

STUDY OF DISCRETE DUALITY FINITE VOLUME SCHEMES FOR THE PEACEMAN MODEL *

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Abstract. In this paper, we are interested in the finite volume approximation of a system describing miscible displacement in porous media. This system is made of two coupled equations: an anisotropic diffusion equation on the pressure and a convection-diffusion-dispersion equation on the concentration of invading fluid. The anisotropic diffusion operators in both equations require a special care while discretizing by a finite volume method. We focus here on the numerical approximation by Discrete Duality Finite Volume methods. After the presentation of the scheme, we establish relevant a priori estimates satisfied by the numerical solution and prove existence and uniqueness of the solution to the scheme. We show the efficiency of the schemes through numerical experiments.

Key words.

finite volume method, porous medium, miscible fluid flows

AMS subject classifications. 65M08, 65N30, 76S05, 76R99

1. Introduction.

1.1. Finite volume methods for diffusion operators. Finite volume methods have been extensively studied for a long time in several engineering fields. Indeed, they are well suited for the numerical approximation of conservation laws appearing for instance in fluid mechanics, petroleum engineering and many other fields. The theoretical analysis of finite volume schemes (convergence analysis, error estimates,...) began at the end of the 1980's and had a rapid expansion during the 1990's : see for instance the book by Eymard, Gallouët, Herbin [24] and all the references therein. However, if finite volume schemes were particularly competitive and popular for the approximation of hyperbolic conservation laws, they were less used for the approximation of diffusion. Indeed, the classical two-points flux approximation for diffusion operators, firstly developed and analyzed, has many drawbacks:

- its use for the Laplace operator is limited to admissible meshes (satisfying some orthogonality condition),
- its extension to anisotropic diffusion operator implies a more restrictive assumption on the meshes which makes it difficult to use,
- its extension to nonlinear diffusion operators like Leray-Lions operators seems not possible because the scheme is based on a two-points approximation of only normal gradients.

For all these reasons, different teams began to work at the end of the 1990's on the development of new finite volume schemes for diffusion equations. Their aim was to provide some discrete reconstruction of gradients and then schemes applicable on almost general meshes and for linear anisotropic diffusion operators or nonlinear Leray-Lions operators. Let us cite for instance the Multi Points Flux Approximation schemes by Aavatsmark, Barkve, Boe and Mannseth[1, 2], the Discrete Duality Finite

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Volume schemes by Domelevo and Omnes [16, 6], based on previous works by Hermeline [29, 30], the Mixed Finite Volume schemes by Droniou and Eymard [21, 19], the Scheme Using Stabilization and Hybrid Interfaces by Eymard, Gallouët and Herbin [25, 26].

All these new methods have been compared in 2008 on a benchmark organized by Herbin and Hubert [28] (the comparison also included finite element methods, discontinuous Galerkin methods, mimetic finite difference methods and other finite volume methods...). The benchmark was devoted to the anisotropic diffusion equation $-\operatorname{div}\mathbb{K}\nabla u = f$ and included many different two-dimensional test cases. A three-dimensional benchmark was also organized in 2011, [27]. In [20], Droniou presents a review on finite volume methods for diffusion equations, with a focus on two important properties of the schemes: coercivity and minimum-maximum principles.

1.2. A system modeling miscible fluid flows in porous media. In this paper, we want to apply one of the finite volume method dedicated to anisotropic diffusion, the Discrete Duality Finite Volume method, to a two-dimensional system arising in petroleum engineering. The Peaceman model describes the single-phase miscible displacement of one fluid by another in a porous medium, in the case where the fluids are assumed incompressible and the gravity is neglected; it has been introduced in [7, 17]. The reservoir is described by a bounded domain $\Omega \subset \mathbb{R}^2$. The unknowns of the problem are p the pressure in the mixture, \mathbf{U} its Darcy velocity and c the concentration of the invading fluid, which are defined on a time interval $(0, T)$. The porous medium is characterized by its porosity $\Phi(x)$ and its permeability $\mathbb{K}(x)$. We denote by $\mu(c)$ the viscosity of the fluid mixture, \hat{c} the injected concentration, q^+ and q^- the injection and the production source terms. The model writes:

$$\begin{cases} \operatorname{div}(\mathbf{U}) = q^+ - q^- & \text{in }]0, T[\times \Omega, \\ \mathbf{U} = -\frac{\mathbb{K}(x)}{\mu(c)} \nabla p & \text{in }]0, T[\times \Omega, \end{cases} \quad (1.1)$$

$$\Phi(x) \partial_t c - \operatorname{div}(\mathbb{D}(x, \mathbf{U}) \nabla c - c \mathbf{U}) + q^- c = q^+ \hat{c} \quad \text{in }]0, T[\times \Omega, \quad (1.2)$$

where \mathbb{D} is the diffusion-dispersion tensor including molecular diffusion and mechanical dispersion. It is given by

$$\mathbb{D}(x, \mathbf{U}) = \Phi(x) \left(d_m \mathbb{I} + |\mathbf{U}| \left(d_l \mathbb{E}(\mathbf{U}) + d_t (\mathbb{I} - \mathbb{E}(\mathbf{U})) \right) \right), \quad (1.3)$$

where \mathbb{I} is the identity matrix, d_m is the molecular diffusion, d_l and d_t are the longitudinal and transverse dispersion coefficients and $\mathbb{E}(\mathbf{U}) = \left(\frac{\mathbf{U}_i \mathbf{U}_j}{|\mathbf{U}|^2} \right)_{1 \leq i, j \leq d}$. The system (1.1)–(1.3) is supplemented with an initial condition on c and Neumann boundary conditions (the boundary $\partial\Omega$ is assumed impermeable). Due to the Neumann boundary conditions, the injection and the production source terms must satisfy a compatibility condition and the pressure is defined up to an arbitrary constant. The viscosity μ is usually determined by the following mixing rule:

$$\mu(c) = \mu(0) \left(1 + \left(M^{1/4} - 1 \right) c \right)^{-4} \quad \text{on } [0, 1], \quad (1.4)$$

where $M = \frac{\mu(0)}{\mu(1)}$ is the mobility ratio (we extend μ to \mathbb{R} by letting $\mu = \mu(0)$ on $] - \infty, 0[$ and $\mu = \mu(1)$ on $]1, \infty[$). The porosity Φ and the permeability \mathbb{K} are in

general assumed to be bounded from above and from below by positive constants (or positive multiples of \mathbb{I} for the tensor \mathbb{K}).

Many different schemes have already been proposed for the Peaceman model since the beginning of the 1980's. In [18], Douglas, Ewing and Wheeler proposed a mixed finite element method for the pressure equation (1.1) and a Galerkin finite element method for the concentration equation (1.2). Russel in [35] introduced a modified method of characteristics for the concentration equation and discretized the pressure equation by a finite element method. Then, Russel, Ewing and Wheeler combined mixed finite element method for (1.1) and a modified method of characteristics for (1.2), see [23, 22]. We also refer to the work by Jaffré and Roberts [32]. In [36, 37], Wang and his coauthors still use a mixed finite element method for the pressure equation but develop a Eulerian Lagrangian Localized Adjoint Method for the concentration equation. In all these papers, the pressure equation is approximated by a finite element scheme. Indeed, due to the anisotropy in the permeability tensor and in the diffusion-dispersion tensor, the system cannot be approached by classical finite volume methods and the development of finite volume schemes dedicated to anisotropy, as explained in the previous section, is new. Therefore the first finite volume methods for the Peaceman model have been developed recently. A mixed finite volume method for both equations (1.1), (1.2) has been proposed by Chainais-Hillairet and Droniou and its convergence has been established [10]. In [3], Amaziane and El Ossmani use a mixed finite element method for (1.1) but propose a finite volume scheme written on a dual mesh for (1.2). In this paper, we will consider the application of the Discrete Duality Finite Volume methods (see [16, 6, 9]) to the Peaceman model. This method seems well-adapted to the given system of equations, due to the anisotropy, and may be applied to almost general meshes.

1.3. Contents and outline of the paper. Let us first rewrite the Peaceman model under the following synthesized form, as in [10]:

$$\begin{aligned} \operatorname{div}(\mathbf{U}) &= q^+ - q^- && \text{in }]0, T[\times \Omega, \\ \mathbf{U} &= -\mathbb{A}(\cdot, c) \nabla p && \text{in }]0, T[\times \Omega, \\ \mathbf{U} \cdot \mathbf{n} &= 0 && \text{on }]0, T[\times \partial\Omega, \\ \int_{\Omega} p(\cdot, x) dx &= 0 && \text{on }]0, T[, \end{aligned} \tag{1.5}$$

$$\begin{aligned} \Phi \partial_t c - \operatorname{div}(\mathbb{D}(\cdot, \mathbf{U}) \nabla c) + \operatorname{div}(c \mathbf{U}) + q^- c &= q^+ \hat{c} && \text{in }]0, T[\times \Omega, \\ \mathbb{D}(\cdot, \mathbf{U}) \nabla c \cdot \mathbf{n} &= 0 && \text{on }]0, T[\times \partial\Omega, \\ c(0, \cdot) &= c_0 && \text{on } \Omega. \end{aligned} \tag{1.6}$$

In the sequel, we assume that Ω is an open bounded connected polygonal domain of \mathbb{R}^2 and that $T > 0$. The assumptions on the data are the following:

$$\begin{aligned} (q^+, q^-) &\in L^\infty(0, T; L^2(\Omega)) \text{ are nonnegative functions such that} \\ \int_{\Omega} q^+(\cdot, x) dx &= \int_{\Omega} q^-(\cdot, x) dx \text{ a.e. on }]0, T[, \end{aligned} \tag{1.7}$$

$$\begin{aligned} \mathbb{A} : \Omega \times \mathbb{R} &\rightarrow \mathcal{M}_2(\mathbb{R}) \text{ is a Caratheodory matrix-valued function satisfying:} \\ \exists \alpha_A > 0 \text{ s. t. } \mathbb{A}(x, s) \boldsymbol{\xi} \cdot \boldsymbol{\xi} &\geq \alpha_A |\boldsymbol{\xi}|^2 \text{ for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R} \text{ and all } \boldsymbol{\xi} \in \mathbb{R}^2, \\ \exists \Lambda_A > 0 \text{ such that } |\mathbb{A}(x, s)| &\leq \Lambda_A \text{ for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R}, \end{aligned} \tag{1.8}$$

$\mathbb{D} : \Omega \times \mathbb{R}^2 \rightarrow \mathcal{M}_2(\mathbb{R})$ is a Caratheodory matrix-valued function satisfying:
 $\exists \alpha_D > 0$ s.t. $\mathbb{D}(x, \mathbf{V}) \xi \cdot \xi \geq \alpha_D (1 + |\mathbf{V}|) |\xi|^2$ for a.e. $x \in \Omega$, all $(\mathbf{V}, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2$, (1.9)
 $\exists \Lambda_D > 0$ such that $|\mathbb{D}(x, \mathbf{V})| \leq \Lambda_D (1 + |\mathbf{V}|)$ for a.e. $x \in \Omega$ and all $\mathbf{V} \in \mathbb{R}^2$,

$$\Phi \in L^\infty(\Omega) \text{ and there exists } \Phi_* > 0 \text{ such that } \Phi_* \leq \Phi \leq \Phi_*^{-1} \text{ a.e. in } \Omega, \quad (1.10)$$

$$\hat{c} \in L^\infty(]0, T[\times \Omega) \text{ satisfies: } 0 \leq \hat{c} \leq 1 \text{ a.e. in }]0, T[\times \Omega, \quad (1.11)$$

$$c_0 \in L^\infty(\Omega) \text{ satisfies: } 0 \leq c_0 \leq 1 \text{ a.e. in } \Omega. \quad (1.12)$$

The DDFV schemes are devoted to the numerical approximation of anisotropic diffusion operators as we have in (1.5) and (1.6). They are based on two fundamental ideas: integration of the equations on a primal and a dual meshes, as suggested by Hermeline [29, 30], reconstruction of discrete gradients on a diamond mesh, as in the work by Coudière, Vila and Villedieu [14]. Developing these two ideas, Domelevo and Omnes in [16] introduced the DDFV schemes for the Laplace equation and established the fundamental duality property between discrete gradient and discrete divergence. Andreianov, Boyer and Hubert extended the DDFV scheme to more general diffusion operators in [6] and proved its convergence. In Section 2, we present a DDFV scheme for the system (1.5)–(1.6). Therefore, we introduce the different meshes, the discrete operators and we recall the discrete duality formula. As there is a convection term in (1.6), we also need to define a discrete convection operator, as in [5] or [12].

In Section 3, we study the stability of the scheme. We prove some a priori estimates on the pressure, the gradient of the pressure, the Darcy velocity (Lemma 3.2) and also a priori estimates on the concentration and the gradient of the concentration (Lemma 3.3). It yields the existence and uniqueness of the approximate solution (Theorem 3.4). Section 4 is devoted to the presentation of some numerical experiments. It shows good results when the permeability is continuous or has discontinuities supported by the edges of the diamond mesh. However, as it is more natural that the discontinuities of the permeability are supported by the edges of the primal mesh, we introduce a modified DDFV scheme in Section 5. The m-DDFV scheme has been proposed by Hermeline in [30] for 2D linear elliptic problems with discontinuities and extended by Boyer and Hubert in [9] for 2D nonlinear elliptic problems with discontinuities. Finally, in Section 5.4, we show numerical experiments obtained with the m-DDFV scheme. We will also compare some qualitative results obtained with the m-DDFV and the MFV schemes.

We only consider here the two-dimensional case but it is worth noticing that DDFV schemes have been successfully extended to the three-dimensional case in [11, 13, 31, 4] for linear anisotropic scalar diffusion equations and in [33] for the Stokes problem. Generalization of the present work to the three-dimensional case would be based on similar ideas.

2. Presentation of the numerical scheme.

2.1. Meshes and notations. In order to define a DDFV scheme, as for instance in [16, 6], we need to introduce three different meshes – the primal mesh, the dual mesh and the diamond mesh – and some associated notations.

The mesh construction starts from the partition of the computational domain Ω , with disjoint open polygonal control volumes $\mathcal{K} \subset \Omega$ such that $\cup \mathcal{K} = \bar{\Omega}$. This

partition is denoted by \mathfrak{M} and called the interior primal mesh. We denote by $\partial\mathfrak{M}$ the set of boundary edges, which are considered as degenerate control volumes. Then, the primal mesh is composed of $\mathfrak{M} \cup \partial\mathfrak{M}$, denoted by $\overline{\mathfrak{M}}$. To construct the two other meshes, we need to associate at each primal cell $\mathcal{K} \in \overline{\mathfrak{M}}$, a point $x_{\mathcal{K}} \in \mathcal{K}$, called the center of the primal cell. Notice that for a degenerate control volume \mathcal{K} , the point $x_{\mathcal{K}}$ is necessarily the midpoint of \mathcal{K} . This family of centers is denoted by $X = \{x_{\mathcal{K}}, \mathcal{K} \in \overline{\mathfrak{M}}\}$ and these will determine the two other meshes.

Let X^* denote the set of the vertices of the primal control volumes in $\overline{\mathfrak{M}}$. Distinguishing the interior vertices from the vertices lying on the boundary, we split X^* into $X^* = X_{int}^* \cup X_{ext}^*$. To any point $x_{\mathcal{K}^*} \in X_{int}^*$, we associate the polygon \mathcal{K}^* , whose vertices are $\{x_{\mathcal{K}} \in X/x_{\mathcal{K}^*} \in \bar{\mathcal{K}}, \mathcal{K} \in \mathfrak{M}\}$. The set of these polygons defines the interior dual mesh denoted by \mathfrak{M}^* . To any point $x_{\mathcal{K}^*} \in X_{ext}^*$, we then associate the polygon \mathcal{K}^* , whose vertices are $\{x_{\mathcal{K}^*}\} \cup \{x_{\mathcal{K}} \in X/x_{\mathcal{K}^*} \in \bar{\mathcal{K}}, \mathcal{K} \in \overline{\mathfrak{M}}\}$. The set of these polygons is denoted by $\partial\mathfrak{M}^*$ called the boundary dual mesh and the dual mesh is $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$, denoted by $\overline{\mathfrak{M}^*}$.

To define the last mesh “the diamond mesh”, we first introduce the notion of edges. For all neighboring primal cells \mathcal{K} and \mathcal{L} of $\overline{\mathfrak{M}}$, we assume that $\partial\mathcal{K} \cap \partial\mathcal{L}$ is a segment, corresponding to an edge of the primal mesh $\overline{\mathfrak{M}}$, denoted by $\sigma = \mathcal{K}|\mathcal{L}$. Let \mathcal{E} be the set of such edges. We similarly define the edges \mathcal{E}^* of the dual mesh $\overline{\mathfrak{M}^*}$: $\sigma^* = \mathcal{K}^*|\mathcal{L}^*$.

REMARK 2.1. *If $\mathcal{K} \in \overline{\mathfrak{M}}$, all its edges belong to \mathcal{E} and if $\mathcal{K}^* \in \mathfrak{M}^*$, all its edges belong to \mathcal{E}^* . But, if $\mathcal{K}^* \in \partial\mathfrak{M}^*$, then it has edges inside the domain and also on its boundary: the interior edges belong to \mathcal{E}^* while the boundary edges belong to \mathcal{E} .*

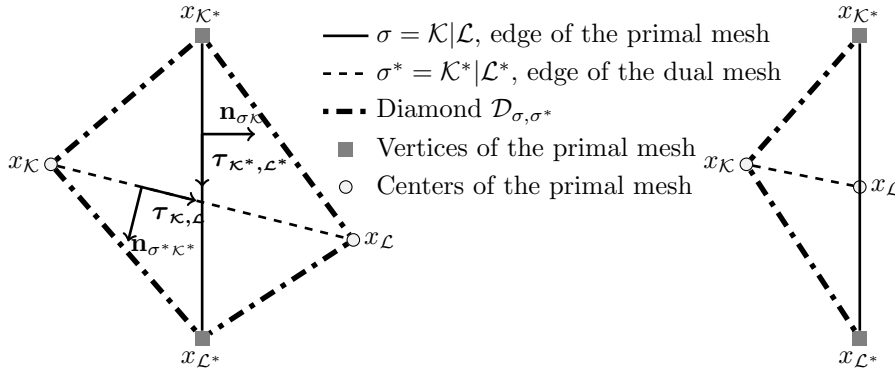


FIG. 2.1. Definition of the diamonds $\mathcal{D}_{\sigma, \sigma^*}$

For each couple $(\sigma, \sigma^*) \in \mathcal{E} \times \mathcal{E}^*$ such that $\sigma = \mathcal{K}|\mathcal{L} = (x_{\mathcal{K}^*}, x_{\mathcal{L}^*})$ and $\sigma^* = \mathcal{K}^*|\mathcal{L}^* = (x_{\mathcal{K}}, x_{\mathcal{L}})$, we define the quadrilateral diamond cell $\mathcal{D}_{\sigma, \sigma^*}$ whose diagonals are σ and σ^* . If $\sigma \in \mathcal{E} \cap \partial\Omega$, we note that the diamond degenerates into a triangle. The set of the diamond cells defines the diamond mesh \mathcal{D} . It verifies $\bar{\Omega} = \bigcup_{\mathcal{D} \in \mathcal{D}} \mathcal{D}$. We have

as many diamond cells as primal edges.

Finally, the DDFV mesh is made of the $\mathcal{T} = (\overline{\mathfrak{M}}, \overline{\mathfrak{M}^*})$ and \mathcal{D} . Let us now introduce some notations associated to the meshes \mathcal{T} and \mathcal{D} . For each primal or dual cell V ($V \in \overline{\mathfrak{M}}$ or $V \in \overline{\mathfrak{M}^*}$), we define m_V the measure of V , \mathcal{E}_V the set of the edges of V (it coincides with the edge $\sigma = V$ if $V \in \partial\mathfrak{M}$), \mathcal{D}_V the set of diamonds $\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}$ such that $m(\mathcal{D}_{\sigma, \sigma^*} \cap V) > 0$, and d_V the diameter of V .

For a diamond $\mathcal{D}_{\sigma, \sigma^*}$, whose vertices are $(x_{\mathcal{K}}, x_{\mathcal{K}^*}, x_{\mathcal{L}}, x_{\mathcal{L}^*})$, we define, as shown on Figure 2.1: $x_{\mathcal{D}}$ the center of the diamond cell \mathcal{D} : $x_{\mathcal{D}} = \sigma \cap \sigma^*$, m_{σ} the length of the primal edge σ , m_{σ^*} the length of the dual edge σ^* , $m_{\mathcal{D}}$ the measure of \mathcal{D} , $d_{\mathcal{D}}$ its diameter, $\alpha_{\mathcal{D}}$ the angle between $(x_{\mathcal{K}}, x_{\mathcal{L}})$ and $(x_{\mathcal{K}^*}, x_{\mathcal{L}^*})$, $m_{\mathcal{D}\mathcal{K}}$ the measure of $\mathcal{D} \cap \mathcal{K}$ and $m_{\mathcal{D}\mathcal{K}^*}$ the measure of $\mathcal{D} \cap \mathcal{K}^*$. We will also use two direct bases $(\boldsymbol{\tau}_{\mathcal{K}^*, \mathcal{L}^*}, \mathbf{n}_{\sigma\mathcal{K}})$ and $(\mathbf{n}_{\sigma^*\mathcal{K}^*}, \boldsymbol{\tau}_{\mathcal{K}, \mathcal{L}})$, where $\mathbf{n}_{\sigma\mathcal{K}}$ is the unit normal to σ , outward \mathcal{K} , $\mathbf{n}_{\sigma^*\mathcal{K}^*}$ is the unit normal to σ^* , outward \mathcal{K}^* , $\boldsymbol{\tau}_{\mathcal{K}^*, \mathcal{L}^*}$ is the unit tangent vector to σ , oriented from \mathcal{K}^* to \mathcal{L}^* , $\boldsymbol{\tau}_{\mathcal{K}, \mathcal{L}}$ is the unit tangent vector to σ^* , oriented from \mathcal{K} to \mathcal{L} .

We introduce now the size of the mesh, $\text{size}(\mathcal{T}) = \max_{\mathcal{D} \in \mathfrak{D}} d_{\mathcal{D}}$. To measure how flat the diamond cells can be, we note $\alpha_{\mathcal{T}}$ the unique real in $]0, \frac{\pi}{2}]$ such that $\sin(\alpha_{\mathcal{T}}) := \min_{\mathcal{D} \in \mathfrak{D}} (|\sin(\alpha_{\mathcal{D}})|)$. We also need some regularity of the mesh, as in [6], which gives the existence of a constant $\zeta > 0$ such that

$$\sum_{\mathcal{D} \in \mathfrak{D}_{\mathcal{K}}} m_{\sigma} m_{\sigma^*} \leq \frac{m_{\mathcal{K}}}{\zeta}, \quad \forall \mathcal{K} \in \mathfrak{M}, \quad \text{and} \quad \sum_{\mathcal{D} \in \mathfrak{D}_{\mathcal{K}^*}} m_{\sigma} m_{\sigma^*} \leq \frac{m_{\mathcal{K}^*}}{\zeta}, \quad \forall \mathcal{K}^* \in \overline{\mathfrak{M}^*}. \quad (2.1)$$

2.2. Discrete operators and duality formula. We define several types of degrees of freedom to represent scalar and vector fields in the discrete setting. We consider :

- $\mathbb{R}^{\mathcal{T}}$ is a linear space of scalar fields constant on the cells of $\overline{\mathfrak{M}}$ and $\overline{\mathfrak{M}^*}$:

$$\mathbb{R}^{\mathcal{T}} = \left\{ u_{\mathcal{T}} = ((u_{\mathcal{K}})_{\mathcal{K} \in \overline{\mathfrak{M}}}, (u_{\mathcal{K}^*})_{\mathcal{K}^* \in \overline{\mathfrak{M}^*}}), \text{ with } u_{\mathcal{K}} \in \mathbb{R}, \forall \mathcal{K} \in \overline{\mathfrak{M}}, \right. \\ \left. \text{and } u_{\mathcal{K}^*} \in \mathbb{R}, \forall \mathcal{K}^* \in \overline{\mathfrak{M}^*} \right\}.$$

- $(\mathbb{R}^2)^{\mathfrak{D}}$ is a linear space of vector fields constant on the cells of \mathfrak{D} :

$$(\mathbb{R}^2)^{\mathfrak{D}} = \{ \boldsymbol{\xi}_{\mathfrak{D}} = (\boldsymbol{\xi}_{\mathcal{D}})_{\mathcal{D} \in \mathfrak{D}}, \text{ with } \boldsymbol{\xi}_{\mathcal{D}} \in \mathbb{R}^2, \forall \mathcal{D} \in \mathfrak{D} \}.$$

In this section, we recall the definition of two discrete operators. The discrete gradient has been introduced in [15] and further with the discrete divergence in [16]. In order to write a discrete duality property (similar to the Green formula at the discrete level), we also introduce some trace operators and scalar products.

DEFINITION 2.2. *The discrete gradient $\nabla^{\mathfrak{D}}$ is a mapping from $\mathbb{R}^{\mathcal{T}}$ to $(\mathbb{R}^2)^{\mathfrak{D}}$ defined for all $u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ by $\nabla^{\mathfrak{D}} u_{\mathcal{T}} = (\nabla^{\mathcal{D}} u_{\mathcal{T}})_{\mathcal{D} \in \mathfrak{D}}$, where for $\mathcal{D} \in \mathfrak{D}$:*

$$\begin{cases} \nabla^{\mathcal{D}} u_{\mathcal{T}} \cdot \boldsymbol{\tau}_{\mathcal{K}^*, \mathcal{L}^*} = \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{m_{\sigma}}, \\ \nabla^{\mathcal{D}} u_{\mathcal{T}} \cdot \boldsymbol{\tau}_{\mathcal{K}, \mathcal{L}} = \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{m_{\sigma^*}}. \end{cases} \\ \iff \nabla^{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{\sin(\alpha_{\mathcal{D}})} \left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{m_{\sigma^*}} \mathbf{n}_{\sigma\mathcal{K}} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{m_{\sigma}} \mathbf{n}_{\sigma^*\mathcal{K}^*} \right)$$

As the measure $m_{\mathcal{D}}$ of the diamond \mathcal{D} verifies : $m_{\mathcal{D}} = \frac{1}{2} m_{\sigma} m_{\sigma^*} \sin(\alpha_{\mathcal{D}})$, the discrete gradient on \mathcal{D} rewrites:

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{2m_{\mathcal{D}}} \left[(u_{\mathcal{L}} - u_{\mathcal{K}}) m_{\sigma} \mathbf{n}_{\sigma\mathcal{K}} + (u_{\mathcal{L}^*} - u_{\mathcal{K}^*}) m_{\sigma^*} \mathbf{n}_{\sigma^*\mathcal{K}^*} \right].$$

Applying the Green formula to a regular function $\boldsymbol{\xi} : \Omega \rightarrow \mathbb{R}^2$, we get:

$$\int_{\mathcal{K}} \operatorname{div}(\boldsymbol{\xi}(x))dx = \sum_{\sigma \in \partial \mathcal{K}} \int_{\sigma} \boldsymbol{\xi}(s) \cdot \mathbf{n}_{\sigma \mathcal{K}} ds, \quad \forall \mathcal{K} \in \mathfrak{M}, \quad (2.2)$$

and a similar equality when integrating over $\mathcal{K}^* \in \overline{\mathfrak{M}^*}$. Therefore, the discrete divergence $\operatorname{div}^{\mathcal{T}}$ is defined by mimicking (2.2) at the discrete level.

DEFINITION 2.3. *The discrete divergence operator $\operatorname{div}^{\mathcal{T}}$ is a mapping from $(\mathbb{R}^2)^{\mathfrak{D}}$ to $\mathbb{R}^{\mathcal{T}}$ defined for all $\boldsymbol{\xi}_{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}$ by*

$$\operatorname{div}^{\mathcal{T}} \boldsymbol{\xi}_{\mathfrak{D}} = \left(\operatorname{div}^{\mathfrak{M}} \boldsymbol{\xi}_{\mathfrak{D}}, \operatorname{div}^{\partial \mathfrak{M}} \boldsymbol{\xi}_{\mathfrak{D}}, \operatorname{div}^{\mathfrak{M}^*} \boldsymbol{\xi}_{\mathfrak{D}}, \operatorname{div}^{\partial \mathfrak{M}^*} \boldsymbol{\xi}_{\mathfrak{D}} \right),$$

with $\operatorname{div}^{\mathfrak{M}} \boldsymbol{\xi}_{\mathfrak{D}} = (\operatorname{div}_{\mathcal{K}} \boldsymbol{\xi}_{\mathfrak{D}})_{\mathcal{K} \in \mathfrak{M}}$, $\operatorname{div}^{\partial \mathfrak{M}} \boldsymbol{\xi}_{\mathfrak{D}} = 0$, $\operatorname{div}^{\mathfrak{M}^*} \boldsymbol{\xi}_{\mathfrak{D}} = (\operatorname{div}_{\mathcal{K}^*} \boldsymbol{\xi}_{\mathfrak{D}})_{\mathcal{K}^* \in \mathfrak{M}^*}$ and $\operatorname{div}^{\partial \mathfrak{M}^*} \boldsymbol{\xi}_{\mathfrak{D}} = (\operatorname{div}_{\mathcal{K}^*} \boldsymbol{\xi}_{\mathfrak{D}})_{\mathcal{K}^* \in \partial \mathfrak{M}^*}$ such that:

$$\forall \mathcal{K} \in \mathfrak{M}, \quad \operatorname{div}_{\mathcal{K}} \boldsymbol{\xi}_{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}}} \sum_{\substack{\mathcal{D} \in \mathfrak{D}_{\mathcal{K}} \\ \mathcal{D} = \mathcal{D}_{\sigma, \sigma^*}}} m_{\sigma} \boldsymbol{\xi}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma \mathcal{K}},$$

$$\forall \mathcal{K}^* \in \mathfrak{M}^*, \quad \operatorname{div}_{\mathcal{K}^*} \boldsymbol{\xi}_{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}^*}} \sum_{\substack{\mathcal{D} \in \mathfrak{D}_{\mathcal{K}^*} \\ \mathcal{D} = \mathcal{D}_{\sigma, \sigma^*}}} m_{\sigma^*} \boldsymbol{\xi}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma^* \mathcal{K}^*},$$

$$\forall \mathcal{K}^* \in \partial \mathfrak{M}^*, \quad \operatorname{div}_{\mathcal{K}^*} \boldsymbol{\xi}_{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}^*}} \left(\sum_{\substack{\mathcal{D} \in \mathfrak{D}_{\mathcal{K}^*} \\ \mathcal{D} = \mathcal{D}_{\sigma, \sigma^*}}} m_{\sigma^*} \boldsymbol{\xi}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma^* \mathcal{K}^*} + \sum_{\substack{\mathcal{D} \in \mathfrak{D}_{\mathcal{K}^*} \cap \mathfrak{D}_{ext} \\ \mathcal{D} = \mathcal{D}_{\sigma, \sigma^*}}} \frac{m_{\sigma}}{2} \boldsymbol{\xi}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma \mathcal{K}} \right).$$

Let us now introduce two trace operators, defined respectively on $\mathbb{R}^{\mathcal{T}}$ and $(\mathbb{R}^2)^{\mathfrak{D}}$. The first one is $\gamma^{\mathcal{T}} : u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}} \mapsto \gamma^{\mathcal{T}}(u_{\mathcal{T}}) = \left(\gamma_{\mathcal{L}}(u_{\mathcal{T}}) \right)_{\mathcal{L} \in \partial \mathfrak{M}} \in \mathbb{R}^{\partial \mathfrak{M}}$, defined by :

$$\gamma_{\mathcal{L}}(u_{\mathcal{T}}) = \frac{u_{\mathcal{K}^*} + 2u_{\mathcal{L}} + u_{\mathcal{L}^*}}{4}, \quad \forall \mathcal{L} = [x_{\mathcal{K}^*}, x_{\mathcal{L}^*}] \in \partial \mathfrak{M}. \quad (2.3)$$

The second one is $\gamma^{\mathfrak{D}} : \boldsymbol{\varphi}^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}} \mapsto (\boldsymbol{\varphi}_{\mathcal{D}})_{\mathcal{D} \in \mathfrak{D}_{ext}} \in (\mathbb{R}^2)^{\mathfrak{D}_{ext}}$.

In order to show the duality property between the discrete gradient and the discrete divergence, we define the scalar products $[\cdot, \cdot]_{\mathcal{T}}$ on $\mathbb{R}^{\mathcal{T}}$ and $(\cdot, \cdot)_{\mathfrak{D}}$ on $(\mathbb{R}^2)^{\mathfrak{D}}$ by

$$\begin{aligned} [v_{\mathcal{T}}, u_{\mathcal{T}}]_{\mathcal{T}} &= \frac{1}{2} \left(\sum_{\mathcal{K} \in \mathfrak{M}} m_{\mathcal{K}} u_{\mathcal{K}} v_{\mathcal{K}} + \sum_{\mathcal{K}^* \in \mathfrak{M}^*} m_{\mathcal{K}^*} u_{\mathcal{K}^*} v_{\mathcal{K}^*} \right), \quad \forall u_{\mathcal{T}}, v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}, \\ (\boldsymbol{\xi}_{\mathfrak{D}}, \boldsymbol{\varphi}_{\mathfrak{D}})_{\mathfrak{D}} &= \sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} \boldsymbol{\xi}_{\mathcal{D}} \cdot \boldsymbol{\varphi}_{\mathcal{D}}, \quad \forall \boldsymbol{\xi}_{\mathfrak{D}}, \boldsymbol{\varphi}_{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}. \end{aligned} \quad (2.4)$$

The corresponding norms are denoted by $\|\cdot\|_{2,\mathcal{T}}$ and $\|\cdot\|_{2,\mathfrak{D}}$. More generally, we set:

$$\begin{aligned} \|u_{\mathcal{T}}\|_{p,\mathcal{T}} &= \left(\frac{1}{2} \sum_{\mathcal{K} \in \mathfrak{M}} m_{\mathcal{K}} |u_{\mathcal{K}}|^p + \frac{1}{2} \sum_{\mathcal{K}^* \in \mathfrak{M}^*} m_{\mathcal{K}^*} |u_{\mathcal{K}^*}|^p \right)^{1/p}, \forall u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}, \forall 1 \leq p < +\infty \\ \|u_{\mathcal{T}}\|_{\infty,\mathcal{T}} &= \max \left(\max_{\mathcal{K} \in \mathfrak{M}} |u_{\mathcal{K}}|, \max_{\mathcal{K}^* \in \mathfrak{M}^*} |u_{\mathcal{K}^*}| \right), \quad \forall u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}, \\ \|\xi_{\mathfrak{D}}\|_{p,\mathfrak{D}} &= \left(\sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} |\xi_{\mathcal{D}}|^p \right)^{1/p}, \quad \forall \xi_{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}, \forall 1 \leq p < +\infty \\ \|\xi_{\mathfrak{D}}\|_{\infty,\mathfrak{D}} &= \max_{\mathcal{D} \in \mathfrak{D}} |\xi_{\mathcal{D}}|, \quad \forall \xi_{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}. \end{aligned} \tag{2.5}$$

We also define $\langle \cdot, \cdot \rangle_{\partial\Omega}$ on $\mathbb{R}^{\mathfrak{D}_{ext}} \times \mathbb{R}^{\partial\mathfrak{M}}$ by

$$\langle \phi_{\mathfrak{D}}, v_{\partial\mathfrak{M}} \rangle_{\partial\Omega} = \sum_{\mathcal{D}, \sigma, \sigma^* \in \mathfrak{D}_{ext}} m_{\sigma} \phi_{\mathcal{D}} v_{\sigma}, \quad \forall \phi_{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}_{ext}}, \forall v_{\partial\mathfrak{M}} = (v_{\sigma})_{\sigma \in \partial\mathfrak{M}} \in \mathbb{R}^{\partial\mathfrak{M}}.$$

The Discrete Duality Finite Volume methods is based on the discrete duality formula recalled in Theorem 2.4 and proved for instance in [16].

THEOREM 2.4. *For all $(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) \in (\mathbb{R}^2)^{\mathfrak{D}} \times \mathbb{R}^{\mathcal{T}}$, we have*

$$\llbracket \operatorname{div}^{\mathcal{T}} \xi_{\mathfrak{D}}, v_{\mathcal{T}} \rrbracket_{\mathcal{T}} = -(\xi_{\mathfrak{D}}, \nabla^{\mathfrak{D}} v_{\mathcal{T}})_{\mathfrak{D}} + \langle \gamma^{\mathfrak{D}}(\xi_{\mathfrak{D}}) \cdot \mathbf{n}, \gamma^{\mathcal{T}}(v_{\mathcal{T}}) \rangle_{\partial\Omega},$$

where \mathbf{n} is the exterior unit normal to Ω .

2.3. A discrete convection operator. In order to treat the convection terms in the concentration equation, we define a discrete convection operator. This operator is similar to the one introduced in [5] but is written here for an upwinding of the convection terms.

DEFINITION 2.5. *The discrete convection operator $\operatorname{divc}^{\mathcal{T}}$ is a mapping from $(\mathbb{R}^2)^{\mathfrak{D}} \times \mathbb{R}^{\mathcal{T}}$ to $\mathbb{R}^{\mathcal{T}}$ defined for all $\xi_{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}$ and $v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ by*

$$\operatorname{divc}^{\mathcal{T}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = \left(\operatorname{divc}^{\mathfrak{M}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}), \operatorname{divc}^{\partial\mathfrak{M}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}), \operatorname{divc}^{\mathfrak{M}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}), \operatorname{divc}^{\partial\mathfrak{M}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) \right),$$

with $\operatorname{divc}^{\mathfrak{M}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = (\operatorname{divc}_{\mathcal{K}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}))_{\mathcal{K} \in \mathfrak{M}}$, $\operatorname{divc}^{\partial\mathfrak{M}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = 0$, $\operatorname{divc}^{\mathfrak{M}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = (\operatorname{divc}_{\mathcal{K}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}))_{\mathcal{K}^* \in \mathfrak{M}^*}$ and $\operatorname{divc}^{\partial\mathfrak{M}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = (\operatorname{divc}_{\mathcal{K}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}))_{\mathcal{K}^* \in \partial\mathfrak{M}^*}$ such that:

$$\begin{aligned} \forall \mathcal{K} \in \mathfrak{M}, \operatorname{divc}_{\mathcal{K}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) &= \frac{1}{m_{\mathcal{K}}} \sum_{\substack{\mathcal{D} \in \mathfrak{D}_{\mathcal{K}} \\ \mathcal{D} = \mathcal{D}_{\sigma, \sigma^*}}} m_{\sigma} \left((\xi_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}})^+ v_{\mathcal{K}} - (\xi_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}})^- v_{\mathcal{L}} \right), \\ \forall \mathcal{K}^* \in \mathfrak{M}^*, \operatorname{divc}_{\mathcal{K}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) &= \frac{1}{m_{\mathcal{K}^*}} \sum_{\substack{\mathcal{D} \in \mathfrak{D}_{\mathcal{K}^*} \\ \mathcal{D} = \mathcal{D}_{\sigma, \sigma^*}}} m_{\sigma^*} \left((\xi_{\mathcal{D}} \cdot \mathbf{n}_{\sigma^*\mathcal{K}^*})^+ v_{\mathcal{K}^*} - (\xi_{\mathcal{D}} \cdot \mathbf{n}_{\sigma^*\mathcal{K}^*})^- v_{\mathcal{L}^*} \right), \\ \forall \mathcal{K}^* \in \partial\mathfrak{M}^*, \operatorname{divc}_{\mathcal{K}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) &= \frac{1}{m_{\mathcal{K}^*}} \sum_{\substack{\mathcal{D} \in \mathfrak{D}_{\mathcal{K}^*} \\ \mathcal{D} = \mathcal{D}_{\sigma, \sigma^*}}} m_{\sigma^*} \left((\xi_{\mathcal{D}} \cdot \mathbf{n}_{\sigma^*\mathcal{K}^*})^+ v_{\mathcal{K}^*} - (\xi_{\mathcal{D}} \cdot \mathbf{n}_{\sigma^*\mathcal{K}^*})^- v_{\mathcal{L}^*} \right) \\ &\quad + \frac{1}{m_{\mathcal{K}^*}} \sum_{\substack{\mathcal{D} \in \mathfrak{D}_{\mathcal{K}^*} \cap \mathfrak{D}_{ext} \\ \mathcal{D} = \mathcal{D}_{\sigma, \sigma^*}}} \frac{m_{\sigma}}{2} \left((\xi_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}})^+ v_{\mathcal{K}} - (\xi_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}})^- v_{\mathcal{L}} \right), \end{aligned}$$

where $x^+ = \max(x, 0)$ and $x^- = -\min(x, 0)$ for all $x \in \mathbb{R}$.

This discrete convection operator satisfies the discrete property stated in Lemma 2.6.

LEMMA 2.6. For all $u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ and $\mathbf{b}_{\mathcal{D}} \in (\mathbb{R}^2)^{\mathcal{D}}$ verifying $\mathbf{b}_{\mathcal{D}} \cdot \mathbf{n} = 0$ for all $\mathcal{D} \in \mathcal{D}_{ext}$, we have:

$$\llbracket \operatorname{div}^{\mathcal{T}}(\mathbf{b}_{\mathcal{D}}, u_{\mathcal{T}}), u_{\mathcal{T}} \rrbracket_{\mathcal{T}} \geq \frac{1}{2} \llbracket \operatorname{div}^{\mathcal{T}}(\mathbf{b}_{\mathcal{D}}), (u_{\mathcal{T}})^2 \rrbracket_{\mathcal{T}}.$$

Proof. Due to the definition of the scalar product $\llbracket \cdot, \cdot \rrbracket_{\mathcal{T}}$ given by (2.4), the quantity $\llbracket \operatorname{div}^{\mathcal{T}}(\mathbf{b}_{\mathcal{D}}, u_{\mathcal{T}}), u_{\mathcal{T}} \rrbracket_{\mathcal{T}}$ is the sum of a primal term T_p and a dual term T_d . We first consider the primal term:

$$\begin{aligned} T_p &= \frac{1}{2} \sum_{\mathcal{K} \in \mathfrak{M}} m_{\mathcal{K}} \operatorname{div}_{\mathcal{K}}(\mathbf{b}_{\mathcal{D}}, u_{\mathcal{T}}) u_{\mathcal{K}}, \\ &= \frac{1}{2} \sum_{\mathcal{K} \in \mathfrak{M}} \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_{\mathcal{K}}} m_{\sigma} \left((\mathbf{b}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}})^+ u_{\mathcal{K}} - (\mathbf{b}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}})^- u_{\mathcal{L}} \right) u_{\mathcal{K}} \end{aligned}$$

Rewriting T_p as a sum on all the primal edges of the mesh and using the relations $|x| = x^+ + x^-$, $x = x^+ - x^-$ and $(a - b)a = \frac{1}{2}((a - b)^2 + a^2 - b^2)$, we get:

$$\begin{aligned} T_p &= \frac{1}{2} \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}} m_{\sigma} \left((\mathbf{b}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}})^+ u_{\mathcal{K}} - (\mathbf{b}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}})^- u_{\mathcal{L}} \right) (u_{\mathcal{K}} - u_{\mathcal{L}}) \\ &= \frac{1}{2} \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}} m_{\sigma} \left(|\mathbf{b}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}}| u_{\mathcal{K}} (u_{\mathcal{K}} - u_{\mathcal{L}}) - (\mathbf{b}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}})^- (u_{\mathcal{K}}^2 - u_{\mathcal{L}}^2) \right) \\ &= \frac{1}{4} \left(\sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}} m_{\sigma} |\mathbf{b}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}}| (u_{\mathcal{K}} - u_{\mathcal{L}})^2 + \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}} m_{\sigma} \mathbf{b}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma\mathcal{K}} (u_{\mathcal{K}}^2 - u_{\mathcal{L}}^2) \right) \end{aligned}$$

The first term in the sum is clearly non-negative and the second term can be rewritten thanks to a discrete integration by parts leading to

$$T_p \geq \frac{1}{4} \sum_{\mathcal{K} \in \mathfrak{M}} m_{\mathcal{K}} u_{\mathcal{K}}^2 \operatorname{div}_{\mathcal{K}}(\mathbf{b}_{\mathcal{D}}). \quad (2.6)$$

For the dual term, we have:

$$\begin{aligned} T_d &= \frac{1}{2} \sum_{\mathcal{K}^* \in \overline{\mathfrak{M}}^*} m_{\mathcal{K}^*} \operatorname{div}_{\mathcal{K}^*}(\mathbf{b}_{\mathcal{D}}, u_{\mathcal{T}}) u_{\mathcal{K}^*} \\ &= \frac{1}{2} \sum_{\mathcal{K}^* \in \overline{\mathfrak{M}}^*} \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_{\mathcal{K}^*}} m_{\sigma^*} \left((\mathbf{b}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma^*\mathcal{K}^*})^+ u_{\mathcal{K}^*} - (\mathbf{b}_{\mathcal{D}} \cdot \mathbf{n}_{\sigma^*\mathcal{K}^*})^- u_{\mathcal{L}^*} \right) u_{\mathcal{K}^*} \end{aligned}$$

because the boundary terms vanish due to the hypothesis $\mathbf{b}_{\mathcal{D}} \cdot \mathbf{n} = 0$ for all $\mathcal{D} \in \mathcal{D}_{ext}$. Then, T_d can be rewritten as a sum on the dual edges and following the same computations as for T_p , we get

$$T_d \geq \frac{1}{4} \sum_{\mathcal{K}^* \in \overline{\mathfrak{M}}^*} m_{\mathcal{K}^*} u_{\mathcal{K}^*}^2 \operatorname{div}_{\mathcal{K}^*}(\mathbf{b}_{\mathcal{D}}). \quad (2.7)$$

Thanks to (2.6), (2.7) and the equality

$$\frac{1}{4} \sum_{\mathcal{K} \in \mathfrak{M}} m_{\mathcal{K}} u_{\mathcal{K}}^2 \operatorname{div}_{\mathcal{K}}(\mathbf{b}_{\mathfrak{D}}) + \frac{1}{4} \sum_{\mathcal{K}^* \in \overline{\mathfrak{M}}^*} m_{\mathcal{K}^*} u_{\mathcal{K}^*}^2 \operatorname{div}_{\mathcal{K}^*}(\mathbf{b}_{\mathfrak{D}}) = \frac{1}{2} \llbracket \operatorname{div}^{\mathcal{T}}(\mathbf{b}_{\mathfrak{D}}), (u_{\mathcal{T}})^2 \rrbracket_{\mathcal{T}},$$

the proof is ended.

□

2.4. The numerical scheme. Let $(\mathcal{T}, \mathfrak{D})$ be a DDFV mesh of Ω (as presented in Section 2.1) and $\delta t > 0$ be a time step. In order to compute a numerical approximation to (1.5)–(1.6) on $]0, T[\times \Omega$, we set $N_T = T/\delta t$ (we always choose time steps such that N_T is an integer) and we define $t_n = n\delta t$ for $n \in \{0, \dots, N_T\}$.

First, we discretize all the data of the problem. Therefore, we introduce $\mathbb{P}_{\mathcal{K}}$ (respectively $\mathbb{P}_{\mathcal{K}^*}$) the L^2 projection over an interior primal cell (respectively a dual cell). We define the projection of the initial data $c_{\mathcal{T}}^0 = ((\mathbb{P}_{\mathcal{K}} c_0)_{\mathcal{K} \in \mathfrak{M}}, 0, (\mathbb{P}_{\mathcal{K}^*} c_0)_{\mathcal{K}^* \in \overline{\mathfrak{M}}^*}) \in \mathbb{R}^{\mathcal{T}}$ and of the porosity $\Phi_{\mathcal{T}} = ((\mathbb{P}_{\mathcal{K}} \Phi)_{\mathcal{K} \in \mathfrak{M}}, 0, (\mathbb{P}_{\mathcal{K}^*} \Phi)_{\mathcal{K}^* \in \overline{\mathfrak{M}}^*}) \in \mathbb{R}^{\mathcal{T}}$. In a similar way, for all $n \geq 1$, we define $(q_{\mathcal{T}}^{+,n}, q_{\mathcal{T}}^{-,n}, \hat{c}_{\mathcal{T}}^n) \in (\mathbb{R}^{\mathcal{T}})^3$ by taking the mean values of q^+ , q^- and \hat{c} on the primal and dual cells crossed with the time interval (t_{n-1}, t_n) :

$$\begin{aligned} q_{\mathcal{T}}^{+,n} &= \frac{1}{\delta t} \int_{t_{n-1}}^{t_n} \left((\mathbb{P}_{\mathcal{K}} q^+(\cdot, t))_{\mathcal{K} \in \mathfrak{M}}, 0, (\mathbb{P}_{\mathcal{K}^*} q^+(\cdot, t))_{\mathcal{K}^* \in \overline{\mathfrak{M}}^*} \right) dt \\ q_{\mathcal{T}}^{-,n} &= \frac{1}{\delta t} \int_{t_{n-1}}^{t_n} \left((\mathbb{P}_{\mathcal{K}} q^-(\cdot, t))_{\mathcal{K} \in \mathfrak{M}}, 0, (\mathbb{P}_{\mathcal{K}^*} q^-(\cdot, t))_{\mathcal{K}^* \in \overline{\mathfrak{M}}^*} \right) dt \\ \hat{c}_{\mathcal{T}}^n &= \frac{1}{\delta t} \int_{t_{n-1}}^{t_n} \left((\mathbb{P}_{\mathcal{K}} \hat{c}(\cdot, t))_{\mathcal{K} \in \mathfrak{M}}, 0, (\mathbb{P}_{\mathcal{K}^*} \hat{c}(\cdot, t))_{\mathcal{K}^* \in \overline{\mathfrak{M}}^*} \right) dt. \end{aligned}$$

At each time step n , the numerical solution will be given by $(p_{\mathcal{T}}^n, \mathbf{U}_{\mathfrak{D}}^n, c_{\mathcal{T}}^n) \in \mathbb{R}^{\mathcal{T}} \times (\mathbb{R}^2)^{\mathfrak{D}} \times \mathbb{R}^{\mathcal{T}}$ and the computation of the pressure is decoupled from the computation of the concentration. From the approximate concentration given on primal and dual cells $c_{\mathcal{T}}^{n-1}$, we reconstruct the approximate values on each diamond cell

$$c_{\mathcal{D}}^{n-1} = \frac{m_{\mathcal{D}_{\mathcal{K}}}}{2m_{\mathcal{D}}} c_{\mathcal{K}}^{n-1} + \frac{m_{\mathcal{D}_{\mathcal{K}^*}}}{2m_{\mathcal{D}}} c_{\mathcal{K}^*}^{n-1} + \frac{m_{\mathcal{D}_{\mathcal{L}}}}{2m_{\mathcal{D}}} c_{\mathcal{L}}^{n-1} + \frac{m_{\mathcal{D}_{\mathcal{L}^*}}}{2m_{\mathcal{D}}} c_{\mathcal{L}^*}^{n-1}.$$

We introduce the new vector $c_{\mathfrak{D}}^{n-1} = (c_{\mathcal{D}}^{n-1})_{\mathcal{D} \in \mathfrak{D}}$, the approximate tensors

$$\mathbb{A}_{\mathcal{D}}(s) = \frac{1}{m_{\mathcal{D}}} \int_{\mathcal{D}} \mathbb{A}(x, s) dx \quad \forall s \in \mathbb{R}, \quad \mathbb{D}_{\mathcal{D}}(\mathbf{W}) = \frac{1}{m_{\mathcal{D}}} \int_{\mathcal{D}} \mathbb{D}(x, \mathbf{W}) dx, \quad \forall \mathbf{W} \in \mathbb{R}^2,$$

and $\mathbb{A}_{\mathfrak{D}}(c_{\mathfrak{D}}^{n-1}) = (\mathbb{A}_{\mathcal{D}}(c_{\mathcal{D}}^{n-1}))_{\mathcal{D} \in \mathfrak{D}}$ and $\mathbb{D}_{\mathfrak{D}}(\mathbf{U}_{\mathfrak{D}}^n) = (\mathbb{D}_{\mathcal{D}}(\mathbf{U}_{\mathcal{D}}^n))_{\mathcal{D} \in \mathfrak{D}}$.

Then, the scheme for (1.5) writes $\forall 1 \leq n \leq N_T$

$$\operatorname{div}^{\mathcal{T}}(\mathbf{U}_{\mathfrak{D}}^n) = q_{\mathcal{T}}^{+,n} - q_{\mathcal{T}}^{-,n}, \quad (2.8)$$

$$\mathbf{U}_{\mathfrak{D}}^n = -\mathbb{A}_{\mathfrak{D}}(c_{\mathfrak{D}}^{n-1}) \nabla^{\mathfrak{D}} p_{\mathcal{T}}^n, \quad (2.9)$$

$$\mathbf{U}_{\mathcal{D}}^n \cdot \mathbf{n} = 0, \quad \forall \mathcal{D} \in \mathfrak{D}_{\text{ext}}, \quad (2.10)$$

$$\sum_{\mathcal{K} \in \mathfrak{M}} m_{\mathcal{K}} p_{\mathcal{K}}^n = \sum_{\mathcal{K}^* \in \overline{\mathfrak{M}}^*} m_{\mathcal{K}^*} p_{\mathcal{K}^*}^n = 0, \quad (2.11)$$

and the scheme for (1.6) writes $\forall 1 \leq n \leq N_T$

$$\Phi_{\mathcal{T}} \frac{c_{\mathcal{T}}^n - c_{\mathcal{T}}^{n-1}}{\delta t} - \operatorname{div}^{\mathcal{T}}(\mathbb{D}_{\mathfrak{D}}(\mathbf{U}_{\mathfrak{D}}^n) \nabla^{\mathfrak{D}} c_{\mathcal{T}}^n) + \operatorname{div}^{\mathcal{T}}(\mathbf{U}_{\mathfrak{D}}^n, c_{\mathcal{T}}^n) + q_{\mathcal{T}}^{-,n} c_{\mathcal{T}}^n = q_{\mathcal{T}}^{+,n} c_{\mathcal{T}}^n, \quad (2.12)$$

$$\mathbb{D}_{\mathfrak{D}}(\mathbf{U}_{\mathfrak{D}}^n) \nabla^{\mathfrak{D}} c_{\mathcal{T}}^n \cdot \mathbf{n} = 0, \forall \mathcal{D} \in \mathfrak{D}_{\text{ext}}. \quad (2.13)$$

The scheme (2.8)–(2.13) boils down to a resolution of two linear systems: starting from $c_{\mathcal{T}}^{n-1}$, $(p_{\mathcal{T}}^n, \mathbf{U}_{\mathfrak{D}}^n)$ is obtained by solving the linear system (2.8)–(2.11) and then $c_{\mathcal{T}}^n$ is computed by solving the linear system (2.12)–(2.13). The existence and uniqueness of the solution to each linear system and therefore the existence and uniqueness of the solution to the scheme will be proved in Section 3.2.

3. A priori estimates and existence and uniqueness result.

3.1. A priori estimates. In this Section, we prove *a priori* estimates satisfied by the solution to the scheme (2.8)–(2.13). Therefore, we need a discrete Poincaré inequality. This inequality is recalled in Theorem 3.1. We refer to [34] or [8] for its proof.

THEOREM 3.1 (Poincaré inequality). *Let Ω be an open bounded connected polygonal domain of \mathbb{R}^2 and \mathcal{T} a DDFV mesh of Ω . There exists $C > 0$ depending only on Ω and on ζ , such that $\forall u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ with $\sum_{\mathcal{K} \in \mathfrak{M}} m_{\mathcal{K}} u_{\mathcal{K}} = \sum_{\mathcal{K}^* \in \mathfrak{M}^*} m_{\mathcal{K}^*} u_{\mathcal{K}^*} = 0$, we have*

$$\|u_{\mathcal{T}}\|_{2,\mathcal{T}} \leq \frac{C}{\sin(\alpha_{\mathcal{T}})} \|\nabla^{\mathfrak{D}} u_{\mathcal{T}}\|_{2,\mathfrak{D}}.$$

We first establish *a priori estimates* on the pressure, the gradient of the pressure and the Darcy's velocity at the discrete level.

LEMMA 3.2. *Let Ω be an open bounded connected polygonal domain of \mathbb{R}^2 and \mathcal{T} a DDFV mesh of this domain. Assume (1.7)–(1.12) hold and that the scheme (2.8)–(2.13) has a solution $(p_{\mathcal{T}}^n, \mathbf{U}_{\mathfrak{D}}^n, c_{\mathcal{T}}^n)_{1 \leq n \leq N_T}$. Then, there exists $C > 0$ depending only on Ω , ζ , $\sin(\alpha_{\mathcal{T}})$, α_A and Λ_A , such that we have for all $n \in [1, \dots, N_T]$:*

$$\|p_{\mathcal{T}}^n\|_{2,\mathcal{T}} + \|\nabla^{\mathfrak{D}} p_{\mathcal{T}}^n\|_{2,\mathfrak{D}} + \|\mathbf{U}_{\mathfrak{D}}^n\|_{2,\mathfrak{D}} \leq C \|q^+ - q^-\|_{L^\infty(0,T;L^2(\Omega))}. \quad (3.1)$$

Proof. We multiply the scheme (2.8) by $p_{\mathcal{T}}^n$. Thanks to the Neumann boundary condition (2.10), the discrete duality formula (Theorem 2.4) yields:

$$\llbracket q_{\mathcal{T}}^{+,n} - q_{\mathcal{T}}^{-,n}, p_{\mathcal{T}}^n \rrbracket_{\mathcal{T}} = \llbracket \operatorname{div}^{\mathcal{T}}(\mathbf{U}_{\mathfrak{D}}^n), p_{\mathcal{T}}^n \rrbracket_{\mathcal{T}} = -(\mathbf{U}_{\mathfrak{D}}^n, \nabla^{\mathfrak{D}} p_{\mathcal{T}}^n)_{\mathfrak{D}}.$$

Then, using (2.9), hypothesis (1.8) and Cauchy-Schwarz inequality, we have

$$\alpha_A \|\nabla^{\mathfrak{D}} p_{\mathcal{T}}^n\|_{2,\mathfrak{D}}^2 \leq \llbracket q_{\mathcal{T}}^{+,n} - q_{\mathcal{T}}^{-,n}, p_{\mathcal{T}}^n \rrbracket_{\mathcal{T}} \leq \|q_{\mathcal{T}}^{+,n} - q_{\mathcal{T}}^{-,n}\|_{2,\mathcal{T}} \|p_{\mathcal{T}}^n\|_{2,\mathcal{T}} \quad (3.2)$$

Applying now the discrete Poincaré inequality (Theorem 3.1), we get simultaneously

$$\begin{aligned} \|\nabla^{\mathfrak{D}} p_{\mathcal{T}}^n\|_{2,\mathfrak{D}} &\leq C \|q_{\mathcal{T}}^{+,n} - q_{\mathcal{T}}^{-,n}\|_{2,\mathcal{T}} \\ \|p_{\mathcal{T}}^n\|_{2,\mathcal{T}} &\leq C \|q_{\mathcal{T}}^{+,n} - q_{\mathcal{T}}^{-,n}\|_{2,\mathcal{T}}, \quad \forall 1 \leq n \leq N_T. \end{aligned} \quad (3.3)$$

The relation (2.9) and the hypothesis (1.8) imply $\|\mathbf{U}_{\mathfrak{D}}^n\|_{2,\mathfrak{D}} \leq \Lambda_A \|\nabla^{\mathfrak{D}} p_{\mathcal{T}}^n\|_{2,\mathfrak{D}}$ and therefore

$$\|\mathbf{U}_{\mathfrak{D}}^n\|_{2,\mathfrak{D}} \leq C \|q_{\mathcal{T}}^{+,n} - q_{\mathcal{T}}^{-,n}\|_{2,\mathcal{T}}, \quad \forall 1 \leq n \leq N_T. \quad (3.4)$$

As

$$\max_{1 \leq n \leq N_T} \|q_{\mathcal{T}}^{+,n} - q_{\mathcal{T}}^{-,n}\|_{2,\mathcal{T}} \leq \|q^+ - q^-\|_{L^\infty(0,T,L^2(\Omega))},$$

the estimate (3.1) is a consequence of (3.3) and (3.4).

□

Lemma 3.3 gives *a priori* estimates on the approximate concentration and its approximate gradient.

LEMMA 3.3. *Let Ω be an open bounded connected polygonal domain of \mathbb{R}^2 and \mathcal{T} a DDFV mesh of this domain. Assume (1.7)–(1.12) hold and that the scheme (2.8)–(2.13) has a solution $(p_{\mathcal{T}}^n, \mathbf{U}_{\mathfrak{D}}^n, c_{\mathcal{T}}^n)_{1 \leq n \leq N_T}$. Then, there exists $C > 0$ depending only on Ω , ζ , $\sin(\alpha_{\mathcal{T}})$, Φ^* and α_D , such that we have for all $N \in [1, \dots, N_T]$:*

$$\begin{aligned} \|c_{\mathcal{T}}^N\|_{2,\mathcal{T}}^2 + \sum_{n=1}^N \delta t \|\nabla^{\mathfrak{D}} c_{\mathcal{T}}^n\|_{2,\mathfrak{D}}^2 + \sum_{n=1}^N \delta t \left\| |\mathbf{U}_{\mathfrak{D}}^n|^{\frac{1}{2}} \nabla^{\mathfrak{D}} c_{\mathcal{T}}^n \right\|_{2,\mathfrak{D}}^2 \\ \leq C \left(\|c_0\|_{L^2(\Omega)}^2 + \|q^+\|_{L^\infty(0,T,L^2(\Omega))}^2 \right). \end{aligned} \quad (3.5)$$

Proof. We multiply the scheme (2.12) by $c_{\mathcal{T}}^n$:

$$\begin{aligned} \left[\Phi_{\mathcal{T}} \frac{c_{\mathcal{T}}^n - c_{\mathcal{T}}^{n-1}}{\delta t}, c_{\mathcal{T}}^n \right]_{\mathcal{T}} - \left[\operatorname{div}^{\mathcal{T}}(\mathbb{D}_{\mathfrak{D}}(\mathbf{U}_{\mathfrak{D}}^n) \nabla^{\mathfrak{D}} c_{\mathcal{T}}^n), c_{\mathcal{T}}^n \right]_{\mathcal{T}} + \left[\operatorname{divc}^{\mathcal{T}}(\mathbf{U}_{\mathfrak{D}}^n, c_{\mathcal{T}}^n), c_{\mathcal{T}}^n \right]_{\mathcal{T}} \\ + \left[q_{\mathcal{T}}^{-,n} c_{\mathcal{T}}^n, c_{\mathcal{T}}^n \right]_{\mathcal{T}} = \left[q_{\mathcal{T}}^{+,n} c_{\mathcal{T}}^n, c_{\mathcal{T}}^n \right]_{\mathcal{T}} \end{aligned} \quad (3.6)$$

Let us analyze successively the different terms in this equality. The relation $(a-b)a = \frac{1}{2}(a^2 - b^2) + \frac{1}{2}(a-b)^2$ ensures

$$\left[\Phi_{\mathcal{T}}(c_{\mathcal{T}}^n - c_{\mathcal{T}}^{n-1}), c_{\mathcal{T}}^n \right]_{\mathcal{T}} \geq \frac{1}{2} \left(\left[\Phi_{\mathcal{T}}, (c_{\mathcal{T}}^n)^2 \right]_{\mathcal{T}} - \left[\Phi_{\mathcal{T}}, (c_{\mathcal{T}}^{n-1})^2 \right]_{\mathcal{T}} \right)$$

Thanks to the Neumann boundary conditions (2.13), Theorem 2.4 and hypothesis (1.9) lead to:

$$\begin{aligned} - \left[\operatorname{div}^{\mathcal{T}}(\mathbb{D}_{\mathfrak{D}}(\mathbf{U}_{\mathfrak{D}}^n) \nabla^{\mathfrak{D}} c_{\mathcal{T}}^n), c_{\mathcal{T}}^n \right]_{\mathcal{T}} &= (\mathbb{D}_{\mathfrak{D}}(\mathbf{U}_{\mathfrak{D}}^n) \nabla^{\mathfrak{D}} c_{\mathcal{T}}^n, \nabla^{\mathfrak{D}} c_{\mathcal{T}}^n)_{\mathfrak{D}} \\ &\geq \alpha_D \left(\|\nabla^{\mathfrak{D}} c_{\mathcal{T}}^n\|_{2,\mathfrak{D}}^2 + \left\| |\mathbf{U}_{\mathfrak{D}}^n|^{\frac{1}{2}} \nabla^{\mathfrak{D}} c_{\mathcal{T}}^n \right\|_{2,\mathfrak{D}}^2 \right). \end{aligned}$$

Due to the boundary conditions (2.10), Lemma 2.6 and (2.8) imply:

$$\left[\operatorname{divc}^{\mathcal{T}}(\mathbf{U}_{\mathfrak{D}}^n, c_{\mathcal{T}}^n), c_{\mathcal{T}}^n \right]_{\mathcal{T}} \geq \frac{1}{2} \left[\operatorname{div}^{\mathcal{T}}(\mathbf{U}_{\mathfrak{D}}^n), (c_{\mathcal{T}}^n)^2 \right]_{\mathcal{T}} = \frac{1}{2} \left[q_{\mathcal{T}}^{+,n} - q_{\mathcal{T}}^{-,n}, (c_{\mathcal{T}}^n)^2 \right]_{\mathcal{T}}$$

and, as $\left[q_{\mathcal{T}}^{-,n} c_{\mathcal{T}}^n, c_{\mathcal{T}}^n \right]_{\mathcal{T}} = \left[q_{\mathcal{T}}^{-,n}, (c_{\mathcal{T}}^n)^2 \right]_{\mathcal{T}}$, we obtain:

$$\left[\operatorname{divc}^{\mathcal{T}}(\mathbf{U}_{\mathfrak{D}}^n, c_{\mathcal{T}}^n), c_{\mathcal{T}}^n \right]_{\mathcal{T}} + \left[q_{\mathcal{T}}^{-,n} c_{\mathcal{T}}^n, c_{\mathcal{T}}^n \right]_{\mathcal{T}} \geq \frac{1}{2} \left[q_{\mathcal{T}}^{+,n} + q_{\mathcal{T}}^{-,n}, (c_{\mathcal{T}}^n)^2 \right]_{\mathcal{T}} \geq 0.$$

Finally, thanks to Cauchy-Schwarz inequality and hypotheses (1.11), we have

$$\left[q_{\mathcal{T}}^{+,n} c_{\mathcal{T}}^n, c_{\mathcal{T}}^n \right]_{\mathcal{T}} \leq \|q_{\mathcal{T}}^{+,n} c_{\mathcal{T}}^n\|_{2,\mathcal{T}} \|c_{\mathcal{T}}^n\|_{2,\mathcal{T}} \leq \|q_{\mathcal{T}}^{+,n}\|_{2,\mathcal{T}} \|c_{\mathcal{T}}^n\|_{2,\mathcal{T}}.$$

Therefore, from equality (3.6), we get:

$$\begin{aligned} & \frac{1}{2\delta t} \left(\llbracket \Phi_{\mathcal{T}}, (c_{\mathcal{T}}^n)^2 - (c_{\mathcal{T}}^{n-1})^2 \rrbracket_{\mathcal{T}} \right) + \alpha_D \left(\left\| \nabla^{\mathfrak{D}} c_{\mathcal{T}}^n \right\|_{2,\mathfrak{D}}^2 + \left\| |\mathbf{U}_{\mathfrak{D}}^n|^{\frac{1}{2}} \nabla^{\mathfrak{D}} c_{\mathcal{T}}^n \right\|_{2,\mathfrak{D}}^2 \right) \\ & \leq \|q_{\mathcal{T}}^{\pm,n}\|_{2,\mathcal{T}} \|c_{\mathcal{T}}^n\|_{2,\mathcal{T}}. \end{aligned} \quad (3.7)$$

Multiplying by $2\delta t$ and summing over $n = 1, \dots, N$ with $1 \leq N \leq N_T$, we get

$$\begin{aligned} & \llbracket \Phi_{\mathcal{T}}, (c_{\mathcal{T}}^N)^2 - (c_{\mathcal{T}}^0)^2 \rrbracket_{\mathcal{T}} + 2\alpha_D \sum_{n=1}^N \delta t \left(\left\| \nabla^{\mathfrak{D}} c_{\mathcal{T}}^n \right\|_{2,\mathfrak{D}}^2 + \left\| |\mathbf{U}_{\mathfrak{D}}^n|^{\frac{1}{2}} \nabla^{\mathfrak{D}} c_{\mathcal{T}}^n \right\|_{2,\mathfrak{D}}^2 \right) \\ & \leq 2 \sum_{n=1}^N \delta t \|q_{\mathcal{T}}^{\pm,n}\|_{2,\mathcal{T}} \|c_{\mathcal{T}}^n\|_{2,\mathcal{T}}. \end{aligned} \quad (3.8)$$

But, the hypotheses on Φ (1.10) ensure:

$$\begin{aligned} \llbracket \Phi_{\mathcal{T}}, (c_{\mathcal{T}}^N)^2 \rrbracket_{\mathcal{T}} & \geq \Phi_* \|c_{\mathcal{T}}^N\|_{2,\mathcal{T}}^2 \\ \llbracket \Phi_{\mathcal{T}}, (c_{\mathcal{T}}^0)^2 \rrbracket_{\mathcal{T}} & \leq \Phi_*^{-1} \|c_{\mathcal{T}}^0\|_{2,\mathcal{T}}^2 \leq \Phi_*^{-1} \|c_0\|_{L^2(\Omega)}^2. \end{aligned}$$

For the right-hand side in (3.8), we have:

$$\begin{aligned} \sum_{n=1}^N \delta t \|q_{\mathcal{T}}^{\pm,n}\|_{2,\mathcal{T}} \|c_{\mathcal{T}}^n\|_{2,\mathcal{T}} & \leq T \|q^+\|_{L^\infty(0,T,L^2(\Omega))} \sup_{n \in [1, \dots, N_T]} \|c_{\mathcal{T}}^n\|_{2,\mathcal{T}} \\ & \leq \frac{T^2}{\Phi_*} \|q^+\|_{L^\infty(0,T,L^2(\Omega))}^2 + \frac{\Phi_*}{4} \sup_{n \in [1, \dots, N_T]} \|c_{\mathcal{T}}^n\|_{2,\mathcal{T}}^2. \end{aligned}$$

Therefore, we deduce from (3.8) that, for all $N \in [1, \dots, N_T]$,

$$\begin{aligned} & \Phi_* \|c_{\mathcal{T}}^N\|_{2,\mathcal{T}}^2 + 2\alpha_D \sum_{n=1}^N \delta t \left(\left\| \nabla^{\mathfrak{D}} c_{\mathcal{T}}^n \right\|_{2,\mathfrak{D}}^2 + \left\| |\mathbf{U}_{\mathfrak{D}}^n|^{\frac{1}{2}} \nabla^{\mathfrak{D}} c_{\mathcal{T}}^n \right\|_{2,\mathfrak{D}}^2 \right) \\ & \leq \Phi_*^{-1} \|c_0\|_{L^2(\Omega)}^2 + \frac{2T^2}{\Phi_*} \|q^+\|_{L^\infty(0,T,L^2(\Omega))}^2 + \frac{\Phi_*}{2} \sup_{n \in [1, \dots, N_T]} \|c_{\mathcal{T}}^n\|_{2,\mathcal{T}}^2. \end{aligned}$$

We first focus on the term $\Phi_* \|c_{\mathcal{T}}^N\|_{2,\mathcal{T}}^2$ of the left hand side of the previous inequality, which is bounded by the right hand side. Taking now the supremum over $N \in [1, \dots, N_T]$ yields the expected inequality (3.5) of this term. Finally, for the other terms of the left hand side of the previous inequality, we directly have the expected inequality (3.5).
□

3.2. Existence and uniqueness of an approximate solution. In this Section, we prove Theorem 3.4. It consists in the proof of existence and uniqueness of the solution to the linear systems (2.8)–(2.11) and (2.12)–(2.13) involved at each time step.

THEOREM 3.4. *Let Ω be an open bounded connected polygonal domain of \mathbb{R}^2 and \mathcal{T} a DDFV mesh of this domain. Assume (1.7)–(1.12) hold. Let $T > 0$ and δt be a time step such that $N_T = T/\delta t$ is an integer. Then, the scheme (2.8)–(2.13) admits a unique solution $(p_{\mathcal{T}}^n, \mathbf{U}_{\mathfrak{D}}^n, c_{\mathcal{T}}^n)_{1 \leq n \leq N_T}$.*

Step 1: Existence and uniqueness of $(p_{\mathcal{T}}^n, \mathbf{U}_{\mathcal{D}}^n)$. Let us first consider the system (2.8)–(2.11) on $(p_{\mathcal{T}}^n, \mathbf{U}_{\mathcal{D}}^n)$. The number of unknowns of the system (2.8)–(2.11) is $M = \text{Card}(\mathfrak{M}) + \text{Card}(\partial\mathfrak{M}) + \text{Card}(\mathfrak{M}^*) + \text{Card}(\partial\mathfrak{M}^*) + \text{Card}(\mathfrak{D})$. Equations (2.8), (2.9) and (2.10) provide respectively $\text{Card}(\mathfrak{M}) + \text{Card}(\mathfrak{M}^*) + \text{Card}(\partial\mathfrak{M}^*)$, $\text{Card}(\mathfrak{D})$ and $\text{Card}(\mathfrak{D}_{ext})$ equations. As $\text{Card}(\mathfrak{D}_{ext}) = \text{Card}(\partial\mathfrak{M})$, the system (2.8)–(2.10) is a square system of size M .

Inequality (3.2) gives us the kernel of the matrix of this square system. Indeed, $q_{\mathcal{T}}^{+,n} - q_{\mathcal{T}}^{-,n} = 0$ implies $\nabla^{\mathfrak{D}} p_{\mathcal{T}}^n = 0$ and, thanks to the definition of the discrete gradient (2.2),

$$\nabla^{\mathfrak{D}} p_{\mathcal{T}}^n = 0 \iff \exists P, \tilde{P} \in \mathbb{R} \text{ such that } p_{\mathcal{K}}^n = P, \forall \mathcal{K} \in \overline{\mathfrak{M}} \text{ and } p_{\mathcal{K}^*}^n = \tilde{P}, \forall \mathcal{K}^* \in \overline{\mathfrak{M}^*}.$$

Therefore, the kernel of the matrix of the square system (2.8)–(2.10) has dimension 2.

Let us now show that the right-hand side of the system belongs to the image of the matrix. Therefore, we multiply the scheme (2.8) by the test function $v_{\mathcal{T}}$ defined by $v_{\mathcal{K}} = 1$ for all $\mathcal{K} \in \overline{\mathfrak{M}}$ and $v_{\mathcal{K}^*} = 0$ for all $\mathcal{K}^* \in \overline{\mathfrak{M}^*}$. We obtain :

$$\begin{aligned} \sum_{\mathcal{K} \in \mathfrak{M}} m_{\mathcal{K}} (q_{\mathcal{K}}^{+,n} - q_{\mathcal{K}}^{-,n}) &= \llbracket q_{\mathcal{T}}^{+,n} - q_{\mathcal{T}}^{-,n}, v_{\mathcal{T}} \rrbracket_{\mathcal{T}} = \llbracket \text{div}^{\mathcal{T}}(\mathbf{U}_{\mathcal{D}}^n), v_{\mathcal{T}} \rrbracket_{\mathcal{T}} \\ &= -(\mathbf{U}_{\mathcal{D}}^n, \nabla^{\mathfrak{D}} v_{\mathcal{T}})_{\mathfrak{D}} = 0. \end{aligned}$$

With the test function $w_{\mathcal{T}}$ defined by $w_{\mathcal{K}} = 0$ for all $\mathcal{K} \in \overline{\mathfrak{M}}$ and $w_{\mathcal{K}^*} = 1$ for all $\mathcal{K}^* \in \overline{\mathfrak{M}^*}$, we get similarly

$$\sum_{\mathcal{K}^* \in \overline{\mathfrak{M}^*}} m_{\mathcal{K}^*} (q_{\mathcal{K}^*}^{+,n} - q_{\mathcal{K}^*}^{-,n}) = \llbracket q_{\mathcal{T}}^{+,n} - q_{\mathcal{T}}^{-,n}, w_{\mathcal{T}} \rrbracket_{\mathcal{T}} = 0.$$

The relations $\sum_{\mathcal{K} \in \mathfrak{M}} m_{\mathcal{K}} (q_{\mathcal{K}}^{+,n} - q_{\mathcal{K}}^{-,n}) = 0$ and $\sum_{\mathcal{K}^* \in \overline{\mathfrak{M}^*}} m_{\mathcal{K}^*} (q_{\mathcal{K}^*}^{+,n} - q_{\mathcal{K}^*}^{-,n}) = 0$ ensure that the right hand-side of the linear system belongs to the image of the matrix of the system.

Therefore, the linear system (2.8)–(2.10) has solutions defined up to the two constants P and \tilde{P} . These values are uniquely determined thanks to the relations (2.11). It implies that the linear system (2.8)–(2.11) has a unique solution.

REMARK 3.5. *In practice, we will eliminate two equations of the linear system (2.8)–(2.10) which are linearly dependent of the other equations (for instance, one equation corresponding to some \mathcal{K}_0 and another one corresponding to some \mathcal{K}_0^*) and replace them by the two linear equations (2.11). The resulting system is an invertible square system, equivalent to (2.8)–(2.11).*

Step 2: Existence and uniqueness of $c_{\mathcal{T}}^n$. We now consider the system (2.12)–(2.13) on $c_{\mathcal{T}}^n$. The number of unknowns is $\text{Card}(\mathfrak{M}) + \text{Card}(\partial\mathfrak{M}) + \text{Card}(\mathfrak{M}^*) + \text{Card}(\partial\mathfrak{M}^*)$, equal to the number of equations. Inequality (3.7) written with $c_{\mathcal{T}}^{n-1} = 0$ and $q_{\mathcal{T}}^{\pm,n} = 0$ imply that the kernel of the linear system is reduced to 0. Therefore, there exists a unique $c_{\mathcal{T}}^n$ solution to (2.12)–(2.13).

It concludes the proof of Theorem 3.4.

4. Numerical convergence of the DDFV scheme. In this section, we illustrate the behavior of DDFV scheme by applying it to the system (1.1)–(1.3), which describes the miscible displacement of one fluid by another in a porous medium.

In all the test cases, the spatial domain is $\Omega = (0, 1000) \times (0, 1000)$ ft² and the time period is $[0, 3600]$ days. The injection well is located at the upper-right corner

(1000, 1000) with an injection rate $q^+ = 30 \text{ ft}^2/\text{day}$ and an injection concentration $\hat{c} = 1.0$. The production well is located at the lower-left corner $(0, 0)$ with a production rate $q^- = 30 \text{ ft}^2/\text{day}$. It means that q^- and q^+ are Dirac masses, which can be taken into account with the scheme. The porosity of the medium is specified as $\Phi(x) = 0.1$ and the initial concentration is $c_0(x) = 0$. The viscosity of the oil is $\mu(0) = 1.0 \text{ cp}$. We choose $\Phi d_l = 5 \text{ ft}$ and $\Phi d_t = 0.5 \text{ ft}$.

In order to compute the numerical order of convergence of the scheme, we introduce a sequence of triangular meshes. For a refinement level $i \in \{1, \dots, 7\}$, the mesh is obtained by dividing the domain into $2^{i+1} \times 2^{i+1}$ equally sized squares and each square is split into 2 triangles along a diagonal. The number of cells for the mesh i is 2^{2i+3} . We present on Figure 4.1 the meshes obtained for $i = 1$ and $i = 3$. We choose this sequence of structured triangular meshes because they fit together and allow the computation of numerical errors. Let us also mention that, even though many choices are possible, we always assume in this paper that $x_{\mathcal{K}}$ is the mass center of $\mathcal{K} \in \mathfrak{M}$.

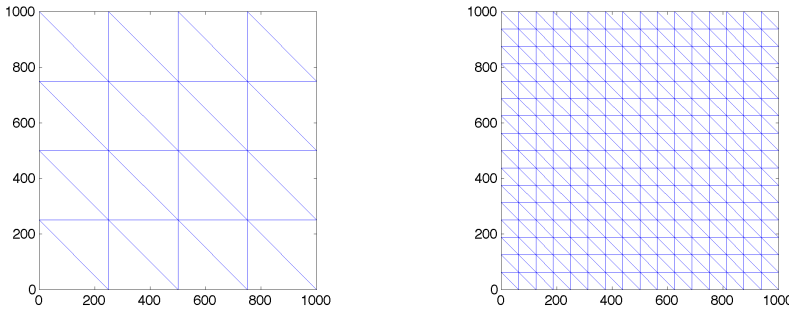


FIG. 4.1. *Triangular meshes with a refinement level $i = 1$ on the left and $i = 3$ on the right.*

We take as the reference solution the solution computed on the mesh $i = 7$. We compute the relative errors in L^1 -norm and L^2 -norm at time $T = 3600$ days. The time step is $\delta t = 36$ days.

Test 1: In this test case, we assume that the viscosity is constant, $\mu(c) = 1 \text{ cp}$. And we set $\Phi d_m = 1 \text{ ft}^2/\text{day}$. We compare the results obtained for two different values of the permeability : a constant one $\mathbb{K} = 80 \mathbb{I}$ and a discontinuous one $\mathbb{K} = 80 \mathbb{I}$ on the subdomain $(0, 1000) \times (0, 500)$ and $\mathbb{K} = 20 \mathbb{I}$ on the subdomain $(0, 1000) \times (500, 1000)$.

refinement level	$\mathbb{K} = 80 \mathbb{I}$			
	error L^1	order L^1	error L^2	order L^2
1	2.72e-02	-	5.15e-02	-
2	7.57e-03	2.14	2.50e-02	1.18
3	2.21e-03	1.93	1.24e-02	1.09
4	6.35e-04	1.88	6.18e-03	1.05
5	1.74e-04	1.91	3.02e-03	1.06
6	4.08e-05	2.12	1.32e-03	1.21

TABLE 4.1

Test 1. Convergence results of the DDFV method on the pressure p at time $t = 3600$ with a continuous permeability.

refinement level	$\mathbb{K} = (80\mathbf{1}_{y < 500} + 20\mathbf{1}_{y > 500})\mathbb{I}$			
	error L^1	order L^1	error L^2	order L^2
1	1.63e-01	-	1.30e-01	-
2	7.95e-02	1.19	6.20e-02	1.21
3	3.83e-02	1.14	2.98e-02	1.13
4	1.78e-02	1.15	1.41e-02	1.13
5	7.64e-03	1.25	6.26e-03	1.19
6	2.55e-03	1.60	2.31e-03	1.46

TABLE 4.2

Test 1. Convergence results of the DDFV method on the pressure p at time $t = 3600$ with a discontinuous permeability.

refinement level	$\mathbb{K} = 80\mathbb{I}$			
	error L^1	order L^1	error L^2	order L^2
1	4.66e-02	-	6.13e-02	-
2	3.28e-02	0.58	4.49e-02	0.52
3	2.03e-02	0.75	2.84e-02	0.72
4	1.11e-02	0.91	1.58e-02	0.89
5	5.20e-03	1.12	7.52e-03	1.09
6	1.83e-03	1.53	2.67e-03	1.51

TABLE 4.3

Test 1. Convergence results of the DDFV method on the concentration c at time $t = 3600$ with a continuous permeability.

In Tables 4.1 and 4.2, we first compare the L^1 -norm and L^2 -norm of the pressure error. When the permeability \mathbb{K} is continuous, we observe an order of convergence in L^1 -norm around 2 while it is around 1 in L^2 -norm. When the permeability \mathbb{K} is discontinuous, we note that the values of the errors in L^1 -norm and L^2 -norm are close and the order of convergence in L^1 -norm is now around 1.2. In fact, this loss of convergence has already been observed by Boyer-Hubert in [9].

In Tables 4.3 and 4.4, we compare the different orders of convergence for the concentration. We observe the same kind of results : order around 1, in L^1 -norm as in L^2 -norm, for the continuous permeability and the discontinuous one.

Let us just mention that we obtain similar results using a sequence of square meshes.

Test 2 : In this test case, we assume that the viscosity really depends on c and is given by (1.4) with $M = 41$. The dependence on c of the viscosity induces a strong coupling in the system. We also assume that there is no molecular diffusion $\Phi d_m = 0$ ft²/day. We still compare the results obtained for two different values of the permeability : a constant one $\mathbb{K} = 80\mathbb{I}$ and a discontinuous one $\mathbb{K} = 80\mathbb{I}$ on the subdomain $(0, 1000) \times (0, 500)$ and $\mathbb{K} = 20\mathbb{I}$ on the subdomain $(0, 1000) \times (500, 1000)$.

In Tables 4.5 and 4.6, we compare the different orders of convergence for the pressure and the concentration. We observe a loss of accuracy for both variables and both values of permeability. Especially we remark that for a continuous permeability we do not preserve the order 2 for the pressure.

refinement level	$\mathbb{K} = (80\mathbf{1}_{y < 500} + 20\mathbf{1}_{y > 500})\mathbb{I}$			
	error L^1	order L^1	error L^2	order L^2
1	4.60e-02	-	6.02e-02	-
2	3.26e-02	0.57	4.53e-02	0.47
3	2.02e-02	0.75	2.88e-02	0.71
4	1.10e-02	0.92	1.60e-02	0.88
5	5.13e-03	1.12	7.66e-03	1.09
6	1.80e-03	1.53	2.73e-03	1.51

TABLE 4.4

Test 1. Convergence results of the DDFV method on the concentration c at time $t = 3600$ with a discontinuous permeability.

refinement level	$\mathbb{K} = 80\mathbb{I}$		$\mathbb{K} = (80\mathbf{1}_{y < 500} + 20\mathbf{1}_{y > 500})\mathbb{I}$	
	error L^1	order L^1	error L^1	order L^1
1	2.78e-01	-	3.71e-01	-
2	2.95e-01	-0.12	2.65e-01	0.56
3	2.67e-01	0.15	1.73e-01	0.66
4	1.95e-01	0.47	1.18e-01	0.57
5	1.01e-01	0.97	7.17e-02	0.74
6	3.49e-02	1.55	3.45e-02	1.07

TABLE 4.5

Test2. Convergence results of the DDFV method on the pressure p at time $t = 3600$.

5. An other scheme: the m-DDFV scheme. As shown in Table 4.1 the DDFV scheme for anisotropic diffusion equation has order 2 in the case where the diffusion tensor is continuous but only order 1 when it is discontinuous (see Table 4.2). It has already been observed in [9], and therefore these authors introduced a new DDFV scheme called m-DDFV.

In this part, we focus on a diffusive tensor $\mathbb{A}(x, c) = \frac{\mathbb{K}(x)}{\mu(c)}$ such that \mathbb{K} is discontinuous across the primal edges. In [30, 9], the authors have proposed to modify the discrete gradient and the numerical fluxes in order to recover a scheme of order two. Note that in [9] the tensor is more general and it is discontinuous across primal and dual edges. First we recall the idea and the definition of the discrete gradient. Then in Section 5.3, we apply this method to our problem (1.5)-(1.6).

5.1. Notations. In this section, we need to introduce additional notations, as shown on Figure 5.1. For any $\mathcal{D} \in \mathfrak{D}$, we define the half-diamonds $\tilde{\mathcal{D}}_{\mathcal{K}} = \mathcal{D} \cap \mathcal{K}$ and $\tilde{\mathcal{D}}_{\mathcal{L}} = \mathcal{D} \cap \mathcal{L}$, such that $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}_{\mathcal{K}} \cup \tilde{\mathcal{D}}_{\mathcal{L}}$ if $\mathcal{D} \in \mathfrak{D}_{int}$ and $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}_{\mathcal{K}}$ if $\mathcal{D} \in \mathfrak{D}_{ext}$. A half-diamond is denoted by $\tilde{\mathcal{D}}$ and the set of all the half-diamonds of the mesh is denoted by $\tilde{\mathfrak{D}}$. For any $\mathcal{D} \in \mathfrak{D}$, we recall that $x_{\mathcal{D}}$ is the center of \mathcal{D} and we introduce the half-edges $\sigma_{\mathcal{K}} = [x_{\mathcal{K}}, x_{\mathcal{D}}]$ and $\sigma_{\mathcal{L}} = [x_{\mathcal{L}}, x_{\mathcal{D}}]$, such that $\sigma = \sigma_{\mathcal{K}} \cup \sigma_{\mathcal{L}}$. Then, $m_{\sigma_{\mathcal{K}}}$ denotes the measure of $\sigma_{\mathcal{K}}$ and $x_{\sigma_{\mathcal{K}}}$ its center (and we define similarly $m_{\sigma_{\mathcal{L}}}$ and $x_{\sigma_{\mathcal{L}}}$).

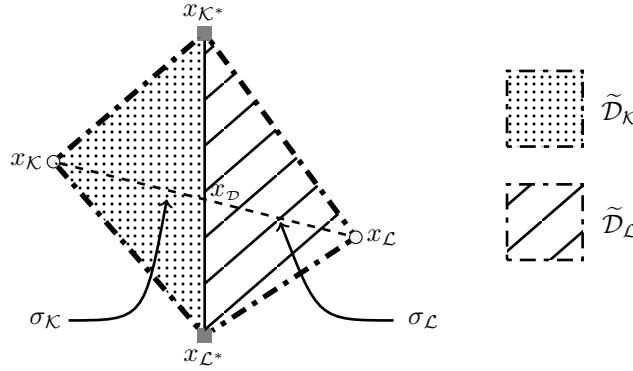
We consider the linear spaces $(\mathbb{R}^2)^{\tilde{\mathfrak{D}}}$ of vector fields constant on the cells of $\tilde{\mathfrak{D}}$:

$$(\mathbb{R}^2)^{\tilde{\mathfrak{D}}} = \left\{ \boldsymbol{\xi}_{\tilde{\mathfrak{D}}} = (\boldsymbol{\xi}_{\tilde{\mathcal{D}}})_{\tilde{\mathcal{D}} \in \tilde{\mathfrak{D}}}, \text{ with } \boldsymbol{\xi}_{\tilde{\mathcal{D}}} \in \mathbb{R}^2, \forall \tilde{\mathcal{D}} \in \tilde{\mathfrak{D}} \right\}.$$

refinement level	$\mathbb{K} = 80 \mathbb{I}$		$\mathbb{K} = (80 \mathbf{1}_{y < 500} + 20 \mathbf{1}_{y > 500}) \mathbb{I}$	
	error L^1	order L^1	error L^1	order L^1
1	4.96e-01	-	4.69e-01	-
2	4.11e-01	0.31	3.67e-01	0.41
3	3.19e-01	0.40	2.62e-01	0.53
4	2.19e-01	0.56	1.68e-01	0.67
5	1.22e-01	0.87	9.44e-02	0.85
6	4.80e-02	1.36	4.16e-02	1.20

TABLE 4.6

Test 2. Convergence results of the DDFV method on the concentration c at time $t = 3600$.

FIG. 5.1. A diamond \mathcal{D} with the half-diamonds $\tilde{\mathcal{D}}_{\mathcal{K}}$ and $\tilde{\mathcal{D}}_{\mathcal{L}}$.

We also define an additional scalar product $(\cdot, \cdot)_{\tilde{\mathcal{D}}}$ on $(\mathbb{R}^2)^{\tilde{\mathcal{D}}}$ by

$$(\xi_{\tilde{\mathcal{D}}}, \varphi_{\tilde{\mathcal{D}}})_{\tilde{\mathcal{D}}} = \sum_{\tilde{\mathcal{D}} \in \tilde{\mathcal{D}}} m_{\tilde{\mathcal{D}}} \xi_{\tilde{\mathcal{D}}} \cdot \varphi_{\tilde{\mathcal{D}}}, \quad \forall \xi_{\tilde{\mathcal{D}}}, \varphi_{\tilde{\mathcal{D}}} \in (\mathbb{R}^2)^{\tilde{\mathcal{D}}},$$

the corresponding norm is denoted by $\|\cdot\|_{2, \tilde{\mathcal{D}}}$:

$$\|\xi_{\tilde{\mathcal{D}}}\|_{2, \tilde{\mathcal{D}}} = \left(\sum_{\tilde{\mathcal{D}} \in \tilde{\mathcal{D}}} m_{\tilde{\mathcal{D}}} |\xi_{\tilde{\mathcal{D}}}|^2 \right)^{1/2}, \quad \forall \xi_{\tilde{\mathcal{D}}} \in (\mathbb{R}^2)^{\tilde{\mathcal{D}}}.$$

5.2. The modified discrete gradient. When the anisotropic tensor \mathbb{K} is discontinuous across the primal edges, the normal component of $\mathbb{K} \nabla u$ must be continuous across these primal edges. For instance, when $\sigma = \mathcal{K} | \mathcal{L}$, σ is also the common edge between the half-diamonds $\tilde{\mathcal{D}}_{\mathcal{K}}$ and $\tilde{\mathcal{D}}_{\mathcal{L}}$ and we need to impose the following consistency at the discrete level:

$$\int_{\sigma} (\mathbb{K}|_{\tilde{\mathcal{D}}_{\mathcal{K}}} \nabla u|_{\tilde{\mathcal{D}}_{\mathcal{K}}}) \cdot \mathbf{n}_{\sigma \mathcal{K}} ds = \int_{\sigma} (\mathbb{K}|_{\tilde{\mathcal{D}}_{\mathcal{L}}} \nabla u|_{\tilde{\mathcal{D}}_{\mathcal{L}}}) \cdot \mathbf{n}_{\sigma \mathcal{L}} ds.$$

Therefore, we introduce a new discrete gradient $\nabla^{\tilde{\mathcal{D}}}$, piecewise constant on the half-diamonds. In the definition of this new discrete gradient, we introduce new unknowns u_{σ} which will be algebraically eliminated in the sequel.

DEFINITION 5.1. *The discrete gradient $\nabla^{\tilde{\mathcal{D}}}$ is a mapping from $\mathbb{R}^{\mathcal{T}}$ to $(\mathbb{R}^2)^{\tilde{\mathcal{D}}}$ defined for all $u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ by $\nabla^{\tilde{\mathcal{D}}} u_{\mathcal{T}} = \left(\nabla^{\tilde{\mathcal{D}}} u_{\mathcal{T}} \right)_{\tilde{\mathcal{D}} \in \tilde{\mathcal{D}}}$, where for $\tilde{\mathcal{D}} \in \tilde{\mathcal{D}}$ whose vertices are $x_{\mathcal{K}}, x_{\mathcal{K}^*}$ and $x_{\mathcal{L}^*}$:*

$$\nabla^{\tilde{\mathcal{D}}} u_{\mathcal{T}} = \frac{1}{\sin(\alpha_{\mathcal{D}})} \left(\frac{u_{\sigma} - u_{\mathcal{K}}}{m_{\sigma\mathcal{K}}} \mathbf{n}_{\sigma\mathcal{K}} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{m_{\sigma}} \mathbf{n}_{\sigma^*\mathcal{K}^*} \right)$$

For any $\tilde{\mathcal{D}} \in \tilde{\mathcal{D}}$, let us define by $\mathbb{K}_{\tilde{\mathcal{D}}}$ the mean value of \mathbb{K} over $\tilde{\mathcal{D}}$: $\mathbb{K}_{\tilde{\mathcal{D}}} = \frac{1}{m_{\tilde{\mathcal{D}}}} \int_{\tilde{\mathcal{D}}} \mathbb{K}(s) ds$; $\mathbb{K}_{\tilde{\mathcal{D}}}$ is a positive definite matrix which approaches \mathbb{K} on the half-diamond. To determine the unknowns u_{σ} , we impose, for all $\mathcal{D} = \mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_{int}$ such that $\mathcal{D} = \tilde{\mathcal{D}}_{\mathcal{K}} \cup \tilde{\mathcal{D}}_{\mathcal{L}}$, the following condition:

$$(\mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{K}}} \nabla^{\tilde{\mathcal{D}}_{\mathcal{K}}} u_{\mathcal{T}}) \cdot \mathbf{n}_{\sigma\mathcal{K}} = (\mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{L}}} \nabla^{\tilde{\mathcal{D}}_{\mathcal{L}}} u_{\mathcal{T}}) \cdot \mathbf{n}_{\sigma\mathcal{K}}. \quad (5.1)$$

This relation rewrites

$$\begin{aligned} \left(\left(\frac{\mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{K}}}}{m_{\sigma\mathcal{K}}} + \frac{\mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{L}}}}{m_{\sigma\mathcal{L}}} \right) \mathbf{n}_{\sigma\mathcal{K}} \right) \cdot \mathbf{n}_{\sigma\mathcal{K}} u_{\sigma} = & u_{\mathcal{K}} \frac{(\mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{K}}} \mathbf{n}_{\sigma\mathcal{K}}) \cdot \mathbf{n}_{\sigma\mathcal{K}}}{m_{\sigma\mathcal{K}}} + u_{\mathcal{L}} \frac{(\mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{L}}} \mathbf{n}_{\sigma\mathcal{K}}) \cdot \mathbf{n}_{\sigma\mathcal{K}}}{m_{\sigma\mathcal{L}}} \\ & + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{m_{\sigma}} \left((\mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{L}}} - \mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{K}}}) \mathbf{n}_{\sigma^*\mathcal{K}^*} \right) \cdot \mathbf{n}_{\sigma\mathcal{K}}. \end{aligned}$$

Since the matrices $\mathbb{K}_{\tilde{\mathcal{D}}}$ are positive definite, we get

$$\begin{aligned} u_{\sigma} = & \frac{m_{\sigma\mathcal{L}} u_{\mathcal{K}} (\mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{K}}} \mathbf{n}_{\sigma\mathcal{K}}) \cdot \mathbf{n}_{\sigma\mathcal{K}}}{((\mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{K}}} m_{\sigma\mathcal{L}} + \mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{L}}} m_{\sigma\mathcal{K}}) \mathbf{n}_{\sigma\mathcal{K}}) \cdot \mathbf{n}_{\sigma\mathcal{K}}} + \frac{m_{\sigma\mathcal{K}} u_{\mathcal{L}} (\mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{L}}} \mathbf{n}_{\sigma\mathcal{K}}) \cdot \mathbf{n}_{\sigma\mathcal{K}}}{((\mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{K}}} m_{\sigma\mathcal{L}} + \mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{L}}} m_{\sigma\mathcal{K}}) \mathbf{n}_{\sigma\mathcal{K}}) \cdot \mathbf{n}_{\sigma\mathcal{K}}} \\ & + \frac{m_{\sigma\mathcal{K}} m_{\sigma\mathcal{L}} (u_{\mathcal{L}^*} - u_{\mathcal{K}^*})}{m_{\sigma} ((\mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{K}}} m_{\sigma\mathcal{L}} + \mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{L}}} m_{\sigma\mathcal{K}}) \mathbf{n}_{\sigma\mathcal{K}}) \cdot \mathbf{n}_{\sigma\mathcal{K}}} \left((\mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{L}}} - \mathbb{K}_{\tilde{\mathcal{D}}_{\mathcal{K}}}) \mathbf{n}_{\sigma^*\mathcal{K}^*} \right) \cdot \mathbf{n}_{\sigma\mathcal{K}} \Big]. \quad (5.2) \end{aligned}$$

Furthermore, for all $\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_{ext}$ we take $u_{\sigma} = u_{\mathcal{L}}$. We remark that on the boundary the diamond $\mathcal{D} \in \mathcal{D}_{ext}$ coincide with the half-diamond $\tilde{\mathcal{D}}$ and therefore $\nabla^{\mathcal{D}} u_{\mathcal{T}} = \nabla^{\tilde{\mathcal{D}}} u_{\mathcal{T}}$.

Thanks to the discrete gradient on half-diamonds, we introduce new numerical velocities as follows.

DEFINITION 5.2. *We define a mapping $\mathbf{K}^{\mathcal{D}}$ from $(\mathbb{R}^2)^{\tilde{\mathcal{D}}}$ to $(\mathbb{R}^2)^{\mathcal{D}}$ such that, for all $\xi_{\tilde{\mathcal{D}}} \in (\mathbb{R}^2)^{\tilde{\mathcal{D}}}$,*

$$\begin{aligned} \mathbf{K}^{\mathcal{D}}(\xi_{\tilde{\mathcal{D}}}) &= (\mathbf{K}_{\mathcal{D}}(\xi_{\tilde{\mathcal{D}}}))_{\mathcal{D} \in \mathcal{D}}, \\ \text{where } \mathbf{K}_{\mathcal{D}}(\xi_{\tilde{\mathcal{D}}}) &= \frac{1}{m_{\mathcal{D}}} \sum_{\tilde{\mathcal{D}} \in \tilde{\mathcal{D}}_{\mathcal{D}}} m_{\tilde{\mathcal{D}}} \mathbb{K}_{\tilde{\mathcal{D}}} \xi_{\tilde{\mathcal{D}}} \quad \forall \mathcal{D} \in \mathcal{D}. \end{aligned}$$

REMARK 5.3.

1. *If \mathbb{K} is smooth on the whole domain, we can replace this definition by*

$$\mathbf{K}_{\mathcal{D}}(\xi_{\tilde{\mathcal{D}}}) = \mathbb{K}(x_{\mathcal{D}}) \xi_{\mathcal{D}}, \quad \forall \mathcal{D} \in \mathcal{D}.$$

2. *If \mathbb{K} is piecewise constant on the primal cells, with $\mathbb{K} = K_{\mathcal{K}}$ on the primal cell \mathcal{K} , we can rewrite $\mathbf{K}_{\mathcal{D}}(\xi_{\tilde{\mathcal{D}}}) = \overline{K}_{\mathcal{D}} \xi_{\mathcal{D}}$ where $\overline{K}_{\mathcal{D}}$ is defined by*

$$\begin{aligned}
(\overline{K}_{\mathcal{D}} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma\kappa} &= \frac{m_{\sigma^*} (K_{\kappa} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma\kappa} (K_{\mathcal{L}} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma\kappa}}{m_{\sigma\mathcal{L}} (K_{\kappa} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma\kappa} + m_{\sigma\kappa} (K_{\mathcal{L}} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma\kappa}}, \\
(\overline{K}_{\mathcal{D}} \mathbf{n}_{\sigma^* \kappa^*}) \cdot \mathbf{n}_{\sigma^* \kappa^*} &= \frac{m_{\sigma\mathcal{L}} (K_{\mathcal{L}} \mathbf{n}_{\sigma^* \kappa^*}) \cdot \mathbf{n}_{\sigma^* \kappa^*} + m_{\sigma\kappa} (K_{\kappa} \mathbf{n}_{\sigma^* \kappa^*}, \mathbf{n}_{\sigma^* \kappa^*})}{m_{\sigma^*}} \\
&\quad - \frac{m_{\sigma\kappa} m_{\sigma\mathcal{L}}}{m_{\sigma^*}} \frac{((K_{\kappa} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma^* \kappa^*} - (K_{\mathcal{L}} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma^* \kappa^*})^2}{m_{\sigma\mathcal{L}} (K_{\kappa} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma\kappa} + m_{\sigma\kappa} (K_{\mathcal{L}} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma\kappa}}, \quad (5.3) \\
(\overline{K}_{\mathcal{D}} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma^* \kappa^*} &= \frac{m_{\sigma\mathcal{L}} (K_{\mathcal{L}} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma^* \kappa^*} (K_{\kappa} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma\kappa}}{m_{\sigma\mathcal{L}} (K_{\kappa} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma\kappa} + m_{\sigma\kappa} (K_{\mathcal{L}} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma\kappa}} \\
&\quad + \frac{m_{\sigma\kappa} (K_{\kappa} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma^* \kappa^*} (K_{\mathcal{L}} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma\kappa}}{m_{\sigma\mathcal{L}} (K_{\kappa} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma\kappa} + m_{\sigma\kappa} (K_{\mathcal{L}} \mathbf{n}_{\sigma\kappa}) \cdot \mathbf{n}_{\sigma\kappa}}.
\end{aligned}$$

3. Each interior diamond \mathcal{D} , whose vertices are x_{κ} , x_{κ^*} , $x_{\mathcal{L}}$, $x_{\mathcal{L}^*}$ and whose center is $x_{\mathcal{D}}$, can be splitted into four triangles $(x_{\kappa} x_{\kappa^*} x_{\mathcal{D}})$, $(x_{\kappa^*} x_{\mathcal{L}} x_{\mathcal{D}})$, $(x_{\mathcal{L}} x_{\mathcal{L}^*} x_{\mathcal{D}})$ and $(x_{\mathcal{L}^*} x_{\kappa} x_{\mathcal{D}})$ which can be called “quarter diamond cells”. In [9], the diffusive tensor \mathbb{K} is assumed to be smooth only on the quarter diamond cells and the discrete gradient operator is defined as piecewise constant on the quarter diamond cells. In the case where the diffusive tensor is only discontinuous across primal edges and then smooth on half-diamonds, the discrete gradient operator introduced here coincides with the discrete gradient operator defined in [9].

5.3. The m-DDFV scheme. In this section, we give the modified DDFV scheme called m-DDFV scheme. It differs from (2.8)-(2.13) in the equation of the Darcy velocity (5.5). The scheme for (1.5) writes $\forall 1 \leq n \leq N_T$

$$\operatorname{div}^{\tau} (\mathbf{U}_{\mathcal{D}}^n) = q_{\mathcal{T}}^{+,n} - q_{\mathcal{T}}^{-,n}, \quad (5.4)$$

$$\mathbf{U}_{\mathcal{D}}^n = - \frac{\mathbf{K}_{\mathcal{D}} (\nabla^{\tilde{\mathcal{D}}} p_{\mathcal{T}}^n)}{\mu(c_{\mathcal{D}}^{n-1})}, \quad (5.5)$$

$$\mathbf{U}_{\mathcal{D}}^n \cdot \mathbf{n} = 0, \quad \forall \mathcal{D} \in \mathcal{D}_{ext}, \quad (5.6)$$

$$\sum_{\mathcal{K} \in \mathfrak{M}} m_{\mathcal{K}} p_{\mathcal{K}}^n = \sum_{\mathcal{K}^* \in \mathfrak{M}^*} m_{\mathcal{K}^*} p_{\mathcal{K}^*}^n = 0, \quad (5.7)$$

and the scheme for (1.6) writes $\forall 1 \leq n \leq N_T$

$$\Phi_{\mathcal{T}} \frac{c_{\mathcal{T}}^n - c_{\mathcal{T}}^{n-1}}{\delta t} - \operatorname{div}^{\tau} (\mathbb{D}_{\mathcal{D}} (\mathbf{U}_{\mathcal{D}}^n) \nabla^{\mathcal{D}} c_{\mathcal{T}}^n) + \operatorname{div}^{\tau} (\mathbf{U}_{\mathcal{D}}^n, c_{\mathcal{T}}^n) + q_{\mathcal{T}}^{-,n} c_{\mathcal{T}}^n = q_{\mathcal{T}}^{+,n} c_{\mathcal{T}}^n, \quad (5.8)$$

$$\mathbb{D}_{\mathcal{D}} (\mathbf{U}_{\mathcal{D}}^n) \nabla^{\mathcal{D}} c_{\mathcal{T}}^n \cdot \mathbf{n} = 0, \quad \forall \mathcal{D} \in \mathcal{D}_{ext}. \quad (5.9)$$

5.4. Numerical experiments. In this Section, we now illustrate the behavior of the m-DDFV scheme by applying it to the Peaceman model (1.1)–(1.3). We study the numerical rates of convergence of this scheme for the pressure and the concentration equations and we compare them to the numerical rates obtained with the DDFV scheme. Therefore, we use the sequence of structured triangular meshes introduced in Section 4.

We first consider the Test 1 introduced in Section 4 with a discontinuous permeability. As the viscosity is constant, this test case is an example where the pressure equation is decoupled from the concentration equation. In Table 5.1, we observe that

Refinement level	DDFV		m-DDFV	
	error L^1	order L^1	error L^1	order L^1
1	1.63e-01	-	3.36e-02	-
2	7.95e-02	1.19	9.50e-03	2.10
3	3.83e-02	1.14	2.69e-03	1.97
4	1.78e-02	1.15	7.60e-04	1.90
5	7.64e-03	1.25	2.07e-04	1.92
6	2.55e-03	1.60	4.79e-05	2.13

TABLE 5.1

Test 1. Convergence results of the DDFV and m-DDFV on the pressure p at time $t = 3600$, with $\mathbb{K} = (80\mathbf{1}_{y < 500} + 20\mathbf{1}_{y > 500})\mathbb{I}$.

Refinement level	DDFV		m-DDFV	
	error L^1	order L^1	error L^1	order L^1
1	4.60e-02	-	4.76e-02	-
2	3.26e-02	0.57	3.31e-02	0.61
3	2.02e-02	0.75	2.04e-02	0.76
4	1.10e-02	0.92	1.10e-02	0.92
5	5.13e-03	1.12	5.17e-03	1.12
6	1.80e-03	1.53	1.81e-03	1.53

TABLE 5.2

Test 1. Convergence results of the DDFV and m-DDFV methods on the concentration c , at time $t = 3600$, with $\mathbb{K} = (80\mathbf{1}_{y < 500} + 20\mathbf{1}_{y > 500})\mathbb{I}$.

the m-DDFV scheme is of order 2 for the pressure equation, while the DDFV scheme is only of order 1. However, we see in Table 5.2 that the values of the errors and the order of convergence obtained with the DDFV and the m-DDFV schemes are similar for the concentration equation: the order of convergence is around 1.

Then, we consider the Test 2 introduced in Section 4 with discontinuous permeability. In this test case, the viscosity is no more constant; it induces a strong coupling between the equations for the pressure and the concentration. We see in Tables 5.3 and 5.4 that the results obtained with the m-DDFV scheme are not really better than the results given by the DDFV scheme. Indeed, the values of the error and of the order of convergence are similar for the concentration equation as for the pressure equation. It would be certainly valuable to use a higher-order scheme for the concentration equation in order to keep the benefit of the use of a m-DDFV scheme for the pressure equation.

In order to compare the results given by the m-DDFV method and the Mixed Finite Volume method presented in [10], we made numerical experiments on three different triangular meshes for Test 2 with the same time step $\Delta t = 36$. The first mesh is the structured triangular mesh introduced in Section 4 with $i = 4$. The second mesh is the structured triangular mesh obtained from the first one by a symmetry with respect to $x = 500$ (so that the squares are divided by 2 along the other diagonal). Both meshes have 2048 triangles. The last mesh is an unstructured triangular mesh made of 5449 triangles. For these two last meshes, we have two triangles in the injection corner as in the production corner.

Figure 5.2 presents the level sets of the concentration obtained with both schemes at

Refinement level	DDFV		m-DDFV	
	error L^1	order L^1	error L^1	order L^1
1	3.71e-01	-	2.50e-01	-
2	2.65e-01	0.56	2.10e-01	0.29
3	1.73e-01	0.66	1.60e-01	0.42
4	1.18e-01	0.57	1.14e-01	0.51
5	7.17e-02	0.74	7.03e-02	0.71
6	3.45e-02	1.07	3.41e-02	1.06

TABLE 5.3

Test 2. Convergence results of the DDFV and m-DDFV methods on the pressure p , at time $t = 3600$, with $\mathbb{K} = (80\mathbf{1}_{y < 500} + 20\mathbf{1}_{y > 500})\mathbb{I}$.

Refinement level	DDFV		m-DDFV	
	error L^1	order L^1	error L^1	order L^1
1	4.69e-01	-	4.70e-01	-
2	3.67e-01	0.41	3.71e-01	0.39
3	2.62e-01	0.53	2.64e-01	0.54
4	1.68e-01	0.67	1.68e-01	0.68
5	9.44e-02	0.85	9.45e-02	0.85
6	4.16e-02	1.20	4.16e-02	1.20

TABLE 5.4

Test 2. Convergence results of the DDFV and m-DDFV methods on the concentration c , at time $t = 3600$, with $\mathbb{K} = (80\mathbf{1}_{y < 500} + 20\mathbf{1}_{y > 500})\mathbb{I}$.

two different times (3 and 10 years); the mesh used for the numerical experiment is the first structured triangular mesh. We observe that both schemes have the same qualitative behavior. Figure 5.3 presents the level sets of the concentration obtained with both schemes after 3 years on the second structured triangular mesh and the unstructured triangular mesh. We still observe that both schemes have the same qualitative behavior but we also observe a dependence of the results with respect to the choice of the mesh. Such observations are confirmed by numerical experiments on other meshes.

In conclusion, we have presented a well-posed scheme for the Peaceman model. We have established relevant a priori estimates and proposed numerical experiments, which show good qualitative properties. They also illustrate the numerical convergence of the scheme. From a qualitative point of view, our scheme gives similar results to the results obtained with the Mixed Finite Volume method. The proof of the convergence of the scheme will be achieved in a forthcoming paper. Another future work should be devoted to the study of higher-order discretization of the convection term in the concentration equation.

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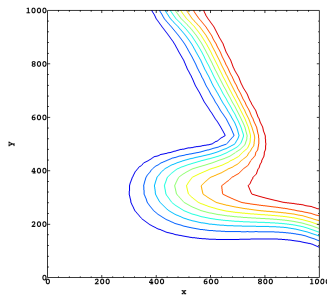
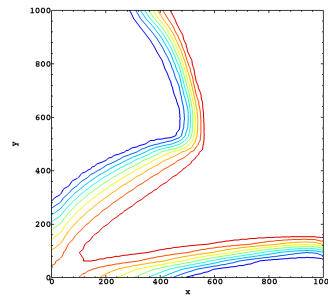
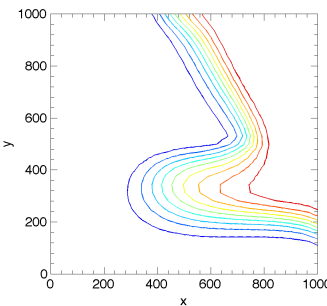
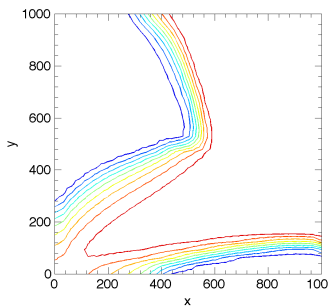
(a) m-DDFV scheme at $t = 3$ years.(b) m-DDFV scheme at $t = 10$ years.(c) MFV scheme at $t = 3$ years.(d) MFV scheme at $t = 10$ years.

FIG. 5.2. *Test2. Comparison of the concentration obtained with the m-DDFV and the MFV schemes on the first structured triangular mesh: level sets of the concentration after 3 and 10 years.*

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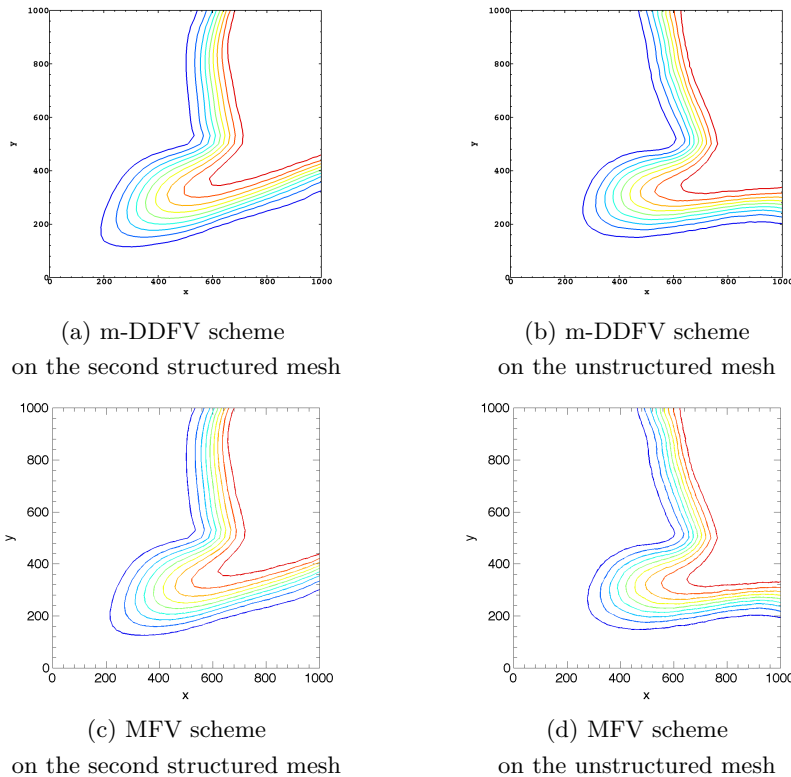


FIG. 5.3. *Test2. Comparison of the concentration obtained with the m -DDFV and the MFV schemes on the second structured triangular mesh and the unstructured triangular mesh: level sets of the concentration after 3 years.*

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