^n_{\mathcal{K}} - c^n_{\mathcal{K}^*})(w_{\mathcal{K}} - w_{\mathcal{K}^*}).$$

Using Cauchy-Schwarz inequality, equality (17) and the definition of $\|w_h\|_{1,\infty\star,\mathcal{T}}$, we obtain

$$\begin{aligned} |T_3| &\le \frac{\lambda}{2h^\beta}\|w_{h,\mathfrak{M}} - w_{h,\overline{\mathfrak{M}^*}}\|_{L^2(\Omega)}\|c^n_{h,\mathfrak{M}} - c^n_{h,\overline{\mathfrak{M}^*}}\|_{L^2(\Omega)} \qquad (32)\\ &\le \frac{\lambda}{\sqrt{2}h^{\frac{\beta}{2}}}\,\|w_h\|_{1,\infty\star,\mathcal{T}}\,\|c^n_{h,\mathfrak{M}} - c^n_{h,\overline{\mathfrak{M}^*}}\|_{L^2(\Omega)}. \qquad (33)\end{aligned}$$

We focus now on the last two terms $T_4 = -\left[\!\left[q^{-,n}_{\mathcal{T}} c^n_{\mathcal{T}}, w_{\mathcal{T}}\right]\!\right]_{\mathcal{T}}$ and $T_5 = \left[\!\left[q^{+,n}_{\mathcal{T}} \hat{c}^n_{\mathcal{T}}, w_{\mathcal{T}}\right]\!\right]_{\mathcal{T}}$. They verify :

$$\begin{aligned} |T_4| &\le \|w_h\|_{1,\infty,\mathcal{T}}\,\|q^{-,n}_{\mathcal{T}}\|_{2,\mathcal{T}}\,\left\|c^n_{\mathcal{T}}\right\|_{2,\mathcal{T}}, \qquad (34)\\ |T_5| &\le \|w_h\|_{1,\infty,\mathcal{T}}\,\|q^{+,n}_{\mathcal{T}}\|_{2,\mathcal{T}}\,\left\|\hat{c}^n_{\mathcal{T}}\right\|_{2,\mathcal{T}}. \qquad (35)\end{aligned}$$

Finally, due to (29), (31), (33), (34) and (35), we obtain that, for all $w_h \in H_{\mathcal{T}}$,

$$\begin{aligned}\left[\!\left[\Phi_{\mathcal{T}}\frac{c^n_{\mathcal{T}} - c^{n-1}_{\mathcal{T}}}{\delta t}, w_{\mathcal{T}}\right]\!\right]_{\mathcal{T}} \le \|w_h\|_{1,\infty\star,\mathcal{T}}\Bigg(&\Lambda_D \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}}\left(1 + \left|\mathbf{U}^n_{\mathcal{D}}\right|\right)\left|\nabla^{\mathcal{D}} c^n_{\mathcal{T}}\right| + \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} \mathrm{m}_{\sigma^*}\mathrm{m}_\sigma |\mathbf{U}^n_{\mathcal{D}}|\,|c^n_{\mathcal{K}}|\\ &+ 2\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}} \mathrm{m}_\sigma\mathrm{m}_{\sigma^*}|\mathbf{U}^n_{\mathcal{D}}|\,|\nabla^{\mathcal{D}} c^n_{\mathcal{T}}| + \frac{\lambda}{\sqrt{2}h^{\frac{\beta}{2}}}\|c^n_{h,\mathfrak{M}} - c^n_{h,\overline{\mathfrak{M}^*}}\|_{L^2(\Omega)} + \|q^{-,n}_{\mathcal{T}}\|_{2,\mathcal{T}}\left\|c^n_{\mathcal{T}}\right\|_{2,\mathcal{T}} + \|q^{+,n}_{\mathcal{T}}\|_{2,\mathcal{T}}\left\|\hat{c}^n_{\mathcal{T}}\right\|_{2,\mathcal{T}}\Bigg).\end{aligned}$$

It gives the bound for $\left\|\Phi_{\mathcal{T}}\partial_{t,\mathcal{T}}c^n_{h,\delta t}\right\|_{1,-1,\mathcal{T}}$. Multiplying by δt and summing over n, we obtain that

$$\sum_{n=1}^{N_T}\delta t\left\|\Phi_{\mathcal{T}}\partial_{t,\mathcal{T}}c^n_{h,\delta t}\right\|_{1,-1,\mathcal{T}} \leq \Lambda_D\sum_{n=1}^{N_T}\delta t\sum_{\mathcal{D}\in\mathfrak{D}}\mathrm{m}_{\mathcal{D}}\left(1+\left|\mathbf{U}^n_{\mathcal{D}}\right|\right)\left|\nabla^{\mathcal{D}}c^n_{\mathcal{T}}\right| + \sum_{n=1}^{N_T}\delta t\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}}\mathrm{m}_{\sigma^*}\mathrm{m}_{\sigma}|\mathbf{U}^n_{\mathcal{D}}|\,|c^n_{\mathcal{K}}|$$
$$+2\sum_{n=1}^{N_T}\delta t\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}}\mathrm{m}_{\sigma}\mathrm{m}_{\sigma^*}|\mathbf{U}^n_{\mathcal{D}}|\,|\nabla^{\mathcal{D}}c^n_{\mathcal{T}}| + \frac{\lambda}{\sqrt{2}h^{\frac{\beta}{2}}}\sum_{n=1}^{N_T}\delta t\|c^n_{h,\mathfrak{M}} - c^n_{h,\overline{\mathfrak{M}^*}}\|_{L^2(\Omega)}$$
$$+\sum_{n=1}^{N_T}\delta t\|q^{-,n}_{\mathcal{T}}\|_{2,\mathcal{T}}\left\|c^n_{\mathcal{T}}\right\|_{2,\mathcal{T}} + \sum_{n=1}^{N_T}\delta t\|q^{+,n}_{\mathcal{T}}\|_{2,\mathcal{T}}\left\|\hat{c}^n_{\mathcal{T}}\right\|_{2,\mathcal{T}}.$$

Applying Cauchy-Schwarz inequality and using the *a priori* estimates (22), (23) and (26), we conclude the proof of (28).

□

4. Spaces of approximate solutions

In order to prove the convergence of a sequence of approximate solutions given by the scheme, we need some compactness properties on the space of approximate solutions $H_{\mathcal{T}}$.

Proposition 4.1. *Let $(\mathcal{T}_m)_m$ be a sequence of DDFV meshes satisfying $h_m = \mathrm{size}(\mathcal{T}_m) \to 0$ when $m \to \infty$ and* (21)*. We consider a sequence of functions $(w_m)_m$ with $w_m = w_{h_m} \in H_{\mathcal{T}_m}$. If the sequence $(\|w_m\|_{1,1,\mathcal{T}_m})_m$ is bounded, then there exists $w \in L^1(\Omega)$ such that, up to a subsequence,*

$$w_m \underset{m\to\infty}{\to} w \text{ in } L^1(\Omega).$$

Proof. The convergence result of Proposition 4.1 is a consequence of an estimate on the space translates of the sequence of approximate solutions. Such an argument is classical in the finite volume framework since [27].

Let us consider one function w_h of the given sequence ($w_h = w_{h_m}$ but we omit the subscript m for ease of presentation). We are looking for an upper bound of $\|w_h(\cdot+\eta) - w_h(\cdot)\|_{L^1(\mathbb{R}^2)}$. But, by construction, $w_h = \frac{1}{2}(w_{h,\mathfrak{M}} + w_{h,\overline{\mathfrak{M}^*}})$. Therefore, we first focus on $\left\|w_{h,\mathfrak{M}}(\cdot+\eta) - w_{h,\mathfrak{M}}(\cdot)\right\|_{L^1(\mathbb{R}^2)}$. The calculations are similar to those followed in [5, Lemma 3.8]; the main difference comes from the fact that we do not impose boundary conditions.

For each primal edge $\sigma = \mathcal{K}|\mathcal{L}$ and for all $x, \eta \in \mathbb{R}^2$, we define

$$\psi_\sigma(x,\eta) \quad = \quad \begin{cases} 1 & \text{where } [x, x+\eta]\cap\sigma \neq \emptyset, \\ 0 & \text{elsewhere.}\end{cases}$$

Then, for $x \in \mathbb{R}^2$ and $\eta \in \mathbb{R}^2\setminus\{0\}$, we have

$$|w_{h,\mathfrak{M}}(x+\eta) - w_{h,\mathfrak{M}}(x)| \leq \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}}\psi_\sigma(x,\eta)|w_{\mathcal{L}} - w_{\mathcal{K}}| + \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}}\psi_\sigma(x,\eta)|w_{\mathcal{K}}|. \tag{36}$$

We treat the first term of the right hand side as in [5, Lemma 3.8]:

$$T_1(x) := \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{int}}\psi_\sigma(x,\eta)|w_{\mathcal{L}} - w_{\mathcal{K}}| \leq \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}}\mathrm{m}_{\sigma^*}\psi_\sigma(x,\eta)\left|\frac{w_{\mathcal{L}} - w_{\mathcal{K}}}{\mathrm{m}_{\sigma^*}}\right|.$$

As $\displaystyle\int_{\mathbb{R}^2}\psi_\sigma(x,\eta)\mathrm{d}x \leq \mathrm{m}_\sigma|\eta|$, we obtain that

$$\int_{\mathbb{R}^2} T_1(x)\mathrm{d}x \leq |\eta|\sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}}\mathrm{m}_{\sigma^*}\mathrm{m}_\sigma\left|\frac{w_{\mathcal{L}} - w_{\mathcal{K}}}{\mathrm{m}_{\sigma^*}}\right| \leq \frac{2}{\sin(\theta_{\mathcal{T}})}|\eta|\sum_{\mathcal{D}\in\mathfrak{D}}\mathrm{m}_{\mathcal{D}}|\nabla^{\mathcal{D}}w_{\mathcal{T}}| \leq \frac{2}{\sin(\theta_{\mathcal{T}})}|\eta|\;\|w_h\|_{1,1,\mathcal{T}}.$$

For the second term of the right hand side in (36), $T_2(x) := \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} \psi_\sigma(x,\eta)|w_{\mathcal{K}}|$, we have

$$\int_{\mathbb{R}^2} T_2(x)\mathrm{d}x \le |\eta| \sum_{\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} \mathrm{m}_\sigma |w_{\mathcal{K}}| \quad \le |\eta| C \|w_h\|_{1,1,\mathcal{T}},$$

thanks to the trace Theorem 7.1 proving in Section 7. Therefore, we get:

$$\left\| w_{h,\mathfrak{M}}(\cdot+\eta) - w_{h,\mathfrak{M}}(\cdot)\right\|_{L^1(\mathbb{R}^2)} \le C|\eta|\ \|w_h\|_{1,1,\mathcal{T}},$$

with C depending only on Ω and the regularity parameters θ and ζ. With the same calculations on the dual mesh, we also get

$$\left\| w_{h,\overline{\mathfrak{M}^*}}(\cdot+\eta) - w_{h,\overline{\mathfrak{M}^*}}(\cdot)\right\|_{L^1(\mathbb{R}^2)} \le C|\eta|\ \|w_h\|_{1,1,\mathcal{T}}.$$

Therefore, since $\|w_m\|_{1,1,\mathcal{T}_m}$ is bounded, there exists C not depending on m such that

$$\|w_m(\cdot+\eta) - w_m(\cdot)\|_{L^1(\mathbb{R}^2)} \le C|\eta|,\ \forall \eta\in\mathbb{R}^2.$$

We conclude thanks to Kolmogorov Theorem: there exists a subsequence of (w_m) which converges towards $w \in L^1(\mathbb{R}^2)$. Furthermore, as w_m vanishes outside Ω for all m, w also vanishes outside Ω: $w\in L^1(\Omega)$. □

Proposition 4.2. *Let $(\mathcal{T}_m)_m$ be a sequence of DDFV meshes satisfying $h_m = \mathrm{size}(\mathcal{T}_m) \to 0$ when $m\to\infty$ and* (21). *We consider a sequence of functions $(w_m)_m$ with $w_m = w_{h_m} \in H_{\mathcal{T}_m}$. If*

$$w_m \underset{m\to\infty}{\to} w \text{ in } L^1(\Omega) \quad \text{and} \quad \|w_m\|_{1,-1,\mathcal{T}_m} \underset{m\to\infty}{\to} 0,$$

then $w = 0$.

Proof. Let us consider one function w_h of the given sequence ($w_h = w_{h_m}$ but we omit the subscript m for ease of presentation). Let $\psi\in C_c^\infty(\Omega)$. We define

$$\psi_{\mathcal{K}} = \frac{1}{\mathrm{m}_{\mathcal{K}}}\int_{\mathcal{K}} \psi(x)dx \quad \forall \mathcal{K}\in\mathfrak{M} \quad \text{and } \psi_{\mathcal{K}} = 0 \quad \forall\mathcal{K}\in\partial\mathfrak{M},$$
$$\psi_{\mathcal{K}^*} = \frac{1}{\mathrm{m}_{\mathcal{K}^*}}\int_{\mathcal{K}^*}\psi(x)dx \quad \forall\mathcal{K}^*\in\overline{\mathfrak{M}^*},$$

and $\psi_{\mathcal{T}} = \left((\psi_{\mathcal{K}})_{\mathcal{K}\in\overline{\mathfrak{M}}}, (\psi_{\mathcal{K}^*})_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}}\right)$. By this way, we can associate to each function $\psi\in C_c^\infty(\Omega)$ a vector $\psi_{\mathcal{T}}$ and a function $\psi_h\in H_{\mathcal{T}}$. For all $\mathcal{D}_{\sigma,\sigma^*}\in\mathfrak{D}$, the Taylor's theorem implies:

$$|\psi_{\mathcal{K}} - \psi_{\mathcal{L}}| \le (d_{\mathcal{K}} + d_{\mathcal{L}})\|\nabla\psi\|_{L^\infty(\Omega)} \quad \text{and} \quad |\psi_{\mathcal{K}^*} - \psi_{\mathcal{L}^*}| \le (d_{\mathcal{K}^*} + d_{\mathcal{L}^*})\|\nabla\psi\|_{L^\infty(\Omega)},$$
$$|\psi_{\mathcal{K}} - \psi_{\mathcal{K}^*}| \le (d_{\mathcal{K}} + d_{\mathcal{K}^*})\|\nabla\psi\|_{L^\infty(\Omega)}.$$

Using the regularity of the mesh, we deduce that there exists C only depending on θ and ζ such that

$$\|\psi_h\|_{1,\infty,\mathcal{T}} \le C\,\|\psi\|_{W^{1,\infty}(\Omega)}\,,$$

and

$$\|\psi_{h,\mathfrak{M}} - \psi_{h,\overline{\mathfrak{M}^*}}\|_{L^2(\Omega)} \le Ch\,\|\psi\|_{W^{1,\infty}(\Omega)}\,. \tag{37}$$

Then, as $\beta < 2$, we deduce, thanks to (17), that

$$\|\psi_h\|_{1,\infty\star,\mathcal{T}} \le C\,\|\psi\|_{W^{1,\infty}(\Omega)}\,.$$

But, for $w_h\in H_{\mathcal{T}}$, we have the following inequality:

$$[\![w_{\mathcal{T}}, \psi_{\mathcal{T}}]\!]_{\mathcal{T}} \le \|w_h\|_{1,-1,\mathcal{T}}\ \|\psi_h\|_{1,\infty\star,\mathcal{T}} \le C\,\|w_h\|_{1,-1,\mathcal{T}}\ \|\psi\|_{W^{1,\infty}(\Omega)}\,.$$

Therefore, if $\psi \in C_c^\infty(\Omega)$ and the sequence (w_m) satisfies $\|w_m\|_{1,-1,\mathcal{T}_m} \underset{m\to\infty}{\longrightarrow} 0$, it yields:

$$[\![w_{\mathcal{T}_m}, \psi_{\mathcal{T}_m}]\!]_{\mathcal{T}_m} \to 0 \text{ as } m \to \infty.$$

Yet, by definition, we have

$$\begin{aligned} [\![w_{\mathcal{T}_m}, \psi_{\mathcal{T}_m}]\!]_{\mathcal{T}_m} &= \frac{1}{2} \sum_{\mathcal{K}\in\mathfrak{M}_m} w_{\mathcal{K}} \int_\Omega \psi(x) \mathbf{1}_{\mathcal{K}}(x) \mathrm{d}x + \frac{1}{2} \sum_{\mathcal{K}^*\in\overline{\mathfrak{M}_m^*}} w_{\mathcal{K}^*} \int_\Omega \psi(x) \mathbf{1}_{\mathcal{K}^*}(x) \mathrm{d}x \\ &= \int_\Omega w_m(x)\psi(x)\mathrm{d}x. \end{aligned} \tag{38}$$

As a consequence, as $w_m \underset{m\to\infty}{\longrightarrow} w$ in $L^1(\Omega)$, we obtain $\int_\Omega w(x)\psi(x)\mathrm{d}x = 0$ for all $\psi \in C_c^\infty(\Omega)$, hence $w = 0$.

□

Proposition 4.3. *Let $(\mathcal{T}_m)_m$ be a sequence of DDFV meshes satisfying $h_m = \mathrm{size}(\mathcal{T}_m) \to 0$ when $m \to \infty$ and (21). We consider a sequence of functions $(v_m)_m$ with $v_m = v_{h_m} \in H_{\mathcal{T}_m}$ such that the sequence $\left(\|v_m\|_{1,2,\mathcal{T}_m}\right)_m$ is bounded. Then, there exists $v \in H^1(\Omega)$ such that, up to a subsequence, we have the following convergence results when $m \to \infty$:*

$$\begin{aligned} &v_m \to v \text{ strongly in } L^2(\Omega), \\ &\nabla^{h_m} v_m \to \nabla v \text{ weakly in } (L^2(\Omega))^2. \end{aligned}$$

Proof. Let us set $w_m = v_m|v_m|$. An adaptation of the proof of Proposition 4.1, with the ideas of [5, Lemma 3.8], leads to

$$\|w_m(\cdot + \eta) - w_m(\cdot)\|_{L^1(\mathbb{R}^2)} \le C|\eta|, \forall \eta \in \mathbb{R}^2.$$

It proves the convergence of (w_m) in $L^1(\Omega)$ and the existence of $v \in L^2(\Omega)$ such that

$$v_m \underset{m\to\infty}{\longrightarrow} v \text{ strongly in } L^2(\Omega).$$

As $\|\nabla^{h_m} v_m\|_2 \le C$, there exists $\chi \in (L^2(\Omega))^2$ such that, up to a subsequence:

$$\nabla^{h_m} v_m \underset{m\to\infty}{\longrightarrow} \chi \text{ weakly in } (L^2(\Omega))^2.$$

It remains to prove that $\chi = \nabla v$, which will also imply $v \in H^1(\Omega)$.

Let $\psi \in (C_c^\infty(\Omega))^2$, we define

$$I_m := \int_\Omega \nabla^{h_m} v_m(z) \cdot \psi(z)\mathrm{d}z + \int_\Omega v_m(z)\mathrm{div}(\psi(z))\mathrm{d}z \underset{m\to\infty}{\longrightarrow} \int_\Omega \chi(z)\cdot\psi(z)\mathrm{d}z + \int_\Omega v(z)\mathrm{div}(\psi(z))\mathrm{d}z.$$

For $\mathcal{D} = \mathcal{D}_{\sigma,\sigma^*}$, we define $\psi_{\mathcal{D}}$, ψ_σ and ψ_{σ^*} respectively as the mean values of ψ over $\mathcal{D}$, σ and σ^*. We consider also $\widetilde{\psi_{\mathcal{D}}}$ defined by

$$\widetilde{\psi_{\mathcal{D}}} \cdot \mathbf{n}_{\sigma\mathcal{K}} = \psi_\sigma \cdot \mathbf{n}_{\sigma\mathcal{K}}, \qquad \widetilde{\psi_{\mathcal{D}}} \cdot \mathbf{n}_{\sigma^*\mathcal{K}^*} = \psi_{\sigma^*} \cdot \mathbf{n}_{\sigma^*\mathcal{K}^*}.$$

We have :

$$\int_\Omega \nabla^{h_m} v_m(z)\cdot\psi(z)\mathrm{d}z = \sum_{\mathcal{D}\in\mathfrak{D}_m} \mathrm{m}_{\mathcal{D}} \nabla^{\mathcal{D}} v_{\mathcal{T}_m} \cdot \psi_{\mathcal{D}} = \sum_{\mathcal{D}\in\mathfrak{D}_m} \mathrm{m}_{\mathcal{D}} \nabla^{\mathcal{D}} v_{\mathcal{T}_m} \cdot \widetilde{\psi_{\mathcal{D}}} + \sum_{\mathcal{D}\in\mathfrak{D}_m} \mathrm{m}_{\mathcal{D}} \nabla^{\mathcal{D}} v_{\mathcal{T}_m} \cdot (\psi_{\mathcal{D}} - \widetilde{\psi_{\mathcal{D}}}).$$

But,

$$\sum_{\mathcal{D}\in\mathfrak{D}_m} \mathrm{m}_{\mathcal{D}} \nabla^{\mathcal{D}} v_{\mathcal{T}_m} \cdot \widetilde{\psi_{\mathcal{D}}} = -\frac{1}{2} \sum_{\mathcal{K}\in\mathfrak{M}_m} v_{\mathcal{K}} \sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}} \\ \mathcal{D}=\mathcal{D}_{\sigma,\sigma^*}}} \mathrm{m}_\sigma \widetilde{\psi_{\mathcal{D}}} \cdot \mathbf{n}_{\sigma\mathcal{K}} - \frac{1}{2} \sum_{\mathcal{K}^*\in\overline{\mathfrak{M}_m^*}} v_{\mathcal{K}^*} \sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*} \\ \mathcal{D}=\mathcal{D}_{\sigma,\sigma^*}}} \mathrm{m}_{\sigma^*} \widetilde{\psi_{\mathcal{D}}} \cdot \mathbf{n}_{\sigma^*\mathcal{K}^*}.$$

Using the definition of $\widetilde{\psi_{\mathcal{D}}}$ and the fact that ψ has a compact support, we get, thanks to Stokes formula,

$$\sum_{\mathcal{D}\in\mathfrak{D}_m} \mathrm{m}_{\mathcal{D}} \nabla^{\mathcal{D}} v_{\mathcal{T}_m} \cdot \widetilde{\psi_{\mathcal{D}}} = -\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}_m} v_{\mathcal{K}} \int_{\mathcal{K}} \mathrm{div}\psi(z)\mathrm{d}z - \frac{1}{2}\sum_{\mathcal{K}^*\in\overline{\mathfrak{M}_m^*}} v_{\mathcal{K}^*} \int_{\mathcal{K}^*} \mathrm{div}\psi(z)\mathrm{d}z = -\int_\Omega v_m(z)\mathrm{div}\psi(z)\mathrm{d}z.$$

It implies that

$$I_m = \sum_{\mathcal{D}\in\mathfrak{D}_m} \mathrm{m}_{\mathcal{D}} \nabla^{\mathcal{D}} v_{\mathcal{T}_m} \cdot (\psi_{\mathcal{D}} - \widetilde{\psi_{\mathcal{D}}}).$$

Since ψ is a smooth function, we have

$$|\psi_{\mathcal{D}} - \widetilde{\psi_{\mathcal{D}}}| \le \frac{1}{\sin(\theta_{\mathcal{T}})}(|\psi_{\mathcal{D}} - \psi_{\sigma^*}| + |\psi_{\mathcal{D}} - \psi_{\sigma}|) \le \frac{2}{\sin(\theta_{\mathcal{T}})} h_m \|\nabla\psi\|_{L^\infty(\Omega)},$$

and we deduce that

$$\left|\sum_{\mathcal{D}\in\mathfrak{D}_m} \mathrm{m}_{\mathcal{D}} \nabla^{\mathcal{D}} v_{\mathcal{T}_m} \cdot (\psi_{\mathcal{D}} - \widetilde{\psi_{\mathcal{D}}})\right| \le \|\nabla^{h_m} v_m\|_{L^2(\Omega)} \sqrt{\mathrm{m}_\Omega} \frac{2}{\sin(\theta_{\mathcal{T}})} h_m \|\nabla\psi\|_{L^\infty(\Omega)},$$

so that I_m tends to 0. We conclude that

$$\int_\Omega \chi(z)\cdot\psi(z)\mathrm{d}z = -\int_\Omega v(z)\mathrm{div}(\psi(z))\mathrm{d}z, \forall \psi \in (C_c^\infty(\Omega))^2,$$

which ends the proof. □

Proposition 4.4. *Let $(\mathcal{T}_m)_m$ be a sequence of DDFV meshes satisfying $h_m = \mathrm{size}(\mathcal{T}_m) \to 0$ when $m \to \infty$ and* (21). *Let $(\delta t_m)_{m\ge1}$ be a sequence of time steps such that $T/\delta t_m$ is an integer and $\delta t_m \underset{m\to\infty}{\longrightarrow} 0$. We consider a sequence of functions $(v_m)_m$ with $v_m = v_{h_m,\delta t_m} \in H_{\mathcal{T}_m,\delta t_m}$ when $m \to \infty$ such that:*

$$v_m \to v \text{ weakly in } L^2((0,T)\times\Omega) \quad (\text{respectively weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)));$$
$$\nabla^{h_m} v_m \to \chi \text{ weakly in } (L^2((0,T)\times\Omega))^2 \quad (\text{respectively weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)));$$

then, we have

$$\nabla v = \chi \text{ and } v \in L^2(0,T;H^1(\Omega)) \quad (\text{respectively } L^\infty(0,T;H^1(\Omega))).$$

Proof. An adaptation of the proof of Proposition 4.3, leads to prove that $\nabla v = \chi$ in the distribution sense on $]0,T[\times\Omega$, and therefore $v \in L^2(0,T;H^1(\Omega))$ or $v \in L^\infty((0,T)\times\Omega)$. □

5. Proof of the convergence of the numerical scheme

5.1. Compactness of the concentration

Proposition 5.1. *Under the assumptions of Theorem 2.6 and the fact that Φ is a constant Φ^*, the sequence $(c_m)_m$ defined by the scheme* (18)–(19) *is relatively compact in $L^1(0,T;L^1(\Omega))$. Let us note by $\bar{c}$ its limit up to a subsequence. Then, $\bar{c}$ lies in $L^2(0,T;H^1(\Omega))$. Furthermore, up to a subsequence, we have, when $m \to \infty$*

$$c_m \to \bar{c} \text{ weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ and strongly in } L^p(0,T;L^q(\Omega)), \forall p < \infty, q < 2;$$
$$\nabla^{h_m} c_m \to \nabla\bar{c} \text{ weakly in } (L^2(0,T;L^2(\Omega)))^2.$$

Proof. The key of the proof is the discrete Aubin-Simon lemma proved by Gallouët and Latché [31, Theorem 3.4]. The family $(H_{\mathcal{T}_m})_m$ is a family of finite dimensional subspaces of $L^1(\Omega)$. Each space $H_{\mathcal{T}_m}$ can be equipped with the norm $\|\cdot\|_{1,1,\mathcal{T}_m}$ or with the norm $\|\cdot\|_{1,-1,\mathcal{T}_m}$. The following properties are satisfied:

- Let consider a sequence $(w_m)_m$with $w_m = w_{h_m} \in H_{\mathcal{T}_m}$. If the sequence $(\|w_m\|_{1,1,\mathcal{T}_m})_m$ is bounded, then there exists $w \in L^1(\Omega)$ such that, up to a subsequence, $(w_m)_m$ converges to w in $L^1(\Omega)$. See Proposition 4.1.

- Let consider a sequence $(w_m)_m$ with $w_m = w_{h_m} \in H_{\mathcal{T}_m}$. If w_m converges towards w in $L^1(\Omega)$ while $(\|w_m\|_{1,-1,\mathcal{T}_m})_m$ tends to 0, then $w = 0$. See Proposition 4.2.

The sequence $(c_m)_m$ verifies $c_m(\cdot, t) = c_m^n \in H_{\mathcal{T}_m}$ for all $t \in [(n-1)\delta t_m, n\delta t_m)$. Furthermore Lemma 3.2 (with Cauchy-Schwarz inequality) ensures that $(c_m)_m$ verifies, for all m,

$$\sum_{n=1}^{N_T(m)} \delta t_m \left\|c_m^n\right\|_{1,1,\mathcal{T}_m} \leq C,$$

and Lemma 3.4 gives, for all m,

$$\sum_{n=1}^{N_T(m)} \delta t_m \left\|\partial_{t,\mathcal{T}_m} c_m^n\right\|_{1,-1,\mathcal{T}_m} \leq C, \tag{39}$$

with C depending only on the data of the problem. Then, Theorem 3.4 in [31] implies that, up to a subsequence, (c_m) converges in $L^1(0,T,L^1(\Omega))$ to a function $\bar{c}$. Furthermore, Lemma 3.2 implies that there exists $w \in (L^2(0,T;L^2(\Omega)))^2$, such that, up to a subsequence, we have, when $m \to \infty$

$$c_m \to \bar{c} \text{ weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)), \text{ weakly in } L^2(0,T;L^2(\Omega)) \text{ and strongly in } L^p(0,T;L^q(\Omega)), \forall p < \infty, q < 2,$$
$$\nabla^{h_m} c_m \to w \text{ weakly in } (L^2(0,T;L^2(\Omega)))^2.$$

We conclude, applying Proposition 4.4:

$$\bar{c} \in L^2(0,T;H^1(\Omega)), \text{ and } \nabla\bar{c} = w.$$

□

Remark 5.2. *We have used the fact that Φ is a constant function in order to get* (39)*. Therefore the compactness of the sequence of approximate concentration is obtained thanks to [31, Theorem 3.4]. If Φ is not a constant, we need to establish some estimates on the time translates of the approximate concentration, as for instance in [9], in order to get the compactness. As the proof is rather technical, we have restricted the proof to the case Φ^*.*

Proposition 5.3. *Under the assumptions of Theorem 2.6 and the fact that Φ is a constant Φ^*, the sequences $(c_{m,\mathfrak{M}})_m$, $(c_{m,\overline{\mathfrak{M}^*}})_m$ and $(c_{m,\mathfrak{D}})_m$, defined by the scheme* (18)–(19) *and* (20)*, are relatively compact in $L^1(0,T;L^1(\Omega))$ and converge to the same limit $\bar{c} \in L^2(0,T;H^1(\Omega))$, defined in Proposition 5.1.*

Proof. We have

$$\|c_{m,\mathfrak{M}} - \bar{c}\|_{L^1(]0,T[\times\Omega)} \leq \|c_m - \bar{c}\|_{L^1(]0,T[\times\Omega)} + \frac{\sqrt{T\mathrm{m}_\Omega}}{2} \left\|c_{m,\mathfrak{M}} - c_{m,\overline{\mathfrak{M}^*}}\right\|_{L^2(]0,T[\times\Omega)}.$$

Lemma 3.3 and Proposition 5.1 imply that

$$\|c_{m,\mathfrak{M}} - \bar{c}\|_{L^1(]0,T[\times\Omega)} \longrightarrow 0, \text{ when } m \to \infty.$$

We do similarly for the convergence of $c_{m,\overline{\mathfrak{M}^*}}$.

For the last convergence, we have

$$\begin{aligned}
\|c_{m,\mathfrak{D}} - \bar{c}\|_{L^1(]0,T[\times\Omega)} &\leq \int_0^T \sum_{\mathcal{D}\in\mathfrak{D}} \int_{\mathcal{D}} \left| \frac{1}{\mathrm{m}_{\mathcal{D}}} \int_{\mathcal{D}} c_m(s,y)\mathrm{d}y - \bar{c}(s,x) \right| \mathrm{d}x\mathrm{d}s \\
&\leq \int_0^T \sum_{\mathcal{D}\in\mathfrak{D}} \int_{\mathcal{D}} |c_m(s,y) - \bar{c}(s,y)|\, \mathrm{d}y\mathrm{d}s \\
&\quad + \int_0^T \sum_{\mathcal{D}\in\mathfrak{D}} \frac{1}{\mathrm{m}_{\mathcal{D}}} \int_{\mathcal{D}} \int_{\mathcal{D}} |\bar{c}(s,y)\mathrm{d}y - \bar{c}(s,x)|\, \mathrm{d}y\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Proposition 5.1 implies that the first term in the right hand side tends to 0. Using the regularity of the mesh and of $\bar{c}$, we have for the second term:

$$\int_0^T \sum_{\mathcal{D}\in\mathfrak{D}} \frac{1}{\mathrm{m}_{\mathcal{D}}} \int_{\mathcal{D}} \int_{\mathcal{D}} |\bar{c}(s,y)\mathrm{d}y - \bar{c}(s,x)|\, \mathrm{d}y\mathrm{d}x\mathrm{d}s \le h_m C \int_0^T \sum_{\mathcal{D}\in\mathfrak{D}} \int_{\mathcal{D}} |\nabla \bar{c}(s,y)|\, \mathrm{d}y\mathrm{d}s,$$

term which tends to 0. We deduce that when $m \to \infty$

$$\|c_{m,\mathfrak{D}} - \bar{c}\|_{L^1(]0,T[\times\Omega)} \longrightarrow 0.$$

$\square$

5.2. Convergence of the pressure

Proposition 5.4. *Under the assumptions of Theorem 2.6, and the fact that Φ is a constant Φ^*, there exists $\bar{p} \in L^\infty(0,T;H^1(\Omega))$ and $\bar{\mathbf{U}} \in L^\infty(0,T;L^2(\Omega))^2$, such that the sequences $(p_m)_m, (\mathbf{U}_m)_m$ defined by the scheme* (18)–(19) *have the following convergence result when $m \to \infty$:*

$$\begin{aligned}
&p_m \to \bar{p} && \textit{weakly-}* \textit{ in } L^\infty(0,T;L^2(\Omega)) \textit{ and strongly in } L^p(0,T;L^q(\Omega)), \forall p<\infty, q<2;\\
&\nabla^{h_m} p_m \to \nabla\bar{p} && \textit{weakly-}* \textit{ in } (L^\infty(0,T;L^2(\Omega)))^2 \textit{ and strongly in } (L^2((0,T)\times\Omega))^2;\\
&\mathbf{U}_m \to \bar{\mathbf{U}} && \textit{weakly-}* \textit{ in } (L^\infty(0,T;L^2(\Omega)))^2 \textit{ and strongly in } (L^2((0,T)\times\Omega))^2;
\end{aligned}$$

and $(\bar{p},\bar{\mathbf{U}})$ is a weak solution to (1)*, with $\bar{c}$ defined in Proposition 5.1.*

Proof. Lemma 3.1 implies that up to a subsequence, we have when $m \to \infty$:

$$\begin{aligned}
&p_m \to \bar{p} \text{ weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega));\\
&\nabla^{h_m} p_m \to v \text{ weakly-}* \text{ in } (L^\infty(0,T;L^2(\Omega)))^2
\end{aligned}$$

and Proposition 4.4 implies

$$\bar{p} \in L^\infty(0,T;H^1(\Omega)), \text{ with } \nabla\bar{p} = v.$$

Furthermore, we have $\int_\Omega p_m(t,.)\mathrm{d}x = 0$ for all $t \in]0,T[$, it gives that $\int_\Omega \bar{p}(t,.)\mathrm{d}x = 0$ for all $t \in]0,T[$. We introduce a new sequence $(\check{c}_m)_m$ defined by

$$\begin{aligned}
\check{c}_m(t,x) &= c^0_{h_m}(x) \in [0,1], \text{ if } t \in [0,\delta t[,\\
\check{c}_m(t,x) &= c_{m,\mathfrak{D}}(t-\delta t, x), \text{ on } [\delta t, T[\times\Omega,
\end{aligned}$$

Thanks to Proposition 5.3, $(c_{m,\mathfrak{D}})_m$ converges to $\bar{c}$ in $L^1(0,T;L^1(\Omega))$. It implies that $(\check{c}_m)_m$ converges also to $\bar{c}$ in $L^1(0,T;L^1(\Omega))$. As in [9, Section 5.2] (working on the diamond mesh instead of the primal mesh), we obtain

$$\mathbf{U}_m = -A_m(.,\check{c}_m)\nabla^{h_m} p_m \underset{m\to\infty}{\to} \bar{\mathbf{U}} = -A(\cdot,\bar{c})\nabla\bar{p} \quad \text{weakly in } (L^2(]0,T[\times\Omega))^2.$$

Let us remark that the *a priori* estimates (Lemma 3.1) gives

$$\mathbf{U}_m \underset{m\to\infty}{\to} \bar{\mathbf{U}} \quad \text{weakly-}* \text{ in } (L^\infty(0,T;L^2(\Omega))^2.$$

It remains to prove (9). Let $\varphi \in C^\infty([0,T]\times\bar{\Omega})$, we define $\varphi^n_{\mathcal{T}_m}$ associated to the discrete values:

$$\varphi^n_{\mathcal{K}} = \frac{1}{\mathrm{m}_{\mathcal{K}}\delta t} \int_{t_{n-1}}^{t_n} \int_{\mathcal{K}} \varphi(s,x)\mathrm{d}x\mathrm{d}s, \forall \mathcal{K} \in \mathfrak{M}_m \text{ and } \varphi^n_{\mathcal{K}} = 0, \forall \mathcal{K} \in \partial\mathfrak{M}_m,\ \forall n \in \{1,\cdots,N\},$$

$$\varphi^n_{\mathcal{K}^*} = \frac{1}{\mathrm{m}_{\mathcal{K}^*}\delta t} \int_{t_{n-1}}^{t_n} \int_{\mathcal{K}^*} \varphi(s,x)\mathrm{d}x\mathrm{d}s,\ \forall \mathcal{K}^* \in \overline{\mathfrak{M}^*_m},\ \forall n \in \{1,\cdots,N\}.$$

We define also the corresponding function φ_m and $\Psi_m = \nabla^{h_m}\varphi_m$. Since p_m is the solution of (18a), the discrete duality formula (Theorem 2.3) gives

$$\sum_{n=1}^{N_T} \delta t [\![q^{+,n}_{\mathcal{T}_m} - q^{-,n}_{\mathcal{T}_m}, \varphi^n_{\mathcal{T}_m}]\!]_{\mathcal{T}_m} = \sum_{n=1}^{N_T} \delta t (\mathbb{A}_{\mathfrak{D}_m}(c^{n-1}_{\mathfrak{D}_m})\nabla^{\mathfrak{D}_m} p^n_{\mathcal{T}_m}, \nabla^{\mathfrak{D}_m}\varphi^n_{\mathcal{T}_m})_{\mathfrak{D}_m}.$$

But, on one hand, thanks to (38), we have

$$\sum_{n=1}^{N_T} \delta t [\![q^{+,n}_{\mathcal{T}_m} - q^{-,n}_{\mathcal{T}_m}, \varphi^n_{\mathcal{T}_m}]\!]_{\mathcal{T}_m} = \int_0^T \int_\Omega (q^+ - q^-)\varphi_m,$$

and on the other hand,

$$\sum_{n=1}^{N_T} \delta t (\mathbb{A}_{\mathfrak{D}_m}(c^{n-1}_{\mathfrak{D}_m})\nabla^{\mathfrak{D}_m} p^n_{\mathcal{T}_m}, \nabla^{\mathfrak{D}_m}\varphi^n_{\mathcal{T}_m})_{\mathfrak{D}_m} = -\int_0^T \int_\Omega \mathbf{U}_m \cdot \Psi_m.$$

We deduce

$$\int_0^T \int_\Omega (q^+ - q^-)\varphi_m = -\int_0^T \int_\Omega \mathbf{U}_m \cdot \Psi_m. \tag{40}$$

The function φ is smooth and then we have the uniform convergence of φ_m and Ψ_m to φ and $\nabla\varphi$, respectively. Therefore, the weak convergence of $\mathbf{U}_m$ to $\bar{\mathbf{U}} = -A(\cdot,\bar{c})\nabla\bar{p}$ in $(L^2((0,T)\times\Omega))^2$ implies (9). As in [9, Section 5.2], using the Minty trick, we deduce the strong convergence of $\nabla^{h_m} p_m$, $\mathbf{U}_m$ and finally of p_m. □

5.3. Convergence of the concentration

Proposition 5.5. *Under the assumptions of Theorem 2.6, and the fact that Φ is a constant Φ^*, the function $\bar{c}$, introduced in Proposition 5.1, and $\bar{\mathbf{U}}$, introduced in Proposition 5.4, satisfy* (10)*.*

Proof. Let $\varphi \in C^\infty([0,T]\times\bar{\Omega})$, we use the same notation as in the proof of Proposition 5.4 in order to define $\varphi^n_{\mathcal{T}_m}$, φ_m and Ψ_m. Since c_m is the solution of (19a), we obtain

$$\sum_{n=1}^{N_T} \delta t \left[\!\!\left[\Phi^*\partial_{t,\mathcal{T}_m} c^n_{h_m} - \mathrm{div}^{\mathcal{T}_m}\left(\mathbb{D}_{\mathfrak{D}_m}\left(\mathbf{U}^n_{\mathfrak{D}_m}\right)\nabla^{\mathfrak{D}_m} c^n_{\mathcal{T}_m}\right), \varphi^n_{\mathcal{T}_m} \right]\!\!\right]_{\mathcal{T}_m}$$
$$+ \sum_{n=1}^{N_T} \delta t \left[\!\!\left[\mathrm{divc}^{\mathcal{T}_m}\left(\mathbf{U}^n_{\mathfrak{D}_m}, c^n_{\mathcal{T}_m}\right) + \lambda\mathcal{P}^{\mathcal{T}_m} c^n_{\mathcal{T}_m} + q^{-,n}_{\mathcal{T}_m} c^n_{\mathcal{T}_m}, \varphi^n_{\mathcal{T}_m} \right]\!\!\right]_{\mathcal{T}_m} = \sum_{n=1}^{N_T} \delta t \left[\!\!\left[q^{+,n}_{\mathcal{T}_m}\hat{c}^n_{\mathcal{T}_m}, \varphi^n_{\mathcal{T}_m} \right]\!\!\right]_{\mathcal{T}_m}.$$

We will pass to the limit separately in each term, denoted by T_i for $0 \le i \le 5$. We start with

$$T_0 := \sum_{n=1}^{N_T} \delta t \Phi^* [\![\partial_{t,\mathcal{T}_m} c^n_{\mathcal{T}_m}, \varphi^n_{\mathcal{T}_m}]\!]_{\mathcal{T}_m}.$$

It rewrites

$$T_0 = -\sum_{n=1}^{N_T - 1} \delta t \Phi^* \left[\!\!\left[c^n_{\mathcal{T}_m}, \frac{\varphi^{n+1}_{\mathcal{T}_m} - \varphi^n_{\mathcal{T}_m}}{\delta t} \right]\!\!\right]_{\mathcal{T}_m} - \Phi^* [\![c^0_{\mathcal{T}_m}, \varphi^1_{\mathcal{T}_m}]\!]_{\mathcal{T}_m},$$

since $\varphi^{N_T}_{\mathcal{T}} = 0$. Applying (38), we get

$$T_0 = -\int_0^T \int_\Omega \Phi^* c_m(s,x) \frac{\varphi(s+\delta t, x) - \varphi(s,x)}{\delta t} \mathrm{d}x\mathrm{d}s - \int_\Omega \Phi^* c_0(x)\varphi_m(\delta t, x)\mathrm{d}x.$$

The function φ is smooth and then we have the uniform convergence of $\dfrac{\varphi(.+\delta t,.)-\varphi(.,.)}{\delta t}$ and $\varphi_m(\delta t,.)$ respectively to $\partial_t\varphi$ and $\varphi(0,\cdot)$. Therefore, the weak convergence of c_m to $\bar{c}$ in $L^\infty(0,T;L^2(\Omega))$ implies that

$$T_0 \longrightarrow -\int_0^T\int_\Omega \Phi^*\bar{c}\partial_t\varphi - \int_\Omega \Phi^* c_0\varphi(0,\cdot).$$

Using the discrete duality formula (Theorem 2.3), T_1 rewrites

$$T_1 := -\sum_{n=1}^{N_T}\delta t \left[\!\left[\mathrm{div}^{\mathcal{T}_m}\left(\mathbb{D}_{\mathfrak{D}_m}\left(\mathbf{U}^n_{\mathfrak{D}_m}\right)\nabla^{\mathfrak{D}_m}c^n_{\mathcal{T}_m}\right), \varphi^n_{\mathcal{T}_m}\right]\!\right]_{\mathcal{T}_m} = \sum_{n=1}^{N_T}\delta t\left(\mathbb{D}_{\mathfrak{D}_m}\left(\mathbf{U}^n_{\mathfrak{D}_m}\right)\nabla^{\mathfrak{D}_m}c^n_{\mathcal{T}_m}, \nabla^{\mathfrak{D}_m}\varphi^n_{\mathcal{T}_m}\right)_{\mathfrak{D}_m}.$$

We deduce

$$T_1 = \int_0^T\int_\Omega \nabla^{h_m}c_m\cdot\left({}^tD(\cdot,\mathbf{U}_m)\Psi_m\right).$$

We have the uniform convergence of Ψ_m to $\nabla\varphi$. Furthermore, we have $\mathbf{U}_m \to \bar{\mathbf{U}} = -A(\cdot,\bar{c})\nabla\bar{p}$ in $(L^2((0,T)\times\Omega))^2$, then we get $D(\cdot,\mathbf{U}_m)\to D(\cdot,\bar{\mathbf{U}})$ in $(L^2((0,T)\times\Omega))^{2\times 2}$. It implies that ${}^tD(\cdot,\mathbf{U}_m)\Psi_m \to {}^tD(\cdot,\bar{\mathbf{U}})\nabla\varphi$ in $(L^2((0,T)\times\Omega))^2$. And finally, the weak convergence of $\nabla^{h_m}c_m$ to $\nabla\bar{c}$ in $(L^2((0,T)\times\Omega))^2$ implies that

$$T_1 \longrightarrow \int_0^T\int_\Omega \nabla\bar{c}\cdot\left({}^tD(\cdot,\bar{\mathbf{U}})\nabla\varphi\right) = \int_0^T\int_\Omega\left(D(\cdot,\bar{\mathbf{U}})\nabla\bar{c}\right)\cdot\nabla\varphi.$$

As in the proof of Lemma 3.4, $T_2 := \sum_{n=1}^{N_T}\delta t\left[\!\left[\mathrm{divc}^{\mathcal{T}_m}\left(\mathbf{U}^n_{\mathfrak{D}_m}, c^n_{\mathcal{T}_m}\right), \varphi^n_{\mathcal{T}_m}\right]\!\right]_{\mathcal{T}_m}$ can be split into the sum of a primal term $T_{2,p}$ and a dual term $T_{2,d}$. Using the relation $\varphi^n_{\mathcal{K}}-\varphi^n_{\mathcal{L}} = \mathrm{m}_{\sigma^*}\nabla^{\mathcal{D}}\varphi^n_{\mathcal{T}_m}\cdot\boldsymbol{\tau}_{\mathcal{K},\mathcal{L}}$, $x^+ = x + x^-$ and (30), the primal part rewrites

$$\begin{aligned} T_{2,p} =& \frac{1}{2}\sum_{n=1}^{N_T}\delta t\sum_{\mathcal{D}\in\mathfrak{D}_m}\mathrm{m}_\sigma\mathrm{m}_{\sigma^*}(\mathbf{U}^n_{\mathcal{D}}\cdot\mathbf{n}_{\sigma\mathcal{K}})c^n_{\mathcal{K}}\nabla^{\mathcal{D}}\varphi^n_{\mathcal{T}_m}\cdot\boldsymbol{\tau}_{\mathcal{K},\mathcal{L}} \\ &+\frac{1}{2}\sum_{n=1}^{N_T}\delta t\sum_{\mathcal{D}\in\mathfrak{D}_m}\mathrm{m}_\sigma\mathrm{m}_{\sigma^*}(\mathbf{U}^n_{\mathcal{D}}\cdot\mathbf{n}_{\sigma\mathcal{K}})^-(c^n_{\mathcal{K}}-c^n_{\mathcal{L}})\nabla^{\mathcal{D}}\varphi^n_{\mathcal{T}_m}\cdot\boldsymbol{\tau}_{\mathcal{K},\mathcal{L}}.\end{aligned}$$

Let set $T_2^* = \int_0^T\int_\Omega c_{m,\mathfrak{D}}\mathbf{U}_m\cdot\Psi_m$. Using the convergence results, we remark that

$$T_2^* \longrightarrow \int_0^T\int_\Omega \bar{c}\bar{\mathbf{U}}\cdot\nabla\varphi.$$

Moreover, T_2^* can also be split into the sum of a primal term $T^*_{2,p}$ and a dual term $T^*_{2,d}$. The primal term is

$$T^*_{2,p} = \sum_{n=1}^{N_T}\delta t\sum_{\mathcal{D}\in\mathfrak{D}_m}\frac{\mathrm{m}_{\mathcal{D}}}{\sin(\theta_{\mathcal{D}})}c^n_{\mathcal{D}}(\mathbf{U}^n_{\mathcal{D}}\cdot\mathbf{n}_{\sigma\mathcal{K}})(\nabla^{\mathcal{D}}\varphi^n_{\mathcal{T}_m}\cdot\boldsymbol{\tau}_{\mathcal{K},\mathcal{L}}),$$

since we have $\mathbf{U}^n_{\mathcal{D}} = \dfrac{1}{\sin(\theta_{\mathcal{D}})}(\mathbf{U}^n_{\mathcal{D}}\cdot\mathbf{n}_{\sigma\mathcal{K}})\boldsymbol{\tau}_{\mathcal{K},\mathcal{L}} + \dfrac{1}{\sin(\theta_{\mathcal{D}})}(\mathbf{U}^n_{\mathcal{D}}\cdot\mathbf{n}_{\sigma^*\mathcal{K}^*})\boldsymbol{\tau}_{\mathcal{K}^*,\mathcal{L}^*}$. Let us prove that $T^*_{2,p}-T_{2,p}$ tends to 0. We obtain

$$\begin{aligned} T^*_{2,p}-T_{2,p} =& \sum_{n=1}^{N_T}\delta t\sum_{\mathcal{D}\in\mathfrak{D}_m}\frac{\mathrm{m}_{\mathcal{D}}}{\sin(\theta_{\mathcal{D}})}(\mathbf{U}^n_{\mathcal{D}}\cdot\mathbf{n}_{\sigma\mathcal{K}})(\nabla^{\mathcal{D}}\varphi^n_{\mathcal{T}_m}\cdot\boldsymbol{\tau}_{\mathcal{K},\mathcal{L}})(c^n_{\mathcal{D}}-c^n_{\mathcal{K}}) \\ &-\sum_{n=1}^{N_T}\delta t\sum_{\mathcal{D}\in\mathfrak{D}_m}\frac{\mathrm{m}_{\mathcal{D}}}{\sin(\theta_{\mathcal{D}})}(\mathbf{U}^n_{\mathcal{D}}\cdot\mathbf{n}_{\sigma\mathcal{K}})^-(\nabla^{\mathcal{D}}\varphi^n_{\mathcal{T}_m}\cdot\boldsymbol{\tau}_{\mathcal{K},\mathcal{L}})(c^n_{\mathcal{K}}-c^n_{\mathcal{L}}).\end{aligned} \tag{41}$$

For the second term in the right hand side of (41), the relation $c^n_{\mathcal{K}} - c^n_{\mathcal{L}} = \mathrm{m}_{\sigma^*}\nabla^{\mathcal{D}} c^n_{\mathcal{T}_m} \cdot \boldsymbol{\tau}_{\mathcal{K},\mathcal{L}}$ and Cauchy-Schwarz inequality imply

$$\left|\sum_{n=1}^{N_T} \delta t \sum_{\mathcal{D}\in\mathfrak{D}_m} \frac{\mathrm{m}_{\mathcal{D}}}{\sin(\theta_{\mathcal{D}})} (\mathbf{U}^n_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}})^- (\nabla^{\mathcal{D}}\varphi^n_{\mathcal{T}_m} \cdot \boldsymbol{\tau}_{\mathcal{K},\mathcal{L}})(c^n_{\mathcal{K}} - c^n_{\mathcal{L}})\right|$$
$$\leq C\sqrt{T} h_m \|\nabla^{\mathfrak{D}_m}\varphi^n_{\mathcal{T}_m}\|_{\infty,\mathfrak{D}_m} \|\mathbf{U}_m\|_{(L^\infty(0,T;L^2(\Omega)))^2} \|\nabla^{h_m} c_m\|_{(L^2((0,T)\times\Omega))^2}. \quad (42)$$

The *a priori* estimates (23) and Lemma 3.5 of [5] give

$$\left|\sum_{n=1}^{N_T} \delta t \sum_{\mathcal{D}\in\mathfrak{D}_m} \frac{\mathrm{m}_{\mathcal{D}}}{\sin(\theta_{\mathcal{D}})} (\mathbf{U}^n_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}})^- (\nabla^{\mathcal{D}}\varphi^n_{\mathcal{T}_m} \cdot \boldsymbol{\tau}_{\mathcal{K},\mathcal{L}})(c^n_{\mathcal{K}} - c^n_{\mathcal{L}})\right| \leq C h_m.$$

This term tends to 0. For the first term in the right hand side of (41), we have similarly

$$\left|\sum_{n=1}^{N_T} \delta t \sum_{\mathcal{D}\in\mathfrak{D}_m} \frac{\mathrm{m}_{\mathcal{D}}}{\sin(\theta_{\mathcal{D}})} (\mathbf{U}^n_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}}) (\nabla^{\mathcal{D}}\varphi^n_{\mathcal{T}_m} \cdot \boldsymbol{\tau}_{\mathcal{K},\mathcal{L}})(c^n_{\mathcal{D}} - c^n_{\mathcal{K}})\right| \leq C\left(\sum_{n=1}^{N_T} \delta t \sum_{\mathcal{D}\in\mathfrak{D}_m} \mathrm{m}_{\mathcal{D}} |c^n_{\mathcal{D}} - c^n_{\mathcal{K}}|^2\right)^{\frac{1}{2}}.$$

We apply Lemma 3.3 to get that this term tends to 0 and finally $T^*_{2,p} - T_{2,p} \longrightarrow 0$. The same convergence result is obtained for the dual part and

$$T_2 \longrightarrow \int_0^T \int_\Omega \bar{c}\bar{\mathbf{U}} \cdot \nabla\varphi.$$

As in the proof of Lemma 3.4, using (32), the penalization term $T_3 := \sum_{n=1}^{N_T} \delta t \left[\!\left[\mathcal{P}^{\mathcal{T}_m} c^n_{\mathcal{T}_m}, \varphi^n_{\mathcal{T}_m}\right]\!\right]_{\mathcal{T}_m}$ verifies

$$|T_3| \leq \frac{1}{2}\frac{1}{h^\beta_m} \|c_{m,\mathfrak{M}} - c_{m,\overline{\mathfrak{M}^*}}\|_{L^2(0,T;L^2(\Omega))} \|\varphi_{m,\mathfrak{M}} - \varphi_{m,\overline{\mathfrak{M}^*}}\|_{L^2(0,T;L^2(\Omega))}.$$

Inequality (26) and (37) imply that:

$$|T_3| \leq C h_m^{1-\frac{\beta}{2}} \longrightarrow 0,$$

since $\beta < 2$.

Thanks to (38), $T_4 := \sum_{n=1}^{N_T} \delta t \left[\!\left[q^{-,n}_{\mathcal{T}_m} c^n_{\mathcal{T}_m}, \varphi^n_{\mathcal{T}_m}\right]\!\right]_{\mathcal{T}_m}$ rewrites

$$T_4 = \frac{1}{2}\int_0^T \int_\Omega c_{m,\mathfrak{M}}(s,x)\varphi_{m,\mathfrak{M}}(s,x) q^-(s,x)\mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_0^T \int_\Omega c_{m,\mathfrak{M}^*}(s,x)\varphi_{m,\mathfrak{M}^*}(s,x) q^-(s,x)\mathrm{d}x\mathrm{d}s$$

The uniform convergence of $\varphi_{m,\mathfrak{M}}$ and $\varphi_{m,\mathfrak{M}^*}$ to φ, the weak convergence of $c_{m,\mathfrak{M}}$ and $c_{m,\mathfrak{M}^*}$ to the same $\bar{c}$ lying in $L^\infty(0,T;L^2(\Omega))$ imply that

$$T_4 \longrightarrow \int_0^T \int_\Omega q^- \bar{c}\varphi.$$

Similarly, $T_5 := \sum_{n=1}^{N_T} \delta t \left[\!\left[q^{+,n}_{\mathcal{T}_m} \hat{c}^n_{\mathcal{T}_m}, \varphi^n_{\mathcal{T}_m}\right]\!\right]_{\mathcal{T}_m}$ rewrites

$$T_5 = \frac{1}{2}\int_0^T \int_\Omega \hat{c}(s,x)\varphi_{m,\mathfrak{M}}(s,x) q^+_{m,\mathfrak{M}}(s,x)\mathrm{d}x\mathrm{d}s + \frac{1}{2}\int_0^T \int_\Omega \hat{c}(s,x)\varphi_{m,\mathfrak{M}^*}(s,x) q^+_{m,\mathfrak{M}^*}(s,x)\mathrm{d}x\mathrm{d}s.$$

The uniform convergence of $\varphi_{m,\mathfrak{M}}$ and $\varphi_{m,\mathfrak{M}^*}$ to φ and the weak convergence of $q^+_{m,\mathfrak{M}}$ and $q^+_{m,\mathfrak{M}^*}$ to q^+ in $L^2((0,T)\times\Omega)$ imply that

$$T_5 \longrightarrow \int_0^T \int_\Omega q^+ \hat{c}\varphi.$$

Passing to the limit in each term, we have proved (10). $\square$

Remark 5.6. *The penalization term in the scheme is useful in order to prove that the sequences* $(c_{m,\mathfrak{M}})_m$, $(c_{m,\overline{\mathfrak{M}^*}})_m$ *and* $(c_{m,\mathfrak{D}})_m$ *converge to the same limit* $\bar{c} \in L^2(0,T;H^1(\Omega))$ *(Lemma 3.3). This is essential when passing to the limit in the convection term* T_2 *and the reaction term* T_4.

6. Numerical experiments

In this section, we define the tensor $\mathbb{A}$, which contains the effect of the permeability $\mathbb{K}$ of the porous medium and the viscosity μ of the fluid mixture:

$$\mathbb{A}(\cdot, c) = \frac{\mathbb{K}(\cdot)}{\mu(c)}.$$

The viscosity μ is usually determined by the following mixing rule:

$$\mu(c) = \mu(0)\Big(1 + \big(M^{1/4} - 1\big)c\Big)^{-4} \text{ on } [0,1], \tag{43}$$

where $M = \frac{\mu(0)}{\mu(1)}$ is the mobility ratio (we extend μ to $\mathbb{R}$ by letting $\mu = \mu(0)$ on $]-\infty, 0[$ and $\mu = \mu(1)$ on $]1,\infty[$). The tensor $\mathbb{D}$ is the diffusion-dispersion tensor; it includes molecular diffusion d_m and mechanical dispersion:

$$\mathbb{D}(x, \mathbf{U}) = \Phi(x)\Big(d_m\mathbb{I} + |\mathbf{U}|\Big(d_l\mathbb{E}(\mathbf{U}) + d_t(\mathbb{I} - \mathbb{E}(\mathbf{U}))\Big)\Big), \tag{44}$$

where $\mathbb{I}$ is the identity matrix, d_l and d_t are the longitudinal and transverse dispersion coefficients and $\mathbb{E}(\mathbf{U}) = \Big(\frac{\mathbf{U}_i\mathbf{U}_j}{|\mathbf{U}|^2}\Big)_{1\le i,j\le d}$.

In [10], we have presented some numerical experiments without penalization to show the efficiency of the DDFV scheme. We have computed the numerical order of convergence of the scheme for the pressure and the concentration equations and obtained good results when the permeability is continuous or has discontinuities supported by the edges of the primal mesh using a modified DDFV scheme. We have also compared some qualitative results, obtained with the m-DDFV and the Mixed Finite Volume scheme of [9], presenting the level sets of the concentration at two different times. Here, we only focus on the influence of the penalization operator in the behavior of DDFV scheme.

The spatial domain is $\Omega = (0,1000) \times (0,1000)$ ft^2 and the time period is $[0,3600]$ days. The injection well is located at the upper-right corner $(1000,1000)$ with an injection rate $q^+ = 30$ ft^2/day and an injection concentration $\hat{c} = 1.0$. The production well is located at the lower-left corner $(0,0)$ with a production rate $q^- = 30$ ft^2/day. It means that q^- and q^+ are Dirac masses, which can be taken into account with the scheme. The porosity of the medium is specified as $\Phi(x) = 0.1$ and the initial concentration is $c_0(x) = 0$. The viscosity of the oil is $\mu(0)$=1.0 cp and $M = 41$. We choose $\Phi d_l = 5$ ft and $\Phi d_t = 0.5$ ft and there is no molecular diffusion $\Phi d_m = 0$ ft^2/day.

Test 1:

We choose a constant permeability $\mathbb{K} = 80\,\mathbb{I}$.

We introduce a sequence of triangular meshes. For a refinement level $i \in \{1, \cdots, 8\}$, the mesh is obtained by dividing the domain into $2^{i+1} \times 2^{i+1}$ equally sized squares and each square is split into 2 triangles along a diagonal. The number of cells for the mesh i is 2^{2i+3}. We present on the left of Figure 6.1 the mesh obtained for $i = 1$. We choose this sequence of structured triangular meshes because they fit together and allow the computation of numerical errors. Let us also mention that, even though many choices are possible, we always assume in this paper that $x_{\mathcal{K}}$ is the mass center of $\mathcal{K} \in \mathfrak{M}$. The time step is $\delta t = 36$ days.

Figures 6.2 and 6.3 present the level sets of the concentration obtained with the DDFV scheme, with the penalization term and without a penalization term, on the structured triangular mesh $i = 5$, at two different times (3 and 10 years). The same qualitative behavior is observed.

The penalization operator is introduced in order to prove that $(c_{m,\mathfrak{M}})_m$ and $(c_{m,\overline{\mathfrak{M}^*}})_m$ have the limit. In Table 6.1, we compute the L^2-norm of the difference between $(c_{m,\mathfrak{M}})_m$ and $(c_{m,\overline{\mathfrak{M}^*}})_m$. We observe that without penalization this difference tends to zero with an order of convergence equal to 0.5.

Let us just mention that we obtain similar results using a sequence of square meshes.

In Table 6.2, we give some information on the overshooting and the undershooting effects of the concentration observed on the corresponding primal and dual mesh of the triangular mesh with refinement level $i = 5$, without

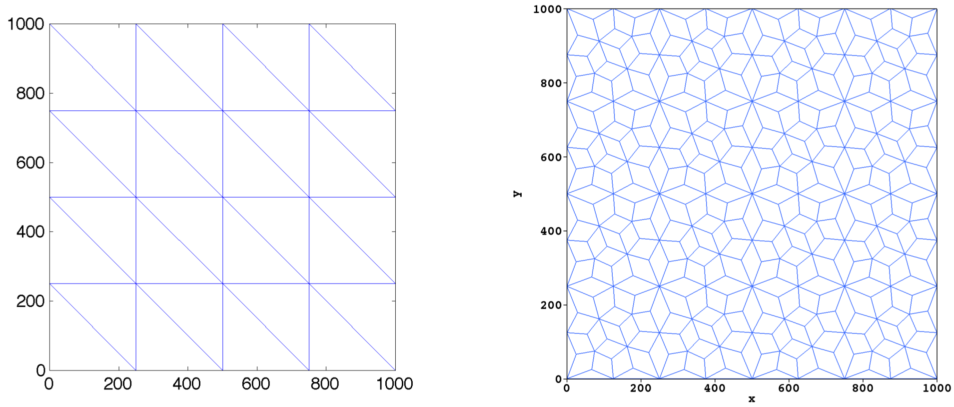
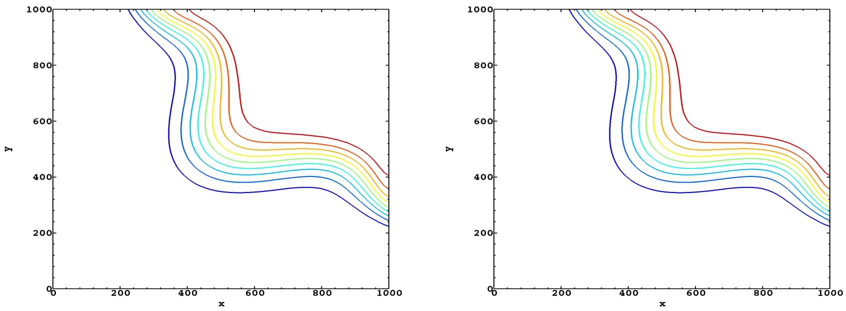


Figure 6.1: Triangular mesh with a refinement level $i = 1$ on the left and quadrangular/triangular mesh with a refinement level $i = 1$ on the right.

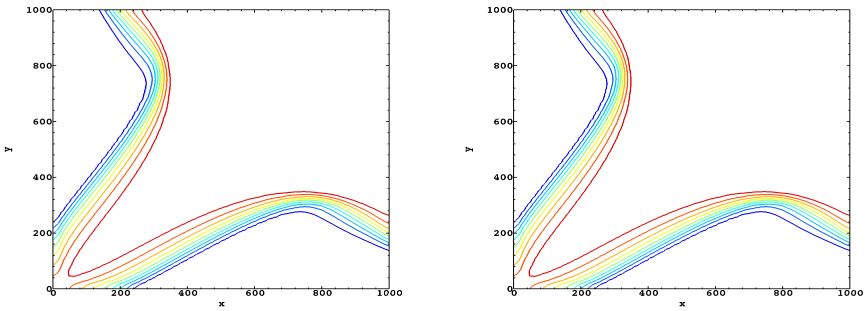
refinement level	error L^2	order L^2
1	1.08e+03	-
2	9.39e+02	0.23
3	7.06e+02	0.45
4	5.24e+02	0.45
5	3.93e+02	0.42
6	2.92e+02	0.44
7	2.11e+02	0.47

Table 6.1: Test 1. The L^2-norm $\|c_{m,\mathfrak{M}} - c_{m,\overline{\mathfrak{M}^*}}\|_{L^2([0,T]\times\Omega)}$without penalization term ($\lambda = 0$).



(a) $\lambda = 10^{-6}$ and $\beta = 1$. (b) $\lambda = 0$.

Figure 6.2: Test 1. Comparison of the concentration obtained with the DDFV schemes, with the penalization term and without, on the structured triangular mesh $i = 5$: level sets of the concentration after 3 years.



(a) $\lambda = 10^{-6}$ and $\beta = 1$. (b) $\lambda = 0$.

Figure 6.3: Test 1. Comparison of the concentration obtained with the DDFV schemes, with the penalization term and without, on the structured triangular mesh $i = 5$: level sets of the concentration after 10 years.

penalization ($\lambda = 0$). The concentration on the dual mesh $c_{\mathfrak{M}^*}$ satisfies the maximum principle whereas on the primal mesh $c_{\mathfrak{M}}$ after 10 years the overshooting and the undershooting are around 10^{-3} for 13% or 7%. As shown in Table 6.3, we obtain essentially similar results on an unstructured triangular mesh.

Test 2:

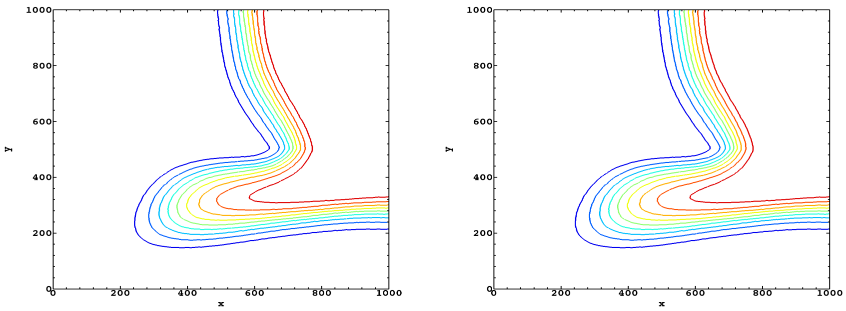
We choose a discontinous permeability $\mathbb{K} = 80\,\mathbb{I}$ on the subdomain $(0, 1000) \times (0, 500)$ and $\mathbb{K} = 20\,\mathbb{I}$ on the subdomain $(0, 1000) \times (500, 1000)$. We introduce a sequence of quadrangular/triangular meshes on the right of Figure 6.1 the mesh obtained for $i = 1$. Let us also mention that, even though many choices are possible, we always assume in this paper that $x_{\mathcal{K}}$ is the mass center of $\mathcal{K} \in \mathfrak{M}$. The time step is $\delta t = 36$ days.

	overshooting		undershooting	
	$\max(c) - 1$	percentage	$\min(c)$	percentage
$t = 3$ years, $c = c_{\mathfrak{M}}$	1.20e-14	0.09%	-2.40e-06	10.65%
$t = 3$ years, $c = c_{\mathfrak{M}^*}$	-3.77e-15	0.00%	3.61e-10	0.00%
$t = 10$ years, $c = c_{\mathfrak{M}}$	1.62e-03	13.00%	-3.76e-03	7.10%
$t = 10$ years, $c = c_{\mathfrak{M}^*}$	2.24e-14	3.27%	8.41e-08	0.00%

Table 6.2: Test 1. Overshooting and undershooting of concentration on the structured triangular mesh with refinement level $i = 5$.

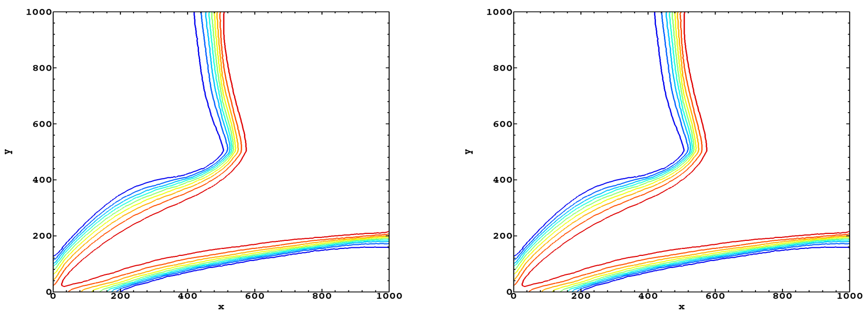
	overshooting		undershooting	
	$\max(c) - 1$	percentage	$\min(c)$	percentage
$t = 3$ years, $c = c_{\mathfrak{M}}$	-5.34e-14	0.00%	-5.11e-06	10.37%
$t = 3$ years, $c = c_{\mathfrak{M}^*}$	-8.78e-14	0.00%	1.61e-10	0.00%
$t = 10$ years, $c = c_{\mathfrak{M}}$	5.45e-04	9.82%	-5.38e-03	8.52%
$t = 10$ years, $c = c_{\mathfrak{M}^*}$	1.24e-14	1.53%	4.96e-09	0.00%

Table 6.3: Test 1. Overshooting and undershooting of concentration on an unstructured triangular mesh made of 5449 triangles.



(a) $\lambda = 10^{-6}$ and $\beta = 1$. (b) $\lambda = 0$.

Figure 6.4: Test 2. Comparison of the concentration obtained with the DDFV schemes, with the penalization term and without, on the quadrangular/triangular mesh $i = 5$: level sets of the concentration after 3 years.



(a) $\lambda = 10^{-6}$ and $\beta = 1$. (b) $\lambda = 0$.

Figure 6.5: Test 2. Comparison of the concentration obtained with the DDFV schemes, with the penalization term and without, on the quadrangular/triangular mesh $i = 5$: level sets of the concentration after 10 years.

Figures 6.4 and 6.5 present the level sets of the concentration obtained with the DDFV scheme, with the penalization term and without a penalization term, on the quadrangular/triangular mesh $i = 5$, at two different times (3 and 10 years). The same qualitative behavior is observed.

In Table 6.4, we set the overshooting and the undershooting effects of the concentration on the corresponding primal and dual mesh of the quadrangular/triangular mesh with refinement level $i = 5$ without penalization. We observe a different behavior than for the test 1. Indeed, the concentration on the primal mesh $c_{\mathfrak{M}}$ almost satisfies the maximum principle, whereas on the dual mesh $c_{\mathfrak{M}^*}$ after 10 years the overshooting (respectively the undershooting) is around 10^{-5} (respectively 10^{-3}) for 13% (respectively for 3%). In this case, contrary to the test 1, the maximum principle is better respected in the primal mesh as the dual mesh.

	overshooting		undershooting	
	$\max(c)-1$	percentage	$\min(c)$	percentage
$t=3$ years, $c=c_{\mathfrak{M}}$	2.68e-13	0.54%	-9.34e-11	0.08%
$t=3$ years, $c=c_{\mathfrak{M}^*}$	4.17e-14	0.26%	-1.64e-06	3.55%
$t=10$ years, $c=c_{\mathfrak{M}}$	1.81e-13	9.98%	-2.08e-19	0.01%
$t=10$ years, $c=c_{\mathfrak{M}^*}$	8.44e-05	13.56%	-2.52e-03	3.14%

Table 6.4: Test 2. Overshooting and undershooting of concentration on quadrangular/triangular mesh with refinement level $i=5$.

In conclusion, we have presented a DDFV scheme for the Peaceman model with a penalization operator and we have established its convergence. The numerical experiments show good qualitative properties with a small penalization or without penalization. We can conclude that the penalization operator can be set to 0 in practice.

We have only considered here the two-dimensional case but it is worth noticing that DDFV schemes have been successfully extended to the three-dimensional case in [11, 14, 34, 3] for linear anisotropic scalar diffusion equations and in [38] for the Stokes problem. Generalization of the present work to the three-dimensional case would be based on similar ideas.

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7. Appendix

First, to a given vector $u_{\mathcal{T}} = \left((u_{\mathcal{K}})_{\mathcal{K}\in\overline{\mathfrak{M}}}, (u_{\mathcal{K}^*})_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}}\right) \in \mathbb{R}^{\mathcal{T}}$ defined on a DDFV mesh $\mathcal{T}$ of size h, we associate the approximate solution on the boundary:

$$u^{\partial\mathfrak{M}\cup\partial\mathfrak{M}^*} = \frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} u_{\mathcal{K}}\mathbf{1}_{\overline{\mathcal{K}}\cap\partial\Omega} + \frac{1}{2}\sum_{\mathcal{K}^*\in\partial\mathfrak{M}^*} u_{\mathcal{K}^*}\mathbf{1}_{\overline{\mathcal{K}^*}\cap\partial\Omega}.$$

With this definition, we use simultaneously the values on the primal mesh and the values on the dual mesh. Indeed, we have $u^{\partial\mathfrak{M}\cup\partial\mathfrak{M}^*} = \frac{1}{2}(u^{\partial\mathfrak{M}} + u^{\partial\mathfrak{M}^*})$, where $u^{\partial\mathfrak{M}}$ and $u^{\partial\mathfrak{M}^*}$ are two different reconstructions based either on the primal values or the dual values:

$$u^{\partial\mathfrak{M}}(x) = \sum_{\mathcal{K}\in\mathfrak{M}} u_{\mathcal{K}}\mathbf{1}_{\overline{\mathcal{K}}\cap\partial\Omega}(x) \quad\text{and}\quad u^{\partial\mathfrak{M}^*}(x) = \sum_{\mathcal{K}^*\in\partial\mathfrak{M}^*} u_{\mathcal{K}^*}\mathbf{1}_{\overline{\mathcal{K}^*}\cap\partial\Omega}(x).$$

Let us now define some norms

$$\|u^{\partial\mathfrak{M}\cup\partial\mathfrak{M}^*}\|_{1,\partial\Omega} = \frac{1}{2}\|u^{\partial\mathfrak{M}^*}\|_{1,\partial\Omega} + \frac{1}{2}\|u^{\partial\mathfrak{M}}\|_{1,\partial\Omega}.$$

Theorem 7.1 (Trace inequality). *Let Ω be a convex polygonal domain of $\mathbb{R}^2$ and $\mathcal{T}$ a DDFV mesh of this domain. There exists $C>0$, depending only on Ω, ζ and θ, such that $\forall\ u_{\mathcal{T}}\in\mathbb{R}^{\mathcal{T}}$:*

$$\|u^{\partial\mathfrak{M}\cup\partial\mathfrak{M}^*}\|_{1,\partial\Omega} = \frac{1}{2}\|u^{\partial\mathfrak{M}^*}\|_{1,\partial\Omega} + \frac{1}{2}\|u^{\partial\mathfrak{M}}\|_{1,\partial\Omega} \le C\left(\|u_{\mathcal{T}}\|_{1,\mathcal{T}} + \left\|\nabla^{\mathfrak{D}}u_{\mathcal{T}}\right\|_{1,\mathfrak{D}}\right). \tag{45}$$

Proof. The calculations are similar to those followed in [27, Lemma 10.5] especially for the primal mesh, the main difference comes from the dual mesh. As a result we detail only this part in the following.

We have, as in [27, Lemma 10.5], by compactness of the boundary $\partial\Omega$, the existence of a finite number of open hyper-rectangles $\{R_i, i=1\cdots N\}$, and normalized vectors of $\mathbb{R}^2$, $\{\eta_i, i=1\cdots N\}$, such that

$$\begin{cases} \partial\Omega\subset\cup_{i=1}^N R_i,\\ (\eta_i,\vec{\nu}(x))\ge\lambda>0 \text{ for all } x\in R_i\cap\partial\Omega, i\in\{1\cdots N\},\\ \{x+t\eta_i, x\in R_i\cap\partial\Omega, t\in\mathbb{R}_+\}\cap R_i\subset\Omega, \end{cases}$$

where λ is some positive number and $\vec{\nu}(x)$ is the normal vector to $\partial\Omega$ at x, inward to Ω (see Figure 7.1). Let $\{\lambda_i, i = 1\cdots N\}$ be a family of functions such that $\sum_{i=1}^{N}\lambda_i(x) = 1$, for all $x \in \partial\Omega$, $\lambda_i \in C_c^\infty(\mathbb{R}^2, \mathbb{R}_+)$ and $\lambda_i = 0$ outside of R_i, for all $i = 1\cdots N$. Let $\partial\Omega_i = R_i \cap \partial\Omega$; we will prove that there exists $C_i > 0$ depending only on λ, ζ, θ and λ_i such that

$$\left(\int_{\partial\Omega_i} \lambda_i(x)|u^{\partial\mathfrak{M}^*}(x)|\mathrm{d}x\right) \leq C_i\left(\|u_{\mathcal{T}}\|_{1,\mathcal{T}} + \left\|\nabla^{\mathfrak{D}}u_{\mathcal{T}}\right\|_{1,\mathfrak{D}}\right).$$

Then, we define $C = \sum_{i=1}^{N} C_i$, depending only on Ω, ζ and θ, to get (45).

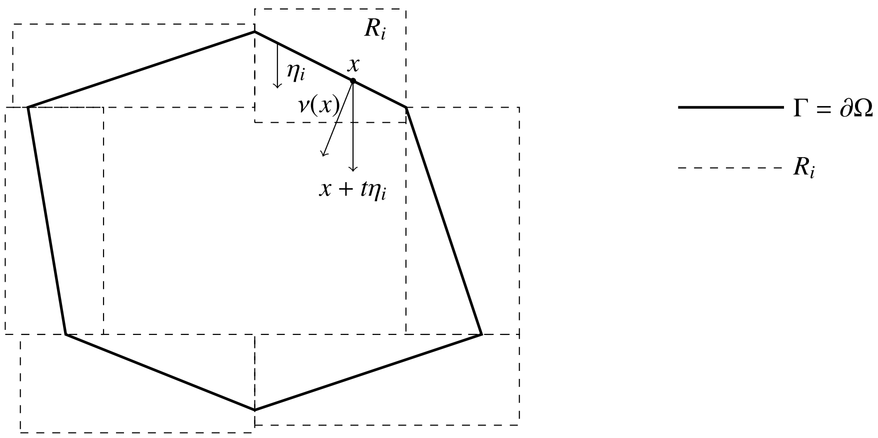

Figure 7.1: Properties of the boundary $\partial\Omega$.

As in [27] we introduce a function which determine the successive neighbours of a cell $u_{\mathcal{K}^*}$: we define, for $x, y \in \Omega$ and $\sigma^* \in \mathcal{E}^*$,

$$\psi_{\sigma^*}(x,y) = \begin{cases} 1 & \text{si } [x,y] \cap \sigma^* \neq \emptyset, \\ 0 & \text{si } [x,y] \cap \sigma^* = \emptyset, \end{cases}$$

and for $\mathcal{K}^* \in \overline{\mathfrak{M}^*}$

$$\psi_{\mathcal{K}^*}(x,y) = \begin{cases} 1 & \text{si } [x,y] \cap \mathcal{K}^* \neq \emptyset, \\ 0 & \text{si } [x,y] \cap \mathcal{K}^* = \emptyset. \end{cases}$$

Let $i \in \{1, \cdots, N\}$ and $x \in \partial\Omega_i$. There exists a unique $t > 0$ such that $x + t\eta_i \in \partial R_i$, let $y(x) = x + t\eta_i$. For $\sigma^* \in \mathcal{E}^*$, when $[x, y(x)] \cap \sigma^* \neq \emptyset$, the intersection is either reduced to a point let then $z_{\sigma^*}(x) = [x, y(x)] \cap \sigma^*$, or a segment $[x, y(x)] \cap \sigma^* = [a(x), b(x)]$ with $(\overrightarrow{a(x)b(x)}, \eta_i) > 0$ and then let $z_{\sigma^*}(x) = b(x)$. For $\mathcal{K}^* \in \overline{\mathfrak{M}^*}$, let $\xi_{\mathcal{K}^*}(x), \eta_{\mathcal{K}^*}(x)$ such that $[x, y(x)] \cap \mathcal{K}^* = [\xi_{\mathcal{K}^*}(x), \eta_{\mathcal{K}^*}(x)]$, if $[x, y(x)] \cap \mathcal{K}^* \neq \emptyset$ and $(\overrightarrow{\xi_{\mathcal{K}^*}(x)\eta_{\mathcal{K}^*}(x)}, \eta_i) > 0$.

Furthermore, let $x \in \mathcal{K}_0^*$ and $y(x) \in \mathcal{L}_0^*$ such that $\sigma_0^* = \mathcal{K}_0^*|\mathcal{L}_0^*$ (see Figure 7.2), we have two cases. Note that in the two cases we have $x = \xi_{\mathcal{K}_0^*}(x)$ and $y(x) = \eta_{\mathcal{L}_0^*}(x)$ we get $\eta_{\mathcal{L}_0^*}(x) \in \partial R_i$, and deduce $\lambda_i(\eta_{\mathcal{L}_0^*}(x)) = 0$.

1. $[x, y(x)] \cap \sigma_0^*$ is reduced to a point then we have $\eta_{\mathcal{K}_0^*}(x) = z_{\sigma_0^*}(x) = \xi_{\mathcal{L}_0^*}(x)$. We obtain

$$\begin{aligned}\lambda_i(x)|u_{\mathcal{K}_0^*}| = &\left(\lambda_i(\xi_{\mathcal{K}_0^*}(x)) - \lambda_i(\eta_{\mathcal{K}_0^*}(x))\right)|u_{\mathcal{K}_0^*}| + \left(\lambda_i(\xi_{\mathcal{L}_0^*}(x)) - \lambda_i(\eta_{\mathcal{L}_0^*}(x))\right)|u_{\mathcal{L}_0^*}| \\ &+ \lambda_i(z_{\mathcal{K}_0^*|\mathcal{L}_0^*}(x))(|u_{\mathcal{K}_0^*}| - |u_{\mathcal{L}_0^*}|).\end{aligned}$$

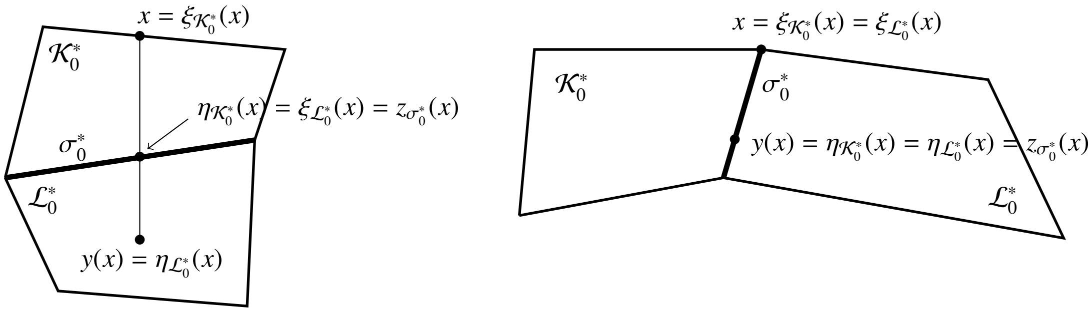

Figure 7.2: (On the left) $[x, y(x)] \cap \sigma_0^*$ is reduced to a point $z_{\sigma_0^*}(x)$. (On the right) $[x, y(x)] \cap \sigma_0^*$ is the segment $[x, y(x)]$.

2. $[x, y(x)] \cap \sigma_0^*$ is a segment, then we have $\eta_{\mathcal{K}_0^*}(x) = y(x)$ and $\lambda_i(\eta_{\mathcal{K}_0^*}(x)) = 0$. We obtain

$$\lambda_i(x)|u_{\mathcal{K}_0^*}| = \left(\lambda_i(\xi_{\mathcal{K}_0^*}(x)) - \lambda_i(\eta_{\mathcal{K}_0^*}(x))\right)|u_{\mathcal{K}_0^*}|.$$

This point is the main difference with [27, Lemma 10.5]. In the two cases we get the same estimates

$$\lambda_i(x)|u_{\mathcal{K}_0^*}| \leq A(x) + B(x),$$

where

$$A(x) = \sum_{\mathcal{D}\in\mathfrak{D}} \psi_{\sigma^*}(x, y(x))\lambda_i(z_{\sigma^*}(x))\,||u_{\mathcal{K}^*}| - |u_{\mathcal{L}^*}||\,,$$

and

$$B(x) = \sum_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}} |\lambda_i(\xi_{\mathcal{K}^*}(x)) - \lambda_i(\eta_{\mathcal{K}^*}(x))|\,|u_{\mathcal{K}^*}|\,\psi_{\mathcal{K}^*}(x, y(x)).$$

We begin with the estimate of A. Using the fact that λ_i is bounded, we get

$$A(x) \leq \|\lambda_i\|_\infty \sum_{\mathcal{D}\in\mathfrak{D}} \psi_{\sigma^*}(x, y(x))\,||u_{\mathcal{K}^*}| - |u_{\mathcal{L}^*}||\,.$$

The following inequality

$$\int_{\partial\Omega_i} \psi_{\sigma^*}(x, y(x))\mathrm{d}x \leq \mathrm{m}_{\sigma^*}\frac{1}{\lambda},$$

implies that

$$\mathcal{A} = \int_{\partial\Omega_i} A(x)\mathrm{d}x \leq \|\lambda_i\|_\infty \sum_{\mathcal{D}\in\mathfrak{D}} \left(\int_{\partial\Omega_i} \psi_{\sigma^*}(x, y(x))\mathrm{d}x\right) ||u_{\mathcal{K}^*}| - |u_{\mathcal{L}^*}|| \leq C \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\sigma^*}\,||u_{\mathcal{K}^*}| - |u_{\mathcal{L}^*}||\,.$$

Since $\big||a| - |b|\big| \leq |a - b|$, we obtain

$$\sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\sigma^*}\big||u_{\mathcal{K}^*}| - |u_{\mathcal{L}^*}|\big| \leq \frac{2}{\sin(\theta_{\mathcal{T}})} \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}}\left|\frac{u_{\mathcal{K}^*} - u_{\mathcal{L}^*}}{\mathrm{m}_\sigma}\right|.$$

Noting that

$$\left|\frac{u_{\mathcal{K}^*} - u_{\mathcal{L}^*}}{\mathrm{m}_\sigma}\right| \leq \left|\nabla^{\mathcal{D}} u_{\mathcal{T}}\right|, \tag{46}$$

we deduce

$$\sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\sigma^*}\big||u_{\mathcal{K}^*}| - |u_{\mathcal{L}^*}|\big| \leq \frac{2}{\sin(\theta_{\mathcal{T}})} \sum_{\mathcal{D}\in\mathfrak{D}} \mathrm{m}_{\mathcal{D}}\left|\nabla^{\mathcal{D}} u_{\mathcal{T}}\right| \leq C\|\nabla^{\mathfrak{D}} u_{\mathcal{T}}\|_{1,\mathfrak{D}} + \|u_{\mathcal{T}}\|_{1,\mathcal{T}}.$$

Finally, we obtain
$$\mathcal{A} \le C_1 \|\nabla^{\mathfrak{D}} u_{\mathcal{T}}\|_{1,\mathfrak{D}} + C_1 \|u_{\mathcal{T}}\|_{1,\mathcal{T}}.$$
Now the bound of B is as follows. Since the function λ_i is smooth, we have
$$B(x) \le \|\nabla \lambda_i\|_\infty \sum_{\mathcal{K}^* \in \overline{\mathfrak{M}^*}} |\xi_{\mathcal{K}^*}(x) - \eta_{\mathcal{K}^*}(x)|\, |u_{\mathcal{K}^*}|\, \psi_{\mathcal{K}^*}(x, y(x)).$$
Furthermore, we have on one hand
$$|\xi_{\mathcal{K}^*}(x) - \eta_{\mathcal{K}^*}(x)| \le d_{\mathcal{K}^*},$$
on the other hand
$$\int_{\partial\Omega_i} \psi_{\mathcal{K}^*}(x, y(x)) \mathrm{d}x \le \frac{d_{\mathcal{K}^*}}{\lambda}.$$
It implies that
$$\int_{\partial\Omega_i} \psi_{\mathcal{K}^*}(x, y(x))\, |\xi_{\mathcal{K}^*}(x) - \eta_{\mathcal{K}^*}(x)|\, \mathrm{d}x \le C \mathrm{m}_{\mathcal{K}^*},$$
with C depending on ζ, θ and λ. We obtain
$$\mathcal{B} = \int_{\partial\Omega_i} B(x)\mathrm{d}x \le C_2 \sum_{\mathcal{K}^* \in \overline{\mathfrak{M}^*}} \mathrm{m}_{\mathcal{K}^*}\, |u_{\mathcal{K}^*}| \le C_2 \|u_{\mathcal{T}}\|_{1,\mathcal{T}}.$$
Finally, we deduce
$$\int_{\partial\Omega_i} \lambda_i(x) |u^{\partial\mathfrak{M}^*}(x)| \mathrm{d}x \le \mathcal{A} + \mathcal{B} \le C_i (\|\nabla^{\mathfrak{D}} u_{\mathcal{T}}\|_{1,\mathfrak{D}} + \|u_{\mathcal{T}}\|_{1,\mathcal{T}}).$$

□

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