Mackey Functors, Generalized Operads and Analytic Monads

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abstract

Let $\mathbb{K}$ be a field. We denote by $\mathrm{Mod}_{\mathbb{K}}$ the category of $\mathbb{K}$-modules. We study a generalization of cohomological Mackey functors defined on $\mathcal{H}Par_n$, a subcategory of the Hecke category of the symmetric group $\mathbb{S}_n$. We denote the category of cohomological Mackey functors defined on $\mathcal{H}Par_n$ by $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ and the category of strict polynomial functors of degree n by $PolFun_n$. We show that $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ is equivalent to $PolFun_n$. An $\mathbb{M}$-module is a collection of objects in $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ parametrized by $n \in \mathbb{N}$. We denote the category of $\mathbb{M}$-modules by $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$. We introduce two monoidal structures on $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$: the tensor product $\boxtimes$ and the composition $\square$ product. A strict analytic functor is a collection of objects in $PolFun_n$ parametrized by $n \in \mathbb{N}$. We denote the category of strict analytic functors by AnFun. We show that the monoidal structures of tensor product and the composition of endofunctors of $\mathrm{Mod}_{\mathbb{K}}$ induce two monoidal structures on the category of strict analytic functors. We call these structures tensor product and composition of strict analytic functors. We show that the equivalence between $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ and $PolFun_n$ induces an equivalence of symmetric monoidal categories between $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \boxtimes)$ and $(\mathrm{AnFun}, \otimes)$ as well an equivalence of monoidal categories $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \square)$ and $(\mathrm{AnFun}, \circ)$. Based on this new constructions we define the concept of an $\mathbb{M}$-Operad, of an $\mathbb{M}$-PROP, and of their categories of algebras. We give examples of categories of algebras governed by $\mathbb{M}$-operads and $\mathbb{M}$-PROPs.

Introduction

We fix a field $\mathbb{K}$ and a non-negative integer n. We denote by $\mathrm{Mod}_{\mathbb{K}}$ the category of $\mathbb{K}$-modules.

Polynomial functors were introduced by Eilenberg and MacLane in [EML54] in the study of homology of Eilenberg-MacLane spaces $K(\pi, n)$. Strict polynomial functors of degree n are particular polynomial functors endowed with an additional structure. They were introduced by Friedlander and Suslin in [FS97a] in the study of the cohomology of finite group schemes. We denote the category of strict polynomial functors of degree n by $PolFun_n$.

We define the category $\mathcal{H}Par_n$, a generalization of the Hecke category associated to the symmetric group $\mathbb{S}_n$. A Cohomological $\mathcal{H}Par_n$-Mackey functor is an additive functor from $\mathcal{H}Par_n$ to $\mathrm{Mod}_{\mathbb{K}}$. We denote the category of Cohomological $\mathcal{H}Par_n$-Mackey functors by $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$. We show that $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ is equivalent to the category of strict polynomial functors of degree n. Our result explicitly reads:

Theorem A (Theorem 2.18)**.** *There exists an equivalence of categories*

$$ev_n : \mathrm{Mac}^{coh}(\mathcal{H}Par_n) \to PolFun_n.$$

A strict analytic functor F is a collection $\{F_n\}_{n \in \mathbb{N}}$ such that F_n is a strict polynomial functor of degree n for each $n \in \mathbb{N}$. We denote the category of strict analytic functors by AnFun. There exists a forgetful functor $\mathcal{U} : \mathrm{AnFun} \to \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$ from the category of strict analytic functors to the category of endofunctor of $\mathrm{Mod}_{\mathbb{K}}$. The tensor product and the composition in $\mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$ extend along $\mathcal{U}$ and define two monoidal structures on the category of strict analytic functors which we denote $(\mathrm{AnFun}, \otimes, \mathbb{K})$ and $(\mathrm{AnFun}, \circ, \mathrm{Id})$.

An $\mathbb{M}$-module is a collection $\{M_n\}_{n \in \mathbb{N}}$ such that $M_n \in \mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ for each $n \in \mathbb{N}$. We denote the category of $\mathbb{M}$-modules by $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$. We endow $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$ with two monoidal structures, the tensor product $\boxtimes$ of $\mathbb{M}$-modules with unit $\mathbb{K}$ and the composition $\square$ of $\mathbb{M}$-modules with unit $\mathbb{I}$. We show the following result:

Theorem B (Theorem 4.28)**.** *The equivalence of Theorem 2.18 extends to an equivalence of symmetric monoidal categories* $ev : (\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \boxtimes, \mathbb{K}) \to (\mathrm{AnFun}, \otimes, \mathbb{K})$ *as well as to an equivalence of monoidal categories* $ev : (\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \square, \mathbb{I}) \to (\mathrm{AnFun}, \circ, \mathrm{Id})$.

We introduce the category of $\mathbb{M}$-operads, denoted by $\mathbb{M}$-Op. An $\mathbb{M}$-operad is defined as a monoid in the category of $\mathbb{M}$-modules with the monoidal product $\square$. To any $\mathbb{M}$-operad we associate a monad and a category of algebras. An $\mathbb{M}$-operad encodes an algebraic structure with polynomial operations. Any operad P defines an $\mathbb{M}$-operad $S_{-}(P)$ such that the category of P-algebras is isomorphic to the category of $S_{-}(P)$-algebras. Moreover, if the operad P is connected then we associate to it two additional $\mathbb{M}$-operads: $\Lambda_{-}(P)$ and $\Gamma_{-}(P)$. The corresponding monads are isomorphic, respectively to $\Lambda(P,-)$ and $\Gamma(P,-)$ (see Appendix **??**).

Let V be a $\mathbb{K}$-module. We define the $\mathbb{M}$-operad $Poly_V$, it replaces the operad End_V in the following sense:

Theorem C (Theorem 5.8)**.** *Let P be an $\mathbb{M}$-operad and V be a $\mathbb{K}$-module. The set of P-algebra structures on V is in bijection with* $\mathrm{Hom}_{\mathbb{M}\text{-Op}}(P, Poly_V)$.

We generalize the construction of $\mathbb{M}$-modules and we define the category of $\mathbb{M}$-PROPs. To any $\mathbb{M}$-PROP we associate a category of algebras. An $\mathbb{M}$-PROP is an object which encodes algebraic structures with polynomial operations with possible multiple inputs and outputs. The category of $\mathbb{M}$-PROPs generalizes the category of PROPs (see Appendix **??**).

We give examples of categories of algebras governed by $\mathbb{M}$-operads and $\mathbb{M}$-PROPs which are not governed by operads nor by PROPs. More precisely we show that the category of p-restricted Poisson algebras, that appears in the theory of quantization of manifolds in positive characteristic (see [BK08]), is governed by an $\mathbb{M}$-operad. The categories of divided power bi-algebras, related to the category of divided powers Hopf algebras (see [And71]), and p-restricted Lie bi-algebras are governed by $\mathbb{M}$-PROPs.

Contents

In Section 1 we introduce the concept of a cohomological Mackey functor from an admissible collection of subgroups. In Section 2 we recall the definition of a strict polynomial functor and we prove the equivalence of categories between $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ and $PolFun_n$. In Section 3 we introduce the category $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$ and the monoidal structures $\boxtimes$, and $\square$. In Section 4 we recall the definition of a strict analytic functor and we prove the equivalence of monoidal categories between $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$ and AnFun. We conclude with applications to operads and PROPs, in Sections 5 and 6.

1 Admissible cohomological Mackey functors on partition subgroups of the symmetric group

We introduce the definition of a cohomological Mackey functor on an admissible collection of subgroups of a finite group. We apply this general definition to a collection of partition subgroups of the symmetric group $\mathbb{S}_n$.

1.1 Admissible cohomological Mackey functors

We fix a finite group G. We introduce the concept of an admissible collection of subgroups of G. Any admissible collection of subgroups $\mathcal{D}$ defines a category denoted by $\mathcal{HD}$ and a category of cohomological Mackey $\mathcal{HD}$-functors.

Definition 1.1 (The Hecke category $\mathcal{H}G$)**.** *We denote by $\mathcal{H}G$ the full subcategory of $\mathbb{K}[G]$-modules whose objects are permutation modules over $\mathbb{K}[G]$, i.e. it is the category defined as follow:*

1. *the objects are direct sums of $\mathbb{K}[G]$-modules of the form $\mathbb{K}[{}^{G}/_{H}]$ where H is a subgroup of G,*
2. *if $\mathbb{K}[{}^{G}/_{H_1}]$ and $\mathbb{K}[{}^{G}/_{H_2}]$ are two objects of $\mathcal{H}G$ then*

$$\mathrm{Hom}_{\mathcal{H}G}(\mathbb{K}[{}^{G}/_{H_1}], \mathbb{K}[{}^{G}/_{H_2}]) = \mathbb{K}[H_1 \backslash {}^{G}/_{H_2}].$$

From this definition, we see that the category $\mathcal{H}G$ is self dual, with an isomorphism $\mathcal{H}G^{op} \to \mathcal{H}G$ which is the identity map on objects, and which is induced by the inversion of G on morphisms.

Definition 1.2 (Admissible collection)**.** *A collection $\mathcal{D}$ of subgroups of G is admissible if it is closed under intersection and conjugation by elements of G.*

Notations 1.3. *Let G be a finite group, $K \leq H$ be subgroups of G and $g \in G$. We use the following notation*

- $\pi_K^H : {}^{G}\!/_{K} \to {}^{G}\!/_{H}$ *is the projection of cosets,*
- ${}^gH = \{ghg^{-1} | h \in H\}$*, and*
- $H^g = \{g^{-1}hg | h \in H\}$.

We associate a category to any admissible collection.

Definition 1.4 (The category $\mathcal{H}\mathcal{D}$)**.** *Let $\mathcal{D}$ be an admissible collection of subgroups of G. We define the category $\mathcal{H}\mathcal{D}$ to be the full subcategory of $\mathcal{H}G$ with objects $\bigoplus_{i=1}^n \mathbb{K}[{}^{G}\!/_{H_i}]$ where H_i is in $\mathcal{D}$.*

Let us mention that $\mathcal{H}\mathcal{D}$ is self dual (like the Hecke category $\mathcal{H}G$).

For any admissible collection we define a category of cohomological Mackey functors.

Definition 1.5 (The category $\mathrm{Mac}^{coh}(\mathcal{H}\mathcal{D})$)**.** *Let $\mathcal{D}$ be an admissible collection of subgroups of G. The category of cohomological $\mathcal{H}\mathcal{D}$-Mackey functors is the category of $\mathbb{K}$-linear functors from $\mathcal{H}\mathcal{D}$ to $\mathrm{Mod}_{\mathbb{K}}$ with natural transformations. We denote this category by $\mathrm{Mac}^{coh}(\mathcal{H}\mathcal{D})$.*

We present an equivalent definition of cohomological $\mathcal{H}\mathcal{D}$-Mackey functors.

Proposition 1.6. *Let $\mathcal{D}$ be an admissible collection of subgroups of G. A cohomological $\mathcal{H}\mathcal{D}$-Mackey functor is equivalent to the following data assignment: a function $A : \mathcal{D} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$; for any inclusion between elements of $\mathcal{D}$, $H_1 \longrightarrow H_2$, a pair of morphisms $\mathrm{Ind}_{H_1}^{H_2} : A(H_1) \longrightarrow A(H_2)$ and $\mathrm{Res}_{H_1}^{H_2} : A(H_2) \longrightarrow A(H_1)$ and for any element $g \in G$ and H in $\mathcal{D}$ an isomorphism $\mathrm{c}_g : A(H) \longrightarrow A({}^gH)$ such that the following relations are satisfied:*

1. $\mathrm{Ind}_{H_2}^{H_3} \mathrm{Ind}_{H_1}^{H_2} = \mathrm{Ind}_{H_1}^{H_3}$,
2. $\mathrm{Res}_{H_1}^{H_2} \mathrm{Res}_{H_2}^{H_3} = \mathrm{Res}_{H_1}^{H_3}$,
3. $\mathrm{c}_g \mathrm{c}_h = \mathrm{c}_{gh}$,
4. $\mathrm{c}_g \mathrm{Ind}_{H_1}^{H_2} = \mathrm{Ind}_{{}^gH_1}^{{}^gH_2} \mathrm{c}_g$,
5. $\mathrm{c}_g \mathrm{Res}_{H_1}^{H_2} = \mathrm{Res}_{{}^gH_1}^{{}^gH_2} \mathrm{c}_g$,
6. $\mathrm{Res}_J^H \mathrm{Ind}_K^H = \sum_{x \in J \backslash H / K} \mathrm{Ind}_{J \cap {}^xK}^J \mathrm{c}_x \mathrm{Res}_{J^x \cap K}^K$,
7. $\mathrm{Ind}_{H_1}^{H_2} \mathrm{Res}_{H_1}^{H_2} = [H_2 : H_1] \mathrm{Id}_{H_2}$,

for all $H_1, H_2, H_3, H, J, K \in \mathcal{D}$ such that $H_1 \leq H_2 \leq H_3$, and $J, K \leq H$.

Proof: Suppose we have an assignment A of this type. It defines a cohomological $\mathcal{H}\mathcal{D}$-Mackey functor M as follows:

1. let $\mathbb{K}[{}^{G}\!/_{H}]$ be an object of $\mathcal{H}\mathcal{D}$, we set $M(\mathbb{K}[{}^{G}\!/_{H}]) = A(H)$,
2. let $\mathbb{K}[{}^{G}\!/_{H_1}]$ and $\mathbb{K}[{}^{G}\!/_{H_2}]$ be two objects of $\mathcal{H}\mathcal{D}$ and $[g]$ an element of

$$\mathrm{Hom}_{\mathcal{H}\mathcal{D}}(\mathbb{K}[{}^{G}\!/_{H_1}], \mathbb{K}[{}^{G}\!/_{H_2}]),$$

   we set

$$M([g])(x) = \mathrm{Ind}_{H_1^g \cap H_2}^{H_2} \mathrm{Res}_{H_1^g \cap H_2}^{H_1^g} \mathrm{c}_g(x).$$

The statement then follows from the Theorem of Yoshida; see [Yos83, Thm. 4.3]. □

From now on we will define cohomological $\mathcal{HD}$-Mackey functors giving their values on the subgroups in $\mathcal{D}$ and the morphisms $\mathrm{Ind}_{H_1}^{H_2}$, $\mathrm{Res}_{H_1}^{H_2}$ and c_g for all $g \in G$, and $H_1, H_2 \in \mathcal{D}$ such that $H_1 \leq H_2$.

Proposition 1.7. *Let $\mathcal{D}$ be an admissible collection of subgroups of G and $K, H \in \mathcal{D}$. We have that $\mathrm{Hom}_{\mathcal{HD}}({}^{G}/_{K}, {}^{G}/_{K})$ is isomorphic to the $\mathbb{K}$-free module generated by the diagram of the form:*

$$\begin{array}{ccc} & {}^{G}/_{K^g \cap H} & \\ {}^{\pi^K_{L^g} \circ c_g} \swarrow & & \searrow {}^{\pi^H_L} \\ {}^{G}/_{K} & & {}^{G}/_{H} \end{array}$$

where $g \in K\backslash G/H$ and $L = K^g \cap H$.

Moreover, let M be a cohomological $\mathcal{HD}$-Mackey functors and suppose $H \leq K$. We have

$$\mathrm{Res}^K_H = M({}^{G}/_{K} \overset{\pi^K_H}{\leftarrow} {}^{G}/_{H} \overset{\mathrm{Id}}{\rightarrow} {}^{G}/_{H}),$$

$$\mathrm{Ind}^K_H = M({}^{G}/_{H} \overset{\mathrm{Id}}{\leftarrow} {}^{G}/_{H} \overset{\pi^K_H}{\rightarrow} {}^{G}/_{K}),$$

and

$$\mathrm{c}_{g,H} = M({}^{G}/_{H} \overset{\mathrm{Id} \circ c_g}{\leftarrow} {}^{G}/_{H^g} \overset{\mathrm{Id}}{\rightarrow} {}^{G}/_{H}).$$

Proof: It follows directly by Proposition **??**. □

1.2 The collection Par_n

Let n be a non-negative integer, we denote by $\mathbb{S}_n$ the symmetric group of n letters set. In this paper we are interested in cohomological Mackey functors for a particular admissible collection of subgroups of $\mathbb{S}_n$ denoted by Par_n.

Definition 1.8 (The collection Par_n). *We define Par_n to be the collection of $\mathbb{S}_n$-subgroups conjugated to*

$$\mathbb{S}_{r_1} \times \ldots \times \mathbb{S}_{r_t} \hookrightarrow \mathbb{S}_n$$

for some non-negative integers $r_1, \ldots, r_t$ such that $r_1 + \ldots + r_t = n$ where the inclusion is induced by the ordering preserving bijection $\coprod_{i \in \{1,\ldots,t\}} \{1, \ldots, r_i\} \to \{1, \ldots n\}$.

These subgroups of $\mathbb{S}_n$ appear in the literature under the name "Young subgroups".

Notations 1.9. *The elements of Par_n are in bijection with the partitions of the set $\mathbf{n} := \{1, \ldots, n\}$. From now on we identify the subgroups $\pi \in Par_n$ with the partitions of $\mathbf{n}$.*

We denote a partition of $\mathbf{n}$ by $(p_1), \ldots, (p_r)$, where p_i is a subset of $\mathbf{n}$ and $\coprod_{i=1}^r p_i = \mathbf{n}$. We denote by δ_n the discrete partition; i.e. the partition associated to the trivial subgroup $\underbrace{\mathbb{S}_1 \times \ldots \times \mathbb{S}_1}_{n}$.

Proposition 1.10. *The set Par_n is an admissible collection of subgroups of $\mathbb{S}_n$.*

Proof: It is easy to check that the collection Par_n is closed by conjugations and intersections. □

In what follows we consider the Hecke category $\mathcal{H}Par_n$ associated to the admissible collection Par_n.

Example 1.11. *Let V be a vector space endowed with an action of $\mathbb{S}_n$. Since the functors $H_k(-, V)$ and $H^k(-, V)$ are cohomological Mackey functors (See [Yos83], Example 2.1) by restriction they are cohomological $\mathcal{H}Par_n$-Mackey functors.*

2 The equivalence between strict polynomial functors and cohomological $\mathcal{H}Par_n$-Mackey functors

In this section we recall the general theory of strict polynomial functors and we show that their category is equivalent to the category of cohomological $\mathcal{H}Par_n$-Mackey functors.

2.1 Strict polynomial functors

We fix a non-negative integer n. We recall the definition of the category of strict polynomial functors of degree n. This category was introduced by Friedlander and Suslin in [FS97b] for the study of group schemes.

Definition 2.1 (The functor $\Gamma_n(-)$)**.** *The functor $\Gamma_n(-) : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ is defined as follows:*

$$\Gamma_n(V) = (\underbrace{V \otimes \ldots \otimes V}_{n})^{\mathbb{S}_n},$$

where $\underbrace{V \otimes \ldots \otimes V}_{n}$ is endowed with the natural $\mathbb{S}_n$-action induced by permutations.

We set:

$$\Gamma_\pi(V) = (\underbrace{V \otimes \ldots \otimes V}_{n})^{\pi},$$

for any $\pi \in Par_n$.

In what follows we use that these functor preserves filtered colimits. This claim follows from the observation that the tensor powers preserve filtered colimits (see for instance [Fre09, Proposition 1.2.3]) and that finite limits commute with filtered colimits in module categories (see [Bor94, Theorem 2.13.4] for the counterpart of this statement in the category of sets).

Notations 2.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$.*

Definition 2.3 (The category $\Gamma_n \mathrm{Mod}_{\mathbb{K}}$)**.** *We denote by $\Gamma_n \mathrm{Mod}_{\mathbb{K}}$ the category defined by:*

1. *the objects are $\mathbb{K}$-modules,*
2. *if V and W are $\mathbb{K}$-modules then*

$$\mathrm{Hom}_{\Gamma_n \mathrm{Mod}_{\mathbb{K}}}(V, W) = \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V, W)),$$

3. *composition is the following:*

$$\Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(W, U)) \otimes \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V, W)) \longrightarrow \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(W, U) \otimes \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V, W)) \longrightarrow \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V, U)).$$

where the first morphism is given by the natural transformation $\zeta_{A,B} : \Gamma_n(A) \otimes \Gamma_n(B) \to \Gamma_n(A \otimes B)$, and the second is given by the composition in $\mathrm{Mod}_{\mathbb{K}}$.

We have a functor $\gamma_n : \mathrm{Mod}_{\mathbb{K}} \to \Gamma_n \mathrm{Mod}_{\mathbb{K}}$ defined as the identity on the objects and for a morphism $f : X \to Y$ in $\mathrm{Mod}_{\mathbb{K}}$ we have $f \mapsto \gamma_n(f) = \underbrace{f \otimes \cdots \otimes f}_{n} \in \Gamma_n(\mathrm{Hom}_{ModK}(X, Y))$.

Definition 2.4 (Strict polynomial functors)**.** *A strict polynomial functor of degree n is a $\mathbb{K}$-linear functor $F : \Gamma_n \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ such that the functor $\mathcal{U}_n(F) = F \circ \gamma_n : \mathrm{Mod}_{\mathbb{K}} \to \mathrm{Mod}_{\mathbb{K}}$ preserves filtered colimits. We denote the category of strict polynomial functors of degree n by $PolFun_n$. The map $\mathcal{U}_n : F \mapsto F \circ \gamma_n$ induces a functor $\mathcal{U}_n : PolFun_n \to \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$. As a consequence to any strict polynomial functor we associate an endofunctor of the category $\mathrm{Mod}_{\mathbb{K}}$.*

Example 2.5. *The following functors have a natural strict polynomial structure of degree n :*

1. *the n-symmetric powers: S_n,*

2. *the n-divided powers:* Γ_n,

3. *the n-external powers:* Λ_n.

Proposition 2.6. *Let $F : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ be a functor. Providing F with the structure of a strict polynomial functor of degree n amounts to giving a natural transformation*

$$\zeta = \zeta_{X,Y} : \Gamma_n(X) \otimes F(Y) \to F(X \otimes Y),$$

for $X, Y \in \mathrm{Mod}_{\mathbb{K}}$ such that the following diagrams commute:

$$\begin{array}{ccccc}
\Gamma_n(X) \otimes \Gamma_n(Y) \otimes F(Z) & & \xrightarrow{\quad id\otimes\zeta_{Y,Z} \quad} & & \Gamma_n(X) \otimes F(Y \otimes Z) \\
\big\downarrow{\scriptstyle \zeta_{X,Y}\otimes \mathrm{Id}_{F(Z)}} & & & & \big\downarrow{\scriptstyle \zeta_{X,Y\otimes Z}} \\
\Gamma_n(X \otimes Y) \otimes F(Z) & \xrightarrow[\zeta_{X\otimes Y,Z}]{} & F((X \otimes Y) \otimes Z) & \xrightarrow[\cong]{} & F(X \otimes (Y \otimes Z)),
\end{array}$$

and

$$\begin{array}{ccc}
\Gamma_n(\mathbb{K}) \otimes F(X) & \xrightarrow{\zeta_{\mathbb{K},X}} & F(\mathbb{K} \otimes X) \\
& {\scriptstyle \cong}\searrow & \big\downarrow{\scriptstyle \cong} \\
& & F(X).
\end{array}$$

Proof: Suppose we have such natural transformation ζ. We have:

$$\begin{array}{ccc}
\Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(X,Y)) \otimes F(X) & \longrightarrow & F(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(X,Y) \otimes X) \\
& {\scriptstyle \exists\alpha}\searrow & \big\downarrow \\
& & F(Y).
\end{array}$$

and we take the adjoint $\alpha_\sharp : \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(X,Y)) \to \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(F(X), F(Y))$. In the converse direction, we assume F is a strict polynomial functor of degree n. We have $\mathrm{Id}_\sharp : X \to \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(Y, X \otimes Y)$ defined by $\mathrm{Id}_\sharp(x) : y \mapsto x \otimes y$ the adjoint of $\mathrm{Id} : X \otimes Y \to X \otimes Y$. We take

$$\begin{array}{ccc}
\Gamma_n(X) \otimes F(Y) & \longrightarrow & \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(Y, X \otimes Y)) \otimes F(X) \\
& {\scriptstyle \exists\zeta}\searrow & \big\downarrow \\
& & F(X \otimes Y).
\end{array}$$

□

We recall some properties of the category of strict polynomial functors.

Proposition 2.7. *Let $\pi \in Par_n$. The functor $\Gamma_\pi(-) : V \mapsto \Gamma_\pi(V)$ is canonically a strict polynomial functor of degree n. The action is given by the following composition*

$$\Gamma_n(X) \otimes \Gamma_\pi(Y) \to \Gamma_\pi(X) \otimes \Gamma_\pi(Y) \to \Gamma_\pi(X \otimes Y),$$

where the first morphism is the restriction $\Gamma_n(X) \hookrightarrow \Gamma_\pi(X)$. □

Proposition 2.8 (Krause [Kra13]). *The set $\{\Gamma_\pi(-)\}_{\pi \in Par_n}$ is a set of small projective generators for the category $PolFun_n$.* □

We recall a result on the Hom-sets between the projective generators $\Gamma_\pi(-)$ in the category of strict polynomial functors of degree n.

Lemma 2.9. *Let $\pi_1 = (p_1) \dots (p_c)$ and $\pi_2 = (q_1), \dots, (q_l)$ be in Par_n. The set B of $l \times c$ $\mathbb{N}$-matrix such that $\sum_{j \in \{1,\dots,c\}} \alpha_{i,j} = |q_i|$ and $\sum_{i \in \{1,\dots,l\}} \alpha_{i,j} = |p_j|$ is in bijection with the set ${}_{\pi_1}\backslash \mathbb{S}_n /_{\pi_2}$.*

Proof: Let $g \in \mathbb{S}_n$ we define the $l \times c$ Set-matrix $m(g)$ by $m(g)_{i,j} = p_i^g \cap q_j$. We have a function $\mathbb{S}_n \to B$ defined by $g \mapsto M(g)_{i,j} = \{|m(i,j)|\}_{i,j}$. Let g_1 and g_2 in $\mathbb{S}_n$. We have $|M_{g_1}| = |M_{g_2}|$ if and only if there exist $h_1 \in \pi_1$ and $h_2 \in \pi_2$ such that $h_1 g_1 h_2 = g_2$. Thus the map pass to the quotient defining an injective function ${}_{\pi_1}\backslash \mathbb{S}_n /_{\pi_2} \to B$.

For the surjectivity suppose that the elements inside (q_j) are ordered by the usual order for every j. Let $b = b_{i,j} \in B$. We take $(q_i)_b = (q_{i,1}), \dots, (q_{i,c})$ a partition of q_i such that $|q_{i,j}| = b_{i,j}$ and we consider the associated matrix $q_{i,j}$. We consider a permutation σ which map the element of p_j in the elements of $\bigsqcup_i q_{i,j}$. We have that $M(\sigma) = b$. □

Example 2.10. *Let $\pi_1 = (1,3),(2,4)$ and $\pi_2 = (1),(2),(3,4) \in Par_4$. We consider $g = (1,2) \in \mathbb{S}_4$ we have $\pi_1^g = (2,3)(1,4)$ and $m(g)$ is*

$\varnothing$	$\{1\}$
$\{2\}$	$\varnothing$
$\{3\}$	$\{4\}$

.

Definition 2.11. *Let $\pi_1 = (p_1)\dots(p_c)$ and $\pi_2 = (q_1),\dots,(q_l)$ be in Par_n. Let $A = \{\alpha_{i,j}\}$ be a $l \times c$ $\mathbb{N}$-matrix such that $\sum_{j\in\{1,\dots,c\}} \alpha_{i,j} = |q_i|$ and $\sum_{i\in\{1,\dots,l\}} \alpha_{i,j} = |p_j|$. Using the permutation of Lemma 2.9 it defines a morphism:*

$$\gamma_A : \Gamma_{\pi_1}(-) \cong \bigotimes_j \Gamma_{p_j}(-) \longrightarrow \bigotimes_j (\bigotimes_i \Gamma_{\alpha_{i,j}}(-)) \cong \bigotimes_i (\bigotimes_j \Gamma_{\alpha_{i,j}}(-)) \longrightarrow \bigotimes_j \Gamma_{q_j}(-) \cong \Gamma_{\pi_2}(-).$$

We call the morphisms defined in this way "standard morphisms".

Lemma 2.12 (Totaro [Tot97], Krause [Kra14])**.** *Let π_1 and π_2 be in Par_n. The set of standard morphisms of Definition 2.11 forms a basis for the $\mathbb{K}$-module*

$$\mathrm{Hom}_{PolFun_n}(\Gamma_{\pi_1}(-), \Gamma_{\pi_2}(-)).$$

2.2 Cohomological $\mathcal{H}Par_n$-Mackey functors and strict polynomial functors

In what follows we prove the equivalence between $\mathrm{Mac}^{coh}(\mathcal{H}Par_n)$ and $PolFun_n$.

We recall the notion of coend.

Definition 2.13. *Let $\mathfrak{C}$ be a small category enriched over $\mathrm{Mod}_{\mathbb{K}}$ (see [Kel05]). Let $F : \mathfrak{C} \times \mathfrak{C}^{op} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ be a $\mathrm{Mod}_{\mathbb{K}}$-enriched functor (a $\mathbb{K}$-linear functor in the terminology used in the previous sections). A extranatural transformation $g : F \longrightarrow x$ with $x \in \mathrm{Mod}_{\mathbb{K}}$, is a collection $\{g_c : F(c,c) \longrightarrow x\}_{c\in\mathfrak{C}}$ of morphisms in $\mathrm{Mod}_{\mathbb{K}}$, such that the following diagram commutes:*

$$\begin{array}{ccc} F(c,d) \otimes \mathrm{Hom}_{\mathfrak{C}}(c,d) & \longrightarrow & F(c,c) \\ \downarrow & & \downarrow \\ F(d,d) & \longrightarrow & x. \end{array}$$

A coend of F is an object $\int^{c\in\mathfrak{C}} F(c,c)$ in $\mathrm{Mod}_{\mathbb{K}}$ with a extranatural transformation $f : F \longrightarrow \int^{c\in\mathfrak{C}} F(c,c)$ such that any extranatural transformation $g : F \longrightarrow x$ factorizes uniquely through f.

A coend of F is equivalent to a coequalizer of the form:

$$\bigoplus_{c,d\in\mathfrak{C}} F(c,d) \otimes \mathrm{Hom}_{\mathfrak{C}}(c,d) \rightrightarrows \bigoplus_c F(c,c) \to \int^{c\in\mathfrak{C}} F(c,c),$$

see [Kel05] for more details on this definition.

Definition 2.14 (The functor ev_n)**.** *Let M_n be a cohomological $\mathcal{H}Par_n$-Mackey functor. It defines a functor:*

$$M_n(-) : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$$

$$V \mapsto \int^{\pi\in Par_n} M_n(\pi) \otimes \Gamma_\pi(V),$$

where we use that the mapping $\pi \mapsto \Gamma_\pi(V)$ gives a covariant functor $\Gamma_-(V) : \mathcal{H}Par_n \to \mathrm{Mod}_\mathbb{K}$ and we compose this functor with the anti-isomorphism $\mathcal{H}Par_n^{op} \to \mathcal{H}Par_n$ of Definition 1.4 to form the contravariant functor $\Gamma_-(V) : \mathcal{H}Par_n^{op} \to \mathrm{Mod}_\mathbb{K}$ of this coend formula.

This mapping is functorial in M_n, we then have:

$$ev_n : \mathrm{Mac}^{coh}(\mathcal{H}Par_n) \longrightarrow \mathrm{Fun}(\mathrm{Mod}_\mathbb{K}, \mathrm{Mod}_\mathbb{K})$$

$$M_n \mapsto ev_n(M_n)(-).$$

Proposition 2.15. *Let M_n be a cohomological $\mathcal{H}Par_n$-Mackey functor. We have that $ev_n(M_n)$ extends canonically to a strict polynomial functor of degree n.*

Proof: We have $ev_n(M_n) = \int^{\pi \in \mathcal{H}Par_n} M_n(\pi) \otimes \Gamma_\pi(-)$. If V and W are two objects in $\Gamma_n \mathrm{Mod}_\mathbb{K}$ then the morphism $\Gamma_\pi(V) \otimes \mathrm{Hom}_{\Gamma_n \mathrm{Mod}_\mathbb{K}}(V, W) \longrightarrow \Gamma_\pi(W)$ induces a morphism:

$$\Big(\int^{\pi \in \mathcal{H}Par_n} M_n(\pi) \otimes \Gamma_\pi(V)\Big) \otimes \mathrm{Hom}_{\Gamma_n \mathrm{Mod}_\mathbb{K}}(V, W) \longrightarrow \int^{\pi} M_n(\pi) \otimes \Gamma_\pi(W).$$

□

Corollary 2.16. *The functor $ev_n : \mathrm{Mac}^{coh}(\mathcal{H}Par_n) \longrightarrow \mathrm{Fun}(\mathrm{Mod}_\mathbb{K}, \mathrm{Mod}_\mathbb{K})$ extends to a functor*

$$\begin{array}{ccc} & & PolFun_n \\ & \overset{ev_n}{\nearrow} & \downarrow \mathcal{U}_n \\ \mathrm{Mac}^{coh}(\mathcal{H}Par_n) & \xrightarrow[ev_n]{} & \mathrm{Fun}(\mathrm{Mod}_\mathbb{K}, \mathrm{Mod}_\mathbb{K}). \end{array}$$

Proposition 2.17. *Let π_1 and π_2 be partitions of n. We have a natural isomorphism:*

$$\mathrm{Hom}_{PolFun_n}(\Gamma_{\pi_1}(-), \Gamma_{\pi_2}(-)) \cong \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1, \pi_2).$$

Proof: We have to check that 2.9 is compatible with composition. This follows by Proposition 1.7 and the observation that a "standard morphism" is the composition of a permutation with $g \in {}_{\pi_1}\backslash \mathbb{S}_n /_{\pi_2}$, a restriction to $\pi_1^g \cap \pi_2$ and an induction to π_2. □

As a direct consequence we have the following theorem.

Theorem 2.18. *The functor $ev_n : \mathrm{Mac}^{coh}(\mathcal{H}Par_n) \to PolFun_n$ induces an equivalence between the category of cohomological $\mathcal{H}Par_n$-Mackey functors, and the category of strict polynomial functors of degree n.*

Proof: The theorem follows applying Yoneda's Lemma, Proposition 2.8 and Lemma 2.17.

We define explicitly an inverse of ev by using Yoneda's Lemma. Let P be a strict polynomial functor of degree n. We define the cohomological $\mathcal{H}Par_n$-Mackey functor:

$$P(\pi) = \mathrm{Hom}_{PolFun_n}(\Gamma_\pi(-), P).$$

Let π_1, and π_2 be in Par_n such that $\pi_1 \leq \pi_2$, and $\sigma \in Par_n$. We recall that by Lemma 2.17 we have a natural isomorphism $\mathrm{Hom}_{PolFun_n}(\Gamma_{\pi_1}(-), \Gamma_{\pi_2}(-)) \cong \mathbb{K}[{}_{\pi_1}\backslash \mathbb{S}_n /_{\pi_2}]$. We define the morphisms $P(\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1, \pi_2))$ by precomposition with $\mathrm{Hom}_{PolFun_n}(\Gamma_{\pi_1}(-), \Gamma_{\pi_2}(-))$. Using the isomorphism $\mathcal{H}Par_n^{op} \to \mathcal{H}Par_n$ we deduce that the relations of cohomological $\mathcal{H}Par_n$-Mackey functors are satisfied. □

3 The category $\mathrm{Mod}_\mathbb{K}^\mathbb{M}$

The aim of this section is to define the category of $\mathbb{M}$-modules, denoted by $\mathrm{Mod}_\mathbb{K}^\mathbb{M}$, and to introduce the two monoidal structures $(\mathrm{Mod}_\mathbb{K}^\mathbb{M}, \boxtimes, \mathbb{K})$ and $(\mathrm{Mod}_\mathbb{K}^\mathbb{M}, \square, \mathbb{I})$.

3.1 $\mathbb{M}$-modules

We introduce the concept of $\mathbb{M}$-module. It generalizes the definition of $\mathbb{S}$-module (see Appendix **??**).

Definition 3.1 ($\mathbb{M}$-module). *An $\mathbb{M}$-module M is a sequence $\{M_n\}_{n\in\mathbb{N}}$ of cohomological $\mathcal{H}Par_n$-Mackey functors. A morphism between two $\mathbb{M}$-modules $\{M_n\}_{n\in\mathbb{N}}$ and $\{N_n\}_{n\in\mathbb{N}}$ is a sequence of natural transformations $\{f_n : M_n \longrightarrow N_n\}_{n\in\mathbb{N}}$. Their category is denoted by $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$.*

We introduce some special classes of $\mathbb{M}$-modules.

Definition 3.2 (The $\Gamma(M)$ and $S(M)$ $\mathbb{M}$-modules). *Let M be a $\mathbb{S}$-module (see Appendix **??**). We set $\Gamma_n(M)(-) = H^0(-, M(n))$ and we consider the $\mathbb{M}$-module $\Gamma(M)$ defined by the collection of these cohomological Mackey functors. We also set $S_n(M)(-) = H_0(-, M(n))$ and consider the $\mathbb{M}$-module $S(M)$ defined by the collection of these cohomological Mackey functors. Remark that $H^k(-, M(n))$ and $H_k(-, M(n))$ are $\mathbb{M}$-module for all k.*

Definition 3.3 (The trace map). *Let M be a $\mathbb{S}$-module (see Appendix **??**). There exists a natural morphism of $\mathbb{M}$-modules $\mathrm{tr}_M : S(M) \longrightarrow \Gamma(M)$ called trace map defined by: for any $n \in \mathbb{N}$ and any $\pi \in Par_n$ we set $\mathrm{tr}_M(\pi) : S_n(M)(\pi) \to \Gamma_n(M)(\pi)$ as $[x] \mapsto \sum_{\sigma\in\pi} \sigma^* x$.*

3.2 The monoidal structures $\boxtimes$ and $\square$

We introduce the two monoidal structures $(\boxtimes, \mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \mathbb{K})$ and $(\square, \mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \mathbb{I})$.

We recall some properties of coends.

Lemma 3.4 (Fubini Theorem for coends). *Let $\mathcal{A}$ and $\mathcal{B}$ be small categories and $F : (\mathcal{A}\times\mathcal{B})^{op} \times (\mathcal{A}\times\mathcal{B}) \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ be a functor. We have, if the coend exists:*

$$\int^{(A,B)\in\mathcal{A}\times\mathcal{B}} F(A,B,A,B) \cong \int^{A\in\mathcal{A}}\int^{B\in\mathcal{B}} F(A,B,A,B) \cong \int^{B\in\mathcal{B}}\int^{A\in\mathcal{A}} F(A,B,A,B).$$

Lemma 3.5 (coYoneda Lemma for coends). *Let $\mathcal{A}$ be a small category enriched over $\mathrm{Mod}_{\mathbb{K}}$ and $F : \mathcal{A} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ be a functor. We have:*

$$F(-) \cong \int^{A\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(A,-) \otimes F(A).$$

Proof: For more details and proofs see [Kel05, Sec. 3.10]. □

We introduce the two monoidal structures on $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$. They correspond to tensor product and composition.

Definition 3.6 (The product $\boxtimes$). *Let M and N be two $\mathbb{M}$-modules. We set:*

$$(M \boxtimes N)_n(\pi) = \bigoplus_{i+j=n} \int^{\pi_1\times\pi_2\in\mathcal{H}Par_i\times\mathcal{H}Par_j} (M(\pi_1)\otimes N(\pi_2)) \otimes \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\pi_2, \pi).$$

for each $\pi \in Par_n$ and for all $n \in \mathbb{N}$.

The action of $\mathcal{H}Par_n$ is given by the action on $\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1 \times \pi_2, \pi)$ inside the coend.

Proposition 3.7. *Let $\mathbb{K}$ be the following $\mathbb{M}$-module:*

$$\mathbb{K}_i := \begin{cases} \mathbb{K} & i = 0; \\ 0 & i \neq 0. \end{cases}$$

The triple $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \boxtimes, \mathbb{K})$ forms a symmetric monoidal category.

Proof: Let A, B, C be $\mathbb{M}$-modules. We consider the following isomorphism:

$$\begin{aligned}
(A \boxtimes (B \boxtimes C))(\pi) &= \bigoplus_{i+j=n} \int^{\pi_1\times\pi_2 \in \mathcal{H}Par_i\times\mathcal{H}Par_j} A(\pi_1)\otimes(B\boxtimes C)(\pi_2)\otimes \mathrm{Hom}_{Par_n}(\pi_1\times\pi_2,\pi)\\
&= \bigoplus_{i+j=n} \int^{\pi_1\times\pi_2} A(\pi_1)\otimes\Big(\bigoplus_{s+t=j} \int^{\rho_1\times\rho_2\in\mathcal{H}Par_s\times\mathcal{H}Par_t} B(\rho_1)\otimes C(\rho_2)\otimes\\
&\qquad \mathrm{Hom}_{\mathcal{H}Par_j}(\rho_1\times\rho_2,\pi_2)\Big)\otimes \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\pi_2,\pi)\\
&\cong \bigoplus_{i+s+t=n} \int^{\pi_1\times\pi_2\times\rho_1\times\rho_2} A(\pi_1)\otimes B(\rho_1)\otimes C(\rho_2)\otimes \mathrm{Hom}_{\mathcal{H}Par_j}(\rho_1\times\rho_2,\pi_2)\otimes\\
&\qquad \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\pi_2,\pi)\\
&\cong \bigoplus_{i+s+t=n} \int^{\pi_1\times\rho_1\times\rho_2} A(\pi_1)\otimes B(\rho_1)\otimes C(\rho_2)\otimes \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\rho_1\times\rho_2,\pi),
\end{aligned}$$

where we first expand the tensor product and then we use the isomorphisms given by Lemma 3.4 and by Lemma 3.5.

We get the same formula for $((A\boxtimes B)\boxtimes C)(\pi)$ hence we have $A\boxtimes(B\boxtimes C)\cong(A\boxtimes B)\boxtimes C$. For the unit $\eta_A : A\boxtimes\mathbb{K}\to A$ morphism we consider the following isomorphism:

$$\begin{aligned}
A\boxtimes\mathbb{K} &= \bigoplus_{i+j} \int^{\pi_1\times\pi_2\in\mathcal{H}Par_i\times\mathcal{H}Par_j} A(\pi_1)\otimes\mathbb{K}(\pi_2)\otimes \mathrm{Hom}_{Par_n}(\pi_1\times\pi_2,\pi)\\
&= \int^{\pi_1\in\mathcal{H}Par_n} A(\pi_1)\otimes\mathbb{K}\otimes \mathrm{Hom}_{Par_n}(\pi_1,\pi)\cong A(\pi),
\end{aligned}$$

where we use the isomorphism of Lemma 3.5.

For the symmetry isomorphism $\beta_{A,B} : A\boxtimes B\to B\boxtimes A$ we consider the following isomorphism:

$$\begin{aligned}
(A\boxtimes B)(\pi) &= \bigoplus_{i+j=n} \int^{\pi_1\times\pi_2\in\mathcal{H}Par_i\times\mathcal{H}Par_j} A(\pi_1)\otimes B(\pi_2)\otimes \mathrm{Hom}_{Par_n}(\pi_1\times\pi_2,\pi)\\
&\cong \bigoplus_{i+j=n} \int^{\pi_2\times\pi_1\in\mathcal{H}Par_j\times\mathcal{H}Par_i} B(\pi_2)\otimes A(\pi_1)\otimes \mathrm{Hom}_{Par_n}(\pi_2\times\pi_1,\pi) = (A\boxtimes B)(\pi).
\end{aligned}$$

□

Definition 3.8 (The product □)**.** *Let M and N be two $\mathbb{M}$-modules we set:*

$$(M\,\square\,N)_n(\pi) = \bigoplus_{r\in\mathbb{N}}\Big(\int^{\rho\in\mathcal{H}Par_r} M(\rho)\otimes(N^{\boxtimes r}(\pi))^{\rho}\Big),$$

for all $\pi\in Par_n$, where we use that $N^{\boxtimes r}(\pi)$ forms a $\mathbb{K}[\mathbb{S}_r]$-module by the symmetry of the tensor product $\boxtimes$ and again we consider the contravariant functor $(N^{\boxtimes r}(\pi))^{-})$ induced by the duality isomorphism $\mathcal{H}Par_n^{op}\to\mathcal{H}Par_n$.

Let $\mathbb{I}$ be the following $\mathbb{M}$-module:

$$\mathbb{I}_i = \begin{cases}\mathbb{K} & i=1,\\ 0 & i\neq 1.\end{cases}$$

The proof that the triple $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}},\square,\mathbb{I})$ forms a monoidal category is postponed to Theorem 4.28.

4 The equivalence between strict analytic functors and $\mathbb{M}$-modules

In this section we recall the definition of AnFun, the category of strict analytic functors. We prove that the equivalence of Theorem 2.18 extends to a monoidal equivalence between $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$ and AnFun.

4.1 Strict analytic functors

We recall the definition of strict analytic functors and we introduce two monoidal structures.

Definition 4.1 (Strict analytic functor)**.** *A strict analytic functor is a collection $\{F_n\}_{n\in\mathbb{N}}$ where F_n is a strict polynomial functor of degree n. Let $\{F_n\}_{n\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ be strict analytic functors. A morphism of strict analytic functors is a collection $\{f_n\} : \{F_n\}_{n\in\mathbb{N}} \to \{G_n\}_{n\in\mathbb{N}}$ where f_n is a morphism of strict polynomial functors. We denote the category of strict analytic functors by* AnFun*. We accordingly have* $\mathrm{AnFun} = \prod_{n\in\mathbb{N}} PolFun_n$.

Definition 4.2 (The functor $\mathcal{U}$)**.** *We define the functor $\mathcal{U} : \mathrm{AnFun} \to \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$. Let $F = \{F_n\}_{n\in\mathbb{N}}$ be a strict analytic functor we set $\mathcal{U}F = \bigoplus_{n\in\mathbb{N}} \mathcal{U}F_n$. This functor $\mathcal{U} : \mathrm{AnFun} \to \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$ is faithful, because this is clearly the case for each functor $\mathcal{U}_n : PolFun_n \to \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$ in Definition 2.4.*

The category AnFun is equipped with two monoidal structures $(\mathrm{AnFun}, \otimes, \mathbb{K})$ and $(\mathrm{AnFun}, \circ, \mathrm{Id})$.

Definition 4.3 (The product $\otimes$)**.** *Let $F = \{F_n\}_{n\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ be strict analytic functors we set:*

$$(F \otimes G)_n(-) = \bigoplus_{i+j=n} F_i(-) \otimes G_j(-).$$

Let $F = \{F_n\}_{n\in\mathbb{N}}, \{G_n\}_{n\in\mathbb{N}}, A = \{A_n\}_{n\in\mathbb{N}}$, and $B = \{B_n\}_{n\in\mathbb{N}}$ be strict analytic functors and $\{f_n\}_{n\in\mathbb{N}} : F \to A$, $\{g_n\}_{n\in\mathbb{N}} : G \to B$ be strict analytic functor morphisms we set:

$$\{f \otimes g\}_n = \sum_{i+j=n} f_i \otimes g_j.$$

Definition 4.4 (The strict analytic functor $\mathbb{K}$)**.** *We define the strict analytic functor $\mathbb{K} = \{\mathbb{K}_n : \Gamma_n \mathrm{Mod}_{\mathbb{K}} \to \mathrm{Mod}_{\mathbb{K}}\}_{n\in\mathbb{N}}$ such that $\mathbb{K}_0 : \Gamma_0 \mathrm{Mod}_{\mathbb{K}} \to \mathrm{Mod}_{\mathbb{K}}$ is the constant functor $V \mapsto \mathbb{K}$, and $\mathbb{K}_n : \Gamma_n \mathrm{Mod}_{\mathbb{K}} \to \mathrm{Mod}_{\mathbb{K}}$ is the constant functor $V \mapsto 0$ when $n \neq 0$.*

Proposition 4.5. *The triple $(\mathrm{AnFun}, \otimes, \mathbb{K})$ forms a symmetric monoidal category. In particular, for F and G strict analytic functors the collection $F \otimes G = \{(F \otimes G)_n\}_{n\in\mathbb{N}}$ is canonically a strict analytic functor. We moreover have a natural isomorphism $\mathcal{U}(F \otimes G) \cong \mathcal{U}(F) \otimes \mathcal{U}(G)$.*

Proof: We show that $(F \otimes G)_n(-)$ is a strict polynomial functor of degree n using the characterization of Proposition 2.6. We have:

$$\Gamma_n(X) \otimes (F \otimes G)_n(Y) = \bigoplus_{i+j=n} \Gamma_n(X) \otimes F_i(Y) \otimes G_j(Y) \overset{(*)}{\to} \bigoplus_{i+j=n} \Gamma_{\mathbb{S}_i \times \mathbb{S}_j}(X) \otimes F_i(Y) \otimes G_j(Y) \cong$$

$$\bigoplus_{i+j=n} \Gamma_i(X) \otimes F_i(Y) \otimes \Gamma_j(X) \otimes G_j(Y) \to \bigoplus_{i+j=n} F_i(X \otimes Y) \otimes G_j(X \otimes Y),$$

where the morphism $(*)$ is given by the restriction map $\Gamma_n(X) = \Gamma_{\mathbb{S}_n}(X) \to \Gamma_{\mathbb{S}_i \times \mathbb{S}_j}(X) = \Gamma_{\mathbb{S}_i}(X) \otimes \Gamma_{\mathbb{S}_j}(X)$. The unit and the associativity property of this action of $\Gamma_n(X)$ on $(F \otimes G)_n$ follows from the commutativity of the following diagrams:

$$\begin{array}{ccc}
\Gamma_n(\mathbb{K}) & \longrightarrow & \bigoplus_{i+j=n} \Gamma_i(\mathbb{K}) \otimes \Gamma_j(\mathbb{K}) \\
\downarrow & & \downarrow \\
\mathbb{K} & \xrightarrow[=]{} & \mathbb{K},
\end{array}$$

and

$$\begin{array}{ccc} \Gamma_n(X)\otimes\Gamma_n(Y) & \longrightarrow & \bigoplus_{i+j=n}\Gamma_i(X)\otimes\Gamma_j(X)\otimes\Gamma_i(Y)\otimes\Gamma_j(Y) \\ \downarrow & & \downarrow \\ \Gamma_n(X\otimes Y) & \xrightarrow{=} & \bigoplus_{i+j=n}\Gamma_i(X\otimes Y)\otimes\Gamma_j(X\otimes Y). \end{array}$$

The relation $\mathcal{U}(F\otimes G)\cong\mathcal{U}(F)\otimes\mathcal{U}(G)$ follows from the distributivity of tensor product with respect to direct sums.

There are evident isomorphisms:

$$(\mathbb{K}\otimes F)_n(-)\cong F_n(-)\cong(F\otimes\mathbb{K})_n(-),$$

and

$$((A\otimes B)\otimes C)_n(-)\cong\bigoplus_{i+j+k=n}A_i(-)\otimes B_j(-)\otimes C_k(-)\cong(A\otimes(B\otimes C))_n(-),$$

the compatibility of these isomorphisms with polynomial structures follows from the unit, associativity and symmetry of the restriction maps used in our definition. □

We recall some relations between polynomial functors, in the sense of Eilenberg-MacLane (see [EML54]), and strict polynomial functors.

Definition 4.6 (Cross-effect). *Let $F:\mathrm{Mod}_{\mathbb{K}}\longrightarrow\mathrm{Mod}_{\mathbb{K}}$ be a functor. We set*

$$\Delta_0(F)=F(0).$$

Let n be a non-negative integer. We define the nth cross-effect $\Delta_n(F):\mathrm{Mod}_{\mathbb{K}}{}^{\times n}\longrightarrow\mathrm{Mod}_{\mathbb{K}}$ by:

$$\Delta_n(F)(V_1,\ldots,V_n)=Ker(F(V_1\oplus\ldots\oplus V_n)\longrightarrow\bigoplus_{i=1}^{n}F(V_1\oplus\ldots\oplus\overset{i}{0}\oplus\ldots\oplus V_n)).$$

Proposition 4.7. *Let F be an endofunctor of the category $\mathrm{Mod}_{\mathbb{K}}$. We have the following canonical decomposition:*

$$F(V_1\oplus\ldots\oplus V_n)=\bigoplus_{r=1}^{n}\bigoplus_{1\le i_1\le\ldots\le i_r\le n}\Delta_r(F)(V_{i_1},\ldots,V_{i_r}).$$

Proof: We refer to [EML54] for a proof of the statement. □

Definition 4.8 (Homogeneous cross-effect). *We assume that $\pi_i:V_1\oplus\cdots\oplus V_s\longrightarrow V_1\oplus\cdots\oplus V_s$ is the endomorphism of $V_1\oplus\cdots\oplus V_s$ induced by the projection on the summand V_i. For $\alpha_1+\ldots+\alpha_s=n$ we consider the following elements of $\Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V_1\oplus\cdots\oplus V_s,V_1\oplus\cdots\oplus V_s))$:*

$$\gamma_{\alpha_1}(\pi_1)\ldots\gamma_{\alpha_s}(\pi_s)=\sum_{\sigma\in\frac{\mathbb{S}_n}{\mathbb{S}_{\alpha_1}\times\cdots\times\mathbb{S}_{\alpha_s}}}\sigma^*(\pi_1^{\otimes\alpha_1}\otimes\cdots\otimes\pi_s^{\otimes\alpha_s}),$$

where the notation γ_α refers to the fact that $\Gamma(-)$ represents the free divided power algebra. In this expression, we use the action of a set of representative of the class $\sigma\in\frac{\mathbb{S}_n}{\mathbb{S}_{\alpha_1}\times\cdots\times\mathbb{S}_{\alpha_s}}$ in the group of permutation $\mathbb{S}_n$ to shuffle the factors $\pi_i^{\otimes\alpha_i}$ in the tensor product $(\pi_1^{\otimes\alpha_1},\ldots,\pi_s^{\otimes\alpha_s})$. We equivalently have:

$$\gamma_{\alpha_1}(\pi_1)\ldots\gamma_{\alpha_s}(\pi_s)=\sum_{|\{i_k=i\}|=\alpha_i}\pi_{i_1}\otimes\cdots\otimes\pi_{i_n},$$

where the sum runs over the set of n-tuples $(i_1,\ldots,i_n)$ with α_i terms such that $i_k=i$ for each i.

The addition formula for divided powers (see Definition A.37) implies that we have the identity:

$$\gamma_n(\mathrm{Id})=\gamma_n(\pi_1+\cdots+\pi_s)=\sum_{\alpha_1+\cdots+\alpha_s=n}\gamma_{\alpha_1}(\pi_1)\cdots\gamma_{\alpha_s}(\pi_s),$$

in $\Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V_1 \oplus \cdots \oplus V_s, V_1 \oplus \cdots \oplus V_s))$*. From the relation*

$$(\pi_{i_1} \otimes \cdots \otimes \pi_{i_n}) \circ (\pi_{j_1} \otimes \cdots \otimes \pi_{j_n}) = (\pi_{i_1}\pi_{j_1} \otimes \cdots \otimes \pi_{i_n}\pi_{j_n}) = \begin{cases} \pi_{i_1} \otimes \cdots \otimes \pi_{i_n}, & \text{if } (i_1, \dots, i_n) = (j_1, \dots, j_n), \\ 0, & \text{otherwise,} \end{cases}$$

in $Hom(V_1 \oplus \cdots \oplus V_s, V_1 \oplus \cdots \oplus V_s)^{\otimes n}$*, we also deduce that:*

$$(\gamma_{\alpha_1}(\pi_1)\dots\gamma_{\alpha_s}(\pi_s)) \circ (\gamma_{\beta_1}(\pi_1)\dots\gamma_{\beta_s}(\pi_s)) = \begin{cases} \gamma_{\alpha_1}(\pi_1)\dots\gamma_{\alpha_s}(\pi_s), & \text{if } (\alpha_1, \dots, \alpha_n) = (\beta_1, \dots, \beta_n), \\ 0, & \text{otherwise,} \end{cases}$$

we also deduce that these elements $(\gamma_{\alpha_1}(\pi_1)\dots\gamma_{\alpha_s}(\pi_s))$ *forms a complete set of orthogonal idempotents in* $\Gamma_n(Hom(V_1 \oplus \cdots \oplus V_s, V_1 \oplus \cdots \oplus V_s))$*. We refer to [Bou67] for this result.*

Let F *be a strict polynomial functor of degree* n*. We define the homogeneous cross-effect of degrees* $(\alpha_1, \dots, \alpha_s)$ *of* F *as follows:*

$$F^{(\alpha_1,\dots,\alpha_s)}(V_1, \dots, V_s) = Im(F(\gamma_{\alpha_1}(\pi_1)\dots\gamma_{\alpha_s}(\pi_s))).$$

Proposition 4.9. *Let* F *be a strict polynomial functor of degree* n*. We have the following canonical decomposition of the* n*th cross-effect:*

$$\Delta_s(\mathcal{U}(F))(V_1, \dots, V_s) = \bigoplus_{\substack{\alpha_1+\ldots+\alpha_s=n \\ \alpha_i>0}} F^{(\alpha_1,\dots,\alpha_s)}(V_1, \dots, V_s).$$

Proof: We refer to [Bou67] for this statement. □

Remark 4.10. *Let* $F : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ *be a functor. We say that* F *is polynomial, in the sense of Eilenberg-MacLane [EML54], of degree lower or equal to* n *if* $\Delta_{n+1}(F) = 0$*. We say that* F *is of degree* n *if it is of degree lower or equal to* n *and* $\Delta_n \neq 0$*.*

Let F *be a strict polynomial functors of degree* n*. The functor* $\mathcal{U}(F) : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ *is a polynomial functor, in the sense of Eilenberg-MacLane [EML54], of degree lower or equal to* n*. The statement is an obvious consequence of the formula of Proposition 4.9 when* $n > s$*.*

On the other hand the functor $\mathcal{U} : PolFun_n \longrightarrow \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$ *does not preserve the polynomial degree. In general if* F *is a strict polynomial functor of degree* n *then* $\mathcal{U}(F)$ *is a polynomial functor of degree* m *where* $m \leq n$*.*

In what follows, we mainly use the following variation on the results of Proposition 4.7 and Proposition 4.9:

Proposition 4.11 (Bousfield [Bou67])**.** *Let* F *be a strict polynomial functor of degree* n*. We have the isomorphism:*

$$F(V_1 \oplus \cdots \oplus V_s) = \bigoplus_{\substack{\alpha_1+\cdots+\alpha_s=n \\ \alpha_i\geq 0}} F^{(\alpha_1,\dots,\alpha_s)}(V_1, \dots, V_s),$$

where the sum runs over all s*-tuples of non-negative integers* $\alpha_i \in \mathbb{N}$ *such that* $\alpha_1 + \cdots + \alpha_s = n$*.*

Proof: The proof follows directly from the decomposition of $\gamma_n(\mathrm{Id})$ in orthogonal idempotents as in Definition 4.8. □

Proposition 4.12. *Let* F *be a strict polynomial functor of degree* n*.*

1. *If* $\alpha_i = 0$ *for some* i*, then we have*

$$F^{(\alpha_1,\dots,\alpha_i,\dots,\alpha_s)}(V_1, \dots, V_i, \dots, V_s) = F^{(\alpha_1,\dots,\widehat{\alpha_i},\dots,\alpha_s)}(V_1, \dots, \widehat{V_i}, \dots, V_s),$$

2. *if we assume* $V_i = \bigoplus_{j=1}^{k_i} V_i^j$ *for each* i*, then we have*

$$F^{(\alpha_1,\dots,\alpha_i,\dots,\alpha_s)}(V_1,\dots,V_i,\dots,V_s) = \bigoplus_{\substack{(\beta_i^j)\\ \sum_j \beta_i^j=\alpha_i}} F^{(\beta_1^1,\dots,\beta_1^{k_1},\dots,\beta_s^1,\dots,\beta_s^{k_s})}(V_1^1,\dots,V_1^{k_1},\dots,V_s^1,\dots,V_s^{k_s}).$$

3. $\Gamma_n^{(\alpha_1,\dots,\alpha_n)}(X_1,\dots,X_r) = \Gamma_{\alpha_1}(X_1)\otimes\cdots\otimes\Gamma_{\alpha_n}(X_n)$.

Proof: The first relation is trivial. The second relation follows from decomposition rules for divided power operations:

$$\gamma_{\alpha_i}(\pi_i) = \gamma_{\alpha_i}(\pi_i^1+\cdots+\pi_i^{k_i}) = \sum_{\beta_i^1+\cdots+\beta_i^{k_i}=\alpha_i} \gamma_{\beta_i^1}(\pi_i^1)\dots\gamma_{\beta_i^{k_i}}(\pi_i^{k_i}),$$

with the obvious notation for the projectors associated to the direct sum $V_i = \bigoplus_{j=1}^{k_i} V_i^j$. To get the third relation, we use the isomorphism:

$$(X_1\oplus\cdots\oplus X_r)^{\otimes n} \cong \bigoplus_{(i_1,\dots,i_n)} X_{i_1}\otimes\cdots\otimes X_{i_n}.$$

The action of a permutation $\sigma\in\mathbb{S}_n$ on the tensor power maps the term $X_{i_1}\otimes\cdots\otimes X_{i_n}$ associated to $(i_1,\dots,i_n)$ to the term $X_{i_{\sigma(1)}}\otimes\cdots\otimes X_{i_{\sigma(n)}}$ in this sum. We then have the relation:

$$im(\Gamma_n(\gamma_{\alpha_1}(\pi_1)\cdots\gamma_{\alpha_s}(\pi_s))) = (\bigoplus_{|\{i_k=i\}|=\alpha_i} X_{i_1}\otimes\cdots\otimes X_{i_n})^{\mathbb{S}_n},$$

from which the requested identity follows. □

Lemma 4.13. *Let* F *be a strict polynomial functor. We have a natural morphism:*

$$\Gamma_{\alpha_1}(X_1)\otimes\dots\otimes\Gamma_{\alpha_r}(X_r)\otimes F^{(\alpha_1,\dots,\alpha_r)}(Y_1,\dots,Y_r)\longrightarrow F^{(\alpha_1,\dots,\alpha_r)}(X_1\otimes Y_1,\dots,X_r\otimes Y_r).$$

This pairing verifies an evident generalization of unit relation of 2.6 when we suppose $X_i=\mathbb{K}$ *for some* i *as well as an evident generalization of associativity relation of Proposition 2.6 when we compose our pairing to get an operation of the form:*

$$(\Gamma_{\alpha_1}(X_1)\otimes\cdots\otimes\Gamma_{\alpha_r}(X_r))\otimes(\Gamma_{\alpha_1}(Y_1)\otimes\cdots\otimes\Gamma_{\alpha_r}(Y_r))\otimes F^{(\alpha_1,\dots,\alpha_r)}(Z_1,\dots,Z_r) \to F^{(\alpha_1,\dots,\alpha_r)}(X_1\otimes Y_1\otimes Z_1,\dots,X_r\otimes Y_r\otimes Z_r)$$

Proof: The morphism is deduced from the following commutative diagram:

$$\begin{array}{ccc}
\Gamma_n(X_1\oplus\dots\oplus X_r)\otimes F(Y_1\oplus\dots\oplus Y_r) & \xrightarrow{(*)} & F(X_1\otimes Y_1\oplus\dots X_r\otimes Y_r)\\
{\scriptstyle \Gamma_n(\gamma_{\alpha_1}(\pi_1)\dots\gamma_{\alpha_r}(\pi_r))\otimes F(\gamma_{\alpha_1}(\pi_1)\dots\gamma_{\alpha_r}(\pi_r))}\downarrow & & \downarrow{\scriptstyle F(\gamma_{\alpha_1}(\pi_1)\dots\gamma_{\alpha_r}(\pi_r))}\\
\Gamma_{\alpha_1}(X_1)\otimes\dots\otimes\Gamma_{\alpha_r}(X_r)\otimes F^{(\alpha_1,\dots,\alpha_r)}(Y_1,\dots,Y_r) & \dashrightarrow & F^{(\alpha_1,\dots,\alpha_r)}(X_1\otimes Y_1,\dots,X_r\otimes Y_r),
\end{array}$$

where $(*)$ is yielded by the morphism of Proposition 2.6 and the projection morphism

$$(X_1\oplus\cdots\oplus X_r)\otimes(Y_1\oplus\cdots\oplus Y_r)\to X_1\otimes Y_1\oplus\cdots\oplus X_r\otimes Y_r.$$

We apply the idempotent construction of Definition 4.8 to $F(X_1\oplus\cdots\oplus X_r)$, $F(Y_1\oplus\cdots\oplus Y_r)$, and $F(X_1\otimes Y_1\oplus\cdots\oplus X_r\otimes Y_r)$ to get the vertical morphisms of this diagram. We actually consider the corestriction of these idempotent morphisms to their image in our diagram. We check that these idempotents commute with the horizontal morphism $(*)$ to establish the existence of the dotted map of our diagram. We deduce this statement from the associativity of Proposition

2.6. To be more precise if we set $X' = X = X_1 \oplus \cdots \oplus X_r$ and $Y' = Y = Y_1 \oplus \cdots \oplus Y_r$, then this associativity property implies that we have a commutative diagram:

$$\begin{array}{ccc}
\Gamma_n(Hom(X,X')) \otimes \Gamma_n(X) \otimes \Gamma_n(Hom(Y,Y')) \otimes F(Y) & \longrightarrow & \Gamma_n(X') \otimes F(Y') . \\
\downarrow & & \\
\Gamma_n(Hom(X,X') \otimes Hom(Y,Y')) \otimes F(X \otimes Y) & & \downarrow \\
\downarrow & & \\
\Gamma_n(Hom(X \otimes Y, X' \otimes Y')) \otimes F(X \otimes Y) & \longrightarrow & F(X' \otimes Y')
\end{array}$$

We take the morphisms induced by the projection of $X \otimes Y = (\bigoplus_i X_i) \otimes (\bigoplus_j Y_j)$ onto $\bigoplus_i X_i \otimes Y_i$ to prolong the vertical morphism of this diagram. We then get a commutative diagram

$$\begin{array}{ccc}
\Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(X \otimes Y, X' \otimes Y')) \otimes F(X \otimes Y) & \longrightarrow & F(X' \otimes Y') \\
\downarrow & & \downarrow \\
\Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\bigoplus_i X_i \otimes Y_i, \bigoplus_i X'_i \otimes Y'_i)) \otimes F(\bigoplus_i X_i \otimes Y_i) & \longrightarrow & F(\bigoplus_i X'_i \otimes Y'_i)
\end{array}$$

We just take $\gamma_{\alpha_1}(\pi_1)\cdots\gamma_{\alpha_r}(\pi_r) \in \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(X,X))$ and $\gamma_{\alpha_1}(\pi_1)\cdots\gamma_{\alpha_r}(\pi_r) \in \Gamma_n(\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(Y,Y))$ to check our assertion.

The associativity of the pairing for $F^{(\alpha_1,\dots,\alpha_r)}$ comes from the associativity of the pairing for F with the direct sum inside. □

We use the observation of the previous proposition to give a sense to homogeneous cross-effects over a countable sequence of variables:

Definition 4.14. *Let F be a strict polynomial functor of degree n. Let $\underline{X} = (X_0, \dots, X_i, \dots)$ be a collection of modules $X_i \in Mod_K$. Let $\underline{\alpha} = (\alpha_0, \dots, \alpha_n, \dots)$ denote a sequence of non-negative integers $\alpha_i \in N$ such that $\alpha_i = 0$ for all but a finite number of indices i and $\sum_i \alpha_i = n$. Let $i_1 < \cdots < i_r$ be the collection of these indices $i = i_k$ such that $\alpha_i > 0$. We set:*

$$F^{\underline{\alpha}}(\underline{X}) = F^{(\alpha_{i_1},\dots,\alpha_{i_r})}(X_{i_1}, \dots, X_{i_r}).$$

We then have the following generalization of the result of Proposition 4.11:

Proposition 4.15. *Let F be a strict polynomial functor of degree n. Let $\underline{X} = (X_0, \dots, X_i, \dots)$ be a collection of modules $X_i \in Mod_K$. We have the isomorphism:*

$$F(X_0 \oplus \cdots \oplus X_i \oplus \dots) = \bigoplus_{\underline{\alpha}} F^{\underline{\alpha}}(\underline{X}),$$

where the sum runs over all the sequences of non-negative integers $\underline{\alpha} = (\alpha_0, \dots, \alpha_i, \dots)$ which satisfy the constraints of the previous definition.

Proof: The statement follows from the fact that F commutes with the filtered colimits (see Definition 2.4) □

Definition 4.16. *Let $(a_1, \dots, a_s)$ be any collection of non-negative integers $a_i \geq 0$. Let $n = a_1 + \cdots + a_s$. For an analytic functor $F = (F_n)_{n \in N}$, we set $F^{(a_1,\dots,a_s)} = F_n^{(a_1,\dots,a_s)}$, where we consider the homogeneous cross effect of the component of F of degree $n = a_1 + \cdots + a_s$. Let $\underline{\alpha} = (\alpha_0, \dots, \alpha_i, \dots)$ be any sequence of non-negative integers such that $\alpha_i = 0$ for all but a finite number of indices $i \geq 0$. Let $n = \sum_i \alpha_i$. We also set $F^{\underline{\alpha}} = F_n^{\underline{\alpha}}$, where we use the construction of Definition 4.14 for the component of F of degree $n = \sum_i \alpha_i$. The formulas of Proposition 4.11 and of Proposition 4.15 have an obvious generalization for analytic functors (we just forget about the constraints $\sum_i \alpha_i = n$ in this case).*

Proposition 4.17. *Let $F = \{F_n\}_{n\in\mathbb{N}}$ and $G = \{G_n\}_{n\in\mathbb{N}}$ be two strict analytic functors. The composition functor $\mathcal{U}F \circ \mathcal{U}G : \mathrm{Mod}_{\mathbb{K}} \longrightarrow \mathrm{Mod}_{\mathbb{K}}$ has a natural structure of strict analytic functor such that:*

$$(F \circ G)_n = \bigoplus_{\substack{s \\ 1\leq t\leq s}} \bigoplus_{\substack{0\leq i_1<\ldots<i_t \\ \alpha_{i_1}+\ldots+\alpha_{i_t}=s \\ i_1\alpha_{i_1}+\ldots+i_t\alpha_{i_t}=n}} F_s^{(\alpha_{i_1},\ldots,\alpha_{i_t})}(G_{i_1},\ldots,G_{i_t}).$$

Proof: Proposition 4.15 implies that the functor $\mathcal{U}(F) \circ \mathcal{U}(G)(X)$ is given by the sum of the expression of the statement. The structure is given by the composition of the following morphisms:

$$\Gamma_n(X) \otimes F_s^{(\alpha_{i_1},\ldots,\alpha_{i_t})}(G_{i_1}(Y),\ldots,G_{i_t}(Y))$$
$$(1)\downarrow$$
$$\Gamma_{\alpha_{i_1}}(\Gamma_{i_1}(X)) \otimes \ldots \otimes \Gamma_{\alpha_{i_t}}(\Gamma_{i_t}(X)) \otimes F_s^{(\alpha_{i_1},\ldots,\alpha_{i_t})}(G_{i_1}(Y),\ldots,G_{i_t}(Y))$$
$$(2)\downarrow$$
$$F_s^{(\alpha_{i_1},\ldots,\alpha_{i_t})}(\Gamma_{i_1}(X)\otimes G_{i_1}(Y),\ldots,\Gamma_{i_t}(X)\otimes G_{i_t}(Y))$$
$$(3)\downarrow$$
$$F_s^{(\alpha_{i_1},\ldots,\alpha_{i_t})}(G_{i_1}(X\otimes Y),\ldots,G_{i_t}(X\otimes Y)),$$

To define our map (1), we use that any composite $\Gamma_k(\Gamma_l(X))$ is identified with the submodule of $X^{\otimes kl}$ spanned by the tensors which are invariant under a certain subgroup of S_{kl}, denoted by $S_k \wr S_l$, and which is classically called the wreath product in the literature. We then have $\Gamma_{\alpha_1}(\Gamma_{i_1}(X)) \otimes \cdots \otimes \Gamma_{\alpha_r}(\Gamma_{i_r}(X)) = (X^{\otimes n})^{S_{\alpha_1}\wr S_{i_1}\times\cdots\times S_{\alpha_r}\wr S_{i_r}}$, and morphism (1) is given by the obvious embedding $\Gamma_n(X) = (X^{\otimes n})^{S_n} \hookrightarrow (X^{\otimes n})^{S_{\alpha_1}\wr S_{i_1}\times\cdots\times S_{\alpha_r}\wr S_{i_r}}$. The morphism (2) is the morphism of Lemma 4.13, and the morphism (3) is induced by the morphism of Proposition 2.6. □

Definition 4.18 (The product $\circ$)**.** *We define the product $\circ$ on* AnFun *by the construction of Proposition 4.17. It is compatible with the usual composition of functors in the sense that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{AnFun}\times\mathrm{AnFun} & \xrightarrow{\quad\circ\quad} & \mathrm{AnFun} \\ {\scriptstyle \mathcal{U}\times\mathcal{U}}\downarrow & & \downarrow{\scriptstyle \mathcal{U}} \\ \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}},\mathrm{Mod}_{\mathbb{K}})\times\mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}},\mathrm{Mod}_{\mathbb{K}}) & \xrightarrow[\circ]{} & \mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}},\mathrm{Mod}_{\mathbb{K}}). \end{array}$$

Lemma 4.19. *Let F, G be analytic functors. We use the short notation $\underline{X} = (X_1,\ldots,X_r)$ for any r-tuple of K-modules X_i. We also use the short notation $\underline{b}$ for any collection $\underline{b} = (b_1,\ldots,b_r) \in \mathbb{N}^r$ and we set $\Gamma_{\underline{b}}(\underline{X}) = \bigotimes_{i=1}^r \Gamma_{b_i}(X_i)$ for short. We equip the set of collections $\mathbb{N}^r$ with a total ordering and we fix $\underline{c} = (c_1,\ldots,c_r) \in \mathbb{N}^r$.*

1. *We have:*

$$(F\circ G)^{(c_1,\ldots,c_r)}(\underline{X}) = \bigoplus_{\substack{\underline{b}^1<\cdots<\underline{b}^l \\ a_i>0 \\ \sum_i a_i b^i_j = c_j\,(\forall j)}} F^{(a_1,\ldots,a_l)}(G^{\underline{b}^1}(\underline{X}),\ldots,G^{\underline{b}^l}(\underline{X})),$$

where the sum runs over all sequences $(a_1,\ldots,a_l)$, $l\geq 0$, of positive integers $a_i > 0$, and over all ordered sequences of collections $\underline{b}^1 < \cdots < \underline{b}^l$ such that we have $\sum_i a_i b^i_j = c_j$, for all $j \in \{1,\ldots,r\}$.

2. *For this object $(F\circ G)^{\underline{c}}(-)$, the pairing of Lemma 4.13 is given by a composite of the form:*

$$\begin{aligned}\Gamma_{\underline{c}}(\underline{X})&\otimes(F\circ G)^{\underline{c}}(\underline{Y})\to\\ &\bigoplus_{\substack{\underline{b}^1<\dots<\underline{b}^l\\ a_i>0\\ \sum_i a_i b^i_j=c_j(\forall j)}}\Gamma_{a_1}(\Gamma_{\underline{b}^1}(\underline{X}))\otimes\dots\otimes\Gamma_{a_l}(\Gamma_{\underline{b}^l}(\underline{X}))\otimes F^{(a_1,\dots,a_l)}(G^{\underline{b}^1}(\underline{Y}),\dots,G^{\underline{b}^l}(\underline{Y}))\\ &\to\bigoplus_{\substack{\underline{b}^1<\dots<\underline{b}^l\\ a_i>0\\ \sum_i a_i b^i_j=c_j(\forall j)}}F^{(a_1,\dots,a_l)}(\Gamma_{\underline{b}^1}(\underline{X})\otimes G^{\underline{b}^1}(\underline{Y}),\dots,\Gamma_{\underline{b}^l}(\underline{X})\otimes G^{\underline{b}^l}(\underline{Y}))\\ &\to\bigoplus_{\substack{\underline{b}^1<\dots<\underline{b}^l\\ a_i>0\\ \sum_i a_i b^i_j=c_j(\forall j)}}F^{(a_1,\dots,a_l)}(G^{\underline{b}^1}(\underline{X}\otimes\underline{Y}),\dots,G^{\underline{b}^l}(\underline{X}\otimes\underline{Y})),\end{aligned}$$

where we use the notation $\underline{Y}=(Y_1,\dots,Y_r)$ for another r-tuple of variables, and we set $\underline{X}\otimes\underline{Y}=(X_1\otimes Y_1,\dots,X_r\otimes Y_r)$. In this composite, the first morphism is given term-wise by a canonical inclusion $\Gamma_{\underline{c}}(\underline{X})\hookrightarrow\Gamma_{a_1}(\Gamma_{\underline{b}^1}(\underline{X}))\otimes\dots\otimes\Gamma_{a_l}(\Gamma_{\underline{b}^l}(\underline{X}))$, and the next morphisms are given by the pairing of Lemma 4.13 for the functors F and G.

Proof: We have by definition:

$$(F\circ G)_n(X_1\oplus\dots\oplus X_r)=\bigoplus_{\substack{n_1<\dots<n_l\\ \alpha_i>0\ (\forall i)\\ \sum n_i\alpha_i=n}}F^{(\alpha_1,\dots,\alpha_l)}(G_{n_1}(X_1+\dots+X_r),\dots,G_{n_l}(X_1+\dots+X_r)).$$

Let $\pi_i:X_1\oplus\dots\oplus X_r\to X_1\oplus\dots\oplus X_r$ be the morphism given by the projection onto the summand X_i in the sum $X=X_1\oplus\dots\oplus X_r$. For any collection $\underline{b}=(b_1,\dots,b_r)$, we set $\gamma_{\underline{b}}(\underline{\pi})=\prod_{i=1}^r\gamma_{b_i}(\pi_i)$ for short.

We use the expansion

$$G_{n_i}(X_1+\dots+X_r)=\bigoplus_{\beta_1+\dots+\beta_r=n_i}G^{(\beta_1,\dots,\beta_r)}(\underline{X})$$

of Proposition 4.13. We adopt the short notation $\Pi^{\underline{\beta}}=\gamma_{\underline{\beta}}(\underline{\pi})$ for the morphism which induces the projection onto the summand $G^{\underline{\beta}}(\underline{X})$ in this sum, where we still write $\underline{\beta}=(\beta_1,\dots,\beta_r)$ for short. We also use the notation $|\underline{\beta}|=\beta_1+\dots+\beta_r$ for any collection $\underline{\beta}=(\beta_1,\dots,\beta_r)$ in what follows.

We aim to determine the image of the element $\gamma_{\underline{c}}(\underline{\pi})\in\Gamma_n(X)$ under the morphism $\Delta:\Gamma_n(X)\to\Gamma_{\alpha_1}(\Gamma_{n_1}(X))\otimes\dots\otimes\Gamma_{\alpha_r}(\Gamma_{n_r}(X))$ which we use in the construction of Proposition 4.17. We explicitly get:

$$\Delta(\gamma_{\underline{c}}(\underline{\pi}))=\sum_{\substack{\underline{b}^{i,1}<\dots<\underline{b}^{i,k_i}\\ s.t.\ |\underline{b}^{i,j}|=n_i\ (\forall i,j)\\ a_i^1,\dots,a_i^{k_i}>0\\ s.t.\ a_i^1+\dots+a_i^{k_i}=\alpha_i\ (\forall i)\\ \sum_{ij}a_i^j b_s^{i,j}=c_s\ (\forall s)}}\underbrace{\gamma_{a_1^1}(\Pi^{\underline{b}^{1,1}})\dots\gamma_{a_1^{k_1}}(\Pi^{\underline{b}^{1,k_1}})}_{\in\Gamma_{\alpha_1}(\Gamma_{n_1}(X))}\otimes\dots\otimes\underbrace{\gamma_{a_l^1}(\Pi^{\underline{b}^{r,1}})\dots\gamma_{a_l^{k_l}}(\Pi^{\underline{b}^{r,k_l}})}_{\in\Gamma_{\alpha_l}(\Gamma_{n_l}(X))},\quad(*)$$

where the sum runs over collections of positive integers $a_i^1,\dots,a_i^{k_i}>0$, $k_i\geq 1$, $i=1,\dots,r$, and over sequences $\underline{b}^{i,1}<\dots<\underline{b}^{i,k_i}$ of collections $\underline{b}^{i,j}=(b_1^{i,j},\dots,b_r^{i,j})$ which satisfy the constraints given in our expression. We put off the verification of this identity until the end of this proof.

We deduce from this result that we have an identity:

$$(F\circ G)^{(c_1,\dots,c_r)}(\underline{X}) = \bigoplus_{\substack{n_1<\dots<n_l\\ \alpha_i>0\,(\forall i)\\ \sum n_i\alpha_i=n}}\left(\bigoplus_{\substack{\underline{b}^{i,1}<\dots<\underline{b}^{i,k_i}\\ s.t.\,|\underline{b}^{i,j}|=n_i\,(\forall i,j)\\ a_i^1,\dots,a_i^{k_i}>0\\ s.t.\,a_i^1+\dots+a_i^{k_i}=\alpha_i\,(\forall i)\\ \sum_{ij}a_i^j b_s^{i,j}=c_s\,(\forall s)}} F^{a_1^1,\dots,a_1^{k_1},\dots,a_l^1,\dots,a_l^{k_l}}(G^{\underline{b}^{1,1}}(\underline{X}),\dots,G^{\underline{b}^{1,k_1}}(\underline{X}),\dots,G^{\underline{b}^{l,1}}(\underline{X}),\dots,G^{\underline{b}^{l,k_l}}(\underline{X}))\right),$$

and we use a straightforward re-indexing of the direct sum which we get in this formula to get the decomposition of the lemma.

The second assertion of the lemma follows from a straightforward expansion of the definition of our pairing in Proposition 4.12 for objects of the form $F^{(\alpha_1,\dots,\alpha_l)}(G_{n_1}(X_1+\dots+X_r),\dots,G_{n_l}(X_1+\dots+X_r))$ and from the expansion of our pairing for the objects $G^{\underline{b}}(\underline{X})$ in Lemma 4.13. We also use that these constructions are compatible with the isomorphisms of Proposition 4.12 which we use to get the expansion of the first assertion of this lemma.

We now explain the proof of Formula $(*)$. We argue as follows. We use a scalar extension $\mathbb{K}[t_1,\dots,t_r]\otimes_{\mathbb{K}}-$, where $(t_1,\dots,t_r)$ denote formal variables and we work in $\mathbb{K}[t_1,\dots,t_r]\otimes_{\mathbb{K}}\Gamma_n(Hom_{\mathrm{Mod}_{\mathbb{K}}}(X,X)) = \Gamma_n(\mathbb{K}[t_1,\dots,t_r]\otimes_{\mathbb{K}} Hom_{\mathrm{Mod}_{\mathbb{K}}}(X,X))$. We have the formula $\gamma_n(t_1\pi_1+\dots+t_r\pi_r) = \sum_{m_1+\dots+m_r=n}\gamma_{m_1}(\pi_1)\dots\gamma_{m_r}(\pi_r)t_1^{m_1}\dots t_r^{m_r}$ by properties of divided powers (see Definition A.37). We can accordingly identify $\gamma_{\underline{c}}(\underline{\pi})$ with the coefficient of $t^{\underline{c}} = t_1^{c_1}\dots t_r^{c_r}$ in the expansion of $\gamma_n(t_1\pi_1+\dots+t_r\pi_r)$. We use that for an element of this form $\gamma_n(\phi)$, where $\phi = t_1\pi_1+\dots+t_r\pi_r$, we have the formula $\Delta(\gamma_n(\phi)) = \gamma_{\alpha_1}(\gamma_{n_1}(\phi))\otimes\dots\otimes\gamma_{\alpha_r}(\gamma_{n_r}(\phi))$ in $\Gamma_{\alpha_1}(\Gamma_{n_1}(X))\otimes\dots\otimes\Gamma_{\alpha_r}(\Gamma_{n_r}(X))$. The terms of Formula $(*)$ correspond to the coefficients of the monomial $t_1^{c_1}\dots t_r^{c_r}$ when we use the properties of the divided powers to expand the factors $\gamma_{\alpha_i}(\gamma_{n_i}(\phi)) = \gamma_{\alpha_i}(\gamma_{n_i}(t_1\pi_1+\dots+t_r\pi_r))$ in this tensor product. □

Lemma 4.20. *Let F be an analytic functor. We have an isomorphism $Id\circ F\simeq F\simeq F\circ Id$ in the category of analytic functors which realizes the obvious identity $\mathrm{Id}\circ\mathcal{U}(F)=\mathcal{U}(F)=\mathcal{U}(F)\circ\mathrm{Id}$ in the category of ordinary functors.*

Let A,B,C be analytic functors. We have an isomorphism $(A\circ B)\circ C\simeq A\circ(B\circ C)$ in the category of analytic functors which realizes the obvious identity $(\mathcal{U}(A)\circ\mathcal{U}(B))\circ\mathcal{U}(C) = \mathcal{U}(A)\circ(\mathcal{U}(B)\circ\mathcal{U}(C))$ in the category of ordinary functors.

Proof: The verification of the unit relation is easy and we focus on the proof of the associativity relation.

We use the following conventions in this proof. We set $\underline{F}(X) = (F_0(X),\dots,F_n(X),\dots)$ for the sequence of modules $F_n(X)$ which we obtain by taking the image of a module X under the components of an analytic functor $F_n\in AnPol_n$. For a sequence $\underline{\alpha}=(\alpha_0,\dots,\alpha_n,\dots)$ such that $\alpha_i=0$ for all but a finite number of indices i, we also set $w(\alpha)=\sum_i i\alpha_i$.

We have a straightforward generalization of the result of the previous lemma in the case where $\underline{X}$ is a countable sequence of modules $\underline{X}=(X_0,\dots,X_n,\dots)$. We then assume that the set of sequences $\underline{b}=(b_0,\dots,b_i,\dots)$ such that $b_i=0$ for all but a finite number of indices i is equipped with a total ordering such that $\underline{b}^1<\underline{b}^2$ if we have $\sum_i b_i^1<\sum_i b_i^2$. We get:

$$(A\circ B)^{\underline{c}}(\underline{X}) = \bigoplus_{\substack{\underline{b}^1<\dots<\underline{b}^l\\ a_i>0\\ \sum_i a_i b_j^i=c_j\,(\forall j)}} A^{(a_1,\dots,a_l)}(B^{\underline{b}^1}(\underline{X}),\dots,B^{\underline{b}^l}(\underline{X})),$$

where the sum runs over all sequences $(a_1,\dots,a_l)$, $l\geq 0$, of positive integers $a_i>0$, and over all ordered sequences of collections $\underline{b}^1<\dots<\underline{b}^l$ such that we have $\sum_i a_i b_j^i = c_j$, for all $j\in\{1,\dots,r\}$.

We use this identity to determine the expansion of $(A \circ B) \circ C$. We explicitly have:

$$
\begin{aligned}
((A \circ B) \circ C)_n(X) &= \bigoplus_{\underline{c}\, s.t.\, w(\underline{c})=n} (A \circ B)^{\underline{c}}(\underline{C}(X)) \\
&= \bigoplus_{\substack{\underline{b}^1 < \cdots < \underline{b}^l \\ a_i > 0 \\ \sum_i a_i b^i_j j = n}} A^{(a_1,\dots,a_l)}(B^{\underline{b}^1}(\underline{C}(X)), \dots, B^{\underline{b}^l}(\underline{C}(X))).
\end{aligned}
$$

We also get that the pairing $\Gamma_n(X) \otimes ((A \circ B) \circ C)_n(Y) \to ((A \circ B) \circ C)_n(X \otimes Y)$ which we associate to the composite functor $((A \circ B) \circ C)$ is carried to the direct sum of the morphisms

$$
\Gamma_n(X) \otimes A^{(a_1,\dots,a_l)}(B^{\underline{b}^1}(\underline{C}(Y)), \dots, B^{\underline{b}^l}(\underline{C}(Y))) \to A^{(a_1,\dots,a_l)}(B^{\underline{b}^1}(\underline{C}(X \otimes Y)), \dots, B^{\underline{b}^l}(\underline{C}(X \otimes Y)))
$$

which we obtain by using the operation of the previous lemma for the composite $A \circ B$, and by using the pairing $\zeta_{X,Y} : \Gamma_i(X) \otimes C_i(Y) \to C_i(X \otimes Y)$ associated to each functor $C_i(-)$ inside the functors $B^{\underline{b}^j}$.

We have on the other hand:

$$
(A \circ (B \circ C))_n(X) = \bigoplus_{\substack{n_1 < \cdots < n_r \\ \alpha_1, \dots, \alpha_r > 0 \\ \alpha_1 n_1 + \cdots + \alpha_r n_r = n \\ r \geq 0}} A^{(\alpha_1,\dots,\alpha_r)}((B \circ C)_{n_1}(X), \dots, (B \circ C)_{n_r}(X)).
$$

We then use the expression of each $(B \circ C)_{n_i}(X)$ as a direct sum of cross-effects in Proposition 4.17, and the result of Proposition 4.12 to get the identity:

$$
\begin{aligned}
(A \circ (B \circ C))_n(X) = \bigoplus_{\substack{n_1 < \cdots < n_r \\ \alpha_1, \dots, \alpha_r > 0 \\ \alpha_1 n_1 + \cdots + \alpha_r n_r = n \\ r \geq 0}} \Bigg(\bigoplus_{\substack{\underline{b}^1_1 < \cdots < \underline{b}^{l_1}_1 < \cdots < \underline{b}^1_1 < \cdots < \underline{b}^{l_r}_1 \\ w(\underline{b}^j_i) = n_i\ (\forall j) \\ a^j_i > 0, \sum_j a^j_i = \alpha_i}} \\
A^{(a^1_1,\dots,a^{l_1}_1,\dots,a^1_r,\dots,a^{l_r}_r)}(B^{\underline{b}^1_1}(\underline{C}(\underline{X})), \dots, B^{\underline{b}^{l_1}_1}(\underline{C}(\underline{X})), \dots, B^{\underline{b}^1_r}(\underline{C}(\underline{X})), \dots, B^{\underline{b}^{l_r}_r}(\underline{C}(\underline{X})))\Bigg).
\end{aligned}
$$

We use a straightforward re-indexing operation in this sum to retrieve the expression of $((A \circ B) \circ C)_n(X)$. We can also check by using the correspondence of Lemma 4.13 and of Proposition 4.17 inside each input of the functor $A^{(\alpha_1,\dots,\alpha_r)}(-,\dots,-)$ that the pairing $\Gamma_n(X) \otimes (A \circ (B \circ C))_n(Y) \to (A \circ (B \circ C))_n(X \otimes Y)$ which we obtain for this expression of the composite $(A \circ (B \circ C))_n(X)$ agrees with the pairing which we obtain for the composite $((A \circ B) \circ C)_n(X)$.

We conclude that we have an isomorphism of strict polynomial functor $((A \circ B) \circ C)_n \simeq (A \circ (B \circ C))_n$, for each $n \in N$. □

Proposition 4.21. *The triple* $(\mathrm{AnFun}, \circ, \mathrm{Id})$ *forms a monoidal category.*

Proof: This statement follows from the result of the previous lemma. Let us simply mention that our structure isomorphisms fulfil the coherence constraints of monoidal categories since we observe that these isomorphisms correspond to the obvious unit and associativity identities of the composition in the category of functors and because the functor $\mathcal{U} : AnFun \to Fun(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$ is faithful. □

4.2 The functor ev

We introduce the equivalence of categories $ev : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}} \longrightarrow \mathrm{AnFun}$ which extends the functor $ev_n : \mathrm{Mac}^{coh}(\mathcal{H}Par_n) \to PolFun_n$ of Definition 2.14. We prove that ev is strongly monoidal; i.e. it reflects the two monoidal structures on $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$ into the tensor product and the composition of functors.

Definition 4.22 (The functor ev). *Let M be an $\mathbb{M}$-module, it defines a strict analytic functor*

$$\{ev_n(M_n)(V)\}_{n\in\mathbb{N}} = \{\int^{\pi\in\mathcal{H}Par_n} M_n(\pi)\otimes\Gamma_\pi(V)\}_{n\in\mathbb{N}}.$$

The mapping ev is functorial in M, so it defines a functor:

$$ev : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}} \longrightarrow \mathrm{AnFun},$$

$$M \mapsto \{ev_n(M)(-)\}_{n\in\mathbb{N}}.$$

Since ev_n is an equivalence of categories for any $n\in\mathbb{N}$, we have that ev is an equivalence of categories as well.

We devote the rest of this section to the study of the image of monoidal structures under the functor ev. We establish a series of intermediate lemmas before formulating our main theorem.

Lemma 4.23. *We have a natural isomorphism*

$$ev(M\boxtimes N)\to ev(M)\otimes ev(N)$$

for any pair $M,N\in\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$, where we consider the functor $ev(M\boxtimes N)\in\mathrm{AnFun}$ associated to $M\boxtimes N$ on the left hand side, the pointwise tensor product of the analytic functors $ev(M), ev(N)\in\mathrm{AnFun}$ such as in Definition 4.3 on the right hand side.

Proof: We prove that there exists a natural isomorphism $ev(M\boxtimes N)\longrightarrow ev(M)\otimes ev(N)$. It follows from a sequence of natural isomorphims given by $\Gamma_{\pi_1\times\pi_2}(V)\cong\Gamma_{\pi_1}(V)\otimes\Gamma_{\pi_2}(V)$, Lemma 3.4, and Lemma 3.5. More precisely:

$$\begin{aligned}
&ev(M\boxtimes N)(V)\\
&\quad=\bigoplus_n\bigoplus_{i+j=n}\int^{\pi\in\mathcal{H}Par_n}\int^{\pi_1\times\pi_2\in\mathcal{H}Par_i\times\mathcal{H}Par_j}(M(\pi_1)\otimes N(\pi_2))\otimes\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\pi_2,\pi)\otimes\Gamma_\pi(V)\\
&\quad\cong\bigoplus_n\bigoplus_{i+j=n}\int^{\pi_1\times\pi_2}(M(\pi_1)\otimes N(\pi_2))\otimes\int^{\pi}\mathrm{Hom}_{Par_n}(\pi_1\times\pi_2,\pi)\otimes\Gamma_\pi(V)\\
&\quad\cong\bigoplus_{i,j}\int^{\pi_1\times\pi_2}(M(\pi_1)\otimes N(\pi_2))\otimes\Gamma_{\pi_1}(V)\otimes\Gamma_{\pi_2}(V)\\
&\quad\cong(\bigoplus_i\int^{\pi_1}M(\pi_1)\otimes\Gamma_{\pi_1}(V))\otimes(\bigoplus_j\int^{\pi_2}N(\pi_2)\otimes\Gamma_{\pi_2}(V)),
\end{aligned}$$

where we use the isomorphisms given by Lemma 3.4 and by Lemma 3.5.

The isomorphism commute with the action of $\Gamma_n(X)$ on $ev_n(M\boxtimes N)(Y)$. This claim follows from the commutativity of the following diagram:

$$\begin{array}{ccc}
\Gamma_n(X)\otimes\int^{\pi\in\mathcal{H}Par_n}\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\pi_2,\pi)\otimes\Gamma_\pi(Y) & \longrightarrow & \int^{\pi}\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\pi_2,\pi)\otimes\Gamma_\pi(X\otimes Y)\\
\uparrow\cong & & \\
\Gamma_n(X)\otimes\Gamma_{\pi_1}(Y)\otimes\Gamma_{\pi_2}(Y) & & \uparrow\cong\\
\downarrow & & \\
\Gamma_i(X)\otimes\Gamma_j(X)\otimes\Gamma_{\pi_1}(Y)\otimes\Gamma_{\pi_2}(Y) & \longrightarrow & \Gamma_{\pi_1}(X\otimes Y)\otimes\Gamma_{\pi_2}(X\otimes Y),
\end{array}$$

where $i+j=n$, $\pi_1\in Par_i$, $\pi_2\in Par_j$ and the morphism $\Gamma_n(X)\to\Gamma_i(X)\otimes\Gamma_j(X)$ is given by the restriction from $\mathbb{S}_n$ to $\mathbb{S}_i\times\mathbb{S}_j$. (We then use that the Fubini isomorphism of Lemma 3.4 is given by the canonical morphism from the object $\Gamma_{\pi_1\times\pi_2}(V)=\mathrm{Id}_{\pi_1\times\pi_2}\otimes\Gamma_{\pi_1\times\pi_2}(V)\subset\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\pi_2,\pi_1\times\pi_2)\otimes\Gamma_n(V)$ into the coend and that $\Gamma_n(X)$ acts on $\Gamma_{\pi_1\times\pi_2}(-)=\Gamma_{\pi_1}(-)\otimes\Gamma_{\pi_2}(-)$ through the diagonal morphism $\Gamma_n(X)\to\Gamma_i(X)\otimes\Gamma_j(X)$.) □

Lemma 4.24. *The isomorphisms of Lemma 4.23 make the unit, associativity, and symmetry isomorphisms of the symmetric monoidal category of $\mathbb{M}$-modules, such as defined in Proposition 3.7, correspond to the unit, associativity and symmetry isomorphisms of the symmetric monoidal category of analytic functors such as defined in Proposition 4.5.*

Proof: The proof of this lemma follows from straightforward verifications. □

We show a similar result for □.

Lemma 4.25. *Let M be an $\mathbb{M}$-module. We have interchange formula:*

$$ev_n((N^{\boxtimes r})^\rho) = (ev_n(N^{\boxtimes r}))^\rho$$

for every ρ subgroup of $\mathbb{S}_r$.

Proof: Since the functor ev_n is an equivalence of category it is an exact functor and hence preserves invariants. □

Lemma 4.26. *We have a natural isomorphism*

$$ev(M \square N) \cong ev(M) \circ ev(N),$$

for every $M, N \in \mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$, where we consider the functor $ev(M \square N) \in \mathrm{AnFun}$ associated to $M \square N$ on the left hand side, the composition product of the analytic functors $ev(M), ev(N) \in \mathrm{AnFun}$ such as in Definition 4.18 on the right hand side.

Proof: We have:

$$\begin{aligned}
ev(M \square N)(V) &\cong \bigoplus_n \int^{\pi \in \mathcal{H}Par_n} \int^{\rho \in \mathcal{H}Par_r} (M(\rho) \otimes (N^{\boxtimes r}(\pi))^\rho) \otimes \Gamma_\pi(V) \\
&\cong \bigoplus_{r,n} \int^{\rho} \int^{\pi} (M(\rho) \otimes (N^{\boxtimes r}(\pi))^\rho) \otimes \Gamma_\pi(V) \\
&\cong \bigoplus_{r,n} \int^{\rho} M(\rho) \otimes \int^{\pi} (N^{\boxtimes r}(\pi))^\rho \otimes \Gamma_\pi(V) \\
&= \bigoplus_{r,n} \int^{\rho} M(\rho) \otimes ev_n((N^{\boxtimes r})^\rho)(V) \\
&\overset{(1)}{\cong} \bigoplus_{r} \int^{\rho} M(\rho) \otimes (ev(N^{\boxtimes r})(V))^\rho \\
&\cong \bigoplus_{r} \int^{\rho} M(\rho) \otimes (ev(N)(V)^{\otimes r})^\rho \\
&\cong \bigoplus_{r} \int^{\rho} M(\rho) \otimes \Gamma_\rho(ev(N)(V)) = ev(M)(ev(N)(V)),
\end{aligned}$$

where we use the isomorphisms given by Lemma 3.4 and by Lemma 3.5, and the isomorphism (1) is given by Lemma 4.25.

To check that this isomorphism commutes with the action of $\Gamma_n(X)$ we use that the isomorphisms inside the coends preserve the natural action of $\Gamma_n(X)$ on our objects. In the final step, we get an action of $\Gamma_n(X)$ on $\Gamma_\rho(ev(N)(-)) \subset ev(N)(-)^{\otimes r}$ which coincides with the action defined in Proposition 4.17 for this composite functor, and the conclusion readily follows. □

Lemma 4.27. *The composition product □ inherits unit and associativity isomorphisms which correspond to the unit and associativity isomorphisms of the composition of analytic functors, such as defined in Definition 4.18. These unit and associativity isomorphisms satisfy the coherence constraints of a monoidal category in $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}$. Thus the triple $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \square, \mathbb{I})$, where $\mathbb{I}$ denotes the obvious $\mathbb{M}$-module which corresponds to the identity functor, forms a monoidal category.*

Proof: This statement follows from the result of Lemma 4.26 and from the observation that ev is an equivalence of categories. □

Theorem 4.28. *The mapping $ev : M \mapsto ev(M)$ defines an equivalence of symmetric monoidal categories $ev : (\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \boxtimes, \mathbb{K}) \to (\mathrm{AnFun}, \otimes, \mathbb{K})$ as well as an equivalence of monoidal categories $ev : (\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \Box, \mathbb{I}) \to (\mathrm{AnFun}, \circ, \mathrm{Id})$.*

Proof: The proof follows from Theorem 2.18, Lemma 4.24 and Lemma 4.27. □

Remark 4.29. *Let A,B and C be three $\mathbb{M}$-modules. We have an isomorphism:*

$$(A \Box C) \boxtimes (B \Box C) \cong (A \boxtimes B) \Box C$$

which reflects the formula:

$$(ev(A) \circ ev(C)) \otimes (ev(B) \circ ev(C)) \cong (ev(A) \otimes ev(B)) \circ ev(C).$$

Recall that by Definition 3.2, to a $\mathbb{S}$-module M we can associate the $\mathbb{M}$-modules $\Gamma(M)$ and $S(M)$. By definition 4.22 we have the strict analytic functors $ev(\Gamma(M))$ and $ev(S(M))$. In the following proposition we identify these strict analytic functors.

Proposition 4.30. *Let $M = \{M_n\}_{n\in\mathbb{N}}$ be a $\mathbb{S}$-module (see Appendix **??**). If V is a free $\mathbb{K}$-module, then we have $ev(S(M))(V) \cong \{S_n(M,V)\}_{n\in\mathbb{N}}$ and $ev(\Gamma(M))(V) \cong \{\Gamma_n(M,V)\}_{n\in\mathbb{N}}$, where $S_n(M,V) = M(n) \otimes_{\mathbb{S}_n} V^n$ and $\Gamma_n(M,V) = M(n) \otimes^{\mathbb{S}_n} V^n$ (see Appendix **??**).*

Proof: We first consider the cohomological $\mathcal{H}Par_n$-Mackey functor $T(M) = T_n(M)$ where $T_n(M) = M_n \otimes \mathbb{I}^{\boxtimes n}$. We have that $ev_n(T_n(M))(V) = ev_n(M_n \otimes \mathbb{I}^{\boxtimes n})(V) \cong M_n \otimes V^{\otimes n}$. The unit object $\mathbb{I}$ is given by $\mathbb{I}_1 = \mathbb{K}$ (the constant functor on the category $\mathcal{H}Par_1$ with object set pt and Hom-object $\mathbb{K}$) and $\mathbb{I}_i = 0$ for $i \neq 1$. Let $\pi \in Par_n$. We accordingly have

$$\begin{aligned}
\mathbb{I}^{\boxtimes r}(\pi) &= \bigoplus_{i_1+\cdots+i_r=n} \int^{\pi_1\times\cdots\times\pi_r\in\mathcal{H}Par_{i_1}\times\cdots\times\mathcal{H}Par_{i_r}} \mathbb{I}(\pi_1)\otimes\cdots\otimes\mathbb{I}(\pi_r)\otimes\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\cdots\times\pi_r,\pi)\\
&= \int^{\pi_1\times\cdots\times\pi_r\in\mathcal{H}Par_1^{\times r}} \mathbb{I}(\pi_1)\otimes\cdots\otimes\mathbb{I}(\pi_1)\otimes\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\cdots\times\pi_r,\pi)\\
&= \mathbb{K}[\mathbb{S}_n/_\pi]
\end{aligned}$$

by the associativity of $\boxtimes$ and the definition of $\mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1,\pi_2)$. We therefore have $T_n(M)(\pi) = M_n \otimes \mathbb{K}[\mathbb{S}_n/_\pi]$.

The mapping $S_n(M)(\pi) \longrightarrow M_n \otimes_{\mathbb{S}_n} \mathbb{K}[\mathbb{S}_n/_\pi]$ defined by $[m] \mapsto [m\otimes e]$ where e is the unit of $\mathbb{S}_n$ induces an isomorphism $S_n(M) \cong M_n \otimes_{\mathbb{S}_n} \mathbb{K}[\mathbb{S}_n/_\pi] \cong (T_n(M))_{\mathbb{S}_n}$. The mapping $\Gamma_n(M)(\pi) \longrightarrow M_n \otimes^{\mathbb{S}_n} \mathbb{K}[\mathbb{S}_n/_\pi]$ defined by $m \mapsto \sum_{\alpha\in\mathbb{S}_n/\pi} \alpha^*(m)\otimes\alpha$ induces an isomorphism $\Gamma_n(M) \cong M_n \otimes^{\mathbb{S}_n} \mathbb{K}[\mathbb{S}_n/_\pi] \cong (T_n(M))^{\mathbb{S}_n}$.

Since ev_n is an equivalence of categories it preserves invariants and coinvariants. The conclusion follows. □

Corollary 4.31. *Let M and N be two $\mathbb{S}$-modules (see Appendix **??**). We have*

- $S(M) \boxtimes S(N) \cong S(M \boxtimes N)$, *and*
- $\Gamma(M) \boxtimes \Gamma(N) \cong \Gamma(M \boxtimes N)$.

Proof: The statement is a direct consequence of Proposition 4.30, of Proposition A.20 and of Theorem 4.28. □

Corollary 4.32. *Let M and N be two $\mathbb{S}$-modules (see Appendix **??**). We have*

- $S(M) \square S(N) \cong S(M \square_{\mathbb{S}} N)$, *and*
- $\Gamma(M) \square \Gamma(N) \cong \Gamma(M \square^{\mathbb{S}} N)$.

Proof: The statement is a direct consequence of Proposition 4.30, of Proposition A.20 and of Theorem 4.28. □

5 $\mathbb{M}$-Operads and their algebras

In this section we introduce the definition of $\mathbb{M}$-operad. Roughly speaking an $\mathbb{M}$-operad is an object governing the category of "type of algebras" with polynomial operations with multiple inputs and one output. Our definition of $\mathbb{M}$-operad is equivalent to the definition of Schur operads introduced by Ekedahl and Salomonsson in [ES04], [Sal03] and studied by Xantcha in [Xan10].

5.1 $\mathbb{M}$-Operads

We introduce the definition of $\mathbb{M}$-operads. They are a generalization of operads (see Appendix **??**).

Definition 5.1 ($\mathbb{M}$-operad)**.** *An $\mathbb{M}$-operad is an $\mathbb{M}$-module P together with two $\mathbb{M}$-module morphisms $\mu : P \square P \longrightarrow P$ and $\eta : \mathbb{I} \longrightarrow P$ such that the following diagrams commute:*

$$\begin{array}{ccc} P \square P \square P & \xrightarrow{\mathrm{Id}_P \square \mu} & P \square P \\ {\scriptstyle \mu \square \mathrm{Id}_P}\downarrow & & \downarrow{\scriptstyle \mu} \\ P \square P & \xrightarrow[\mu]{} & P, \end{array} \qquad \text{(associativity)}$$

$$\begin{array}{ccccc} P \square \mathbb{I} & \xrightarrow{\mathrm{Id}_P \square \eta} & P \square P & \xleftarrow{\eta \square \mathrm{Id}_P} & \mathbb{I} \square P \\ & {\scriptstyle \pi_1}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \pi_2} & \\ & & P, & & \end{array} \qquad \text{(unity)}$$

i.e. (P, μ, η) *is a monoid in the monoidal category* $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \square, \mathbb{I})$.

A morphism of $\mathbb{M}$-operad is a morphism of monoid in the monoidal category $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{M}}, \square, \mathbb{I})$. *We denote the category of $\mathbb{M}$-operad by* $\mathbb{M}$-Op.

Proposition 5.2. *Let P be a connected operad (see Appendix* **??***). The $\mathbb{M}$-modules $S(P)$, $\Gamma(P)$ and $\Lambda(P)$ are $\mathbb{M}$-operads.*

Proof: Let $\mu : P \square_{\mathbb{S}} P \longrightarrow P$ and $\eta : \mathbb{I} \longrightarrow P$ be the structure maps of the operad P. We have two induced morphisms $S(\mu) : S(P \square_{\mathbb{S}} P) \longrightarrow S(P)$ and $\eta : \mathbb{I} \longrightarrow S(P)$. From Proposition 4.32 we have isomorphisms $S(P \square_{\mathbb{S}} P) \cong S(P) \square S(P)$ and $\Gamma(P \square^{\mathbb{S}} P) \cong \Gamma(P) \square \Gamma(P)$. □

Proposition 5.3. *Let (P, μ, η) be an $\mathbb{M}$-operad. The endofunctor $ev(P)$ endowed with the morphisms $ev(\mu)$ and $ev(\eta)$ is a monad.*

Proof: It is a consequence of Theorem 4.27. □

Definition 5.4 (P-algebra)**.** *Let (P, μ, η) be an $\mathbb{M}$-operad. The category of P-algebras is the category of algebras governed by the monad $ev(P)$. More explicitly, a P-algebra is a pair (V, γ), where V is an object of* $\mathrm{Mod}_{\mathbb{K}}$ *and $\gamma : ev(P)(V) \longrightarrow V$ is a morphism in* $\mathrm{Mod}_{\mathbb{K}}$ *such that the following diagrams commute:*

$$\begin{array}{ccc} ev(P)(ev(P)(V)) & \xrightarrow{\gamma} & ev(P)(V) \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \gamma} \\ ev(P)(V) & \xrightarrow[\gamma]{} & V, \end{array} \qquad \text{(associativity)}$$

$$\begin{array}{ccc} V & \xrightarrow{\eta} & ev(P)(V) \\ & \searrow^{\mathrm{Id}_A} & \downarrow \gamma \\ & & V. \end{array} \qquad \text{(unity)}$$

5.2 The $\mathbb{M}$-operad $Poly_V$

Let (P,μ,η) be an $\mathbb{M}$-operad and V be a $\mathbb{K}$-module. The set of P-algebra structures over V is governed by the set of morphisms of $\mathbb{M}$-operads between P and an $\mathbb{M}$-operad denoted by $Poly_V$.

Lemma 5.5. *Let M be an $\mathbb{M}$-module and V be a $\mathbb{K}$-module. We denote by $\bar{M} : \mathcal{H}Par_n^{op} \to \mathrm{Mod}_{\mathbb{K}}$ the functor obtained by the composition of M with the isomorphism $\mathcal{H}Par_n^{op} \to \mathcal{H}Par_n$. The V-dual of M is the $\mathbb{M}$-module defined by $\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\bar{M}(-),V) : \mathcal{H}Par_n \to \mathrm{Mod}_{\mathbb{K}}$.*

Proof: It follows from the linearity of $\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(-,V)$. □

Definition 5.6 (The $\mathbb{M}$-module $Poly_V$). *Let V be a $\mathbb{K}$-module. We define the $\mathbb{M}$-module $Poly_V$ to be the V-dual of $\Gamma_-(V)$, explicitly:*

1. *let π be an object of $\mathcal{H}Par_n$, we set $Poly_V(\pi) \coloneqq \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\pi(V),V)$,*
2. *let π_1 and π_2 be objects in $\mathcal{H}Par_n$ such that π_1 is a subgroup of π_2, we set $\mathrm{Ind}_{\pi_1}^{\pi_2} \coloneqq (\mathrm{Res}_{\pi_1}^{\pi_2})^*$ and $\mathrm{Res}_{\pi_1}^{\pi_2} \coloneqq (\mathrm{Ind}_{\pi_1}^{\pi_2})^*$.*

Proposition 5.7. *Let V be a $\mathbb{K}$-module. The $\mathbb{M}$-module $Poly_V$ inherits the structure of an $\mathbb{M}$-operad.*

Proof: We aim to define

$$(Poly_V \,\square\, Poly_V)(\pi) \to Poly_V(\pi),$$

which is equivalent to give a morphism as follows:

$$(Poly_V \,\square\, Poly_V)(\pi) \otimes \Gamma_\pi(V) \to V.$$

We have:

$$\begin{aligned}
&(Poly_V \,\square\, Poly_V)(\pi) \otimes \Gamma_\pi(V) \\
&\quad = \Big(\bigoplus_r \int^{\rho\in\mathcal{H}Par_r} Poly_V(\rho) \otimes \Big(\bigoplus_{n_1+\cdots+n_r=n} \int^{\pi_1\times\ldots\times\pi_r\in\mathcal{H}Par_{n_1}\times\cdots\times\mathcal{H}Par_{n_r}} Poly_V(\pi_1)\otimes\cdots\otimes Poly_V(\pi_r)\otimes \\
&\qquad \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\ldots\pi_r,\pi)\Big)^\rho\Big) \otimes \Gamma_\pi(V) \\
&\quad \cong \bigoplus_r \int^{\rho} Poly_V(\rho) \otimes \Big(\bigoplus_{n_1+\cdots+n_r=n} \int^{\pi_1\times\ldots\times\pi_r} Poly_V(\pi_1)\otimes\cdots\otimes Poly_V(\pi_r)\otimes \\
&\qquad \mathrm{Hom}_{\mathcal{H}Par_n}(\pi_1\times\ldots\pi_r,\pi)\otimes\Gamma_\pi(V)\Big)^\rho\Big) \\
&\quad \overset{(1)}{\cong} \bigoplus_r \int^{\rho} Poly_V(\rho) \otimes \Big(\bigoplus_{n_1+\cdots+n_r=n} \int^{\pi_1\times\ldots\times\pi_r} Poly_V(\pi_1)\otimes\cdots\otimes Poly_V(\pi_r)\otimes \\
&\qquad \Gamma_{\pi_1}(V)\otimes\cdots\Gamma_{\pi_r}(V)\Big)^\rho\Big) \\
&\quad \overset{(2)}{\to} \bigoplus_r \int^{\rho} Poly_V(\rho)\otimes (V^{\otimes r})^\rho\Big) \cong \bigoplus_r \int^{\rho\in\mathcal{H}Par_r} Poly_V(\rho)\otimes\Gamma_\rho(V)\Big) \overset{(3)}{\to} V
\end{aligned}$$

where we first expand the composite, the isomorphism (1) is given by $\Gamma_{\pi_1\times\cdots\times\pi_r}(V) \cong \Gamma_{\pi_1}(V)\otimes\cdots\otimes\Gamma_{\pi_r}(V)$, and the morphisms (2) and (3) by the maps $Poly_V(\pi)\otimes\Gamma_\pi(V) = \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\pi(V),V)\otimes\Gamma_\pi(V) \to V$.

Unit and associativity follow from straightforward verifications. □

Theorem 5.8. *Let P be an $\mathbb{M}$-operad and V be in $\mathrm{Mod}_{\mathbb{K}}$, the set of P-algebra structures over V is in bijection with $\mathrm{Hom}_{\mathbb{M}\text{-}\mathrm{Op}}(P, Poly_V)$.*

Proof: We define a function between the set of monoids morphisms between P and $Poly_V$ and the set of P-algebra structures of V:

$$\phi : \mathrm{Hom}_{\mathbb{M}\text{-}\mathrm{Op}}(P, Poly_V) \longrightarrow \{\gamma : ev(P)(V) \longrightarrow V | \gamma \text{ P-Algebra structure}\}.$$

Let $f : P(-) \to \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_{-}(V), V)$ be an $\mathbb{M}$-operad morphism between P and $Poly_V$. We denote by $f^* : \bigoplus_\pi P(\pi) \otimes \Gamma_\pi(V) \to V$ the morphism defined by the adjoint of f. We set $\phi(f) : P(V) = \int^{\pi \in \mathcal{H}Par_n} P(\pi) \otimes \Gamma_\pi(V) \longrightarrow V$ by the universal property of the coend:

$$\begin{array}{ccccc}
\bigoplus_{\alpha \to \beta} P(\alpha) \otimes \Gamma_\beta(V) & \longrightarrow & \bigoplus_\pi P(\pi) \otimes \Gamma_\pi(V) & & \\
\downarrow & & \downarrow & \searrow^{f^*} & \\
\bigoplus_\pi P(\pi) \otimes \Gamma_\pi(V) & \longrightarrow & P(V) & & \\
& \searrow_{f^*} & & \searrow & \\
& & & & V
\end{array}$$

The Theorem follows from the following sequences of isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}\Big(\bigoplus_n \int^{\pi \in \mathcal{H}Par_n} P(\pi) \otimes \Gamma_\pi(V), V\Big) &\cong \bigoplus_n \int^\pi \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(P(\pi) \otimes \Gamma_\pi(V), V) \\
&\cong \bigoplus_n \int^\pi \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(P(\pi), \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\pi(V), V)).
\end{aligned}$$

More precisely an $\mathbb{M}$-operad morphism between P and $Poly_V$ is a morphism of $\mathbb{M}$-modules $g : P \longrightarrow Poly_V$ such that the following diagram commutes:

$$\begin{array}{ccc}
\bigoplus_r \int^{\rho \in \mathcal{H}Par_r} P(\rho) \otimes (P^{\boxtimes r}(\pi))^\rho & \xrightarrow{\mathrm{Id}_P \,\square\, g} & \bigoplus_r \int^\rho P(\rho) \otimes (Poly_V^{\boxtimes r}(\pi))^\rho \\
& & \downarrow \\
& & \bigoplus_r \int^\rho P(\rho) \otimes \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\pi(V), \Gamma_\rho(V)) \\
\downarrow \mu & & \downarrow f \square \mathrm{Id}_P \\
& & \bigoplus_r \int^\rho \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\rho(V), V) \otimes \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\pi(V), \Gamma_\rho(V)) \\
& & \downarrow \\
P(\pi) & \xrightarrow[g]{} & Poly_V(\pi) = \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\pi(V), V).
\end{array}$$

Applying the isomorphism we get the following commutative diagram:

$$\begin{array}{ccc}
\bigoplus_n \int^{\pi \in \mathcal{H}Par_n} \Big(\bigoplus_r \int^{\rho \in \mathcal{H}Par_r} P(\rho) \otimes (P^{\boxtimes r}(\pi))^\rho\Big) \otimes \Gamma_\pi(V) & \longrightarrow & \bigoplus_n \int^\pi \bigoplus_r \int^\rho P(\rho) \otimes \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\pi(V), \Gamma_\rho(V)) \otimes \Gamma_\pi(V) \\
\downarrow & & \downarrow \\
\bigoplus_n \int^\pi P(\pi) \otimes \Gamma_\pi(V) & \xrightarrow{=} & \bigoplus_r \int^\rho P(\rho) \otimes \Gamma_\rho(V).
\end{array}$$

□

5.3 Examples

We present some examples of categories of algebras governed by $\mathbb{M}$-operads.

We only aim to give an idea of future applications of our constructions in this example section. We therefore posit the existence of free objects in the category of $\mathbb{M}$-operads, which generalize the ordinary free operads, without giving further details on the construction of such objects.

Proposition 5.9. *Let P be a connected operad. We have that the category of $S(P)$-algebras is isomorphic to the category of $S(P,-)$-algebras, and the category of $\Gamma(P)$-algebras is isomorphic to the category of $\Gamma(P,-)$-algebras (see Appendix **??**).*

Proof: By Proposition 4.30 $S(P)$ is an $\mathbb{M}$-operad such that $ev(S(P)) \cong S(P,-)$ and the structure maps are induced by the structure maps of P. The same argument works for $\Gamma(P)$. □

Example 5.10. *A $\Gamma(Com)$-algebra structure corresponds to a divided power algebra. That is a triple $(V,\mu,\{\gamma_i\}_{i\in\mathbb{N}})$ such that (V,μ) is a commutative algebra and $\gamma_i : V \longrightarrow V$ are set-theoretical functions such that:*

$$\gamma_n(x+y) = \sum_{i=0}^{n} \gamma_{n-i}(x)\gamma_i(y),$$
$$\gamma_i(\lambda x) = \lambda^i \gamma_i(x),$$
$$\gamma_1(x) = x,$$
$$\gamma_m(x)\gamma_n(x) = \binom{m+n}{n}\gamma_{m+n}(x),$$
$$\gamma_m(\gamma_n(x)) = \frac{mn!}{(n!)^m m!}\gamma_{mn}(x).$$

Let $\mathbb{K}$ be a field of positive characteristic p. A $\Gamma(Lie)$-algebra structure corresponds to a p-restricted Lie algebra (see [Fre00]) . That is a triple $(V,[-,-],-^{[p]})$ such that $(V,[-,-])$ is a Lie algebra, and $-^{[p]} : V \longrightarrow V$ is a set-theoretical function such that:

$$(\lambda x)^{[p]} = \lambda^p (x)^{[p]},$$
$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i(x,y)}{i},$$
$$ad(x^{[p]}) = (ad(x))^{[p]}.$$

There are explicit descriptions for $\Gamma(Pois)$-algebras (see [Fre00]) and for $\Lambda(PreLie)$ and $\Gamma(PreLie)$-algebras (see the preprint of the author [Ces15]), where $Pois$ is the operad governing Poisson algebras, and $PreLie$ is the operad governing the category of pre-Lie algebras.

Definition 5.11 (2-restricted Poisson algebra)**.** *Let $\mathbb{K}$ be a field of characteristic 2. A 2-restricted Poisson algebra is a triple*

$$(A,[-,-] : A\otimes A \longrightarrow A, (-)^2 : A \longrightarrow A)$$

where A is a commutative algebra and $(A,[-,-] : A\otimes A \longrightarrow A, (-)^{[2]} : A \longrightarrow A)$ is a 2-restricted Lie algebra structure, such that:

1. *$[x,yz] = y[x,z] + [x,y]z$, and*
2. *$(xy)^{[2]} = x^2(y)^{[2]} + x[x,y]y + (x)^{[2]}y^2$.*

Proposition 5.12. *Let $\mathbb{K}$ a field of characteristic 2. The $\mathbb{M}$-module $S(Com)\,\square\,\Gamma(Lie)$ is an $\mathbb{M}$-operad, denoted by 2-Pois, which encodes the category of 2-restricted Poisson algebras.*

Sketch: For the partition $(1)(2)$ of the set $\mathbf{2} = \{1,2\}$ let $\mu \in Com((1)(2))$ and $[-,-] \in Lie((1)(2))$ be respectively the generators of the operads Com and Lie. Consider the $\mathbb{M}$-module $S(Com)\,\square\,\Gamma(Lie)$. We show that the relation 1 of Definition 5.11 defines a distributive law of monads in the sense of Beck [Bec69]. We define the morphism of $\mathbb{M}$-modules $\rho_- : \Gamma(Lie)\,\square\,S(Com) \longrightarrow S(Com)\,\square\,\Gamma(Lie)$ using this relation. □

Remark 5.13. *Let $\mathbb{K}$ be a field of positive characteristic $p > 2$. The $\mathbb{M}$-module $S(Com)\square\Gamma(Lie)$ still forms an $\mathbb{M}$-operad by using the distributive law of monads induced by relation 1 of Definition 5.11. In this case the relation 2 of Definition 5.11 is replaced by the more complicated:*

$$(xy)^{[p]} = x^p y^{[p]} + x^{[p]} y^p + P(x,y)$$

where $P(x,y)$ is a Poisson polynomial that can be made explicit. This structure was first introduced by Bezrukavnikov and Kaledin in [BK08] in the study of quantization of algebraic manifolds in positive characteristic.

6 $\mathbb{M}$-PROPs and their algebras

In this section we introduce the definition of $\mathbb{M}$-PROPs. A $\mathbb{M}$-PROP is an algebraic object which governs algebraic structures with (polynomial) operations with multiple inputs and multiple outputs.

6.1 The category $\mathrm{Mod}_{\mathbb{K}}^{\mathrm{Bi}\mathbb{M}}$

Definition 6.1 (Cohomological $(\mathcal{H}Par_n, \mathcal{H}Par_m)$-Mackey bifunctor)**.** *Let n and m be two non-negative integers. A cohomological $(\mathcal{H}Par_n, \mathcal{H}Par_m)$-Mackey bifunctor M is a biadditive bifunctor:*

$$M : \mathcal{H}Par_n \times \mathcal{H}Par_m \longrightarrow \mathrm{Mod}_{\mathbb{K}}\,.$$

Definition 6.2 (Bi$\mathbb{M}$-module)**.** *A* Bi$\mathbb{M}$*-module $M_{\bullet,\bullet}$ is a collection $\{M_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ of cohomological $(\mathcal{H}Par_n, \mathcal{H}Par_m)$-Mackey bifunctors. A morphism between two* Bi$\mathbb{M}$*-modules is a collection of natural transformations. Their category is denoted by $\mathrm{Mod}_{\mathbb{K}}^{\mathrm{Bi}\mathbb{M}}$.*

We define two monoidal structures $(\mathrm{Mod}_{\mathbb{K}}^{\mathrm{Bi}\mathbb{M}}, \boxdot, \mathbb{K})$ and $(\mathrm{Mod}_{\mathbb{K}}^{\mathrm{Bi}\mathbb{M}}, \boxminus, \mathbb{I})$, respectively the horizontal and the vertical composition.

Definition 6.3 (The product $\boxdot$)**.** *For any M and N* Bi$\mathbb{M}$*-modules we set:*

$$(M \boxdot N)(\pi,\rho) = \bigoplus_{\substack{i_1+j_1=n_1\\ i_2+j_2=n_2}} \int^{\substack{\pi_1\times\pi_2\in\mathcal{H}Par_{i_1}\times\mathcal{H}Par_{i_2}\\ \rho_1\times\rho_2\in\mathcal{H}Par_{j_1}\times\mathcal{H}Par_{j_2}}} M(\pi_1,\pi_2)\otimes N(\rho_1,\rho_2)\otimes \mathrm{Hom}_{\mathcal{H}Par_{n_1}\times\mathcal{H}Par_{n_2}}((\pi_1\times\rho_1, \pi_2\times\rho_2),(\pi,\rho)).$$

Proposition 6.4. *The product $\boxdot$ forms a symmetric monoidal structure together with the* Bi$\mathbb{M}$*-module $\mathbb{K}$:*

$$\mathbb{K}_{i_1,i_2} := \begin{cases} \mathbb{K} & (i_1,i_2) = (0,0), \\ 0 & (i_1,i_2) \neq (0,0) \end{cases}$$

as unit.

Proof: A prove similar to the one for Definition 3.6 works. □

Definition 6.5 (The product $\boxminus$)**.** *Let M and N be two* Bi$\mathbb{M}$*-modules we define*

$$(M \boxminus N)(\pi,\rho) = \bigoplus_{w} \int^{\upsilon\in\mathcal{H}Par_w} M(\pi,\upsilon)\otimes N(\upsilon,\rho).$$

Proposition 6.6. *The product $\boxminus$ forms a monoidal structure together with the* Bi$\mathbb{M}$*-module $\mathbb{I}$:*

$$\mathbb{I}_{i_1,i_2} := \begin{cases} \mathbb{K} & i_1 = i_2, \\ 0 & i_1 \neq i_2) \end{cases}$$

as unit.

Proof: It follows directly from the monoidal structure of the tensor product of $\mathbb{K}$-modules. □

6.2 $\mathbb{M}$-PROPs

We introduce the concept of an $\mathbb{M}$-PROP which generalizes the concept of an $\mathbb{M}$-operad.

Definition 6.7 ($\mathbb{M}$-PROP)**.** *An $\mathbb{M}$-PROP is a* Bi$\mathbb{M}$*-module P endowed with two associative multiplication maps $\mu_h : P \boxbar P \longrightarrow P$, $\mu_v : P \boxminus P \longrightarrow P$ and a unit $\eta : \mathbb{I} \to P$ for μ_v such that:*

- *the restriction of $\eta : \mathbb{I} \to P$ to $\mathbb{K} \hookrightarrow \mathbb{I}$ is a unit for μ_h,*
- *for any $f_1 \in P(\pi_1, \upsilon_1), f_2 \in P(\pi_2, \upsilon_2)$ we have*

$$\mu_h(f_2), f_1) = c_{\sigma,\tau}(\mu_h(f_1, f_2)))$$

  where σ (resp. τ) is the permutation in $\mathbb{S}_{n_1+n_2}$ (resp. $\mathbb{S}_{m_1+m_2}$) which permutes the blocks $\{1,\dots,n_1\}$ and $\{n_1+1,\dots,n_1+n_2\}$ (resp. $\{1,\dots,m_1\}$ and $\{m_1+1,\dots,m_1+m_2\}$) and fix the orders inside the blocks.

- *for any $f_1 \in P(\pi_1, \upsilon_1), f_2 \in P(\pi_2, \upsilon_2), g_1 \in P(\upsilon_1, \rho_1), g_2 \in P(\upsilon_2, \rho_2)$ we have:*

$$\mu_h(\mu_v(f_1, g_1), \mu_v(f_2, g_2)) = \mu_v(\mu_h(f_1, f_2), \mu_h(g_1, g_2)).$$

A morphism of $\mathbb{M}$-PROPs is a natural transformation compatible with this structure.

Example 6.8. *Let P be an $\mathbb{M}$-operad then it is, in particular, an $\mathbb{M}$-PROP.*

Proposition 6.9. *Let P be a PROP (see Appendix* **??***). It defines different $\mathbb{M}$-PROPs as follows:*

- *$S(P)$ is defined by:*

$$S_{n,m}(P)(\pi,\rho) = {}_{\pi}(P(n,m))_{\rho}$$

  where $\pi \in \mathcal{H}Par_n$ and $\rho \in \mathcal{H}Par_m$, and

- *if P is biconnected, $\Gamma(P)$ is defined by:*

$$\Gamma_{n,m}(P)(\pi,\rho) = {}^{\pi}(P(n,m))^{\rho}$$

  where $\pi \in \mathcal{H}Par_n$ and $\rho \in \mathcal{H}Par_m$.

Proof: We show how the composition on P induces a composition on $\Gamma(P)$.

$$\int^{\pi \in \mathcal{H}Par_n} \Gamma_{n,m}(P)(\pi,\rho) \otimes \Gamma_{s,n}(P)(\sigma,\pi) \cong$$

$$\Gamma_{s,m}(\int^{\pi} {}^{\pi}P(n,m) \otimes P(s,n)^{\pi})(\sigma,\rho) \to \Gamma_{s,m}(\int^{\pi} (P(n,m) \otimes P(s,n))^{\pi})(\sigma,\rho) \cong$$

$$\Gamma_{s,m}(P(n,m) \otimes^{\mathbb{S}_n} P(s,n))(\sigma,\rho) \to \Gamma_{s,m}(P(s,m))(\sigma,\rho) =$$

$$\Gamma_{s,m}(P)(\sigma,\rho)$$

a similar proof works for $S(P)$. □

6.3 Algebras over an $\mathbb{M}$-PROP

Fix a $\mathbb{K}$-module V. We define an $\mathbb{M}$-PROP denoted by $BiPoly_V$ which we use to define the category of algebras over an $\mathbb{M}$-PROP.

Definition 6.10 (The $\mathbb{M}$-PROP $BiPoly_V$)**.** *We define the $\mathbb{M}$-PROP $BiPoly_V$ by:*

$$BiPoly_V(\pi,\rho) = \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(\Gamma_\pi(V), \Gamma_\rho(V)).$$

The horizontal composition is induced by the tensor product of morphism in $\mathrm{Mod}_{\mathbb{K}}$ and the vertical composition by the composition of morphisms in $\mathrm{Mod}_{\mathbb{K}}$.

Definition 6.11 ($\mathbb{M}$-PROP algebras)**.** *Let P be a $\mathbb{M}$-PROP. A P-algebra over the $\mathbb{K}$-module V is a morphism of $\mathbb{M}$-PROPs $\gamma : P \longrightarrow BiPoly_V$.*

6.4 Examples

Let P be a PROP (see Appendix **??**). We prove that the category of $S(P)$-algebras is equivalent to the category of P-algebras. We prove that the category of p-restricted Lie bialgebras and the category of divided power bialgebras are governed by two $\mathbb{M}$-PROPs. These two categories are not governed by any PROPs.

We again only aim to give an idea of future applications of our constructions in this example section. We still posit the existence of free objects in the category of $\mathbb{M}$-PROPs, which generalize the ordinary free PROPs, without giving further details on the construction of such objects.

Proposition 6.12. *Let P be an $\mathbb{M}$-operad. It defines an $\mathbb{M}$-PROP where*

$$P(\pi,\rho) \coloneqq (P^{\boxtimes r}(\pi))^{\rho},$$

for all $\pi \in Par_n$ and $\rho \in Par_r$.

Proposition 6.13. *Let P be a PROP (see Appendix **??**). The category of algebras associated to the $\mathbb{M}$-PROP $S(P)$ is equivalent to the category of P-algebras.*

Proof: Let V be a $\mathbb{K}$-module. Let $\phi : S(P) \longrightarrow BiPoly_V$ be a $S(P)$-algebra then if restricted to the discrete partitions it defines a P-algebra structure. Vice-versa since inductions are epimorphisms in $S(P)$ any P-algebra structure can be extended to a unique $S(P)$-algebra structure. □

Definition 6.14 (2-restricted Lie bialgebra)**.** *Let $\mathbb{K}$ be a field of characteristic 2. We say that $(A, [-,-], -^{[2]}, \delta)$, where*

- $A \in \mathrm{Mod}_{\mathbb{K}}$,
- $[-,-] : A \otimes A \longrightarrow A$,
- $-^{[2]} : A \longrightarrow A$,
- $\delta : A \longrightarrow A \otimes A$,

is a 2-restricted Lie bialgebra if $(A, [-,-], (-)^{[2]})$ is a 2-restricted Lie algebra, $(A, [-,-], \delta)$ is a Lie bialgebra and

$$\delta(-^{[2]}) = 0.$$

Proposition 6.15. *Let $\mathbb{K}$ be a field of characteristic 2. There exists an $\mathbb{M}$-PROP, denoted by $\Gamma BiLie$, which encodes the category of 2-restricted Lie bialgebras.*

Sketch: Let $BiLie$ be the PROP which governs the category of Lie bialgebras. We consider the $\mathbb{M}$-PROP $\Gamma(BiLie)$.

We prove that the $\Gamma(BiLie)$-algebras correspond to the 2-restricted Lie bialgebras.

Let $\phi : \Gamma(BiLie) \longrightarrow Poly_V$ be a $\Gamma(BiLie)$-algebra. There exists a monomorphism from the $\mathbb{M}$-PROP defined by the $\mathbb{M}$-operad $\Gamma(Lie)$ and $\Gamma(BiLie)$ that we denote i. From this inclusion ϕ defines a 2-restricted Lie algebra $(V, [-,-], -^{[2]})$. The restriction of ϕ to the discrete partitions is equivalent to a $BiLie$-algebra $(V, [-,-], \delta)$ where $[-,-] = \phi(m)$ and $\delta = \phi(c)$. For $i = 2$ we have that:

$$c(m) = e \otimes m(c \otimes e) + m \otimes e(e \otimes c) + m \otimes e((2,3)c \otimes e) + e \otimes m((1,2)e \otimes c).$$

By applying this relation to the image by ϕ of $\Gamma_{2,2}(BiLie)((1,2),(1)(2))$ we obtain:

$$\delta(-^{[2]}) = 0.$$

Let $(V, [-,-], -^{[2]}, \delta)$ be a 2-restricted Lie bialgebra. In particular (V, μ, δ) is a bialgebra, that is equivalent to a morphism $\psi : BiLie \longrightarrow \mathrm{BiEnd}_V$. We identify indexes of these two PROPs with the discrete partitions and partially extend the morphism ψ by the inductions morphisms. The 2-restricted Lie bialgebra $(V, [-,-], -^{[2]})$ is in particular a $\Gamma(Lie)$-algebra. Extending ϕ by the inclusion of the $\mathbb{M}$-PROP defined by $\Gamma(Lie)$ into $\Gamma(BiLie)$ we obtain a $\Gamma(BiLie)$-algebra structure. □

Remark 6.16. *Let $\mathbb{K}$ be a field of positive characteristic $p > 2$. It is possible to define p-restricted Lie bialgebra.*

Definition 6.17 (Divided power bialgebra)**.** *We say that $(A, \mu, \{\gamma_i\}_{i\in\mathbb{N}}, \Delta)$, where:*

- $A \in \mathrm{Mod}_{\mathbb{K}}$,
- $\mu : A \otimes A \longrightarrow A$,
- $\gamma_i : A \longrightarrow A$
- $\gamma_i : A \longrightarrow A$,
- $\Delta : A \longrightarrow A \otimes A$,

is a divided power bialgebra if $(A, \mu, \{\gamma_i\}_{i\in\mathbb{N}})$ is a divided powers algebra, Δ is co-associative and a map of divided power algebras. In particular (A, μ, Δ) is a commutative bialgebra.

The notion of divided power bialgebras have been studied by André in [And71], Bulliksen and Levin in [GL69], and Block in [Blo85] for its relations with the enveloping algebra of a Lie algebra over a field of positive characteristic and the Hopf algebra associated.

Proposition 6.18. *There exists an $\mathbb{M}$-PROP, denoted by $\Gamma BiAlg_{Com}$, which encodes the category of divided power bialgebras.*

Sketch: Let $BiAlg_{Com}$ be the PROP which governs the category of commutative bialgebras. We denote $m \in BiAlg_{Com}(2,1)$ and $c \in BiAlg_{Com}(1,2)$ the generating elements. We consider the $\mathbb{M}$-PROP $\Gamma(BiAlg_{Com})$.

We prove that $\Gamma(BiAlg_{Com})$-algebras correspond to divided power bialgebras.

Let $\phi : \Gamma(BiAlg_{Com}) \longrightarrow BiPoly_V$ be a $\Gamma(BiAlg_{Com})$-algebra. There exists a monomorphism from the $\mathbb{M}$-PROP defined by the $\mathbb{M}$-operad $\Gamma(Com)$ and $\Gamma(BiAlg_{Com})$ that we denote i. By this inclusion ϕ defines a divided power algebra $(V, \mu, \{\gamma_i\}_{i\in\mathbb{N}})$. The restriction of ϕ to the discrete partitions is equivalent to a $BiAlg_{Com}$-algebra (V, μ, Δ) where $\mu = \phi(m)$ and $\Delta = \phi(c)$. For $i = 2$ we have that:

$$c(m) = m \otimes m((2,3)c \otimes c).$$

Applying this relation to the image by ϕ of $\Gamma_{2,2}(BiAlg_{Com})((1,2),(1)(2))$ we obtain:

$$\Delta(\gamma_2) = \gamma_2 \otimes \gamma_2((2,3)\Delta \otimes \Delta).$$

This is equivalent to say that Δ is compatible with γ_2. Similar computations work for the general γ_i.

Let $(V, \mu, \{\gamma_i\}_{i\in\mathbb{N}}, \Delta)$ be a divided power bialgebra. In particular (V, μ, Δ) is a bialgebra. This is equivalent to a morphism $\psi : BiAlg_{Com} \longrightarrow \mathrm{BiEnd}_V$. We identify indexes of these two PROPs with the discrete partitions and partially extend the morphism ψ by the inductions morphisms. The divided power bialgebra $(V, \mu, \{\gamma_i\}_{i\in\mathbb{N}})$ is in particular a $\Gamma(Com)$-algebra. Extending ϕ by the inclusion of the $\mathbb{M}$-PROP defined by $\Gamma(Com)$ into $\Gamma(BiAlg_{Com})$ we obtain a $\Gamma(BiAlg_{Com})$-algebra structure. □

A Operads and PROPs

The aim of this section is to recall the basic definitions and notions of the theory of operads and PROPS.

We fix a commutative ring $\mathbb{K}$. We denote the category of $\mathbb{K}$-modules by $\mathrm{Mod}_{\mathbb{K}}$. In this section we recall the definitions and properties of symmetric modules, of (symmetric) operads and of (symmetric) PROPs in the category $\mathrm{Mod}_{\mathbb{K}}$.

A.1 Symmetric modules

We recall the definition of the notion of a symmetric module.

Definition A.1 (Symmetric modules)**.** *A symmetric module A is a collection $\{A_n\}_{n\in\mathbb{N}}$ of $\mathbb{K}$-modules with an action of the symmetric group $\mathbb{S}_n$ on A_n for all $n \in \mathbb{N}$.*

A morphism of symmetric modules $f : A \to B$ is a collection of $\mathbb{K}$-morphisms $f_n : A_n \to B_n$ commuting with the symmetric group actions.

We denote the category of symmetric modules by $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}$.

A symmetric module $A = \{A_n\}_{n\in\mathbb{N}}$ is said to be connected if $A_0 = 0$.

The category $\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}$ has three important monoidal structures, namely $\boxtimes, \square_{\mathbb{S}}$ and $\square^{\mathbb{S}}$. The first two correspond to the classical tensor product and to the composition of symmetric modules. They are used to define the notions of operads and algebras over an operad. The product $\square^{\mathbb{S}}$ was introduced by Fresse in [Fre00] and it is used to define the categories of ΓP-algebras or algebras with divided symmetries for any connected operad P.

We recall the definition of unit objects which we associate to these monoidal structures in the paragraph. We explain the definition of the operations $\boxtimes$, $\square_{\mathbb{S}}$, and $\square^{\mathbb{S}}$ afterwards.

Definition A.2. *1. The tensor unit symmetric module $\mathbb{K}$ is the symmetric module*

$$\mathbb{K}_n = \begin{cases} \mathbb{K} & n = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

2. The composition unit symmetric module $\mathbb{I}$ is the symmetric module

$$\mathbb{I}_n = \begin{cases} \mathbb{K} & n = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Definition A.3 (The product $\boxtimes$)**.** *Let $A = \{A_n\}_{n\in\mathbb{N}}$ and $B = \{B_n\}_{n\in\mathbb{N}}$ be symmetric modules. We define the symmetric module $A \boxtimes B$ as follows:*

$$A \boxtimes B = \bigoplus_{i+j=n} \mathrm{Ind}_{\mathbb{S}_i\times\mathbb{S}_j}^{\mathbb{S}_n} A_i \otimes B_j,$$

where $\mathrm{Ind}_{\mathbb{S}_i\times\mathbb{S}_j}^{\mathbb{S}_n} A_i \otimes B_j$ stands for the $\mathbb{K}[\mathbb{S}_n]$-module induced by the $\mathbb{K}[\mathbb{S}_i \times \mathbb{S}_j]$-module $A_i \otimes B_j$.

The product $\boxtimes$ forms a bifunctor. To be explicit, let $f : A \to B$ and $g : A' \to B'$ be symmetric module morphisms. We define $f \boxtimes g$ to be the collection

$$(f \boxtimes g)_n = \bigoplus_{i+j=n} \mathrm{Ind}_{\mathbb{S}_i\times\mathbb{S}_j}^{\mathbb{S}_n} f_i \otimes g_j.$$

Proposition A.4. *The triple $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \boxtimes, \mathbb{K})$ forms a symmetric monoidal category.*

Proof: See [Fre00, Proposition 1.1.6]. □

Definition A.5 (The product $\otimes_{\mathbb{S}_n}$)**.** *Let R and S be $\mathbb{K}[\mathbb{S}_n]$-modules. We denote by $R \otimes_{\mathbb{S}_n} S$ the $\mathbb{K}$-module of coinvariants of the $\mathbb{K}[\mathbb{S}_n]$-module $R \otimes S$ endowed with the diagonal action of $\mathbb{S}_n$. In what follows we use the notation $[x \otimes y]$ for the class of a tensor $x \otimes y \in R \otimes S$ in this quotient.*

Definition A.6 (The product $\square_{\mathbb{S}}$). *Let $A = \{A_n\}_{n\in\mathbb{N}}$ and $B = \{B_n\}_{n\in\mathbb{N}}$ be symmetric modules. We define the symmetric module $A \square_{\mathbb{S}} B$ by:*

$$(A \square_{\mathbb{S}} B)_n = \bigoplus_{r\in\mathbb{N}} A_r \otimes_{\mathbb{S}_r} (B^{\boxtimes r})_n.$$

The product $\square_{\mathbb{S}}$ forms a bifunctor. To be explicit, let $f : A \to B$ and $g : A' \to B'$ be symmetric module morphisms. We define $f \square_{\mathbb{S}} g$ to be the collection

$$(f \square_{\mathbb{S}} g)_n = \sum_{r\in\mathbb{N}} f_r \otimes_{\mathbb{S}_r} (\sum_{t_1+\cdots+t_r=n} g_{t_1} \otimes \cdots \otimes g_{t_r}).$$

Proposition A.7. *The triple $(\mathrm{Mod}^{\mathbb{S}}_{\mathbb{K}}, \square_{\mathbb{S}}, \mathbb{I})$ forms a monoidal category.*

Proof: We refer to [Fre00, Proposition 1.1.9]. □

Definition A.8 (The product $\otimes^{\mathbb{S}_n}$). *Let A and B be $\mathbb{K}[\mathbb{S}_n]$-modules. We denote by $A \otimes^{\mathbb{S}_n} B$ the $\mathbb{K}$-module of invariants of the $\mathbb{K}[\mathbb{S}_n]$-module $A \otimes B$ endowed with the diagonal action of $\mathbb{S}_n$.*

Definition A.9 (The product $\square^{\mathbb{S}}$). *Let $A = \{A_n\}_{n\in\mathbb{N}}$ and $B = \{B_n\}_{n\in\mathbb{N}}$ be symmetric modules. We define the symmetric module $A \square^{\mathbb{S}} B$ by:*

$$(A \square^{\mathbb{S}} B)_n = \bigoplus_{r\in\mathbb{N}} A_r \otimes^{\mathbb{S}_r} (B^{\boxtimes r})_n.$$

The product $\square^{\mathbb{S}}$ forms a bifunctor. To be explicit, let $f : A \to B$ and $g : A' \to B'$ be symmetric module morphisms. We define $f \square^{\mathbb{S}} g$ as the collection

$$(f \square^{\mathbb{S}} g)_n = \sum_{r\in\mathbb{N}} f_r \otimes^{\mathbb{S}_r} (\sum_{t_1+\cdots+t_r=n} g_{t_1} \otimes \cdots \otimes g_{t_r}).$$

Proposition A.10. *The triple $(\mathrm{Mod}^{\mathbb{S}}_{\mathbb{K}}, \square^{\mathbb{S}}, \mathbb{I})$ forms a monoidal category.*

Proof: We refer to [Fre00, Proposition 1.1.9]. □

Let G be a finite group and X be a $\mathbb{K}[G]$-module. We consider the $\mathbb{K}$-module of coinvariant X_G and the $\mathbb{K}$-module of invariant X^G. There is a natural map, called trace or norm map, $tr : X_G \to X^G$ defined by $[x] \mapsto \sum_{g\in G} g^* x$, for any $x \in X$. We apply this observation to our composition product:

Definition A.11 (The natural transformation tr). *Let $A = \{A_n\}_{n\in\mathbb{N}}$ and $B = \{B_n\}_{n\in\mathbb{N}}$ be symmetric modules. We define the morphism of symmetric modules*

$$tr : A \square_{\mathbb{S}} B \to A \square^{\mathbb{S}} B,$$

by

$$tr([a \otimes b_1 \otimes \cdots \otimes b_r]) = \sum_{\sigma\in\mathbb{S}_n} \sigma^*(a \otimes b_1 \otimes \cdots \otimes b_r),$$

for any $[a \otimes b_1 \otimes \cdots \otimes b_r] \in A_r \otimes_{\mathbb{S}_r} (B^{\boxtimes r})_n$.

We use the epi-mono factorization of tr to define a third product $\square_{\mathrm{tr}}$ intermediate between $\square_{\mathbb{S}}$ and $\square^{\mathbb{S}}$:

Proposition A.12. *The natural transformation tr is monoidal, i.e. it preserves unit and associativity isomorphisms.*

Proof: See [Fre00, Lemma 1.1.19]. □

Definition A.13 (The product $\square_{\mathrm{tr}}$). *Let $A = \{A_n\}_{n\in\mathbb{N}}$ and $B = \{B_n\}_{n\in\mathbb{N}}$ be symmetric modules. We define the symmetric module $A \square_{\mathrm{tr}} B$ by:*

$$(A \square_{\mathrm{tr}} B)_n = Im(tr : (A \square_{\mathbb{S}} B)_n \to (A \square^{\mathbb{S}} B)_n),$$

for each $n \in \mathbb{N}$.

The product $\square_{\mathrm{tr}}$ forms a bifunctor. To be explicit, let $f : A \to B$ and $g : A' \to B'$ be symmetric module morphisms. We define $f \square_{\mathrm{tr}} g$ as the collection

$$(f \square_{\mathrm{tr}} g)_n = (f \square^{\mathbb{S}} g)_n\big|_{(A \square_{\mathrm{tr}} B)_n},$$

the restrictions of $(f \square^{\mathbb{S}} g)_n$ to $(A \square_{\mathrm{tr}} B)_n$.

Proposition A.14. *Let $\mathbb{K}$ be a field. The triple $(\mathrm{Mod}^{\mathbb{S}}_{\mathbb{K}}, \square_{\mathrm{tr}}, \mathbb{I})$ forms a monoidal category.*

Proof: We use that tr is monoidal and the observations that $- \square_{\mathbb{S}} -$ preserves the epimorphisms and $- \square^{\mathbb{S}} -$ preserves the monomorphisms to obtain a diagram of the form:

$$\begin{array}{ccccc}
(A \square_{\mathbb{S}} B) \square_{\mathbb{S}} C & \twoheadrightarrow & (A \square_{\mathrm{tr}} B) \square_{\mathrm{tr}} C & \hookrightarrow & (A \square^{\mathbb{S}} B) \square^{\mathbb{S}} C \\
\cong\downarrow & & \exists\,\downarrow & & \downarrow\cong \\
A \square_{\mathbb{S}} (B \square_{\mathbb{S}} C) & \twoheadrightarrow & A \square_{\mathrm{tr}} (B \square_{\mathrm{tr}} C) & \hookrightarrow & A \square^{\mathbb{S}} (B \square^{\mathbb{S}} C).
\end{array}$$

We deduce the associativity diagram for $\square_{\mathrm{tr}}$, the unit follows easily. □

Let G be a group of cardinality n and X be a $\mathbb{K}[G]$-module. If $\mathbb{K}$ is a field of characteristic 0 then the natural map $tr^{-1} : X^G \to X_G$ defined as follows $x \mapsto \frac{1}{n}[x]$ is the inverse of the trace map. Thus, the natural transformation tr is an isomorphism of bifunctors.

Proposition A.15. *If $\mathbb{K}$ is a field of characteristic 0 then the trace induces an isomorphism of monoidal categories*

$$(\mathrm{Mod}^{\mathbb{S}}_{\mathbb{K}}, \square_{\mathbb{S}}, \mathbb{I}) \cong (\mathrm{Mod}^{\mathbb{S}}_{\mathbb{K}}, \square_{\mathrm{tr}}, \mathbb{I}) \cong (\mathrm{Mod}^{\mathbb{S}}_{\mathbb{K}}, \square^{\mathbb{S}}, \mathbb{I}).$$

□

If $\mathbb{K}$ does not contain $\mathbb{Q}$ we still have:

Proposition A.16 (Fresse [Fre00], Proposition 1.1.15). *Let $A = \{A_n\}_{n\in\mathbb{N}}$ and $B = \{B_n\}_{n\in\mathbb{N}}$ be symmetric modules. If B is connected then $tr : A \square_{\mathbb{S}} B \to A \square^{\mathbb{S}} B$ is an isomorphism of symmetric modules.* □

We are interested in symmetric modules because they are combinatorial models of a special kind of endofunctors of the category $\mathrm{Mod}_{\mathbb{K}}$. We explain this correspondence in the following definition.

Definition A.17 (The functors $S(A,-)$, $\Gamma(A,-)$ and $\Lambda(A,-)$). *Let $A = \{A_n\}_{n\in\mathbb{N}}$ be a symmetric module. We have an obvious inclusion $in : \mathrm{Mod}_{\mathbb{K}} \hookrightarrow \mathrm{Mod}^{\mathbb{S}}_{\mathbb{K}}$ such that:*

$$in(V)_n = \begin{cases} V & n = 0, \\ 0 & otherwise. \end{cases}$$

We then consider the functors $S(A,-)$, $\Lambda(A,-)$, and $\Gamma(A,-) : \mathrm{Mod}_{\mathbb{K}} \to \mathrm{Mod}_{\mathbb{K}}$ such that:

$$S(A,V) = A \square_{\mathbb{S}} in(V),$$

$$\Lambda(A,V) = A \square_{\mathrm{tr}} in(V),$$

$$\Gamma(A,V) = A \square^{\mathbb{S}} in(V).$$

We have natural transformations:

$$S(A,-) \to \Lambda(A,-) \to \Gamma(A,-)$$

given by the epi-mono factorization of the trace map on these composition products.

The functor $S(A,-)$ is the standard functor of the theory of operads and is usually called the Schur functor associated to A.

Let A be a symmetric module. In general the functors $S(A,-)$, $\Lambda(A,-)$ and $\Gamma(A,-)$ are not isomorphic. But we have the following statement:

Proposition A.18 (Fresse [Fre00], Proposition 1.1.2). *Let $A = \{A_n\}_{n\in\mathbb{N}}$ be a symmetric module. If A is projective as a symmetric sequence then*

$$tr : S(A,-) \to \Gamma(A,-)$$

is an isomorphism. □

Corollary A.19. *We have that $S(As,-)$ is isomorphic to $\Gamma(As,-)$.* □

The functors $S(-,-)$, $\Lambda(-,-)$, and $\Gamma(-,-)$ are compatible with the monoidal structures $\boxtimes$, $\square_{\mathbb{S}}$, $\square_{\mathrm{tr}}$, and $\square^{\mathbb{S}}$:

Proposition A.20 (Fresse [Fre00], Propositions 1.1.6 and 1.1.9). *The bifunctors $S(-,-)$, $\Gamma(-,-)$ and $\Lambda(-,-) : \mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}} \to Fun(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}})$ are:*

- *(strongly) symmetric monoidal functors with respect to the two symmetric monoidal structures $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \boxtimes, \mathbb{K})$ and $(\mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}}), \otimes, \mathbb{K})$, hence we have:*

$$S(A \boxtimes B,-) \cong S(A,-) \otimes S(B,-), \Gamma(A \boxtimes B,-) \cong \Gamma(A,-) \otimes \Gamma(B,-),$$

  if $\mathbb{K}$ is a field

$$\Lambda(A \boxtimes B,-) \cong \Lambda(A,-) \otimes \Lambda(B,-),$$

- *(strongly) monoidal functors with respect to the two monoidal structures $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \square, \mathbb{I})$ and $(\mathrm{Fun}(\mathrm{Mod}_{\mathbb{K}}, \mathrm{Mod}_{\mathbb{K}}), \circ, \mathrm{Id}_{\mathrm{Mod}_{\mathbb{K}}})$, hence we have:*

$$S(A \square_{\mathbb{S}} B,-) \cong S(A,-) \circ S(B,-), \Gamma(A \square^{\mathbb{S}} B,-) \cong \Gamma(A,-) \circ \Gamma(B,-),$$

  if $\mathbb{K}$ is a field

$$\Lambda(A \square_{\mathrm{tr}} B,-) \cong \Lambda(A,-) \circ \Lambda(B,-).$$

□

A.2 Operads and their associated monads

We recall the definitions and the properties of operads and of the categories of algebras associated to operads.

A.2.1 Operads and algebras over an operad

Since $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \square_{\mathbb{S}}, \mathbb{I})$ is a monoidal category we can define the category of monoids with respect to this structure.

Definition A.21 (Operads). *Let $P = \{P_n\}_{n\in\mathbb{N}}$ be a symmetric module. Let $\mu : P \square_{\mathbb{S}} P \to P$ and $\eta : \mathbb{I} \to P$ be morphisms of symmetric modules. The triple (P, μ, η) is an operad if it is a monoid in the monoidal category $(\mathrm{Mod}_{\mathbb{K}}^{\mathbb{S}}, \square_{\mathbb{S}}, \mathbb{I})$. More explicitly the triple (P, μ, η) is an operad if the following diagrams commute:*

$$\begin{array}{ccc} P \square_{\mathbb{S}} P \square_{\mathbb{S}} P & \xrightarrow{\mathrm{Id}_P \square_{\mathbb{S}} \mu} & P \square_{\mathbb{S}} P \\ {\scriptstyle \mu \square_{\mathbb{S}} \mathrm{Id}_P}\downarrow & & \downarrow{\scriptstyle \mu} \\ P \square_{\mathbb{S}} P & \xrightarrow[\mu]{} & P, \end{array} \qquad \text{(Associativity)}$$

and

$$\xymatrix{\mathbb{I}\,\square_{\mathbb{S}}\,P \ar[r]^{\eta\,\square_{\mathbb{S}}\,\mathrm{Id}_P} \ar[dr] & P\,\square_{\mathbb{S}}\,P \ar[d]^{\mu} & P\,\square_{\mathbb{S}}\,\mathbb{I} \ar[l]_{\mathrm{Id}_P\,\square_{\mathbb{S}}\,\eta} \ar[dl] \\ & P. & } \tag{Unit}$$

Let $(P = \{P_n\}_{n\in\mathbb{N}}, \mu, \eta)$ and $(P' = \{P'_n\}_{n\in\mathbb{N}}, \mu', \eta')$ be operads. A morphism of operads is a morphism of symmetric modules $\phi : P \to P'$ such that the following diagrams commute:

$$\xymatrix{P\,\square_{\mathbb{S}}\,P \ar[r]^{\phi\,\square_{\mathbb{S}}\,\phi} \ar[d]_{\mu} & P'\,\square_{\mathbb{S}}\,P' \ar[d]^{\mu'} \\ P \ar[r]_{\phi} & P',}$$

and

$$\xymatrix{ & P \ar[dd]^{\phi} \\ \mathbb{K} \ar[ur]^{\eta} \ar[dr]_{\eta'} & \\ & P'.}$$

We denote the category of operads by Op*.*

We use that $P\,\square_{\mathbb{S}}\,P$ is spanned by tensors of the form $[p \otimes q_1 \otimes \cdots \otimes q_n]$ with $p \in P_n$ and $q_1, \ldots, q_n \in P$ to give an explicit definition of μ.

Remark A.22. *The general theory of operads allows us to define the free operad generated by a symmetric module, and the ideals of an operad. We can present any operad by generators and relations. Since this theory goes beyond the purpose of this section we do not give more details. For the interested reader we refer to the books of Fresse [Fre09, Section 3.1], Loday and Vallette [LV12, Section 5.5], and Markl, Schnider and Stasheff [MSS02].*

Let $(P = \{P_n\}_{n\in\mathbb{N}}, \mu, \eta)$ be an operad, the elements $p \in P_n$ can be interpreted as n-ary operations and μ as the rule for composing them. The morphism η represents the identity operation. We can present operads by generating operations and relations.

We introduce a different and useful definition of operad structure on a symmetric module.

Definition A.23 (System of partial compositions)**.** *Let $P = \{P_n\}_{n\in\mathbb{N}}$ be a symmetric module. A system of partial compositions $(\{\circ_i\}_{i\in\mathbb{N}^*}, \eta)$ is a collection of $\mathbb{K}$-modules morphisms $- \circ_i - : P_n \otimes P_m \to P_{n+m-1}$ and a morphism of $\mathbb{K}$-module $\eta : \mathbb{K} \to P_1$ such that:*

1. *$- \circ_i - : P_n \otimes P_m \to P_{n+m-1}$ is the zero map if $i > n$,*
2. *$- \circ_i (- \circ_j -) = (- \circ_i -) \circ_{i+j-1} -$, and*
3. *$p \circ_i (\eta(1)) = (\eta(1)) \circ_1 p = p$ for any $p \in P_n$ and $i \leq n$,*

and which respect the symmetric action. That is:

$$x \circ_i \sigma^*(y) = \overline{\sigma}^*(x \circ_i y)$$

for all $\sigma \in \mathbb{S}_n$ where $\overline{\sigma}$ is the $\mathbb{S}_{n+m-1}$ permutation that act as the identity on the set $\{1, \ldots, i-1, i+n, \ldots, n+m\}$ and as σ on the set $\{i, \ldots, i+n-1\}$, and

$$\rho^*(x) \circ_i y = \underline{\rho}^*(x \circ_i y)$$

for all $\rho \in \mathbb{S}_m$ where $\underline{\sigma}$ is the $\mathbb{S}_{n+m-1}$ permutation that act as ρ on the blocks $\{(1), \ldots, (i-1), (i, \ldots, i+n-1), (i+n), \ldots, (n+m)\}$ and identity inside the block $(i, \ldots, i+n-1)$.

Proposition A.24. *Let $P = \{P_n\}_{n\in\mathbb{N}}$ be a symmetric module. An operad structure (P, μ, η) is equivalent to a system of partial compositions $(P, \{\circ_i\}^*_{i\in\mathbb{N}}, \eta)$.*

Proof: For more details see [LV12, Section 5.3.7] □

The compatibility of $S(-,-)$ with the composition products $\square_{\mathbb{S}}$ and $\circ$ has an important consequence. Any monoid with respect to $\square_{\mathbb{S}}$ defines a monoid in the category of endofunctor of $\mathrm{Mod}_{\mathbb{K}}$ with respect to the composition of functors $\circ$, a monad is the usual terminology of category theory:

Proposition A.25. *Let $(P = \{P_n\}_{n\in\mathbb{N}}, \mu, \eta)$ be an operad. The triple $(S(P,-), S(\mu,-), S(\eta,-))$ is a monad.*

Proof: The statement is a direct consequence of Proposition A.20. □

To any monad we associate a category of algebras. Thus, to any operad we associate a category of algebras.

Definition A.26 (P-algebra)**.** *Let $(P = \{P_n\}_{n\in\mathbb{N}}, \mu, \eta)$ be an operad. The category of P-algebras is the category of algebras over the monad $(S(P,-), S(\mu,-), S(\eta,-))$. It is denoted by Alg_P. More explicitly an object of Alg_P is a couple (V,γ) such that the following diagrams commute:*

$$\begin{array}{ccccc} S(P\,\square_{\mathbb{S}}\,P, V) & \xrightarrow{\cong} & S(P, S(P,V)) & \xrightarrow{\mathrm{Id}_P\,\square_{\mathbb{S}}\,\gamma} & S(P,V) \\ \Big\downarrow{\scriptstyle \mu\circ \mathrm{Id}_V} & & & & \Big\downarrow{\scriptstyle \gamma} \\ S(P,V) & & \xrightarrow{\quad\quad\gamma\quad\quad} & & V, \end{array}$$

and

$$\begin{array}{ccc} S(\mathbb{I}, V) & \xrightarrow{\eta\circ\mathrm{Id}_V} & S(P,V) \\ & \searrow & \Big\downarrow{\scriptstyle\gamma} \\ & & V. \end{array}$$

Example A.27. *1. The symmetric module As defined by $As_0 = 0$ and $As_n = \mathbb{K}[\mathbb{S}_n]$ with multiplicative actions for all $n > 0$ is an operad with the composition product such that:*

$$\mu([\rho \otimes \tau_1 \otimes \cdots \otimes \tau_r]) = \tau_{\rho(1)} \oplus \cdots \oplus \tau_{\rho(r)}$$

for $\rho \in \mathbb{S}_r$ and $\tau_i \in \mathbb{S}_{n_i}$ and $\eta = \mathrm{Id}_{\mathbb{K}}$. Alternatively the operad As can be defined as the free operad generated by a binary operation m quotient by the ideal generated by the relation $m(m(-,-),-) = m(-,m(-,-))$.

The category of As-algebras is isomorphic to the category of non unital associative algebras.

2. *The symmetric module Com is defined by $Com_0 = 0$ and $Com_n = \mathbb{K}$ with trivial action for all $n > 0$, is an operad if endowed with the morphisms $\mu = \mathrm{Id}_{\mathbb{K}}$ and $\eta = \mathrm{Id}_{\mathbb{K}}$. Alternatively the operad Com can be defined as the free operad generated by a commutative binary operation c quotient by the ideal generated by the relation $c(c(-,-),-) = c(-,c(-,-))$.*

   The category of Com-algebras is isomorphic to the category of associative commutative algebras,

3. *The symmetric module Lie is defined by $Lie_0 = 0$ and $Lie_n = \mathrm{Ind}^{\mathbb{S}_n}_{\mathbb{Z}/n\mathbb{Z}}(\rho)$, where ρ is the one dimensional representation of the n-cyclic group given by an irreducible nth-root for all $n > 0$. We can define an operad structure on the symmetric module Lie as the free operad generated by an anti-symmetric binary operation $[-,-]$ quotient by the ideal generated by the relation $[[1,2],3] + [[2,3],1] + [[3,1],2] = 0$.*

   The category of Lie-algebras is isomorphic to the category of Lie algebras.

Not every monad is in the image of $S(-,-)$. Operads correspond, in some sense, to the category of monads presented by multilinear operations and multilinear relations between them. The advantage of working with the category of operads instead of the whole category of monads is their combinatorial nature that allows us to make explicit computations.

Definition A.28 (The operad End_V)**.** *Let V be a $\mathbb{K}$-module. We define the symmetric module End_V by:*

$$\mathrm{End}_{V,n} = \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V^{\otimes n}, V),$$

with the symmetric group action induced by the permutation action on $V^{\otimes n}$ for any $n \in \mathbb{N}$. The composition of morphisms in the category $\mathrm{Mod}_{\mathbb{K}}$ and the identity of V gives an operad structure on End_V. We explicit set:

$$\mu([f(\underbrace{-,\dots,-}_{r}) \otimes g_1(\underbrace{-,\dots,-}_{n_1}) \otimes \cdots \otimes g_r(\underbrace{-,\dots,-}_{n_r})]) = f(g_1,\dots g_r)(\underbrace{-,\dots,-}_{n_1+\dots+n_r}),$$

for $f \in \mathrm{End}_{V,r}$, and $g_i \in \mathrm{End}_{V,n_i}$ and

$$\eta(1) = \mathrm{Id}_V.$$

Remark A.29. *The construction of the symmetric module End_V is not functorial on V.*

Proposition A.30. *Let $(P = \{P_n\}_{n\in\mathbb{N}}, \mu, \eta)$ be an operad and V be a $\mathbb{K}$-module. We have the following bijection:*

$$\{\gamma | (V,\gamma) \in \mathrm{Alg}_P\} \cong \mathrm{Hom}_{\mathrm{Op}}(P, \mathrm{End}_V).$$

Let (V,γ) and (V',γ') be P-algebras and $f : V \to V'$ be a $\mathbb{K}$-morphism. It is a morphism of P-algebras if and only if

$$\gamma'(p)(\underbrace{f \otimes \cdots \otimes f}_{n}(\underbrace{-,\dots,-}_{n})) = f(\gamma(p)(\underbrace{-,\dots,-}_{n})),$$

for all $p \in P_n$ and $n \in \mathbb{N}$.

Proof: We refer to Fresse [Fre09, Proposition 3.4.2] and Loday and Vallette [LV12, Proposition 5.2.13]. □

A.2.2 ΛP and ΓP-algebras

Let $\mathbb{K}$ be a field. o Since the map $P \square_{\mathbb{S}} P \to P \square^{\mathbb{S}} P$ induced by the trace is an isomorphism for connected symmetric modules. Hence the category of connected operads coincides with the category of connected monoids with respect to $\square_{\mathrm{tr}}$ and $\square^{\mathbb{S}}$. Let P be a connected operad. We use the compatibility of the functors $\Lambda(-,-)$ and $\Gamma(-,-)$ with the composition to define other two monads associated to P.

Proposition A.31. *Let $(P = \{P_n\}_{n\in\mathbb{N}}, \mu, \eta)$ be a connected operad. The triples*

$$\Lambda(P,-), \Lambda(\mu,-), \Lambda(\eta,-)), \qquad (\Gamma(P,-), \Gamma(\mu,-), \Gamma(\eta,-))$$

are monads such that the morphisms given by the epi-mono factorization of tr:

$$S(P,-) \to \Lambda(P,-) \to \Gamma(P,-)$$

are monad morphisms. □

Definition A.32 (ΛP-algebras)**.** *Let $(P = \{P_n\}_{n\in\mathbb{N}}, \mu, \eta)$ be a connected operad. We define the category of ΛP-algebras as the category of algebras over the monad $(\Lambda(P,-), \Lambda(\mu,-), \Lambda(\eta,-))$.*

Proposition A.33. *Let $(P = \{P_n\}_{n\in\mathbb{N}}, \mu, \eta)$ be a connected operad and A be a ΛP-algebra. The monad $S(P,-)$ acts on A through the morphism $S(P,-) \to \Lambda(P,-)$ so that A inherits a natural P-algebra structure.* □

Let P be a connected operad. Since the functor $\Lambda(P,-)$ is, in general, different from the functor $S(P,-)$ the category of ΛP-algebras is, in general, not equivalent to the category of P-algebras. The category of ΛP-algebras can be interpreted as the subcategory of P-algebras satisfying some additional non-linear relations.

Example A.34. *We have:*

1. *let $\mathbb{K}$ be a field of positive characteristic p, a Com-algebra C is a ΛCom-algebra if $c^p = 0$ for any $c \in C$,*

2. *let $\mathbb{K}$ be a field of characteristic 2, a Lie-algebra L is a ΛLie-algebra if $[l, l] = 0$ for any $l \in L$,*

see [Fre04, Proposition 1.2.15-1.2.16].

Definition A.35 (ΓP-algebras)**.** *Let $(P = \{P_n\}_{n \in \mathbb{N}}, \mu, \eta)$ be a connected operad. We define the category of ΓP-algebras as the category of algebras over the monad $(\Gamma(P, -), \Gamma P(\mu, -), \Gamma P(\eta, -))$.*

Proposition A.36. *Let $(P = \{P_n\}_{n \in \mathbb{N}}, \mu, \eta)$ be a connected operad and A be a ΓP-algebra. The monad $\Lambda(P, -)$ acts on A through the morphism $\Lambda(P, -) \to \Gamma(P, -)$ so that A inherits a natural ΛP-algebra structure.* □

As for ΛP-algebras, since $\Gamma(P, -)$ is, in general, different from $S(P, -)$ the category of ΓP-algebras is, in general, not equivalent to the categories of P-algebras. The category of ΓP-algebras can be interpreted as the category of ΛP-algebras with an additional structure.

Definition A.37 (Divided power algebras)**.** *A divided power algebra is a commutative algebra C endowed with a collection of operations $\gamma_i : C \longrightarrow C$ such that:*

$$\gamma_n(x + y) = \sum_{i=0}^{n} \gamma_{n-i}(x)\gamma_i(y),$$
$$\gamma_i(\lambda x) = \lambda^i \gamma_i(x),$$
$$\gamma_1(x) = x,$$
$$\gamma_m(x)\gamma_n(x) = \binom{m+n}{n}\gamma_{m+n}(x),$$
$$\gamma_m(\gamma_n(x)) = \frac{mn!}{(n!)^m m!}\gamma_{mn}(x).$$

Let C and D be divided power algebras. A commutative algebra morphism $\phi : C \to D$ is a morphism of divided power algebras if

$$\phi(\gamma_i(-)) = \gamma_i(\phi(-))$$

for any $i \in \mathbb{N}$.

Definition A.38 (p-restricted Lie algebras)**.** *Let $\mathbb{K}$ be a field of positive characteristic p. A p-restricted Lie algebra is a Lie algebra L equipped with an operation $-^{[p]} : L \longrightarrow L$ such that:*

$$(\lambda x)^{[p]} = \lambda^p (x)^{[p]},$$
$$(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i(x, y)}{i},$$
$$ad(x^{[p]}) = (ad(x))^{[p]},$$

where $s_i(x, y)$ is the coefficient of t^{i-1} on the expression of $ad_{(tx+y)^{p-1}}(x)$.

Let L and G be p-restricted Lie algebras. A Lie algebra morphism $\phi : L \to G$ is a p-restricted Lie algebra morphism if

$$\phi((-)^{[p]}) = (\phi(-))^{[p]}.$$

Example A.39. *We have:*

1. *the category of ΓCom-algebras is isomorphic to the category of divided power algebras (see Fresse [Fre00, Proposition 1.2.3]),*

2. *let $\mathbb{K}$ be a field of positive characteristic p; the category of ΓLie-algebras is isomorphic to the category of p-restricted Lie algebras (see Fresse [Fre00, Theorem 1.2.5]).*

A.3 PROPs and their algebras

We recall the notion of a PROP and of the category of algebras associated to a PROP. These notions were first introduced by MacLane. We first introduce the concept of symmetric bimodule.

Definition A.40 ((G,H)-modules)**.** *Let G and H be groups. A (G,H)-module is a $\mathbb{K}$-module V endowed with a left G-action and a right H-action such that the two actions commute with each other. A morphism of (G,H)-modules is a morphism of left $\mathbb{K}[G]$-modules and right $\mathbb{K}[H]$-modules.*

Definition A.41 (Symmetric bimodule)**.** *A symmetric bimodule $A = \{A_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ is a collection of $(\mathbb{S}_n,\mathbb{S}_m)$-modules.*

Let A and B be symmetric bimodules. A morphism of symmetric bimodules $f : A \to B$ is a collection $\{f_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ of $(\mathbb{S}_n,\mathbb{S}_m)$-module morphisms.

We denote their category by $\mathrm{BiMod}_{\mathbb{K}}^{\mathbb{S}}$.

We define two monoidal structures, namely $\boxplus$ and $\boxminus$. They correspond to tensor and composition products. We recall the definition of unit objects for these monoidal structures.

Definition A.42.

1. *the horizontal tensor unit $\mathbb{K}$ is the symmetric bimodule defined as follows:*
$$\mathbb{K}_{n,m} = \begin{cases} \mathbb{K} & n = 0 \text{ and } m = 0, \\ 0 & \text{otherwise}, \end{cases}$$
2. *the vertical tensor unit $\mathbb{K}$ is the symmetric bimodule defined as follows:*
$$\mathbb{I}_{n,m} = \begin{cases} \mathbb{K} & n = m, \\ 0 & \text{otherwise}, \end{cases}$$
where we take the trivial action of symmetric groups on $\mathbb{K}$.

Definition A.43 (The product $\boxplus$)**.** *Let $A = \{A_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ and $B = \{B_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ be symmetric bimodules. We define the symmetric bimodule $A \boxplus B$ by:*

$$(A \boxplus B)_{n,m} = \bigoplus_{\substack{n_1+n_2=n \\ m_1+m_2=m}} \mathrm{Ind}_{\mathbb{S}_{n_1}\times\mathbb{S}_{n_2},\mathbb{S}_{m_1}\times\mathbb{S}_{m_2}}^{\mathbb{S}_n,\mathbb{S}_m} A_{n_1,m_1} \otimes A_{n_2,m_2},$$

where

$$\mathrm{Ind}_{\mathbb{S}_{n_1}\times\mathbb{S}_{n_2},\mathbb{S}_{m_1}\times\mathbb{S}_{m_2}}^{\mathbb{S}_n,\mathbb{S}_m}(-) = \mathrm{Ind}_{\mathbb{S}_{n_1}\times\mathbb{S}_{n_2}}^{\mathbb{S}_n}(\mathrm{Ind}_{\mathbb{S}_{m_1}\times\mathbb{S}_{m_2}}^{\mathbb{S}_m}(-)) = \mathrm{Ind}_{\mathbb{S}_{m_1}\times\mathbb{S}_{m_2}}^{\mathbb{S}_m}(\mathrm{Ind}_{\mathbb{S}_{n_1}\times\mathbb{S}_{n_2}}^{\mathbb{S}_n}(-)).$$

The product $\boxplus$ is a bifunctor. To be explicit let $f : A \to B$ and $g : A' \to B'$ be symmetric bimodule morphisms. We define $f \boxplus g : A \boxplus A' \to B \boxplus B'$ by:

$$(f \boxplus g)_{n,m} = \sum_{\substack{n_1+n_2=n \\ m_1+m_2=m}} \mathrm{Ind}_{\mathbb{S}_{n_1}\times\mathbb{S}_{n_2},\mathbb{S}_{m_1}\times\mathbb{S}_{m_2}}^{\mathbb{S}_n,\mathbb{S}_m} f_{n_1,m_1} \otimes g_{n_2,m_2}.$$

Proposition A.44. *The triple $(\mathrm{BiMod}_{\mathbb{K}}^{\mathbb{S}}, \boxplus, \mathbb{K})$ forms a symmetric monoidal category.*

Proof: It easily follow by adapting the proof [Fre00, Proposition 1.1.6]. □

Definition A.45 (The product $\boxminus$)**.** *Let $A = \{A_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ and $B = \{B_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}$ be symmetric bimodules. We define the symmetric bimodule $A \boxminus B$ by:*

$$(A \boxminus B)_{n,m} = \bigoplus_{r\in\mathbb{N}} A_{n,r} \otimes_{\mathbb{S}_r} B_{r,m}.$$

The product $\boxminus$ is a bifunctor. To be explicit let $f : A \to B$ and $g : A' \to B'$ be symmetric bimodule morphisms. We define $f \boxminus g : A \boxminus A' \to B \boxminus B'$ to be the morphism defined such that:

$$(f \boxminus g)_{n,m} = \sum_{r\in\mathbb{N}} f_{n,r} \otimes_{\mathbb{S}_r} g_{r,m}.$$

Proposition A.46. *The triple* $(\mathrm{BiMod}_{\mathbb{K}}^{\mathbb{S}}, \boxminus, \mathbb{I})$ *forms a monoidal category.*

Proof:[Sketch] The unit is given by

$$(A \boxminus \mathbb{I})_{n,m} = \bigoplus_{r \in \mathbb{N}} A_{n,r} \otimes_{\mathbb{S}_r} \mathbb{I}_{r,m} = A_{n,n} \otimes_{\mathbb{S}_n} \mathbb{K} = A_{n,m},$$

the associativity morphism is given by:

$$\begin{aligned}
(A \boxminus B) \boxminus C)_{n,m} &= \bigoplus_{r} (A \boxminus B)_{n,r} \otimes_{\mathbb{S}_r} C_{r,m} \\
&= \bigoplus_{r} (\bigoplus_{s} A_{n,s} \otimes_{\mathbb{S}_s} B_{s,r}) \otimes_{\mathbb{S}_r} C_{r,m} \\
&= \bigoplus_{r} \bigoplus_{s} A_{n,s} \otimes_{\mathbb{S}_s} (B_{s,r} \otimes_{\mathbb{S}_r} C_{r,m}) \\
&= (A \boxminus (B \boxminus C))_{n,m}
\end{aligned}$$

□

Let $A = \{A_{n,m}\}_{(n,m) \in \mathbb{N} \times \mathbb{N}}$ be a symmetric bimodule. As for operads, we want to identify the elements of $A_{n,m}$ with some abstract operations with n inputs and m outputs. A PROP is a symmetric bimodule endowed with a structure that encodes the composition of these abstract operations.

Definition A.47 (PROP)**.** *Let* $P = \{P_{n,m}\}_{(n,m) \in \mathbb{N} \times \mathbb{N}}$ *be a symmetric bimodule,* $\mu_h : P \boxbar P \to P$, $\mu_v : P \boxminus P \to P$, *and* $\eta : \mathbb{I} \to P$ *be symmetric bimodule morphisms. The set of data* (P, μ_h, μ_v, η) *is a PROP if the following diagrams commute:*

$$\begin{array}{ccc}
P \boxbar P \boxbar P & \xrightarrow{\mathrm{Id}_P \,\square_{\mathbb{S}}\, \mu_h} & P \boxbar P \\
{\scriptstyle \mu_h \,\square_{\mathbb{S}}\, \mathrm{Id}_P}\downarrow & & \downarrow{\scriptstyle \mu_h} \\
P \boxbar P & \xrightarrow[\mu_h]{} & P,
\end{array} \qquad \text{(Horizontal associativity)}$$

$$\begin{array}{ccccc}
\mathbb{I}\,\mathbb{K} \boxbar P & \xrightarrow{\eta|_{\mathbb{K}} \boxbar \mathrm{Id}_P} & P \boxbar P & \xleftarrow{\mathrm{Id}_P \boxbar \eta|_{\mathbb{K}}} & P \boxbar \mathbb{K} \\
& \searrow & \downarrow{\scriptstyle \mu_v} & \swarrow & \\
& & P, & &
\end{array} \qquad \text{(Horizontal unit)}$$

$$\begin{array}{ccc}
P \boxminus P \boxminus P & \xrightarrow{\mathrm{Id}_P \,\square_{\mathbb{S}}\, \mu_v} & P \boxminus P \\
{\scriptstyle \mu_v \,\square_{\mathbb{S}}\, \mathrm{Id}_P}\downarrow & & \downarrow{\scriptstyle \mu_v} \\
P \boxminus P & \xrightarrow[\mu_v]{} & P,
\end{array} \qquad \text{(Vertical associativity)}$$

$$\begin{array}{ccccc}
\mathbb{I} \boxminus P & \xrightarrow{\eta \boxminus \mathrm{Id}_P} & P \boxminus P & \xleftarrow{\mathrm{Id}_P \boxminus \eta} & P \boxminus \mathbb{I} \\
& \searrow & \downarrow{\scriptstyle \mu_v} & \swarrow & \\
& & P, & &
\end{array} \qquad \text{(Vertical unit)}$$

and the following equations holds:

$$\mu_h(f \otimes g) = \sigma^*(\mu_h(g \otimes f))\tau_*, \qquad \text{(Horizontal commutativity)}$$

for all $f \in P_{n_1,m_1}$ *and* $g \in P_{n_2,m_2}$ *where* σ *(resp.* τ*) is the permutation in* $\mathbb{S}_{n_1+n_2}$ *(resp.* $\mathbb{S}_{m_1+m_2}$*) which permutes the blocks* $\{1, \ldots, n_1\}$ *and* $\{n_1 + 1, \ldots, n_1 + n_2\}$ *(resp.* $\{1, \ldots, m_1\}$ *and* $\{m_1 + 1, \ldots, m_1 + m_2\}$*) and fix the orders inside the blocks.*

$$\mu_h(\mu_v(f_1, g_1), \mu_v(f_2, g_2)) = \mu_v(\mu_h(f_1, f_2), \mu_h(g_1, g_2)). \qquad \text{(Distributivity)}$$

A morphism of symmetric bimodules $f : P \to Q$ *is a morphism of PROPs if it commutes with all* μ_v, μ_h *and* η.

We denote the category of PROPs by $PROP$.

Proposition A.48. *The category of PROPs is equivalent to the category of symmetric monoidal categories* $(P, \odot, S, e)$ *enriched over* $\mathrm{Mod}_{\mathbb{K}}$ *such that:*

1. *the class of objects is identified with the set of natural numbers* $\mathbb{N}$*,*

2. *the product on objects is defined by* $m \odot n = m + n$ *for any* $m, n \in \mathbb{N}$*.*

Proof: We refer to Markl [Mar08, Section 8] for more details. □

Remark A.49. *As for operads, the general theory of PROPs allows us to define the free PROP generated by a set of operations and ideals generated by relations. Any PROP can be presented by generators and relations.*

Example A.50. *We can define the following PROPs:*

1. *the PROP* $BiAlg$ *is the PROP generated by a product* $m \in BiAlg_{2,1}$ *and a coproduct* $\Delta \in BiAlg_{1,2}$*, quotiented by the ideal generated by the following relations:*

$$m(m(-,-),-) = m(-,m(-,-)), \qquad (\Delta \otimes \mathrm{Id})\Delta(-) = (\mathrm{Id} \otimes \Delta)(\Delta(-)),$$
$$\Delta(m(1,2)) = (m \otimes m)((2,3)^*(\Delta(1), \Delta(2)))$$

2. *the PROP* $Frob$ *is the PROP generated by a product* $m \in Frob_{2,1}$*, a unit* $e \in Frob_{0,1}$*, a coproduct* $\Delta \in Frob_{1,2}$ *and a counit* $c \in Frob_{1,0}$*, quotiented by the ideal generated by the following relations:*

$$m(m(-,-),-) = m(-,m(-,-)), \qquad m(-,e) = m(e,-) = \mathrm{Id}(-),$$
$$(\Delta \otimes \mathrm{Id})\Delta(-) = (\mathrm{Id} \otimes \Delta)(\Delta(-)), \qquad (\mathrm{Id} \otimes c)(\Delta(-)) = (c \otimes \mathrm{Id})(\Delta(-)) = \mathrm{Id}(-),$$

and the Frobenius relation:

$$(\mathrm{Id} \otimes m)(\Delta \otimes \mathrm{Id})(-,-) = (m \otimes \mathrm{Id})(\mathrm{Id} \otimes \Delta)(-,-) = \Delta(m(-,-)),$$

3. *if* $\mathbb{K}$ *has characteristic different from* 2*, the PROP* $BiLie$ *is the PROP generated by an antisymmetric product* $[-,-] \in BiLie_{2,1}$ *and an antisymmetric product* $\delta \in BiLie_{1,2}$*, quotiented by the ideal generated by the following relations:*

$$[[1,2],3] + [[2,3],1] + [[3,1],2] = 0,$$

$$(1,2,3)(\delta \otimes \mathrm{Id})(\delta(-)) + (2,3,1)(\delta \otimes \mathrm{Id})(\delta(-)) + (3,1,2)(\delta \otimes \mathrm{Id})(\delta(-)) = 0,$$

and

$$(1,2)\delta([1,2]) - (1,2)([-,-] \otimes \mathrm{Id})(\mathrm{Id} \otimes \delta)(1,2) - (2,1)([-,-] \otimes \mathrm{Id})(\mathrm{Id} \otimes \delta)(1,2)$$
$$- (2,1)([-,-] \otimes \mathrm{Id})(\mathrm{Id} \otimes \delta)(2,1) - (1,2)([-,-] \otimes \mathrm{Id})(\mathrm{Id} \otimes \delta)(2,1) = 0.$$

Definition A.51 (The PROP End_V)**.** *Let* V *be a* $\mathbb{K}$*-module. The PROP* End_V *is the strict symmetric monoidal category* $(\mathrm{End}_V, \odot, S, e)$ *such that:*

$$\mathrm{End}_{V;n,m} = \mathrm{Hom}_{\mathrm{Mod}_{\mathbb{K}}}(V^{\otimes n}, V^{\otimes m}).$$

The PROP structure is given by the permutation action on $V^{\otimes n}$ *and* $V^{\otimes m}$*, the tensor product and the composition of morphisms in* $\mathrm{Mod}_{\mathbb{K}}$*.*

As for operads, PROPs are combinatorial objects that govern categories of algebras which are described by multilinear operations and multilinear relations. The major difference between operads and PROPs is that PROPs allow operations with more than one outputs. Another important difference between operads and PROPs is that, in general, a PROP is not associated to any monad. To define the category of algebras we are forced to use the PROP End_V.

Definition A.52 (P-algebras)**.** *Let $(P = \{P_{n,m}\}_{(n,m)\in\mathbb{N}\times\mathbb{N}}, \mu_h, \mu_v, \eta)$ be a PROP. A P-algebra is a pair $(V, \phi : P \to \mathrm{End}_V)$ where V is a $\mathbb{K}$-modules and ϕ a morphism of PROPs.*

Let (V, ϕ) and (V', ϕ') be P-algebras. A morphisms of $\mathbb{K}$-module $f : V \to V'$ is a P-algebra morphism if

$$\underbrace{f \otimes \cdots \otimes f}_{m}(\phi(p)(\underbrace{-, \dots, -}_{n})) = \phi'(p)(\underbrace{f \otimes \cdots \otimes f}_{n})(\underbrace{-, \dots, -}_{n}),$$

for any $p \in P_{n,m}$.

We denote the category of algebras over the PROP P by Alg_P.

Example A.53. *We have:*

1. *the category of $BiAlg$-algebras is equivalent to the category of associative, coassociative bialgebras,*
2. *the category of $Frob$-algebras is equivalent to the category of Frobenius algebras,*
3. *the category of $BiLie$-algebras is equivalent to the category of Lie bialgebras.*

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