LECTURE, SUMMER SCHOOL, AUTRANS, JULY 2013

QUANTUM CORRELATIONS IN BIPARTITE QUANTUM SYSTEMS

Dominique SPEHNER, Institut Fourier & Laboratoire de Physique des Milieux Condensés, Grenoble, France.

I). ENTANGLED VS SEPARABLE STATES : PURE STATES

- Consider a quantum system composed of 2 subsystems A and B with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Assume $n = \dim \mathcal{H}_A \leq \dim \mathcal{H}_B < \infty$.

- A <u>pure state</u> of AB is a unit vector $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Any such $|\Psi\rangle$ admits a <u>Schmidt decomposition</u> $\boxed{|\Psi\rangle = \sum_{i=1}^{n} \sqrt{\mu_i}\, |\varphi_i^A\rangle \otimes |\varphi_i^B\rangle}$

  same i

  with $\{|\varphi_i^A\rangle\}_{i=1}^{n}$, $\{|\varphi_j^B\rangle\}_{j=1}^{\dim \mathcal{H}_B}$ ONB. of $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mu_i \geq 0$, $\sum_i \mu_i = 1$

- <u>Reduced states</u> :

$$\boxed{\rho_A = \operatorname{tr}_B |\Psi\rangle\langle\Psi| = \sum_{i=1}^{n} \mu_i \, |\varphi_i^A\rangle\langle\varphi_i^A|}\,, \quad \text{same for } \rho_B$$

$\hookrightarrow$ unicity of the Schmidt decomposition.

$$\boxed{\text{A pure state } |\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \text{ is \underline{separable} if } |\Psi\rangle = \underset{\mathcal{H}_A}{\underset{\cap}{|\alpha\rangle}} \otimes \underset{\mathcal{H}_B}{\underset{\cap}{|\beta\rangle}} \text{ (product state). Otherwise, } |\Psi\rangle \text{ is \underline{entangled}.}}$$

Def. 1 A pure state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is <u>separable</u> if $|\Psi\rangle = |\alpha\rangle \otimes |\beta\rangle$, with $|\alpha\rangle \in \mathcal{H}_A$, $|\beta\rangle \in \mathcal{H}_B$ (product state). Otherwise, $|\Psi\rangle$ is <u>entangled</u>.

- Hence $|\Psi\rangle$ is separable <u>iff</u> all Schmidt coefficients μ_i save one vanish

  <u>iff</u> $\rho_A = |\alpha\rangle\langle\alpha|$ is a pure state (same for ρ_B)

- $|\Psi\rangle$ is <u>maximally entangled</u> <u>iff</u> all μ_i are equal, $\mu_i = \frac{1}{n}$

  <u>iff</u> $\rho_A = \frac{\mathbb{1}}{n}$

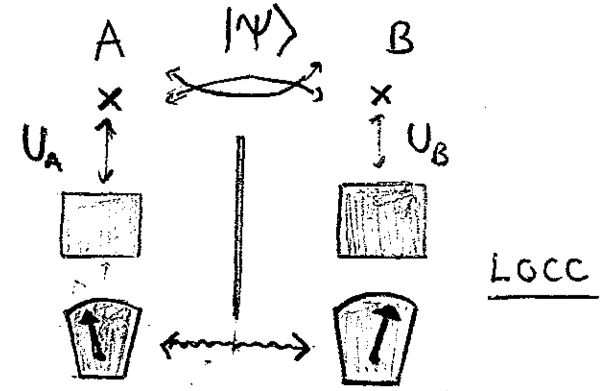
e.g. singlet $|\phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$ (Bell state)

Entanglement measures

(1) Entanglement of Formation

$$\boxed{E_{EoF}(|\Psi\rangle) = S(\rho_A) = S(\rho_B) = -\sum_{i=1}^{n} \mu_i \ln \mu_i}$$

with $S(\rho)$ = von Neumann entropy



- $|\Psi\rangle$ separable iff $E_{EoF}(|\Psi\rangle) = 0$
- $|\Psi\rangle$ max. entangled iff $E_{EoF}(|\Psi\rangle) = \ln n$ maximum

N.B. $\lim_{N\to\infty} \frac{1}{N} E_{EoF}(|\Psi\rangle^{\otimes N})$ = fraction of singlets $|\phi_-\rangle \in \mathbb{C}^4$ per copy of $|\Psi\rangle$ that can be produced from N copies of $|\Psi\rangle$ by Local Operations & Classical Communication when $N \to \infty$

(2) Fubini-Study distance to the set $\mathcal{S}_p$ of pure separable states

$$\boxed{\begin{aligned} E(|\Psi\rangle) = d_{FS}(|\Psi\rangle, \mathcal{S}_p)^2 &= \min_{|\phi\rangle \in \mathcal{S}_p} 2\left(1 - |\langle\Psi|\phi\rangle|\right) \\ &= 2(1-\mu_{max}) \end{aligned}}$$

(3) For any concave function $f : \rho \mapsto f(\rho) \in \mathbb{R}_+$ which is symmetric in the eigenvalues of ρ and such that $f(\lambda_1, \ldots \lambda_k, 0) = f(\lambda_1, \ldots \lambda_k)$, $f(1, 0, \ldots 0) = 0$,

$E_f(|\Psi\rangle) = f(\rho_A) = f(\rho_B)$ is a "good" entanglement measure [VIDAL 2000]

II). ENTANGLED VS SEPARABLE : MIXED STATES

- Pure state convex decompositions of a density matrix ρ

$$\boxed{\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|}, \qquad |\Psi_k\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \ \||\Psi_k\rangle\| = 1, \ p_k \geq 0, \ \sum_k p_k = 1$$

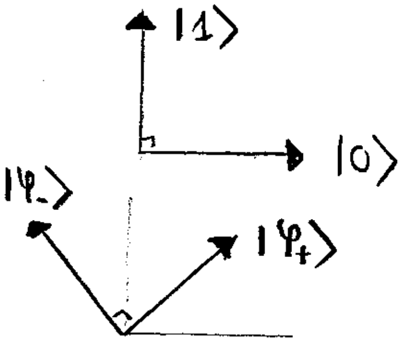
NOT NECESSARILY $\perp$

Physically: system AB prepared in state $|\Psi_k\rangle$ with proba p_k.

There are ∞-many such decompositions ! (related by $\sqrt{p'_j}\,|\Psi'_j\rangle = \sum_k u_{jk} \sqrt{p_k}\,|\Psi_k\rangle$)

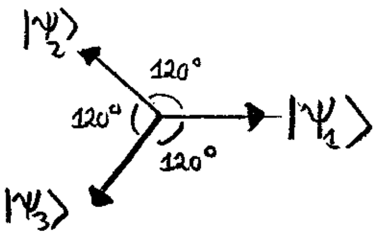
unitary matrix

Ex: $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$



$= \frac{1}{2}|\varphi_+\rangle\langle\varphi_+| + \frac{1}{2}|\varphi_-\rangle\langle\varphi_-|$

$= c\left(|\Psi_1\rangle\langle\Psi_1| + |\Psi_2\rangle\langle\Psi_2| + |\Psi_3\rangle\langle\Psi_3|\right)$



QUANTUM AMBIGUITY

Def 2 [WERNER '89] A mixed state ρ of AB is separable if it admits a pure state convex decomposition with $|\Psi_k\rangle = |\alpha_k\rangle \otimes |\beta_k\rangle \quad \forall k$

- <u>PERES-HOREDECKI criterium</u>: ρ separable $\Rightarrow \rho^{T_B} \geq 0$

with $T_B =$ <u>partial transpose</u> ($\langle \varphi_i^A \otimes \varphi_j^B | \rho^{T_B} | \varphi_k^A \otimes \varphi_e^B \rangle \equiv \langle \varphi_i^A \otimes \varphi_e^B | \rho | \varphi_k^A \otimes \varphi_j^B \rangle$)

Transposition is a positive but not 2-positive map.

For <u>$n = 2 \leq \dim \mathcal{H}_B \leq 3$</u>, converse is also true: $\rho^{T_B} \geq 0 \Rightarrow \rho$ separable

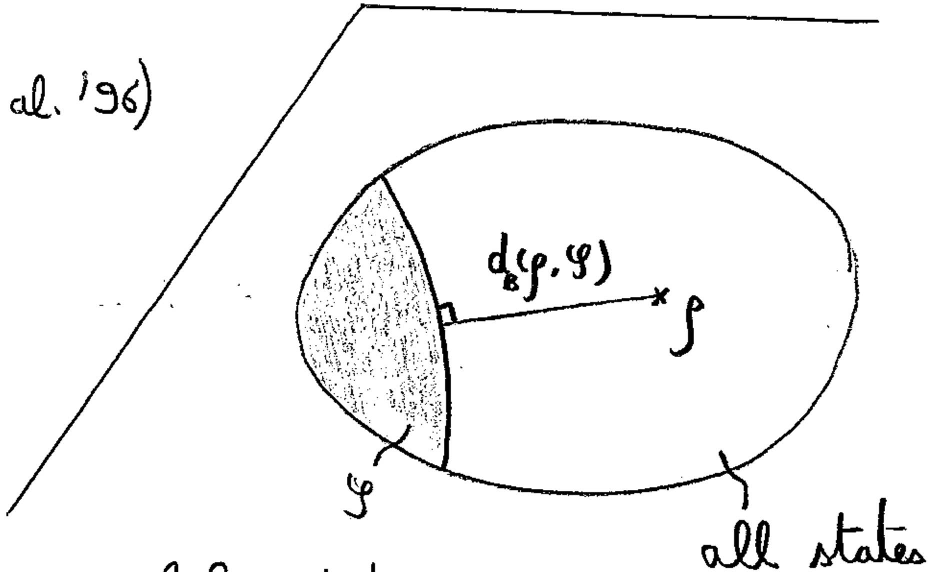
- For $n \times \dim \mathcal{H}_B > 6$, no simple separability criterium!

Entanglement measures

① **Entanglement of formation** (BENNETT et al. '96)

$$\boxed{E_{EoF}(\rho) = \min_{\{|\Psi_k\rangle, p_k\}} \left\{ \sum_k p_k \, E_{EoF}(|\Psi_k\rangle) \right\}}$$

(convex roof)



② **Bures distance to the (convex) set of separable states $\mathcal{S}$:**

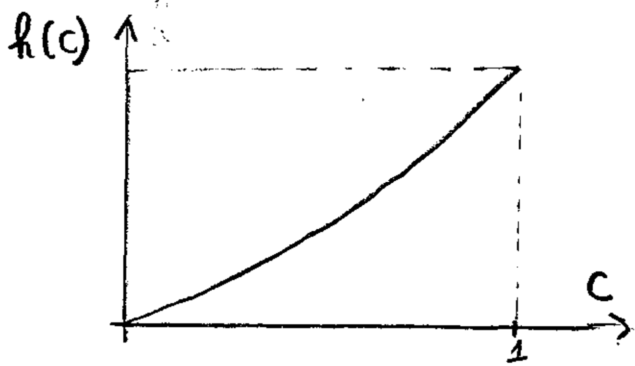
$$\boxed{E(\rho) = d_B(\rho,\mathcal{S})^2 \equiv \min_{\sigma\in\mathcal{S}} 2\Big(1 - \underbrace{\mathrm{tr}\sqrt{\sqrt{\rho}\,\sigma\,\sqrt{\rho}}}_{F(\rho,\sigma)}\Big)}$$

BURES distance

$F(\rho,\sigma)$ = UHLMANN's fidelity ['75]

$$F(\rho,\mathcal{S}) \equiv \max_{\sigma\in\mathcal{S}} F(\rho,\sigma) = \max_{\{|\Psi_k\rangle, p_k\}} \left\{ \sum_k p_k \, F(|\Psi_k\rangle, \mathcal{S}_p) \right\}$$ [STRELTSOV, KAMPERMANN, BRUß '10]

③ The 2-qubit case : Concurrence



Prop [WOOTTERS '97]

If $\mathcal{H}_A \simeq \mathcal{H}_B \simeq \mathbb{C}^2$, then $E_{EoF}(\rho) = h(C(\rho))$ with

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0$ are the square roots of the eigenvalues of the 4×4 matrix $\rho\ \sigma_y \otimes \sigma_y\ \bar{\rho}\ \sigma_y \otimes \sigma_y$ with $\sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

complex conjugation in canonical basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$

Moreover, $C(\rho) = \max\limits_{\{|\Psi_k\rangle, p_k\}} \sum_k p_k\, C(|\Psi_k\rangle)$ with $C(|\Psi\rangle) = |\langle \Psi | \sigma_y \otimes \sigma_y | \bar{\Psi} \rangle|$

N.B.: Similarly, the geometric meas. of entanglement $E(\rho)$ is a simple function of $C(\rho)$

III). NON-CLASSICAL VS CLASSICAL STATES

- Classical stochastic processes:

↳ Correlations between A and B characterized by the mutual information

$$\boxed{I(A:B) = H(A) + H(B) - H(A,B) \geqslant 0}$$ with H = Shannon entropy

$I(A:B) = 0 \Leftrightarrow$ A and B independent.

Then $I(A:B) = \underbrace{H(B) - H(B|A)}_{\text{GAIN OF INFORMATION ON B BY MEASURING A}}$ with $H(B|A) = \sum_i p_A(i)\, H(B|A=i)$

(conditionnal entropy)