Von Neumann algebras and ergodic theory of group actions

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Today’s talk

**Functional analysis**
- Hilbert space operators,
- Von Neumann algebras.

**Group theory**
- free groups,
- hyperbolic groups.

**Ergodic theory**
- of group actions,
- of equivalence relations.
Examples of **bounded operators** on a Hilbert space:

- Operators on $\mathbb{C}^n \leftrightarrow n \times n$ matrices.
- Given $f \in L^\infty(X)$, we have the multiplication operator on $L^2(X)$ given by $\xi \mapsto f\xi$.
- The one-sided shift on $\ell^2(\mathbb{N})$ given by $\delta_n \mapsto \delta_{n+1}$.
  
  Notation : $(\delta_n)_{n \in \mathbb{N}}$ is the canonical orthonormal basis of $\ell^2(\mathbb{N})$.

Every bounded operator $T$ has an adjoint $T^*$ determined by

$$\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle.$$

Self-adjoint operators : $T = T^*$,

unitary operators : $U^*U = 1 = UU^*$,

normal operators : $T^*T = TT^*$. 
The weak topology on $B(H)$. Let $T_n, T \in B(H)$.

We say that $T_n \to T$ weakly if $\langle T_n \xi, \eta \rangle \to \langle T \xi, \eta \rangle$ for all $\xi, \eta \in H$.

Example. If $u \in B(\ell^2(\mathbb{N}))$ is the one-sided shift, then $u^n \to 0$ weakly.

Definition

A **von Neumann algebra** is a weakly closed, self-adjoint algebra of operators on a Hilbert space.

Examples: $B(H)$, in particular $M_n(\mathbb{C})$, $L^\infty(X)$ as multiplication operators on $L^2(X)$.

Von Neumann’s bicommutant theorem (1929)

Let $M \subset B(H)$ be a self-adjoint algebra of operators with $1 \in M$. TFAE

- Analytic property: $M$ is weakly closed.
- Algebraic property: $M = M''$, where $A'$ is the commutant of $A$. 
Group von Neumann algebras

Let $\Gamma$ be a countable group.

- A unitary representation of $\Gamma$ on a Hilbert space $H$ is a map $\pi : \Gamma \to B(H) : g \mapsto \pi_g$ satisfying
  - all $\pi_g$ are unitary operators,
  - $\pi_g \pi_h = \pi_{gh}$ for all $g, h \in \Gamma$.

- The regular representation of $\Gamma$ is the unitary representation $\lambda$ on $\ell^2(\Gamma)$ given by $\lambda_g \delta_h = \delta_{gh}$, where $(\delta_h)_{h \in \Gamma}$ is the natural orthonormal basis of $\ell^2(\Gamma)$.

Definition (Murray - von Neumann, 1943)

The group von Neumann algebra $L\Gamma$ is defined as the weakly closed linear span of $\{\lambda_g \mid g \in \Gamma\}$.

As we shall see, the relation between the group $\Gamma$ and its von Neumann algebra $L\Gamma$ is extremely subtle.
The free groups

The free group $\mathbb{F}_2$ is defined as “the group generated by $a$ and $b$ subject to no relations”.

- Elements of $\mathbb{F}_2$ are reduced words in the letters $a, a^{-1}, b, b^{-1}$, like $aba^{-1}a^{-1}b$, or like $bbbbbbba^{-1}bbbb$.
- Reduced means: no $aa^{-1}$, no $b^{-1}b$, ... in the word, because they “simplify”. So $bbaa^{-1}a$ is not reduced. It reduces to $bba$.
- Group operation: concatenation followed by reduction.

Similarly: the free group $\mathbb{F}_n$ with $n = 2, 3, ...$ free generators.

Big open problem (Murray - von Neumann, 1943)

Are the free group factors $\mathbb{L}\mathbb{F}_n$ isomorphic for different $n = 2, 3, ...$?

(Voiculescu 1990, Radulescu 1993)

They are either all isomorphic, or all non-isomorphic.
**II$_1$ factors**

- **A factor** is a von Neumann algebra $M$ with trivial center. Equivalently: $M$ is not the direct sum of two von Neumann algebras.

- **A II$_1$ factor** is a factor $M$ that admits a trace, i.e. a linear functional $\tau : M \to \mathbb{C}$ such that $\tau(xy) = \tau(yx)$ for all $x, y \in M$.

In a way, all von Neumann algebras can be assembled from II$_1$ factors.

The group von Neumann algebra $L\Gamma$ is a factor iff $\Gamma$ has infinite conjugacy classes: for every $g \neq e$, the set $\{hgh^{-1} \mid h \in \Gamma\}$ is infinite.

$L\Gamma$ always has a trace, namely $\tau(x) = \langle x\delta_e, \delta_e \rangle$, satisfying $\tau(\lambda_g) = 0$ for all $g \neq e$ and $\tau(1) = \tau(\lambda_e) = 1$.

All icc groups $\Gamma$ give us II$_1$ factors $L\Gamma$, but their structure is largely non-understood.
Amenable groups and Connes’s theorem

Definition (von Neumann, 1929)

A group $\Gamma$ is called **amenable** if there exists a finitely additive probability $m$ on all the subsets of $\Gamma$ that is translation invariant: $m(gU) = m(U)$ for all $g \in \Gamma$ and $U \subset \Gamma$.

Non-amenable groups $\iff$ Banach-Tarski paradox.

Theorem (Connes, 1976)

All amenable groups $\Gamma$ with infinite conjugacy classes have isomorphic group von Neumann algebras $L\Gamma$.

Actually: there is a unique “amenable” $II_1$ factor.

The following groups are amenable.

- Abelian groups, solvable groups.
- Stable under subgroups, extensions, direct limits.
▶ **Open problem**: are the group von Neumann algebras $L(SL(n, \mathbb{Z}))$ for $n = 3, 4, \ldots$, non-isomorphic?

▶ Conjecturally, they are all non-isomorphic.

▶ The groups $SL(n, \mathbb{Z})$ with $n = 3, 4, \ldots$, have Kazhdan’s property (T).

**Conjecture (Connes, 1980)**

Let $\Gamma, \Lambda$ be icc groups with Kazhdan’s property (T). Then $L\Gamma \cong L\Lambda$ if and only if $\Gamma \cong \Lambda$. 
**Theorem (Ioana-Popa-V, 2010)**

There are countable groups $G$ such that $LG$ entirely remembers $G$: if $Λ$ is an arbitrary countable group with $LG \cong LΛ$, then $G \cong Λ$.

These groups are of the form $G = (\mathbb{Z}/2\mathbb{Z})^{(I)} \rtimes Γ$.

**Theorem (Berbec-V, 2012)**

The same is true for $G = (\mathbb{Z}/2\mathbb{Z})^{(Γ)} \rtimes (Γ \times Γ)$, for many groups $Γ$, including the free groups and arbitrary free product groups $Γ = Γ_1 * Γ_2$. 
Recall: it is wide open whether the $\mathbb{L}F_n$ with $n = 2, 3, \ldots$, are isomorphic.

**Theorem (Popa-V, 2011)**

The group measure space $\text{II}_1$ factors $L^\infty(X) \rtimes F_n$, arising from free ergodic probability measure preserving actions $F_n \curvearrowright X$, are non-isomorphic for different values of $n = 2, 3, \ldots$, independently of the actions.

Group $\Gamma$: von Neumann algebra $L\Gamma$.

Group action $\Gamma \curvearrowright X$: von Neumann algebra $L^\infty(X) \rtimes \Gamma$.

We will explain all these concepts, and some ideas behind the theorem.
We study actions of countable groups $\Gamma$ on probability spaces $(X, \mu)$ by
- measurable transformations,
- that preserve the measure $\mu$.

We call $\Gamma \curvearrowright (X, \mu)$ a **pmp action**.

**Favorite examples:**

- **Irrational rotation** $\mathbb{Z} \curvearrowright \mathbb{T}$ given by $n \cdot z = \exp(2\pi i \alpha n) z$ for a fixed $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

- **Bernoulli action** $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ given by $(g \cdot x)_h = x_{hg}$.

- The action $\text{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$.

- The action $\Gamma \curvearrowright G/\Lambda$ for **lattices** $\Gamma, \Lambda < G$ in a Lie group $G$. 
Group measure space construction (MvN, 1943)

Data: a pmp action $\Gamma \curvearrowright (X, \mu)$.

Output: von Neumann algebra $M = L^\infty(X) \times \Gamma$ with trace $\tau: M \to \mathbb{C}$.

- **Notations:** $g \in \Gamma$ acts on $x \in X$ as $g \cdot x$.
  
  Then, $g$ acts on a function $\xi: X \to \mathbb{C}$ as $(g \cdot \xi)(x) = \xi(g^{-1} \cdot x)$.

- $L^\infty(X) \rtimes \Gamma$ is generated by unitaries $(u_g)_{g \in \Gamma}$ and a copy of $L^\infty(X)$.

- We have the algebraic relations (cf. semidirect products of groups)
  
  - $u_g u_h = u_{gh}$ and $u_g^* = u_{g^{-1}}$,
  
  - $u_g F u_g^* = g \cdot F$.

- Trace: $\tau(F) = \int F \, d\mu$ and $\tau(F u_g) = 0$ for $g \neq e$. 
Essential freeness and ergodicity

We do not want trivial actions.

$\Gamma \curvearrowright (X, \mu)$ is called (essentially) free if for all $g \neq e$ and almost every $x \in X$, we have that $g \cdot x \neq x$.

Equivalently, $L^\infty(X)$ is a maximal abelian subalgebra of $L^\infty(X) \rtimes \Gamma$.

We do not want the union of two actions.

$\Gamma \curvearrowright (X, \mu)$ is called ergodic if $\Gamma$-invariant subsets of $X$ have measure 0 or 1.

If $\Gamma \curvearrowright (X, \mu)$ is free and ergodic, then $L^\infty(X) \rtimes \Gamma$ is a $\text{II}_1$ factor.
Theorem (Popa-V, 2011)

If $F_n \curvearrowright X$ and $F_m \curvearrowright Y$ are free ergodic pmp actions with $n \neq m$, then $L^\infty(X) \rtimes F_n \not\cong L^\infty(Y) \rtimes F_m$.

Approach to this theorem:

- Special role of $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$: a Cartan subalgebra.
- Uniqueness problem of Cartan subalgebras.
- Countable pmp equivalence relations.
- Gaboriau’s work on orbit equivalence relations for free groups.
**Definition**

A **Cartan subalgebra** $A$ of a II$_1$ factor $M$ is

- a maximal abelian subalgebra: $A' \cap M = A$,
- such that the normalizer $\mathcal{N}_M(A) := \{ u \in \mathcal{U}(M) \mid uAu^* = A \}$ spans a weakly dense subalgebra of $M$.

**Typical example** : $L^\infty(X)$ is a Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$ whenever $\Gamma \bowtie X$ is a free, ergodic, pmp action.

**Main questions**, given a II$_1$ factor $M$ : 

- Does $M$ have a Cartan subalgebra?
- If yes, is it unique?
  
  - up to unitary conjugacy: $A = uBu^*$ for some $u \in \mathcal{U}(M)$,
  - up to automorphic conjugacy: $A = \alpha(B)$ for some $\alpha \in \text{Aut}(M)$. 
Non-uniqueness and non-existene of Cartan subalgebras

Connes-Jones, 1981: \( \text{II}_1 \) factors with at least two Cartan subalgebras that are non-conjugate by an automorphism.

Uniqueness of Cartan subalgebras seemed hopeless.

Voiculescu, 1995: \( \text{L}^\infty F_n, 2 \leq n \leq \infty \), has no Cartan subalgebra.

Ozawa-Popa, 2007: the \( \text{II}_1 \) factor \( M = \text{L}^\infty (\mathbb{Z}_p^2) \rtimes (\mathbb{Z}^2 \rtimes \text{SL}(2, \mathbb{Z})) \) has two non-conjugate Cartan subalgebras, namely \( \text{L}^\infty (\mathbb{Z}_p^2) \) and \( \text{L}(\mathbb{Z}^2) \).

Speelman-V, 2011: \( \text{II}_1 \) factors with many Cartan subalgebras (where “many” has a descriptive set theory meaning).
Uniqueness of Cartan subalgebras

**Theorem (Popa-V, 2011)**

Let $\mathbb{F}_n \curvearrowright X$ be an arbitrary free ergodic pmp action. Then $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$, up to unitary conjugacy.

But before that very general result, there was:

**Theorem (Ozawa-Popa, 2007)**

Same conclusion if $\mathbb{F}_n \curvearrowright X$ is a profinite free ergodic pmp action.

Profinite actions: $\Gamma \curvearrowright \varprojlim \Gamma/\Gamma_n$ where $\Gamma_n$ is a decreasing sequence of finite index subgroups of $\Gamma$.

Profinite crossed products $L^\infty(X) \rtimes \mathbb{F}_n$ have a very special approximation property. Several conceptual novelties were needed to deal with non profinite actions.
Proposition (Singer, 1955). The following are equivalent.

- There exists an isomorphism \( \pi : \mathbb{L}^\infty(X) \rtimes \Gamma \rightarrow \mathbb{L}^\infty(Y) \rtimes \Lambda \) with \( \pi(\mathbb{L}^\infty(X)) = \mathbb{L}^\infty(Y) \).
- The actions \( \Gamma \curvearrowright X \) and \( \Lambda \curvearrowright Y \) are orbit equivalent.

Orbit equivalence: existence of a measure preserving isomorphism \( \Delta : X \rightarrow Y \) such that \( \Delta(\Gamma \cdot x) = \Lambda \cdot \Delta(x) \) for a.e. \( x \in X \).

Corollary of our uniqueness of Cartan theorem: if \( \mathbb{L}^\infty(X) \rtimes \mathbb{F}_n \) is isomorphic with \( \mathbb{L}^\infty(Y) \rtimes \mathbb{F}_m \), then the actions \( \mathbb{F}_n \curvearrowright X \) and \( \mathbb{F}_m \curvearrowright Y \) must be orbit equivalent.

Remaining question: do free groups \( \mathbb{F}_n, \mathbb{F}_m \) with \( n \neq m \) admit orbit equivalent actions?
Gaboriau’s notion of cost

Fix a pmp action $\Gamma \acts (X, \mu)$ and denote by $\mathcal{R}$ the orbit equivalence relation: $x \sim y$ iff $x \in \Gamma \cdot y$.

A graphing of $\mathcal{R}$ is a family $(\varphi_n)$ of partial measure preserving transformations with domain $D(\varphi_n) \subset X$ and range $R(\varphi_n) \subset X$ satisfying:

- $(x, \varphi_n(x)) \in \mathcal{R}$ for all $n$ and almost every $x \in D(\varphi_n)$,
- up to measure zero, $\mathcal{R}$ is the smallest equivalence relation that contains the graphs of all the $\varphi_n$.

The cost of a graphing is defined as $\sum_n \mu(D(\varphi_n))$.

The cost of the equivalence relation $\mathcal{R}$ is defined as the infimum of the costs of all graphings.

**Theorem (Gaboriau, 1999)**

The cost of $\mathcal{R}(\mathbb{F}_n \acts X)$ is $n$. In particular, the free groups $\mathbb{F}_n$, $\mathbb{F}_m$ with $n \neq m$ do not admit orbit equivalent actions.
We call $\Gamma$ a **C-rigid group** if for all free ergodic pmp actions, the $\mathbb{II}_1$ factor $L^\infty(X) \rtimes \Gamma$ has a unique Cartan subalgebra, up to unitary conjugacy.

We have seen (Popa-V, 2011) : the free groups $\mathbb{F}_n, n \geq 2$, are $C$-rigid.

**Theorem (Popa-V, 2012)**

All of the following groups are $C$-rigid.

- Gromov hyperbolic groups.
- Discrete subgroups of rank one simple Lie groups, like $\text{SO}(n, 1)$ or $\text{SU}(n, 1)$ or $\text{Sp}(n, 1)$.
- The limit groups of Sela.
- Direct products of the above groups.

Unique Cartan for **profinite** actions of the same groups : Chifan-Sinclair, 2011.
Which groups are $C$-rigid?

A characterization seems even difficult go guess!

**Conjecture**

If for some $n \geq 1$, we have $\beta_n^{(2)}(\Gamma) > 0$, then $\Gamma$ is $C$-rigid.

- (Popa-V, 2011) If $\Gamma$ is weakly amenable and has $\beta_1^{(2)}(\Gamma) > 0$, then $\Gamma$ is $C$-rigid.

- The conjecture is not giving a characterization: lattices in $\text{SO}(d, 1)$ are $C$-rigid, but have all $\beta_n^{(2)}$ zero if $d$ is odd.

**Ozawa-Popa 2008, Popa-V 2009:**

If $\Gamma = H \rtimes \Lambda$ with $H$ infinite abelian, then $\Gamma$ is not $C$-rigid.

**Question:** give an example of a group $\Gamma$ that has no almost normal infinite amenable subgroups and that is not $C$-rigid either!