Production d'intrication et complexité

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Quantum Systems

• **Pure state**: unit vector in Hilbert space
$\psi \in \mathcal{H}$, $\|\psi\| = 1$, dynamics $\psi_t = \mathrm{e}^{-\mathrm{i}tH}\psi_0$, Hamiltonian H, observables $A = A^* \in \mathcal{B}(\mathcal{H})$, expectation of observable $\langle A \rangle_t = \langle \psi_t, A\psi_t \rangle$

• **Mixed state**: density matrix
$\rho = \rho^* \geq 0$ trace class operator on $\mathcal{H}$, $\mathrm{Tr}\rho = 1$, dynamics $\rho_t = \mathrm{e}^{-\mathrm{i}tH}\rho_0\mathrm{e}^{\mathrm{i}tH}$, expectation of observable $\langle A \rangle_t = \mathrm{Tr}(\rho_t A)$, if ρ is rank-one projection onto $\mathbf{C}\psi$ then it is pure state, otherwise it is a statistical mixture

• **Algebraic formulation**: state on a C^*-algebra (or von Neumann algebra) algebra of observables $\mathcal{A}$, state ω on $\mathcal{A}$, dynamics τ^t a group of *-automorphisms of $\mathcal{A}$, dynamics of observable $A_t = \tau^t(A)$ $(A \in \mathcal{A})$, dynamics of state $\omega_t = \omega \circ \tau^t$, Hilbert space description via GNS construction

Often, physics determines the Hilbert space, e.g. $\mathbf{C}^2$, $L^2(\mathbf{R}^d, \mathrm{d}^d x)$, Fock space. However, sometimes only expectations of (rich class of) observables is known and Hilbert space has to be constructed (Araki-Woods ‘63, Araki-Wyss ‘64, Hellmich-Honegger-Köstler-Kümmerer-Rieckers ‘02)

Quantum (bipartite) Entanglement

• Joint system Hilbert space: $\mathcal{H}_{\rm AB} = \mathcal{H}_{\rm A} \otimes \mathcal{H}_{\rm B}$

• A pure state $\psi \in \mathcal{H}_{\rm AB}$ is called **entangled** if it is NOT a product state. A product state $\psi = \psi_{\rm A} \otimes \psi_{\rm B}$ is called **disentangled**

• Entanglement crucial in quantum theory, it encodes quantum correlations between subsystems A and B

• Pure states $\psi \in \mathcal{H}_{\rm AB}$: **von Neumann entropy** $S(\rho) = -{\rm Tr}(\rho \ln \rho)$ quantifies entanglement (${\rm Tr}_{\rm B}$ is **partial trace**)

$$\psi \text{ product state} \Leftrightarrow S({\rm Tr}_{\rm B}|\psi\rangle\langle\psi|) = 0$$

• **Entanglement of formation** (Bennett et al. '96)

$$\mathcal{E}(\rho) = \inf_{\{p_j,\psi_j\}} \sum_j p_j S({\rm Tr}_{\rm B}|\psi_j\rangle\langle\psi_j|)$$

Infimum taken over constraint $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$

• $\mathcal{E}(\rho)$ very difficult to analyze! However, for **spins** $1/2$ (or **qubits**), $\mathcal{H}_{\rm A} = \mathcal{H}_{\rm B} = \mathbf{C}^2$, Wooters ('98) found the **concurrence** $C(\rho)$, which is related strictly monotonically to $\mathcal{E}(\rho)$

• $C(\rho)$ given explicitly by a function of eigenvalues of a 4×4 matrix built from ρ

• $0 \leq C(\rho) \leq 1$, value zero for disentangled (separable) state ρ, value one for 'maximally entangled state'

• Dynamics of entanglement $t \mapsto C(\rho_t)$ for qubits subject to noise studied in variety of settings: classical and quantum noises, explicitly solvable models, markovian approximations, numerical simulations

Effects of noise on entanglement evolution

- **Entanglement decay and sudden death**

– Yu-Eberly ('04): $C(\rho_t) \le \mathrm{e}^{-\gamma t} C(\rho_0)$, some $\gamma > 0$

Markovian approx., spins coupled to local zero-temperature cavity reservoir via energy exchange

– *Sudden death of entanglement*: $\exists \rho_0$ s.t. $C(\rho_t) = 0 \; \forall t \ge t_D$

– But also: for purely energy conserving interactions and classical (commutative, stochastic) noises can have entanglement-free subspaces.

- **Entanglement sudden death and revival**

Bellomo et al. ('07): $\exists \rho_0$ s.t. initial entanglement dies out, then reappears with smaller amplitude

Explicitly solvable, non markovian energy-exchange model. Entanglement is shifted from spins to reservoirs (initially unentangled), then shifted back to spins with some loss

• **Entanglement creation due to common noise**

Braun ('02), Benatti et al. ('03): Initially unentangled states of the spins acquire entanglement (for intermediate times), due to the interaction with a *common reservoir*

• Above works consider explicitly solvable models, markovian approximations or do numerical analysis.

We derive entanglement survival/death times rigorously (Berman-Borgonovi-Merkli ('11)) for *not* explicitly solvable models, starting with *true Hamiltonian description*

COMPLEXITY? (quantum registers, quantum biology...)

Instead of $N = 2$, consider $N \gg 1$ qubits coupled to a common heat bath (quantum noise).

How does entanglement creation depend on N?

Short answer: complexity (N large) suppresses entanglement creation

Complexity and entanglement creation

Model of N qubits coupled to common heat bath
(Berman-Borgonovi-Merkli-Tsifrinovich '12)

– Hilbert space

$$\mathcal{H}_N = \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2 \otimes \mathcal{F}_\beta$$

$\mathbf{C}^2$: spin (qubit) Hilbert space
$\mathcal{F}_\beta$: GNS Hilbert space of bosonic heat bath at temperature $1/\beta > 0$
– Dynamics: unitary group $\mathrm{e}^{\mathrm{i}tH_N}$ on $\mathcal{H}_N$ generated by Hamiltonian

$$H_N = -\omega \sum_{n=1}^{N} S_n^z + H_{\mathrm{Reservoir}} + \kappa \sum_{n=1}^{N} S_n^z \otimes \phi(f)$$

("energy conserving model") Coupling constant $\kappa \in \mathbf{R}$.

- Reduced two-spin density matrix (4×4 matrix)

$$\rho_N(t) = \mathrm{Tr}_{3,\ldots,n,\mathrm{R}}\big(\mathrm{e}^{-\mathrm{i}tH_N}\ (\sigma \otimes \sigma \cdots \otimes \sigma \otimes \sigma_{\mathrm{R},\beta})\ \mathrm{e}^{\mathrm{i}tH_N}\big)$$

- Matrix elements in energy basis: $[\rho_N(t)]_{jj} = [\rho_N(0)]_{jj}$ and e.g.

$$[\rho_N(t)]_{12} = [\rho_N(0)]_{12}\ \mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{e}^{-\mathrm{i}\kappa^2 S(t)}\ \mathrm{e}^{-\kappa^2\Gamma(t)}\ P_N(t)$$

– $S(t) \geq 0$: phase, $\Gamma(t) \geq 0$: decoherence function; both $\propto t$ for t large
– $P_N(t)$: oscillatory term due to $N-2$ traced-out spins, $|P_N(t)| \leq 1$
– as $N \to \infty$, $P_N(t) \to 0$ except for discrete peak times t_p, width of peaks are of order $(\kappa^2\sqrt{N})^{-1}$

$\Rightarrow$ **scale κ with N:** $\kappa \to \kappa/N^\eta$ **for some** $\eta \geq 0$

Proposition. *Let $\eta \geq 0$ and $t \geq 0$ be fixed. The concurrence of the reduced two-spin state satisfies*

$$\lim_{N\to\infty} C(\rho_N(t)) = 0.$$

Conclusion: No entanglement can be created (at any time) for $N \to \infty$. However, for $N = 2$, entanglement **is** created (Braun). So we expect entanglement creation to decrease as N increases.

Let

$$C_{\max}(N) = \sup_{t\geq 0} C(\rho_N(t)).$$

Numerical calculation: $C_{\max}(N)$ decreases exponentially in N, for all $\eta \geq 0$.

We are lead to the fact that **complexity can cause independence** (factorization). In physics, this effect appears in **mean field theories**.

From wikipedia:

"The main idea of MFT is to replace all interactions to any one body with an average or effective interaction, sometimes called a molecular field. This reduces any multi-body problem into an effective one-body problem."

Goal: Prove that MFT is correct for open quantum systems

Mean field evolution of open quantum systems

An exactly solvable model (Berman-Merkli '12): N identical quantum systems interacting indirectly via common quantum heat bath (noise)

Hilbert space

$$\mathcal{H}_N = \mathcal{H} \otimes \cdots \otimes \mathcal{H} \otimes \mathcal{F}_\beta$$

$\dim \mathcal{H} = d < \infty$, $\mathcal{F}_\beta$ is GNS Hilbert space of spatially infinitely extended KMS state of free Bose field (heat bath)

Dyanmics $\mathrm{e}^{\mathrm{i}tH_N}$ generated by **mean field scaled Hamiltonian**

$$H_N = \sum_{j=1}^N A_j + K + \frac{\kappa}{\sqrt{N}} \sum_{j=1}^N W_j \otimes \phi(f)$$

$A_j = A$ acting on jth factor, $W_j = W$ on jth factor and reservoir, energy-conserving property: $AW = WA$

Reduced n-body density matrix

$$\rho_{n,N}(t) = \mathrm{Tr}_{[n+1,N],R}\, \mathrm{e}^{-\mathrm{i}tH_N} \rho_0 \otimes \cdots \otimes \rho_0 \otimes \rho_\beta\, \mathrm{e}^{\mathrm{i}tH_N}$$

ρ_0: initial state of systems, ρ_β: KMS state of reservoir

Theorem (Convergence to Hartree-Lindblad dynamics) *Fix* $t \in \mathbf{R}$ *and* $n \geq 1$. *Then*

$$\lim_{N\to\infty} \mathrm{Tr}\Big|\rho_{n,N}(t) - \rho_1(t) \otimes \cdots \otimes \rho_1(t)\Big| = 0.$$

The single particle density matrix $\rho_1(t)$ *satisfies*

$$\mathrm{i}\dot\rho_1 = [A, \rho_1] + \kappa^2 \mathrm{Tr}_2[W_{\mathrm{eff}}(t), \rho_1 \otimes \rho_1]$$

where $W_{\mathrm{eff}}(t) = 2\dot S(t)\ W \otimes W$ *is a time-dependent effective two-body interaction (with explicit function* $S(t) \in \mathbf{R}$*).*

Derivation of nonlinear dynamics for C^* dynamical systems

Local dynamical systems

- L: a countable infinite set of 'sites' x
- At each $x \in L$ there is a single-site C^*-dynamical system $(\mathcal{A}, \tau_0^t)$

  $\mathcal{A}$ is the **algebra of observables**, a unital C^*-algebra

  τ_0^t is the **dynamics**, a $*$-automorphism group of $\mathcal{A}$, strongly continuous:

$$\lim_{t\to 0} \|\tau_0^t(A) - A\| = 0 \qquad \forall A \in \mathcal{A}$$

- For a finite region $\Lambda \subset L$ we have the C^*-dynamical system $(\mathcal{A}_\Lambda, \tau_{0,\Lambda}^t)$

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{A} \qquad \tau_{0,\Lambda}^t = \bigotimes_{x\in\Lambda} \tau_0^t$$

- The **interaction** is given by a ν-site interaction operator

$$\Phi_\Lambda = \frac{1}{|\Lambda|^{\nu-1}} \sum_{x_1,\dots,x_\nu \in \Lambda} \varphi_\Lambda(x_1,\dots,x_\nu) \in \mathcal{A}_\Lambda$$

where the sum is over all $(x_1,\dots,x_\nu) \in \Lambda^\nu$. Here, $\varphi_\Lambda(x_1,\dots,x_\nu) \in \mathcal{A}_\Lambda$ is a self-adjoint operator which acts non-trivially only on the factors of $\mathcal{A}_\Lambda$ with indices $x_1,\dots,x_\nu$, is permutation invariant and vanishes if two x are the same. $\varphi_\Lambda(x_1,\dots,x_\nu)$ is identified with

$$\varphi \in \otimes_{k=1}^{\nu} \mathcal{A} \quad \text{such that} \quad \Pi_\beta(\varphi) = \varphi$$

for any bijection β of $\{1,\dots,\nu\}$, where

$$\Pi_\beta(A_1 \otimes \cdots \otimes A_\nu) = A_{\beta(1)} \otimes \cdots \otimes A_{\beta(\nu)}$$

• **Interacting dynamics:** strongly continuous $*$-automorphism group of $\mathcal{A}_\Lambda$, defined by the norm-convergent Araki-Dyson series

$$\begin{aligned}\tau_\Lambda^t(A) &= \tau_{0,\Lambda}^t(A) + \sum_{n=1}^\infty \mathrm{i}^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \\ &\qquad \left[\tau_{0,\Lambda}^{t_n}(\Phi_\Lambda), \left[\tau_{0,\Lambda}^{t_{n-1}}(\Phi_\Lambda), \ldots \left[\tau_{0,\Lambda}^{t_1}(\Phi_\Lambda), \tau_{0,\Lambda}^t(A)\right] \ldots\right]\right]\end{aligned}$$

• Let Λ be finite and let ω_Λ be a state on $\mathcal{A}_\Lambda$. For $\Lambda' \subset \Lambda$ we define the **reduction** of ω_Λ to $\mathcal{A}_{\Lambda'}$ by

$$\omega_{\Lambda',\Lambda}^t(A) = \omega_\Lambda\left(\tau_\Lambda^t(A \otimes \mathbb{1}_{\Lambda\setminus\Lambda'})\right)$$

GOALS:

1. Show that $\omega_{\Lambda',\Lambda}^t$ has a limit as $|\Lambda| \to \infty$, for any fixed finite Λ' and t.
2. Derive the evolution equation for the limit state.

Theorem 1. *Let $\Lambda^{(k)}$, $k \geq 1$, be an increasing sequence of finite regions such that $\lim_{k\to\infty} |\Lambda^{(k)}| = \infty$. For each $k \geq 1$, let $\omega_{\Lambda^{(k)}}$ be a symmetric state on $\mathcal{A}_{\Lambda^{(k)}}$, such that $\lim_{k\to\infty} \omega_{\Lambda',\Lambda^{(k)}} \equiv \omega_{\Lambda',\infty}$ exists for any finite Λ' (limit in norm of states). Then*

$$\lim_{k\to\infty} \omega^t_{\Lambda',\Lambda^{(k)}} \equiv \omega^t_{\Lambda',\infty}$$

exists for any finite Λ' and any $t \geq 0$ (limit in norm of states).

Any product state of the form

$$\omega_{\Lambda^{(k)}} = \bigotimes_{x\in\Lambda^{(k)}} \mu,$$

where μ is a fixed state on $\mathcal{A}$, satisfies the conditions of the theorem.

What is the dynamical equation of $\omega^t_{\Lambda',\infty}$?

• Let δ be the derivation of the single site dynamics,

$$-\mathrm{i}\partial_t \tau_0^t(A) = \tau_0^t\big(\delta(A)\big)$$

• Suppose μ^t, $0 \le t \le T$ (some $T > 0$), is a state on $\mathcal{A}$ solving the nonlinear differential equation

$$-\mathrm{i}\partial_t \mu^t = \mu^t \circ \delta + \nu \left(\otimes_{k=1}^{\nu} \mu^t\right)([\varphi, \cdot\,]),$$

with initial condition $\mu^0 \equiv \mu$.

Theorem 2. *If $\omega_{\Lambda^{(k)}} = \bigotimes_{x\in\Lambda^{(k)}} \mu$, for all $k \ge 1$, then the initial reduced state $\omega_{\Lambda',\infty} = \bigotimes_{x\in\Lambda'} \mu$ stays a product state for all $0 \le t \le T$, and for all finite Λ',*

$$\omega^t_{\Lambda',\infty} = \bigotimes_{x\in\Lambda'} \mu^t.$$

Motivation and outlook

Last two theorems are based on a paper by H. Spohn (Rev.Mod.Phys. '80)

Spohn derives the Hartree equation using the BBGKY hierarchy, for single site systems given by density matrices on a Hilbert space. Spohn's dynamics is given by a bounded single-site Hamiltonian and a bounded two-site interaction. Here, we

– dispose of the Hilbert space representation of all quantum systems

– allow for ν-site interactions

A motivation. Complex open systems: each site hosts a system-reservoir(s) type open quantum system. Reservoir Hilbert space representations are known only in special cases (e.g. Araki-Woods, Araki-Wyss representations for free quantum gases).

Outlook. Develop the mean field theory for indirectly interacting systems (coupling via reservoirs) and analyze entanglement production as function of N.